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Hall conductivity of flavor fields from AdS/CFT correspondence

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We use the AdS/CFT correspondence to compute a conductivity associated with massive $\mathcal{N} = 2$ supersymmetric hypermultiplet fields at finite baryon density, propagating through an $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills plasma in the large N_c, large 't Hooft coupling limit. We do so by introducing external electric and magnetic fields coupled to baryon number and computing the resulting induced current, from which we extract the conductivity tensor. At large hypermultiplet mass we compute the drag force on the charge carriers. We also compute the product of the drag coefficient with the kinetic mass, and find that the answer is unchanged from the zero-density case. The gravitational dual is a probe D7-brane, with a nontrivial world volume gauge field configuration, in an AdS-Schwarzschild background. We identify an effective horizon on the D7-brane world volume analogous to the world sheet horizon observed for strings moving in the same background. We generalize our results to a class of theories described by probe D-branes in various backgrounds.

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I. INTRODUCTION

The conductivity tensor σ_{ij} measures the response of a conducting medium to externally applied fields. It is defined by

$$\langle J_i \rangle = \sigma_{ij} E_j,$$

where E are externally applied electric fields and $\langle J \rangle$ are the currents induced in the medium. An external magnetic field B produces off-diagonal elements in σ_{ij}: the induced current is perpendicular to both E and B. This is the Hall effect. For a rotationally invariant system with E in the x direction and B perpendicular to the xy plane, $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{xy} = -\sigma_{yx}$. The component σ_{xx} is called the Ohmic conductivity and σ_{xy} the Hall conductivity.

Our goal in this paper is to compute a conductivity associated with massive $\mathcal{N} = 2$ supersymmetric hypermultiplet flavor fields propagating in an $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills (SYM) plasma at temperature T. We work in the limits $N_c \to \infty$ and 't Hooft coupling $\lambda \equiv g_{YM}^2 N_c \gg 1$. We take the number N_f of flavor fields to be $N_f \ll N_c$, so that for massless hypermultiplets the theory is conformal to leading order in N_f/N_c.

The flavor fields have a global $U(N_f)$ symmetry whose $U(1)_B$ subgroup we identify as baryon number. We work at finite $U(1)_B$ density. If we introduce nondynamical E and B fields that couple to $U(1)_B$ charge, then the flavor degrees of freedom will be accelerated. The $\mathcal{N} = 4$ SYM plasma provides resistance, allowing for a steady-state $U(1)_B$ current J^μ. This is the origin of the conductivity we will compute. We extend the result of Ref. [1], where only E was included, to nonzero B and hence nonzero σ_{xy}.

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Our main tool will be the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, which equates the $\mathcal{N} = 4$ SYM theory in the limits described above with supergravity on the ten-dimensional spacetime $AdS_5 \times S^5$ [2–4]. The SYM theory in thermal equilibrium is dual to supergravity on an AdS-Schwarzschild spacetime, where the SYM theory temperature is identified with the Hawking temperature of the AdS-Schwarzschild black hole [5,6]. This conjectured correspondence originated from analysis of the black D3-brane solution in type IIB string theory [2].

The N_f $\mathcal{N} = 2$ hypermultiplet fields appear in the supergravity description as N_f D7-branes [7]. When we introduce only $N_f \ll N_c$ of them, we may neglect their backreaction on the geometry: they are probes. The D7-brane action is then the Dirac-Born-Infeld (DBI) action. The hypermultiplet mass m is dual to the geometry of the D7-brane in a way we will make precise in the sequel. The global $U(1)_B$ symmetry is dual to the $U(1)$ world volume gauge invariance of the D7-branes.

More specifically, if we wish to study the field theory with finite baryon number density $\langle J^t \rangle$, we introduce a nontrivial time component $A_t(z)$ of the D7-brane gauge field, with z the AdS radial coordinate [8]. Following the usual AdS/CFT prescription [3,4], $A_t(z)$'s behavior near the AdS boundary gives the $U(1)_B$ chemical potential, μ_B, and density, $\langle J^t \rangle$, of the SYM theory. In the field theory we also want background electric and magnetic fields $F^{tx} = E$ and $F^{xy} = B$ and induced currents $\langle J^x \rangle$ and $\langle J^y \rangle$. We introduce these in the supergravity theory as nontrivial gauge field components $A_x(z, t) = -Et + f_x(z)$, which produces E and $\langle J^x \rangle$, and $A_y(z, x) = Bx + f_y(z)$, which produces B and $\langle J^y \rangle$.

As shown in Ref. [8], when $A_t(z)$ is nontrivial the only physically allowed D7-brane embeddings are the so-called "black hole" embeddings. These are D7-branes extended in the $AdS_5 \times S^3$ directions and intersecting the AdS-Schwarzschild horizon. They thus possess a world volume horizon themselves. As we want a nontrivial $A_t(z)$, we will work only with black hole D7-brane embeddings. We review D7-brane embeddings in more detail below.

To illuminate salient features of our system we will compare to Refs. [9–12], where the conductivity tensor of a strongly coupled, finite-temperature CFT in $2+1$ dimensions was computed using gauge-gravity duality. An example of such a theory is the $\mathcal{N}=8$ SYM theory in $2+1$ dimensions, with a $U(1)$ subgroup of the $SO(8)$ R-symmetry playing the role of electromagnetism. The gravitational dual of this theory is 11-dimensional supergravity on $\mathrm{AdS}_4 \times S^7$, consistently truncated to Einstein-Maxwell theory on AdS_4. Electric and magnetic fields in the field theory are described in the gravity theory by a dyonic black hole [9]. In Refs. [11,12], the external fields were given harmonic time dependence. We include only static external fields in our setup, so we will compare to the zero-frequency result of Refs. [9–11], which was, in fact, identical to the result for a Lorentz-invariant system [9] obeying linear (Maxwell) electrodynamics.

Our SYM theory differs from that of Refs. [9–12] in two important ways. First, our theory is not a CFT. Our hypermultiplet fields have the mass m. Second, our system effectively has energy and momentum dissipation. The flavor fields contribute an order $N_f N_c$ term to the stress-energy tensor. When $N_f \ll N_c$, this is dwarfed by the order N_c^2 contribution from the $\mathcal{N}=4$ SYM plasma. Our moving charges may thus transfer their energy and momentum into the plasma at a constant rate, without producing any significant motion of the plasma, for at least a time of order N_c. This is why a time-independent, steady-state solution appears in the limit of large N_c with $N_f \ll N_c$.

Additionally, our method differs from that of Refs. [9–12]. We will not use Kubo formulas to compute the conductivity, as in Refs. [9–12]. Kubo formulas are only valid in the regime of linear response. For flavor fields, we can capture some nonlinear effects. This is because, in the supergravity description, we use a DBI action rather than a Yang-Mills action. We calculate the conductivity simply by demanding reality of the on-shell DBI action [1].

If we take a limit in which m is finite but arbitrarily larger than any other scale, for example, the scale $\Delta m = \frac{1}{2}\sqrt{\lambda}T$ of zero-density thermal corrections to m [13], we expect the flavor excitations to behave as quasiparticles. We denote this limit $m \to \infty$. In this limit we can compute the drag force on the charge carriers, and, in particular, we can compute μM, where μ is the drag coefficient and M is the kinetic mass of the quasiparticles, distinct from the Lagrangian mass m at finite temperature and density.[1] μM was computed in the $m \to \infty$ limit for $\langle J^t \rangle = 0$ in Refs. [13–15] and at finite $\langle J^t \rangle$ in Ref. [1]. The result in both cases was $\mu M = \frac{\pi}{2}\sqrt{\lambda}T^2$. We will find the same answer at finite B. We will argue that the result is independent of the density and external fields simply because we are working to leading order in large N_c. We will draw an instructive comparison, however, between our calculation and the calculation of μM from a single moving string [15,16]. In particular, we identify an effective horizon on the D7-brane world volume analogous to the world sheet horizon on a single string [17,18].

Everything we will do comes with a caveat: the phase diagram in the parameter space of T, $\langle J^t \rangle$, E, and B (in units of m) is not fully known. At $E = B = 0$, a region of instability is known to exist in the plane of $\langle J^t \rangle$ versus T, and for sufficiently large chemical potential the hypermultiplet scalars may undergo Bose-Einstein condensation [8]. In such regions of parameter space our D7-brane solutions do not represent the ground state of the theory and must be discarded. Our results are valid only when D7-brane black hole embeddings are the appropriate supergravity description.[2]

As in Ref. [1], we may also generalize our results to theories whose gravitational duals are probe Dq-branes in backgrounds of Dp-branes. This is possible when the Dq-brane has a world volume horizon and the Dq-brane's dynamics is described by the DBI term alone. Wess-Zumino couplings will, in general, introduce new terms into the equation of motion for the Dq-brane world volume gauge field that may render our solution inapplicable.

This paper is organized as follows. In Sec. II we briefly review some results from classical electromagnetism. In Sec. III we solve for the probe D7-brane gauge field. In Sec. IV we compute the conductivity. In Sec. V we compute μM in the $m \to \infty$ limit. In Sec. VI we generalize our results to Dq-brane probes in Dp-brane backgrounds. We conclude in Sec. VII. In the appendix we use holographic renormalization to compute $\langle J^\mu \rangle$.

II. PRELIMINARIES

We first review two results from classical electromagnetism that we will reproduce from our supergravity calculation in appropriate limits.

Imagine filling the vacuum with a charge density $\langle J^t \rangle$. In the lab frame we may introduce a magnetic field $\vec{B}$. In a frame moving with velocity $-\vec{v}$ relative to the lab frame we will find a current $\vec{J} = \langle J^t \rangle \vec{v}$ and an electric field

$$\vec{E} = -\vec{v} \times \vec{B} = -\frac{1}{\langle J^t \rangle} \vec{J} \times \vec{B}. \tag{2.1}$$

If we take $\vec{B} = (0, 0, B)$ we find the conductivity

[1] At zero density, for $m \gg \Delta m$, we know $M = m(1 - \frac{\Delta m}{m} + O(\frac{\Delta m}{m})^2)$ [13]. In our $m \to \infty$ limit, M and m are therefore indistinguishable. We will continue to use the symbol M, however, to remind ourselves of the distinction outside of this limit.

[2] At finite B with $T = E = \langle J^t \rangle = 0$ the field theory exhibits spontaneous breaking of a chiral symmetry even at $m = 0$ and a Zeeman-like splitting in the meson spectrum [19,20].

$$\sigma_{xx} = 0, \qquad \sigma_{xy} = \langle J^t \rangle / B. \tag{2.2}$$

Notice that this argument does not require that the charge density be comprised of quasiparticle charge carriers. Indeed, this argument relies only on Lorentz invariance. This was the result found in Refs. [9–11] for a (2 + 1)-dimensional CFT at finite temperature.

Now imagine a density $\langle J^t \rangle$ of massive quasiparticles propagating nonrelativistically through an isotropic, homogeneous, neutral medium. In the rest frame of the medium we introduce an electric field E in the $\hat{x}$ direction in addition to the magnetic field. The force on a quasiparticle is then

$$\frac{d\vec{p}}{dt} = \vec{E} + \vec{v} \times \vec{B} - \mu \vec{p}, \tag{2.3}$$

where our quasiparticle has charge +1 and μ is the drag coefficient. We replace the momentum with the velocity using $\vec{p} = M\vec{v}$ for quasiparticle mass M. We then replace the velocity with the induced current using $\vec{v} = \langle \vec{J} \rangle / \langle J^t \rangle$. Imposing the steady-state condition $\frac{d\vec{p}}{dt} = 0$ and solving for $\langle \vec{J} \rangle$ yields

$$\sigma_{xx} = \frac{\sigma_0}{(B/\mu M)^2 + 1}, \qquad \sigma_{xy} = \frac{\sigma_0 (B/\mu M)}{(B/\mu M)^2 + 1}, \tag{2.4}$$

where $\sigma_0 = \langle J^t \rangle / \mu M$ is the conductivity when $B = 0$.

III. THE PROBE D7-BRANE SOLUTION

In type IIB string theory, we consider a system of N_c nonextremal D3-branes and N_f D7-branes aligned in flat ten-dimensional space as

	X_0	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
D3	×	×	×	×						
D7	×	×	×	×	×	×	×	×		

(3.1)

The X_8 and X_9 directions are orthogonal to both stacks of D-branes, which thus appear as points in the X_8-X_9 plane. If we separate these points, an open string may stretch between the two stacks. The mass of this string is its length times its tension. This mass appears in the SYM theory on the D3-brane world volume as the hypermultiplet mass m.

We take the usual AdS/CFT limit, $N_c \to \infty$, $g_s \to 0$ with $g_s N_c$ fixed and $g_s N_c \gg 1$ [2]. We obtain the near-horizon geometry of nonextremal D3-branes, five-dimensional AdS-Schwarzschild times S^5. We use an AdS-Schwarzschild metric, in units where the AdS radius is one,

$$ds^2 = \frac{dz^2}{z^2} - \frac{1}{z^2}\frac{(1 - z^4/z_H^4)^2}{1 + z^4/z_H^4} dt^2 + \frac{1}{z^2}(1 + z^4/z_H^4) d\vec{x}^2, \tag{3.2}$$

where z is the radial coordinate, t the time coordinate, and $d\vec{x}^2$ is the metric of three-dimensional Euclidean space. The boundary is at $z = 0$ and the black hole horizon is at $z = z_H$ with $z_H^{-1} = \frac{\pi}{\sqrt{2}} T$. Our S^5 metric is

$$d\Omega_5^2 = d\theta^2 + \sin^2\theta d\psi^2 + \cos^2\theta d\Omega_3^2, \tag{3.3}$$

where $d\Omega_3^2$ is the standard S^3 metric and θ runs from zero to $\pi/2$. We have chosen coordinates such that $X_8 = \frac{1}{z}\sin\theta$. In our units, string theory and SYM quantities are related by $\alpha'^{-2} = 4\pi g_s N_c = g_{YM}^2 N_c \equiv \lambda$.

In the near-horizon geometry the D7-branes extend along $AdS_5 \times S^3$ [7]. Nonzero separation in the X_8-X_9 plane appears in the near-horizon geometry as a D7-brane with nontrivial embedding. Specifically, the position of the world volume S^3 on the S^5 will be described by an embedding function $\theta(z)$ [7]. $\theta(z)$ is dual holographically to the hypermultiplet mass operator[3] [7]. $\theta(z)$'s leading asymptotic value, denoted θ_0 in the appendix, is simply the separation between the D3-branes and the D7-branes, hence $m = \frac{\theta_0}{2\pi\alpha'}$.

$\theta(z)$ is determined by an equation of motion derived from the D7-brane action and a boundary condition, the value of m. At one extreme is $m = 0$, which produces $\theta(z) = 0$, the trivial solution to the equation of motion. In this case the D7-brane wraps the maximum-volume equatorial $S^3 \subset S^5$ for all z. At zero temperature, nonzero m produces the so-called Minkowski embeddings, in which the world volume S^3 shrinks as we move away from $z = 0$ and eventually collapses to zero volume: $\theta(z') = \frac{\pi}{2}$ and $\cos\theta(z') = 0$ at some z'. The D7-brane then does not extend past z' in the radial direction, rather, it appears to end abruptly at z' [7]. At the other extreme is $m = \infty$, which produces $\theta(z) = \frac{\pi}{2}$ for all z. This effectively eliminates the D7-brane, which ends right at the boundary.

In the AdS-Schwarzschild background, with no gauge field excited on the D7-brane world volume, two classes of embedding are possible. The first are Minkowski embeddings that end outside the horizon, $z' < z_H$. These do not possess a horizon on their world volume. The second class of embeddings are black hole embeddings, in which the S^3 never collapses to zero volume and the D7-brane intersects the AdS-Schwarzschild horizon. The D7-brane thus possesses a horizon on its world volume. These embeddings are depicted in Fig. 1.

If we introduce a world volume gauge field $A_t(z)$, the resulting radial electric field lines must have some place to end. For a Minkowski embedding no such place exists. We may introduce point sources, strings stretching from the D7-brane to the horizon, to accommodate the radial field lines. As shown in Ref. [8], however, the force that the strings exert on the D7-brane will overcome the tension of the D7-brane, so the D7-brane will be drawn into the

[3] $\theta(z)$ is dual to the operator given by taking $\frac{\partial}{\partial m}$ of the SYM theory Lagrangian. This operator includes the mass operator as well as couplings to adjoint scalars. The exact operator is written in Ref. [8]. Thinking in terms of the mass operator will be sufficient for our purposes.

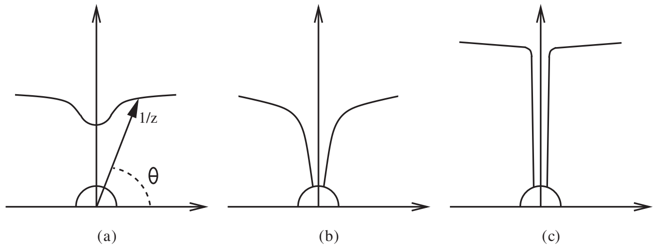

FIG. 1. Cartoons of D7-brane embeddings, with the coordinates z and θ indicated. We can imagine that the D3-branes sit at the origin. The semicircle about the origin represents the horizon at $z = z_H$. The boundary is $z = 0$. θ runs from 0 to $\frac{\pi}{2}$. The horizontal axis is a direction transverse to the D3-branes but parallel to the D7-branes, i.e. one of X_4, X_5 X_6, or X_7. The vertical axis is X_8. (a) A Minkowski embedding. (b) A black hole embedding. (c) A black hole embedding with a spike in the $m \to \infty$ limit.

horizon, producing a black hole embedding. We will therefore work only with black hole embeddings, for which the field lines may end on the horizon.

With nonzero $A_t(z)$, in the SYM theory limit $m \to \infty$, the D7-brane black hole embedding resembles a "spike": the S^3 *almost* collapses to zero volume at some value $z \equiv z_{\text{spike}}$, but then remains at constant finite volume all the way to the horizon.[4] In fact, the action of the spike is identical to the action of a bundle of strings [8]. This makes sense intuitively: a finite baryon density in the SYM theory should appear in the supergravity description as very many strings. What is perhaps surprising is that the D7-brane alone, with no strings introduced explicitly, manifests these strings itself via the spike.

As in Ref. [1], we will not solve for $\theta(z)$ but we will consider limits. The $m = 0$ limit is $\theta(z) = 0$. For $m \to \infty$ we may approximate $\theta(z) \approx \pi/2$ or $\cos\theta(z) \approx 0$ when $z > z_{\text{spike}}$. In particular, we will use this for z near the horizon.

We will now solve for the D7-brane world volume gauge fields. The D7-brane action is

$$\begin{aligned} S_{D7} &= -N_f T_{D7} \int L \\ &= -N_f T_{D7} \int d^8\zeta \sqrt{-\det(g_{ab} + (2\pi\alpha')F_{ab})} \end{aligned} \tag{3.4}$$

plus Wess-Zumino terms that will be zero in what we do. T_{D7} is the D7-brane tension, ζ are world volume coordinates, g_{ab} is the induced metric, and F_{ab} is the $U(1)$ field strength. In our conventions, a string end point couples to this gauge field with coupling $+1$. In the SYM theory we also want fields E and B, a charge density $\langle J^t\rangle$, and induced currents $\langle J^x\rangle$ and $\langle J^y\rangle$. We thus introduce world volume gauge field components $A_t(z)$ and

$$A_x(z, t) = -Et + f_x(z), \qquad A_y(z, x) = Bx + f_y(z) \tag{3.5}$$

so that at the boundary we have electric and magnetic fields $F^{tx} = E$ and $F^{xy} = B$. As part of our gauge choice we take $A_z = 0$. As our gauge fields only depend on (z, t, x, y), the D7-brane action is simply a $(3+1)$-dimensional Born-Infeld action, with some "extra" factors in front from the S^3 and the extra spatial direction,

$$\begin{aligned} S_{D7} = &-\mathcal{N} \int d^4x \cos^3\theta g_{xx}^{1/2} \\ &\times \sqrt{-g - (2\pi\alpha')^2 \frac{1}{2} g F^2 - (2\pi\alpha')^4 \frac{1}{4}(F \wedge F)^2}. \end{aligned} \tag{3.6}$$

The overall prefactor is, using $T_{D7} = \frac{\alpha'^{-4} g_s^{-1}}{(2\pi)^7} = \frac{\lambda N_c}{2^5\pi^6}$,

$$\mathcal{N} \equiv N_f T_{D7} 2\pi^2 = \frac{\lambda}{(2\pi)^4} N_f N_c \tag{3.7}$$

with $2\pi^2$ the volume of a unit S^3. We have divided both sides of Eq. (3.6) by the volume of $\mathbb{R}$, defined $d^4x = dzdtdxdy$, and defined $g = g_{zz}g_{tt}g_{xx}^2$ as the determinant of the induced metric in the (z, t, x, y) subspace, with $g_{zz} = 1/z^2 + \theta'(z)^2$. Writing $F^2 = F^{\mu\nu}F_{\mu\nu}$, where Greek indices run over (z, t, x, y), and $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ for totally antisymmetric $\epsilon^{\mu\nu\alpha\beta}$ with $\epsilon^{ztxy} = +1$, we have explicitly

$$\begin{aligned} \tfrac{1}{2} g F^2 = &\, g_{xx}^2 A_t'^2 + g_{tt}g_{xx}A_x'^2 + g_{tt}g_{xx}A_y'^2 \\ &+ g_{zz}g_{xx}\dot{A}_x^2 + g_{zz}g_{tt}\bar{A}_y^2, \end{aligned} \tag{3.8a}$$

$$\begin{aligned} \tfrac{1}{4}(F \wedge F)^2 = (\tfrac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu})^2 = &\, \bar{A}_y^2 A_t'^2 + \dot{A}_x^2 A_y'^2 \\ &+ 2\bar{A}_y A_t' \dot{A}_x A_y', \end{aligned} \tag{3.8b}$$

where dots, $\dot{A}$, denote derivatives with respect to t, primes, A', denote derivatives with respect to z, and bars, $\bar{A}$, denote derivatives with respect to x.

The action only depends on the derivatives of $A_t(z)$, $f_x(z)$, and $f_y(z)$, so we will have three conserved charges. In the appendix we identify these as $\langle J^t\rangle$, $\langle J^x\rangle$, and $\langle J^y\rangle$,

[4]For $E = B = 0$, the position where corrections to the constant-volume solution are non-negligible is, defining $\theta(z) = \frac{\pi}{2} - \varepsilon$ with $\varepsilon \ll 1$ and using SYM quantities, $z_{\text{spike}}/z_H \sim \varepsilon(\frac{\langle J^t\rangle}{\sqrt{\lambda}N_f N_c T^3})^{-1/3}$ [8].

$$\mathcal{N}(2\pi\alpha')^2 g_{xx}^{1/2}\cos^3\theta \frac{-g_{xx}^2 A_t' - (2\pi\alpha')^2(\bar{A}_y^2 A_t' + \bar{A}_y \dot{A}_x A_y')}{\sqrt{-g - (2\pi\alpha')^2 \frac{1}{2} g F^2 - (2\pi\alpha')^4 \frac{1}{4}(F\wedge F)^2}} = \langle J^t \rangle, \tag{3.9a}$$

$$\mathcal{N}(2\pi\alpha')^2 g_{xx}^{1/2}\cos^3\theta \frac{|g_{tt}| g_{xx} A_x'}{\sqrt{-g - (2\pi\alpha')^2 \frac{1}{2} g F^2 - (2\pi\alpha')^4 \frac{1}{4}(F\wedge F)^2}} = \langle J^x \rangle, \tag{3.9b}$$

$$\mathcal{N}(2\pi\alpha')^2 g_{xx}^{1/2}\cos^3\theta \frac{|g_{tt}| g_{xx} A_y' - (2\pi\alpha')^2(\dot{A}_x^2 A_y' + \bar{A}_y \dot{A}_x A_t')}{\sqrt{-g - (2\pi\alpha')^2 \frac{1}{2} g F^2 - (2\pi\alpha')^4 \frac{1}{4}(F\wedge F)^2}} = \langle J^y \rangle. \tag{3.9c}$$

Notice that the density and currents are order $\mathcal{N}(2\pi\alpha')^2 \propto N_f N_c$.

With a little algebra we solve for the gauge fields from Eq. (3.9),

$$A_t'(z) = -\frac{\sqrt{g_{zz}|g_{tt}|}}{g_{xx}} \frac{\langle J^t\rangle \xi - Ba}{\sqrt{\xi\chi - a^2}}, \tag{3.10}$$

where we have introduced the coefficients

$$\begin{aligned}\xi &= |g_{tt}| g_{xx}^2 - (2\pi\alpha')^2 \tilde{F}^{z\mu}\tilde{F}^z_\mu \\ &= |g_{tt}| g_{xx}^2 + (2\pi\alpha')^2(|g_{tt}|B^2 - g_{xx}E^2), \end{aligned}\tag{3.11a}$$

$$\begin{aligned}\chi &= |g_{tt}| g_{xx}^2[\mathcal{N}^2(2\pi\alpha')^4 g_{xx}\cos^6\theta] - (2\pi\alpha')^2\langle J_\mu\rangle\langle J^\mu\rangle \\ &= |g_{tt}| g_{xx}^2[\mathcal{N}^2(2\pi\alpha')^4 g_{xx}\cos^6\theta] \\ &\quad + (2\pi\alpha')^2(|g_{tt}|\langle J^t\rangle^2 - g_{xx}(\langle J^x\rangle^2 + \langle J^y\rangle^2)), \end{aligned}\tag{3.11b}$$

$$a = -(2\pi\alpha')^2 \tilde{F}^{z\mu}\langle J_\mu\rangle = (2\pi\alpha')^2(|g_{tt}|\langle J^t\rangle B + g_{xx}\langle J^y\rangle E). \tag{3.11c}$$

Notice that ξ is simply $-det(g_{ab} + (2\pi\alpha')F_{ab})$ in the (t, x, y) subspace, and that $\cos\theta(z)$ appears only in χ. We have written χ in a way that will make generalizing to Dp/Dq systems in Sec. VI more transparent. We also have

$$\begin{aligned}A_x'(z) &= \sqrt{\frac{g_{zz}}{|g_{tt}|}} \frac{\langle J^x\rangle \xi}{\sqrt{\xi\chi - a^2}}, \\ A_y'(z) &= \sqrt{\frac{g_{zz}}{|g_{tt}|}} \frac{\langle J^y\rangle \xi + Ea}{\sqrt{\xi\chi - a^2}}. \end{aligned}\tag{3.12}$$

In the original action we may now replace the gauge fields with the conserved charges. The resulting effective action has only the single dynamical field $\theta(z)$,

$$S_{D7} = -\mathcal{N}^2(2\pi\alpha')^2 \int d^4x \cos^6\theta g_{xx}^2 \sqrt{g_{zz}|g_{tt}|} \frac{\xi}{\sqrt{\xi\chi - a^2}}. \tag{3.13}$$

We may obtain the equation of motion for $\theta(z)$ in two ways. We may derive it from the original action equation (3.6) and then plug in our gauge field solutions Eqs. (3.10) and (3.12), or we may Legendre transform to eliminate the gauge fields at the level of the action. The Legendre-transformed action $\hat{S}_{D7}$ is

$$\begin{aligned}\hat{S}_{D7} &= S_{D7} - \int d^4x \left(F_{zt}\frac{\delta S_{D7}}{\delta F_{zt}} + F_{zx}\frac{\delta S_{D7}}{\delta F_{zx}} + F_{zy}\frac{\delta S_{D7}}{\delta F_{zy}}\right) \\ &= -\frac{1}{(2\pi\alpha')^2}\int d^4x g_{zz}^{1/2}|g_{tt}|^{-1/2} g_{xx}^{-1}\sqrt{\xi\chi - a^2} \end{aligned}\tag{3.14}$$

where $\frac{\delta\hat{S}_{D7}}{\delta\langle J^t\rangle} = A_t'(z)$, $\frac{\delta\hat{S}_{D7}}{\delta\langle J^x\rangle} = A_x'(z)$ and $\frac{\delta\hat{S}_{D7}}{\delta\langle J^y\rangle} = A_y'(z)$ reproduce Eqs. (3.10) and (3.12).

Specifying the boundary conditions will then determine the D7-brane solution completely. First notice that, at the horizon, the gauge field must obey $A_t(z_H) = 0$ to be well defined as a one-form. We are free to choose the leading asymptotic values of the fields near the boundary $z \to 0$. We first choose the asymptotic value θ_0 of $\theta(z)$. The gauge fields asymptotically approach the boundary as

$$A_t(z) = \mu_B - \frac{1}{2}\frac{\langle J^t\rangle}{\mathcal{N}(2\pi\alpha')^2} z^2 + O(z^4), \tag{3.15a}$$

$$A_x(z) = -Et + c_x + \frac{1}{2}\frac{\langle J^x\rangle}{\mathcal{N}(2\pi\alpha')^2} z^2 + O(z^4), \tag{3.15b}$$

$$A_y(z) = Bx + c_y + \frac{1}{2}\frac{\langle J^y\rangle}{\mathcal{N}(2\pi\alpha')^2} z^2 + O(z^4), \tag{3.15c}$$

where μ_B, c_x, and c_y are constants of integration. The leading asymptotic value μ_B is the $U(1)_B$ chemical potential. For A_x and A_y we impose the boundary condition $c_x = c_y = 0$.

IV. THE CONDUCTIVITY

We focus now on the quantity $\sqrt{\xi\chi - a^2}$ appearing in the effective action equation (3.13). As in Ref. [1], we will find that demanding reality of the effective action allows us to solve for $\langle J^x\rangle$ and $\langle J^y\rangle$, and hence the conductivity, in terms of E, B, and $\langle J^t\rangle$.

In Eq. (3.11a) we see that, as a function of z, ξ has a zero: $\xi < 0$ at the horizon where $|g_{tt}| = 0$, whereas $\xi > 0$ near the boundary $z \to 0$. We denote the zero of ξ as z_*,

$$\frac{z_*^4}{z_H^4} = e^2 - b^2 + \sqrt{(e^2-b^2)^2 + 2(e^2+b^2) + 1} - \sqrt{((e^2-b^2) + \sqrt{(e^2-b^2)^2 + 2(e^2+b^2)+1})^2 - 1}, \tag{4.1}$$

where we have defined the dimensionless quantities

$$e = \frac{1}{2}(2\pi\alpha')Ez_H^2 = \frac{E}{\frac{\pi}{2}\sqrt{\lambda}T^2}, \quad b = \frac{1}{2}(2\pi\alpha')Bz_H^2 = \frac{B}{\frac{\pi}{2}\sqrt{\lambda}T^2} \tag{4.2}$$

and converted to field theory quantities. Knowing that ξ is the (t, x, y) part of $-det(g_{ab} + (2\pi\alpha')F_{ab})$, we will interpret z_* as an effective horizon on the D7-brane world volume. Notice that $z_* = z_H$ when $E = 0$. We will also need $g_{xx}^2(z_*) = \pi^4 T^4 \mathcal{F}(e, b)$ where

$$\mathcal{F}(e,b) = \tfrac{1}{2}(1 + e^2 - b^2 + \sqrt{(e^2-b^2)^2 + 2(e^2+b^2)+1}). \tag{4.3}$$

For later use notice that $\mathcal{F}(e, 0) = e^2 + 1$ and $\mathcal{F}(0, b) = 1$.

In fact all three functions, ξ, χ, and a must share the same zero z_*. From Eq. (3.11b) we see that at the horizon $\chi < 0$ while at the boundary $\chi > 0$, so χ also has a zero. In particular $\xi\chi > 0$ at the horizon and at the boundary. If ξ and χ have distinct zeroes, then in the region between those zeroes one would change sign while the other would not, hence in that region $\xi\chi < 0$ and the effective action would be imaginary. The only consistent possibility is for ξ and χ to share the zero at z_*. We must also have $a^2 < \xi\chi \rightarrow 0$ as $z \rightarrow z_*$, so that $a \rightarrow 0$ at z_* as well.

We thus set all of Eqs. (3.11) to zero at z_* and solve for $\langle J^x \rangle$ and $\langle J^y \rangle$,

$$\langle J^x \rangle = \frac{Eg_{xx}}{g_{xx}^2 + (2\pi\alpha')^2B^2}\sqrt{(g_{xx}^2 + (2\pi\alpha')^2B^2)\mathcal{N}^2(2\pi\alpha')^4 g_{xx}\cos^6\theta(z_*) + (2\pi\alpha')^2\langle J^t\rangle^2}, \tag{4.4a}$$

$$\langle J^y \rangle = -\frac{(2\pi\alpha')^2\langle J^t\rangle B}{g_{xx}^2 + (2\pi\alpha')^2B^2}E \tag{4.4b}$$

with all functions of z evaluated at z_*. Converting to field theory quantities, we find

$$\sigma_{xx} = \sqrt{\frac{N_f^2N_c^2T^2}{16\pi^2}\frac{\mathcal{F}^{3/2}}{b^2+\mathcal{F}}\cos^6\theta(z_*) + \frac{\rho^2\mathcal{F}}{(b^2+\mathcal{F})^2}}, \tag{4.5a}$$

$$\sigma_{xy} = \frac{\rho b}{b^2+\mathcal{F}}, \tag{4.5b}$$

where we have defined ρ similarly to e and b,

$$\rho = \frac{\langle J^t\rangle}{\frac{\pi}{2}\sqrt{\lambda}T^2}, \tag{4.6}$$

but while e and b are dimensionless, ρ has dimension one.

As in Ref. [1], we interpret our result as follows. Two types of charge carriers contribute to the conductivity. The first are the charge carriers we have introduced explicitly in ρ. Taking $\rho = 0$ leaves a nonzero σ_{xx}, however, so we must have another source of charge carriers. We will guess that these come from pair production in the plasma. Such pair production should depend on m via a Boltzmann factor $e^{-m/T}$. The mass m, or equivalently θ_0, appears implicitly in Eq. (4.5) in $\cos\theta(z_*)$, which should thus behave as $e^{-m/T}$. Notice $\cos\theta(z_*)$ has the correct limiting behavior: $\cos\theta(z_*) \rightarrow 0$ as $m \rightarrow \infty$, and $\cos\theta(z_*) = 1$ for $m = 0$. We are currently investigating whether $\cos\theta(z_*)$ produces the Boltzmann factor [21].

We will check our answer in three limits. The first is simply to take $b \rightarrow 0$ where $\mathcal{F}(e, 0) = e^2 + 1$ and we immediately recover the result of Ref. [1].

To recover Eq. (2.2), we linearize in the electric field. In practical terms this means setting $e = 0$, and hence $\mathcal{F}(0, b) = 1$, in Eq. (4.5). We also restore Lorentz invariance by taking $T \rightarrow 0$. We find $\sigma_{xx} = 0$ and $\sigma_{xy} = \langle J^t\rangle/B$, as expected.

To recover Eq. (2.4), we return to finite T and again linearize in the electric field. We additionally take the $m \rightarrow \infty$ limit $\cos\theta(z_*) \approx 0$. The conductivity becomes

$$\sigma_{xx} = \frac{\rho}{b^2+1}, \qquad \sigma_{xy} = \frac{\rho b}{b^2+1}. \tag{4.7}$$

As shown in Sec. V, in the $m \rightarrow \infty$ limit we identify $\frac{\pi}{2} \times \sqrt{\lambda}T^2 = \mu M$. We thus have $\rho = \frac{\langle J^t\rangle}{\mu M}$ and $b = \frac{B}{\mu M}$, and the conductivity indeed has the form expected for quasiparticles propagating through an isotropic, homogeneous medium, Eq. (2.4).

V. THE DRAG FORCE

In the $m \rightarrow \infty$ limit where $\cos\theta \approx 0$, we expect the flavor excitations to be well described as a collection of quasiparticles, with equation of motion

$$\frac{d\vec{p}}{dt} = \vec{E} + \vec{v} \times \vec{B} - \mu\vec{p}, \tag{5.1}$$

with v the quasiparticle velocity and μ the drag coefficient. Our first goal is to compute the magnitude of the drag force, $\mu|\vec{p}|$. In the steady state, $\frac{dp}{dt} = 0$. We then have

$$\mu|\vec{p}| = \sqrt{E^2 + v^2B^2 + 2\vec{E}\cdot(\vec{v}\times\vec{B})}. \tag{5.2}$$

As $m \to \infty$, we expect pair creation to be suppressed, so only the charge carriers in $\langle J^t\rangle$ should contribute to $\langle\vec{J}\rangle$, hence $\langle\vec{J}\rangle = \langle J^t\rangle\vec{v}$. We immediately read off $v^2 = |g_{tt}|/g_{xx}$ by setting χ to zero at z_* and dropping the $\cos\theta(z_*)$ term. Setting $\xi = 0$ at z_* gives us

$$\begin{aligned} E^2 &= \frac{1}{(2\pi\alpha')^2}|g_{tt}|g_{xx} + \frac{|g_{tt}|}{g_{xx}}B^2 \\ &= \frac{1}{(2\pi\alpha')^2}g_{xx}^2v^2 + v^2B^2. \end{aligned} \tag{5.3}$$

Setting $a = 0$ at z_* gives us the component of $\vec{v}$ in the $\hat{y}$ direction,

$$v_y = \frac{\langle J^y\rangle}{\langle J^t\rangle} = -\frac{|g_{tt}|}{g_{xx}}\frac{B}{E} = -v^2\frac{B}{E}. \tag{5.4}$$

We then have $2\vec{E}\cdot(\vec{v}\times\vec{B}) = 2EBv_y = -2B^2v^2$. The drag force is then

$$\mu|\vec{p}| = \frac{1}{2\pi\alpha'}g_{xx}(z_*)v. \tag{5.5}$$

We can now compute μM. To compare to Refs. [13–15], we employ the relativistic relation $|\vec{p}| = \gamma M v$ with $\gamma = \frac{1}{\sqrt{1-v^2}}$, and find

$$\mu M = \frac{1}{2\pi\alpha'}\sqrt{g_{xx}(z_*)^2 - |g_{tt}(z_*)|g_{xx}(z_*)}, \tag{5.6}$$

which evaluates to $\frac{1}{\pi\alpha'}z_H^{-2} = \frac{\pi}{2}\sqrt{\lambda}T^2$. This is identical to the zero-density result of Refs. [13,14] and finite density result of Ref. [1], but now with nonzero B.

The $\langle J^t\rangle$ independence is easy to understand.[5] The plasma contains order N_c^2 adjoint degrees of freedom and order N_fN_c flavor degrees of freedom. The flavor excitations are thus dilute in the large-N_c limit. In a perturbative analysis, the flavor excitations will be more likely to scatter off of adjoint degrees of freedom than other flavor excitations. Scatterings with adjoint degrees of freedom will thus be the flavor excitations' primary mechanism for the microscopic energy loss that results in the macroscopic drag force. Introducing a density $\langle J^t\rangle$ of order N_fN_c will not change this to leading order in large N_c. Increasing the strength of the coupling muddies the picture of isolated scatterings but does not affect the argument, which relies only on large-N_c counting. Taking $m \to \infty$, and in particular $m \gg \mu_B$, serves only to dilute the charge carriers further. We therefore expect to recover the zero-density result at leading order in the $N_f \ll N_c$ limit.

The B independence follows from this, simply because the zero-density result $\frac{\pi}{2}\sqrt{\lambda}T^2$ was already, curiously, independent of the quasiparticle momentum, or equivalently of m and v [13]. As v is determined by E and B, and is the only place where E and B could appear in the answer, we expect the answer to be independent of E and B.

The result for the drag force, Eq. (5.5), is identical in form to the drag force computed at zero density via single-string calculations. Let us summarize the story that emerges from these single-string calculations [15–18]. Consider a Minkowski-embedded D7-brane that ends far from the horizon. Attach the end point of a string to this D7-brane. An electric field E will cause this end point to move with velocity v. The body of the string will dangle into the bulk of AdS, trailing behind the end point (see Fig. 2). The string will be long and heavy, and thus behave as a classical object. Such a configuration is the single-string manifestation of our $m \to \infty$ limit. In the SYM theory we interpret the end point as a single moving "quark," i.e. flavor excitation.

This "trailing string" in fact has a horizon on its world sheet: a point along its length at which the time component of the induced world volume metric vanishes [16–18]. Let z_{WH} denote this world sheet horizon. z_{WH} is fixed by v. As $v \to 1$ the horizon moves up the string, towards the boundary, while as $v \to 0$ the horizon moves down the string, towards the AdS-Schwarzschild horizon. At $v = 0$, the string stretches straight from the D7-brane to the horizon. The world sheet horizon then coincides with the AdS-Schwarzschild horizon.

The drag force computed from such trailing strings is given by Eq. (5.5), with z_* replaced by z_{WH}. Our effective horizon z_* thus appears to be the generalization of the world sheet horizon to the D7-brane. This makes sense intuitively when $m \to \infty$ because the dynamics of the D7-brane spike is identical to that of a bundle of strings [8].

In fact, for a single string, Eq. (5.5) with z_* replaced by z_{WH} is the result for any asymptotically AdS geometry with a horizon [15]. In this sense Eq. (5.5) is "universal," when written in terms of the supergravity quantity g_{xx}. The conversion to SYM quantities will not always reproduce $\frac{\pi}{2}\sqrt{\lambda}T^2$, however. For example, the charged AdS-Schwarzschild black hole background, dual to $\mathcal{N} = 4$

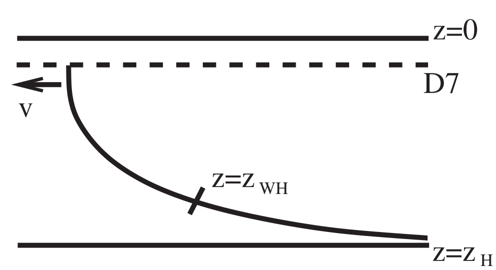

FIG. 2. Cartoon of the trailing string. The AdS boundary $z = 0$ is at the top. The AdS-Schwarzschild horizon $z = z_H$ is at the bottom. The dashed line is the position where the D7-brane ends. The world sheet horizon on the string, z_{WH}, is indicated.

[5] We thank L. Yaffe for the following argument.

SYM with nonzero R-charge density [22,23], will produce a μM that depends on R-charge chemical potentials. That Eq. (5.5) could be "universal" also for the D7-brane seems plausible, but to show this would require a more general analysis. Notice, however, that the argument showing that ξ, χ, and a share a zero at z_* required only that the relevant part of the D7-brane metric be asymptotically AdS and possess a horizon.

VI. GENERALIZATION TO DP/DQ SYSTEMS

As in Ref. [1], we can compute the conductivity for a class of field theories whose holographic duals are probe Dq-branes in a background of Dp-branes [24,25]. This is possible because we required only that the DBI action be a reliable effective action and that the Dq-brane had a horizon. The dual field theories will be large-N_c Yang-Mills theories with $N_f \ll N_c$ fundamental-representation fields that, in some cases, may be confined to a defect.

The Dp-brane solution includes coordinates parallel to the Dp-branes and spherical coordinates for directions transverse to the Dp-branes. In this background we may generically write the induced Dq-brane metric as

$$ds^2_{Dq} = g_{zz}dz^2 + g_{tt}dt^2 + g_{xx}d\vec{x}^2 + g_{SS}d\Omega_n^2, \tag{6.1}$$

where z is the radial coordinate. We assume this induced metric depends only on z and parameters like T. The Dq-brane wraps some n-sphere S^n with metric component g_{SS} in the space transverse to the Dp-branes. The Dq-brane world volume then includes $\mathbb{R}^d$ with $d = q - n - 1$. A magnetic field is only possible for $d \geq 2$. We assume the Dq-brane world volume has a horizon z_H defined by $g_{tt}(z_H) = 0$. The Dp-brane background may also include a nontrivial dilaton $\phi(z)$ and nontrivial Ramond-Ramond (RR) form fields.

We now introduce $A_t(z)$, $A_x(z, t)$, and $A_y(z, x)$. The Dq-brane action includes the Born-Infeld term and Wess-Zumino couplings to background RR fields. The Born-Infeld term is again a (3 + 1)-dimensional Born-Infeld action with an extra factor,

$$S_{Dq} = -\int d^4x \frac{c(z)}{(2\pi\alpha')^2} \times \sqrt{-g - (2\pi\alpha')^2\frac{1}{2}gF^2 - (2\pi\alpha')^4\frac{1}{4}(F\wedge F)^2}, \tag{6.2}$$

where we have divided both sides by the volume of $\mathbb{R}^{d-2}$, and now the extra factor is

$$c(z) = \mathcal{N}_q(2\pi\alpha')^2e^{-\phi(z)}g_{xx}^{(d/2)-1}(z)g_{SS}^{n/2}(z), \tag{6.3}$$

where $\mathcal{N}_q \equiv N_f T_{Dq} V_n$, with T_{Dq} the Dq-brane tension and V_n the volume of a unit S^n. Comparing Eqs. (3.6) and (6.2) we see that everything is identical to what we have already done, but with

$$\mathcal{N}(2\pi\alpha')^2 g_{xx}^{1/2}\cos^3\theta \to c(z). \tag{6.4}$$

In particular, the only change in Eq. (3.11) is in χ,

$$\chi = |g_{tt}|g_{xx}^2c(z)^2 + (2\pi\alpha')^2(|g_{tt}|\langle J^t\rangle^2 - g_{xx}(\langle J^x\rangle^2 + \langle J^y\rangle^2)). \tag{6.5}$$

In the appendix we show that the identification of $\langle J^t\rangle$, $\langle J^x\rangle$, and $\langle J^y\rangle$ is valid for any probe Dq-brane satisfying our assumptions, so taking $\xi = \chi = a = 0$ at z_* we find

$$\sigma_{xx} = \frac{g_{xx}}{g_{xx}^2 + (2\pi\alpha')^2B^2} \times\sqrt{(g_{xx}^2 + (2\pi\alpha')^2B^2)c(z_*)^2 + (2\pi\alpha')^2\langle J^t\rangle^2}, \tag{6.6a}$$

$$\sigma_{xy} = \frac{(2\pi\alpha')^2\langle J^t\rangle B}{g_{xx}^2 + (2\pi\alpha')^2B^2}. \tag{6.6b}$$

The Dq-brane action also includes Wess-Zumino couplings to RR fields. Generically, these introduce additional terms in the gauge field equation of motion. Whether our solution remains valid must be determined on a case-by-case basis. For example, in the D4/D8/$\bar{\text{D}}$8 system [26], the D8-brane action includes $\int dC_3 \wedge A \wedge F \wedge F$, with dC_3 proportional to the volume form of S^4. This coupling introduces an additional term in the equation of motion that invalidates our gauge field solution.

A flat C_{q-3} form, $dC_{q-3} = 0$, will leave the Dp-brane background unchanged and produce a term, $\int C_{q-3} \wedge F \wedge F$, in the Dq-brane action that leaves the gauge field equation of motion unchanged. Our solution thus remains valid. Integrating C_{q-3} produces a θ parameter,

$$S^\theta_{Dq} = -\frac{\theta}{8\pi^2}\int F\wedge F. \tag{6.7}$$

For our gauge field solutions this shifts $\langle J^\mu\rangle \to \langle J^\mu\rangle + \Delta\langle J^\mu\rangle$ with

$$\Delta\langle J^t\rangle = +\frac{\theta}{4\pi^2}B, \qquad \Delta\langle J^x\rangle = +\frac{\theta}{4\pi^2}E_y, \qquad \Delta\langle J^y\rangle = -\frac{\theta}{4\pi^2}E_x, \tag{6.8}$$

which we implement in Eq. (6.6) by taking $\langle J^t\rangle \to \langle J^t\rangle + \frac{\theta}{4\pi^2}B$ and $\sigma_{xy} \to \sigma_{xy} + \frac{\theta}{4\pi^2}$.

VII. CONCLUSION

Using the AdS/CFT correspondence, we computed the Hall conductivity of a finite baryon number density of $\mathcal{N} = 2$ hypermultiplet excitations in an $\mathcal{N} = 4$ SYM plasma in the limits of large N_c and large 't Hooft coupling. Our method is valid for any values of m, $\langle J^t\rangle$, T, B, and E for which the supergravity description as a probe D7-brane with world volume horizon is valid. We also computed the drag force on flavor excitations in the plasma in the $m \to \infty$

limit, and identified the D7-brane analogue of the trailing string world sheet horizon.

Electric-magnetic self-duality, or S-duality, of $U(1)$ Yang-Mills theory in AdS_4, and its interpretation in the dual $(2+1)$-dimensional CFT, was studied in Refs. [27–31]. Put briefly, S-duality appears in the CFT as particle-vortex duality. S-duality may be extended to $SL(2, \mathbb{Z})$ if a T transformation can be found. For Abelian Yang-Mills in AdS_4, this arises as a 2π shift of the bulk θ angle, which appears in the dual field theory as a shift in the two-point function of the dual current by a contact term [27]. The transformation of the conductivity (and other transport coefficients) under S- and T-duality was studied in Refs. [9–12].

A similar analysis should be possible for probe Dq-branes using the well-known extension of S-duality to $(3+1)$-dimensional Born-Infeld theory [32–35]. Indeed, the Dq-brane action equation (6.2) is simply the $(3+1)$-dimensional Born-Infeld action with the extra factor $c(z)$. The θ angle we identified in the Dq-brane action produces the T-transformation in the same fashion as for Yang-Mills theory.

In the condensed matter physics literature, an $SL(2, \mathbb{Z})$ duality transformation has been proposed to relate transitions between quantum Hall plateaux. As a small sampling of this literature see Refs. [36–39]. We note in passing that in Ref. [39] the $SL(2, \mathbb{Z})$ action was shown to persist unaltered even beyond the linear response regime.

We reiterate the comment of Ref. [11], however, that how a quantum Hall effect may occur in gauge-gravity duality is currently unclear. The fundamental problem seems to be how to describe a Fermi surface using gauge-gravity duality.[6] This is perhaps the most exciting direction for future research.

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APPENDIX: HOLOGRAPHIC RENORMALIZATION

In AdS/CFT, we equate the on-shell supergravity action with the generating functional of field theory correlation functions. The on-shell action, however, is divergent due to the radial integration. In holographic renormalization (holo-rg) [41–44] we introduce a regulator $z = \epsilon$, add counterterms at $z = \epsilon$ to cancel the divergences, and then take $\epsilon \to 0$.

We find from its equation of motion that $\theta(z)$ has the asymptotic expansion

[6]For recent work in this direction, see Ref. [40].

$$\theta(z) = \theta_0 z + \theta_2 z^3 + \ldots \tag{A1}$$

The leading coefficient θ_0 is the source for the dual operator, given by taking $\frac{\partial}{\partial m}$ of the SYM Lagrangian. In other words θ_0 gives the hypermultiplet mass. If we separate the D3-branes and the D7-branes by a distance L in the X_8 direction, then $m = \frac{L}{2\pi\alpha'}$ and $L = \lim_{z\to 0} \frac{1}{z}\sin\theta(z) = \theta_0$ allows us to identify $\theta_0 = (2\pi\alpha')m$.

Plugging Eq. (A1) into the regulated action we find the divergences

$$\begin{aligned} S_{\rm reg} &= -\int_\epsilon^{z_H} dz L \\ &= -\mathcal{N}\int_\epsilon^{z_H} dz\Big(z^{-5} - \theta_0^2 z^{-3} + \frac{1}{2}(2\pi\alpha')^2 \\ &\quad \times (B^2 - E^2)z^{-1} + O(z)\Big). \end{aligned} \tag{A2}$$

The counterterms we need are [45]

$$L_1 = \tfrac{1}{4}\mathcal{N}\sqrt{-\gamma}, \qquad L_2 = -\tfrac{1}{2}\mathcal{N}\sqrt{-\gamma}\theta(\epsilon)^2, \qquad L_f = \mathcal{N}\tfrac{5}{12}\sqrt{-\gamma}\theta(\epsilon)^4 \tag{A3}$$

with γ_{ij} the induced metric at $z = \epsilon$ and γ its determinant. Notice that $\sqrt{-\gamma} = \epsilon^{-4} + O(\epsilon^4)$. Supersymmetry requires the finite counterterm L_f [45]. We suppress $\int dtdxdy$ unless stated otherwise. The last divergence requires a counterterm

$$\begin{aligned} L_F &= -\tfrac{1}{4}\mathcal{N}(2\pi\alpha')^2\sqrt{-\gamma}F^{ij}F_{ij}\log\epsilon \\ &= -\tfrac{1}{2}\mathcal{N}(2\pi\alpha')^2(B^2 - E^2)\log\epsilon + O(\epsilon^4\log\epsilon). \end{aligned} \tag{A4}$$

The generating functional of the field theory is the $\epsilon \to 0$ limit of $S = S_{\rm reg} + \sum_i L_i$. We want the expectation values $\langle J^t\rangle$, $\langle J^x\rangle$, and $\langle J^y\rangle$. In holo-rg, $\langle J^\mu\rangle$ is

$$\langle J^\mu\rangle = \lim_{\epsilon\to 0}\frac{1}{\epsilon^4}\frac{1}{\sqrt{-\gamma}}\frac{\delta S}{\delta A_\mu(\epsilon)}. \tag{A5}$$

For $\langle J^t\rangle$, we need

$$\begin{aligned} \delta S &= -\int_\epsilon^{z_H} dz\frac{\delta L}{\delta\partial_z A_t}\partial_z\delta A_t = -\frac{\delta L}{\delta\partial_z A_t}\int_\epsilon^{z_H} dz\partial_z\delta A_t \\ &= -\frac{\delta L}{\delta\partial_z A_t}(\delta A_t(z_H) - \delta A_t(\epsilon)), \end{aligned} \tag{A6}$$

where we have used the fact that $\frac{\delta L}{\delta\partial_z A_t}$ is z-independent on shell. Enforcing $\delta A_t(z_H) = 0$ we find $\frac{\delta S}{\delta A_t(\epsilon)} = \frac{\delta L}{\delta\partial_z A_t}$ and hence $\langle J^t\rangle = \frac{\delta L}{\delta\partial_z A_t}$.

For $\langle J^x\rangle$, we reinstate $\int dt$ because A_x is time dependent,

$$\delta S = -\int dzdt\Big(\frac{\delta L}{\delta\partial_z A_x}\partial_z\delta A_x + \frac{\delta L}{\delta\partial_t A_x}\partial_t\delta A_x\Big). \tag{A7}$$

We employ precisely the same argument as before for the first term. For the second term we observe that $\frac{\delta L}{\delta\partial_t A_x}$ is t-independent on shell and hence

$$\int dt \frac{\delta L}{\delta \partial_t A_x} \partial_t \delta A_x = \frac{\delta L}{\delta \partial_t A_x} \int dt \partial_t \delta A_x = 0, \qquad (A8)$$

where we demand that the fluctuation be well behaved (vanishing) at $t = \pm\infty$. The counterterm L_F gives a vanishing contribution to $\langle J^x \rangle$ for the same reason,

$$\begin{aligned} \delta L_F &= -\frac{1}{4}\mathcal{N}(2\pi\alpha')^2\sqrt{-\gamma}\gamma^{ij}\gamma^{kl} \\ &\quad \times \int dt \frac{\delta}{\delta \partial_t A_x}(F_{ik}F_{jl})\partial_t \delta A_x \log\epsilon \\ &= +\frac{1}{2}\mathcal{N}(2\pi\alpha')^2 \int dt \dot{A}_x(\epsilon)\partial_t \delta A_x \log\epsilon \\ &\quad + O(\epsilon^4 \log\epsilon) \\ &= O(\epsilon^4 \log\epsilon). \end{aligned} \qquad (A9)$$

We then have $\frac{\delta S}{\delta A_x(\epsilon)} = \frac{\delta L}{\delta \partial_z A_x}$ and hence $\langle J^x \rangle = \frac{\delta L}{\delta \partial_z A_x}$.

$\langle J^y \rangle$ is very similar. A_y depends on x so we reinstate $\int dx$. We have

$$\delta S = -\int dz dx \left(\frac{\delta L}{\delta \partial_z A_y} \partial_z \delta A_y + \frac{\delta L}{\delta \partial_x A_y} \partial_x \delta A_y \right). \qquad (A10)$$

The same argument as above applies for the first term, and for the second term we observe that $\frac{\delta L}{\delta \partial_x A_y}$ is x-independent on shell. Demanding that the fluctuation be well behaved at $x = \pm\infty$ gives $\int dx \partial_x \delta A_y = 0$ and no contribution from L_F. We thus have $\langle J^y \rangle = \frac{\delta L}{\delta \partial_z A_y}$.

As in Ref. [1], we claim that these results are valid for any probe Dq-brane with a world volume horizon in a Dp-brane background. The identification of $\langle J^t \rangle$ depended only on the difference in the value of A_t at the horizon and its asymptotic value. This behavior will be true for any probe brane with horizon. Similar statements apply for the identifications of $\langle J^x \rangle$ and $\langle J^y \rangle$. Additional counterterms may appear for different systems but no such counterterms can change these results. Any counterterm must be built from gauge- and Lorentz-invariant combinations of the field strength. The only components of the field strength that could contribute are F_{tx} and F_{xy}, which in our solution are constants, so we will always end up with $\int dt \partial_t \delta A_x = 0$ and $\int dx \partial_x \delta A_y = 0$, as above.

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