

Nonrelativistic conformal field theories

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We study representations of the Schrödinger algebra in terms of operators in nonrelativistic conformal field theories. We prove a correspondence between primary operators and eigenstates of few-body systems in a harmonic potential. Using the correspondence we compute analytically the energy of fermions at unitarity in a harmonic potential near two and four spatial dimensions. We also compute the energy of anyons in a harmonic potential near the bosonic and fermionic limits.

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I. INTRODUCTION

Conformal field theories (CFTs) form a special class of relativistic quantum field theories, where the Poincaré symmetry group is enlarged to the group of conformal transformations. One element of the conformal algebra is dilatation: CFTs are always scale invariant. The conformal algebra and its representation have been studied extensively [1].

In this paper we study nonrelativistic counterparts of relativistic conformal field theories. There are several examples of such theories beside the trivial noninteracting theories. Nonrelativistic particles interacting through a $1/r^2$ potential is one such example. The physically most important example in three spatial dimensions is the theory describing spin-1/2 fermions with pointlike interaction fine-tuned to infinite scattering length (fermions at unitarity) [2]. Such fermionic systems have been created and studied experimentally. Theories describing anyons present another class of nonrelativistic CFTs, but in two spatial dimensions.

The nonrelativistic analog of the conformal algebra is the so-called Schrödinger algebra [3,4]. While the Schrödinger algebra has been studied before [5–7], we are mostly interested in the representations of the Schrödinger algebra in terms of operators. We show that the concept of primary operators can be directly transferred to nonrelativistic theories. In addition, we show that there is an operator-state correspondence: a primary operator (with some exceptions) corresponds to an eigenstate of a few-particle system in a harmonic potential. The scaling dimension of the operator coincides with the energy of the corresponding eigenstate, divided by the oscillator frequency.

The operator-state correspondence allows us to translate the problem of finding the energy eigenvalues of a few-fermion state at unitarity, or a few-anyon state, in a harmonic potential to another problem of finding the anomalous dimensions of composite operators in the nonrelativistic conformal field theory in free space. The latter problem is amenable to standard diagrammatic techniques for fermions at unitarity near two or four spatial dimensions, or for anyons near the bosonic and fermionic

limits. We present a few examples of such calculations in this paper. In particular, we compute the ground state energy of up to six fermions at unitarity in a harmonic potential near two and four dimensions, and interpolate the results to find the energy in three dimensions. We also compute the ground state energy of up to four anyons in a harmonic potential.

II. SCHRÖDINGER ALGEBRA**A. Derivation of the algebra**

We briefly review the Schrödinger algebra [3,4]. For definiteness, consider a nonrelativistic theory described by a second-quantized field $\psi_\alpha(\mathbf{x})$ (where α is the spin index) which satisfies the commutation or anticommutation relation

$$[\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})]_\pm = \delta(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta}. \quad (1)$$

Throughout this paper, we use nonrelativistic natural units $\hbar = m = 1$ where m is a particle mass. We consider a general spatial dimension d . Define the number density and momentum density,

$$\begin{aligned} n(\mathbf{x}) &= \psi^\dagger(\mathbf{x})\psi(\mathbf{x}), \\ j_i(\mathbf{x}) &= -\frac{i}{2}(\psi^\dagger(\mathbf{x})\partial_i\psi(\mathbf{x}) - \partial_i\psi^\dagger(\mathbf{x})\psi(\mathbf{x})) \end{aligned} \quad (2)$$

(summation over spin indices is implied). Their commutators are

$$[n(\mathbf{x}), n(\mathbf{y})] = 0, \quad [n(\mathbf{x}), j_i(\mathbf{y})] = -in(\mathbf{y})\partial_i\delta(\mathbf{x} - \mathbf{y}), \quad (3a)$$

$$[j_i(\mathbf{x}), j_j(\mathbf{y})] = -i(j_j(\mathbf{x})\partial_i + j_i(\mathbf{y})\partial_j)\delta(\mathbf{x} - \mathbf{y}). \quad (3b)$$

The Schrödinger algebra is formed by the following operators:

$$\begin{aligned} N &= \int d\mathbf{x} n(\mathbf{x}), & P_i &= \int d\mathbf{x} j_i(\mathbf{x}), \\ M_{ij} &= \int d\mathbf{x} (x_i j_j(\mathbf{x}) - x_j j_i(\mathbf{x})), \end{aligned} \quad (4)$$

$$\begin{aligned}
K_i &= \int d\mathbf{x} x_i n(\mathbf{x}), & C &= \int d\mathbf{x} \frac{x^2}{2} n(\mathbf{x}), \\
D &= \int d\mathbf{x} x_i j_i(\mathbf{x}),
\end{aligned} \tag{5}$$

and the Hamiltonian H . The operators in Eq. (4) have simple physical interpretation: N is the particle number, P_i is the momentum, and M_{ij} is the orbital angular momentum. In a scale-invariant theory like unitary fermions, these operators form a closed algebra. All commutators except those that involve H can be found from Eqs. (3). First N commutes with all other operators:

$$[N, \text{any}] = 0. \tag{6}$$

The commutator of the angular momentum M_{ij} with an operator is determined by the transformation properties of the latter under rotations,

$$[M_{ij}, N] = [M_{ij}, C] = [M_{ij}, D] = 0, \tag{7}$$

$$\begin{aligned}
[M_{ij}, P_k] &= i(\delta_{ik} P_j - \delta_{jk} P_i), \\
[M_{ij}, K_k] &= i(\delta_{ik} K_j - \delta_{jk} K_i),
\end{aligned} \tag{8}$$

$$[M_{ij}, M_{kl}] = i(\delta_{ik} M_{jk} - \delta_{jk} M_{il} + \delta_{il} M_{kj} - \delta_{jl} M_{ki}). \tag{9}$$

The remaining commutators are

$$[P_i, P_j] = [K_i, K_j] = [K_i, C] = 0, \quad [K_i, P_j] = i\delta_{ij} N, \tag{10}$$

$$[D, P_i] = iP_i, \quad [D, K_i] = -iK_i, \quad [D, C] = -2iC. \tag{11}$$

Now let us find the commutators of the Hamiltonian H with other operators. Conservation of particle number, momentum, and angular momentum implies that

$$[H, N] = [H, P_i] = [H, M_{ij}] = 0. \tag{12}$$

On the other hand, the continuity equation implies that

$$[H, n] = -i\partial_i n = i\partial_i j_i, \tag{13}$$

from which it follows that

$$[H, K_i] = -iP_i, \quad [H, C] = -iD. \tag{14}$$

The computation of the last commutator $[H, D]$ requires the condition of scale invariance. For definiteness, let us consider fermions at unitarity, described by the Hamiltonian

$$\begin{aligned}
H &= \int d\mathbf{x} \frac{1}{2} \partial_i \psi_\alpha^\dagger \partial_i \psi_\alpha + \frac{1}{2} \int d\mathbf{x} d\mathbf{y} \psi_\alpha^\dagger(\mathbf{x}) \psi_\beta^\dagger(\mathbf{y}) \\
&\quad \times V(|\mathbf{x} - \mathbf{y}|) \psi_\beta(\mathbf{y}) \psi_\alpha(\mathbf{x}),
\end{aligned} \tag{15}$$

where $V(|\mathbf{x} - \mathbf{y}|)$ is a short-range potential with infinite scattering length. We note that D is an operator of dilata-

tion,

$$e^{-i\lambda D} \psi(\mathbf{x}) e^{i\lambda D} = e^{d\lambda/2} \psi(e^\lambda \mathbf{x}), \tag{16}$$

from which one finds

$$e^{-i\lambda D} H e^{i\lambda D} = e^{2\lambda} H', \tag{17}$$

where H' is the same as H but the potential V is replaced with a new potential:

$$V(r) \rightarrow V'(r) = e^{-2\lambda} V(e^{-\lambda} r). \tag{18}$$

If V corresponds to infinite scattering length, then V' also corresponds to infinite scattering length. From the point of view of long-distance physics, $H' = H$. Therefore, we find

$$[D, H] = 2iH. \tag{19}$$

It is clear that Eq. (19) is simply the condition of scale invariance, and hence must hold for noninteracting anyons in two spatial dimensions.

A more lengthy proof, which can be given for particles at unitarity, is to use the momentum conservation equation

$$\partial_i j_i + \partial_j \Pi_{ij} = 0, \tag{20}$$

where Π_{ij} is the stress tensor, which can be defined for a generic potential $V(|\mathbf{x} - \mathbf{y}|)$ (see Appendix B). The commutator is then

$$[D, H] = i \int d\mathbf{x} \Pi_{ii}(\mathbf{x}). \tag{21}$$

At unitarity, one can show that (see Appendix B)

$$\int d\mathbf{x} \Pi_{ii}(\mathbf{x}) = 2H, \tag{22}$$

and Eq. (19) follows.

The Schrödinger algebra is summarized in Appendix A.

B. Local operators and representations of the Schrödinger algebra

We introduce the notation of local operators $\mathcal{O}(t, \mathbf{x})$ as operators which depend on the position in time and space t, \mathbf{x} so that

$$\mathcal{O}(t, \mathbf{x}) = e^{iHt - iP_i x_i} \mathcal{O}(0) e^{-iHt + iP_i x_i}. \tag{23}$$

A local operator \mathcal{O} is said to have scaling dimension $\Delta_{\mathcal{O}}$ if

$$[D, \mathcal{O}(0)] = i\Delta_{\mathcal{O}} \mathcal{O}(0), \tag{24}$$

and to have particle number $N_{\mathcal{O}}$ if

$$[N, \mathcal{O}(0)] = N_{\mathcal{O}} \mathcal{O}(0). \tag{25}$$

We shall consider only operators with well-defined particle number and scaling dimension. Examples of such operators are ψ and ψ^\dagger ; $\Delta_\psi = \Delta_{\psi^\dagger} = d/2$ and $-N_\psi = N_{\psi^\dagger} = 1$. In the case of fermions at unitarity, a more complicated local operator is

$$\phi(\mathbf{x}) = \lim_{\mathbf{y} \rightarrow \mathbf{x}} |\mathbf{x} - \mathbf{y}|^{d-2} \psi_1(\mathbf{x}) \psi_1(\mathbf{y}). \quad (26)$$

The presence of the prefactor $|\mathbf{x} - \mathbf{y}|^{d-2}$ guarantees that the matrix elements of the operator $\phi(\mathbf{x})$ between two states in the Hilbert space are finite.¹ The scaling dimension of ϕ is $\Delta_\phi = 2$. This can be found applying elementary dimension counting to Eq. (26):

$$\Delta_\phi = 2\Delta_\psi + (d-2)\Delta_x = 2\frac{d}{2} + (d-2)(-1) = 2. \quad (27)$$

Let us look at the set of all local operators $\mathcal{O}_a(t, \mathbf{x})$. These operators, put at $t = 0$ and $\mathbf{x} = 0$, form a representation of the Schrödinger algebra: for any operator A in the algebra

$$[A, \mathcal{O}_a(0)] = A_{ab} \mathcal{O}_b(0). \quad (28)$$

We shall discuss the irreducible representations of the Schrödinger algebra.

First we notice that if \mathcal{O} has dimension $\Delta_\mathcal{O}$, then $[P_i, \mathcal{O}]$ has dimension $\Delta_\mathcal{O} + 1$:

$$\begin{aligned} [D, [P_i, \mathcal{O}]] &= [P_i, [D, \mathcal{O}]] + [[D, P_i], \mathcal{O}] \\ &= [P_i, i\Delta_\mathcal{O} \mathcal{O}] + [iP_i, \mathcal{O}] \\ &= i(\Delta_\mathcal{O} + 1)[P_i, \mathcal{O}]. \end{aligned} \quad (29)$$

Analogously the dimensions of $[K_i, \mathcal{O}]$, $[C, \mathcal{O}]$, and $[H, \mathcal{O}]$ are $\Delta_\mathcal{O} - 1$, $\Delta_\mathcal{O} - 2$, and $\Delta_\mathcal{O} + 2$, respectively:

$$[D, [K_i, \mathcal{O}]] = i(\Delta_\mathcal{O} - 1)[K_i, \mathcal{O}], \quad (30)$$

$$[D, [C, \mathcal{O}]] = i(\Delta_\mathcal{O} - 2)[C, \mathcal{O}], \quad (31)$$

$$[D, [H, \mathcal{O}]] = i(\Delta_\mathcal{O} + 2)[H, \mathcal{O}]. \quad (32)$$

Assuming that the dimensions of operators are bounded from below, if one starts with a given operator and repeatedly takes its commutator with K_i and C , one lowers the dimension of the operator until it cannot be lowered further. The last operator \mathcal{O} obtained this way has the property

$$[K_i, \mathcal{O}] = [C, \mathcal{O}] = 0. \quad (33)$$

Operators that commute with K_i and C will be called *primary operators* (quasiprimary operators in the terminology of Ref. [7]). In general, $[K_i, \mathcal{O}] = 0$ does not imply that $[C, \mathcal{O}] = 0$, and vice versa.

Starting with a primary operator \mathcal{O} , one can build up a tower of operators by taking commutators with P_i and H . In other words, starting with primary operators one can build up whole towers of operators by taking their space and time derivatives. For example, the operators with

dimension $\Delta_\mathcal{O} + 1$ in the tower are $[P_i, \mathcal{O}] \equiv i\partial_i \mathcal{O}$. At the next level (dimension $\Delta_\mathcal{O} + 2$), the following are possible:

$$[H, \mathcal{O}] = -i\partial_t \mathcal{O}, \quad [P_i, [P_j, \mathcal{O}]] \equiv -\partial_i \partial_j \mathcal{O}. \quad (34)$$

Commuting the operators in Eq. (34) with K_i and C , we get back the operators in the lower rungs of the tower.

It is easy to see that the operators built from a primary operator by taking space and time derivatives form an irreducible representation of the Schrödinger algebra. It is also possible to show that the full set of all local operators can be decomposed into irreducible representations, each of which is built upon a single primary operator. The task of finding the spectrum of dimensions of all local operators reduces to finding the dimensions of primary operators.

For an operator $\mathcal{O}(t, \mathbf{x})$ with dimension $\Delta_\mathcal{O}$ at an arbitrary spacetime point, the following commutation relations hold,

$$[P_i, \mathcal{O}] = i\partial_i \mathcal{O}, \quad [H, \mathcal{O}] = -i\partial_t \mathcal{O}, \quad (35)$$

$$[D, \mathcal{O}] = i(2t\partial_t + x_i \partial_i + \Delta_\mathcal{O}) \mathcal{O}. \quad (36)$$

Moreover, if \mathcal{O} is a primary operator then

$$[K_i, \mathcal{O}] = (-it\partial_i + N_{\mathcal{O}} x_i) \mathcal{O}, \quad (37)$$

$$[C, \mathcal{O}] = -i(t^2 \partial_t + tx_i \partial_i + t\Delta_\mathcal{O}) \mathcal{O} + \frac{x^2}{2} N_{\mathcal{O}} \mathcal{O}. \quad (38)$$

The exponentiated version of Eq. (36) is

$$e^{-i\lambda D} \mathcal{O}(t, \mathbf{x}) e^{i\lambda D} = e^{\lambda \Delta_\mathcal{O}} \mathcal{O}(e^{2\lambda} t, e^\lambda \mathbf{x}). \quad (39)$$

For any set of n operators, one can define an n -point correlation function,

$$\begin{aligned} G_n(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2, \dots, t_n, \mathbf{x}_n) \\ = \langle 0 | T \mathcal{O}_1(t_1, \mathbf{x}_1) \mathcal{O}_2(t_2, \mathbf{x}_2) \dots \mathcal{O}_n(t_n, \mathbf{x}_n) | 0 \rangle, \end{aligned} \quad (40)$$

where T is time ordering. Clearly for G_n to be nonzero it is necessary that $N_{\mathcal{O}_1} + N_{\mathcal{O}_2} + \dots + N_{\mathcal{O}_n} = 0$. If all \mathcal{O}_i have definite dimensions then this correlation function has a scaling property

$$G_n(e^{2\lambda} t_i, e^\lambda \mathbf{x}_i) = \exp\left(-\lambda \sum_{i=1}^n \Delta_{\mathcal{O}_i}\right) G_n(t_i, \mathbf{x}_i), \quad (41)$$

which follows from Eq. (39) and $e^{i\lambda D} |0\rangle = |0\rangle$.

The correlation functions of primary operators are further constrained [7]. As an example, consider the two-point correlation function of a primary operator \mathcal{O} with its Hermitian conjugate:

$$G(t, \mathbf{x}) = \langle 0 | T \mathcal{O}(t, \mathbf{x}) \mathcal{O}^\dagger(0) | 0 \rangle. \quad (42)$$

Using $\langle 0 | [K_i, T \mathcal{O}(x) \mathcal{O}^\dagger(y)] | 0 \rangle = 0$ and Eq. (37) one obtains

¹The condition of unitarity requires that the wave function of N spin-up and M spin-down fermions $\Psi(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N; \mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_M)$ behaves like $|\mathbf{x}_i - \mathbf{y}_j|^{2-d}$ when $|\mathbf{x}_i - \mathbf{y}_j| \rightarrow 0$ for any pair of fermions with opposite spins i, j .

$$(-it\partial_i + N_{\mathcal{O}}x_i)G(t, \mathbf{x}) = 0. \quad (43)$$

Combining with the scale invariance, the two-point correlation function is determined up to an overall coefficient,

$$G(t, \mathbf{x}) = Ct^{-\Delta_{\mathcal{O}}} \exp\left(-iN_{\mathcal{O}} \frac{|\mathbf{x}|^2}{2t}\right). \quad (44)$$

C. Correspondence to states in a harmonic potential

We now show that each primary operator corresponds to an energy eigenstate of a system in a harmonic potential. We set the oscillator frequency of the harmonic potential ω to 1. The total Hamiltonian of the system in a harmonic potential is

$$H_{\text{osc}} = H + C. \quad (45)$$

Consider a primary operator \mathcal{O} put at $t = 0$ and $\mathbf{x} = 0$. Let \mathcal{O} be constructed from annihilation operators, so that \mathcal{O}^\dagger acts nontrivially on the vacuum $|0\rangle$. Consider the following state

$$|\Psi_{\mathcal{O}}\rangle = e^{-H}\mathcal{O}^\dagger|0\rangle. \quad (46)$$

If the particle number of \mathcal{O}^\dagger is $N_{\mathcal{O}^\dagger}$, then $|\Psi_{\mathcal{O}}\rangle$ is an $N_{\mathcal{O}^\dagger}$ -body state. Let us show that $|\Psi_{\mathcal{O}}\rangle$ is an eigenstate of the Hamiltonian: $H_{\text{osc}} = H + C$. Indeed

$$H_{\text{osc}}|\Psi_{\mathcal{O}}\rangle = e^{-H}(e^H H_{\text{osc}} e^{-H})\mathcal{O}^\dagger|0\rangle. \quad (47)$$

We now use the formula

$$e^H H_{\text{osc}} e^{-H} = H_{\text{osc}} + [H, H_{\text{osc}}] + \frac{1}{2}[H, [H, H_{\text{osc}}]] + \cdots. \quad (48)$$

Using the commutation relations in Appendix A, we find that all terms in the \cdots vanish, and the right-hand side is equal to $C - iD$. Therefore

$$\begin{aligned} H_{\text{osc}}|\Psi_{\mathcal{O}}\rangle &= e^{-H}(C - iD)\mathcal{O}^\dagger|0\rangle \\ &= e^{-H}\mathcal{O}^\dagger(C - iD)|0\rangle + e^{-H}[C - iD, \mathcal{O}^\dagger]|0\rangle. \end{aligned} \quad (49)$$

However, both C and D annihilate the vacuum, $C|0\rangle = D|0\rangle = 0$, and since \mathcal{O} is a primary operator, $[C, \mathcal{O}^\dagger] = 0$. Thus, using $[D, \mathcal{O}^\dagger] = -[D, \mathcal{O}]^\dagger = i\Delta_{\mathcal{O}}\mathcal{O}^\dagger$, we obtain

$$H_{\text{osc}}|\Psi_{\mathcal{O}}\rangle = e^{-H}\Delta_{\mathcal{O}}\mathcal{O}^\dagger|0\rangle = \Delta_{\mathcal{O}}|\Psi_{\mathcal{O}}\rangle, \quad (50)$$

i.e., $|\Psi_{\mathcal{O}}\rangle$ is an eigenstate of the system of $N_{\mathcal{O}^\dagger}$ particles in a harmonic potential, with the energy eigenvalue $\Delta_{\mathcal{O}}$ (times $\hbar\omega$).

It is known that the eigenstates of H_{osc} are organized into ladders with spacing between steps equal to 2 [8,9]. The raising and lowering operators within a ladder are [8]

$$L_+ = H - C + iD, \quad (51)$$

$$L_- = H - C - iD. \quad (52)$$

Let us show that the state $|\Psi_{\mathcal{O}}\rangle$ is annihilated by L_- and hence is the lowest state in its ladder. Indeed, using the identity

$$e^H L_- e^{-H} = -C, \quad (53)$$

we find

$$\begin{aligned} L_-|\Psi_{\mathcal{O}}\rangle &= e^{-H}e^H L_- e^{-H}\mathcal{O}^\dagger|0\rangle = -e^{-H}C\mathcal{O}^\dagger|0\rangle \\ &= -e^{-H}\mathcal{O}^\dagger C|0\rangle = 0. \end{aligned} \quad (54)$$

Clearly, in order to correspond to a nontrivial eigenstate of H_{osc} , \mathcal{O}^\dagger must not annihilate the vacuum: $\mathcal{O}^\dagger|0\rangle \neq 0$. We shall consider the operators \mathcal{O} that are built from the fundamental annihilation operators of the field theories.

D. Simple examples: One and two-body operators/states

Let us illustrate this correspondence using one-particle and two-particle operators ψ and ϕ at unitarity. The operator ψ has scaling dimension $d/2$, which matches the ground state energy of one particle in a harmonic potential in spatial dimension d . The operator ϕ has scaling dimension 2. The ground state of two particles at unitarity in a harmonic potential has the wave function

$$\phi(\mathbf{x}, \mathbf{y}) \propto \frac{e^{-(x^2+y^2)/2}}{|\mathbf{x} - \mathbf{y}|^{d-2}}, \quad (55)$$

and the ground state energy is also 2.

III. EXAMPLE 1: FERMIONS AT UNITARITY

In this section, we compute the scaling dimensions of some operators in the theory describing spin-1/2 fermions at unitarity. In order to have a small parameter for perturbative expansions, we shall work near two and four spatial dimensions, and then, interpolate the results to the physical three spatial dimensions. Since the energy eigenvalues of two and three fermions in a harmonic potential can be found exactly, we can use these cases to test our expansions and interpolation schemes. In the cases of more than three fermions, only numerical results exist. Our analytical calculations, as we will see, are consistent with the numerical ones.

There are two field-theoretical representations of fermions at unitarity, one becoming weakly coupled as $d \rightarrow 4$ and the other becoming weakly coupled as $d \rightarrow 2$ [10,11]. We shall consider these two cases separately.

A. Near four spatial dimensions

1. Fixed point

In the first representation, the Lagrangian density describing fermions at unitarity is

$$\begin{aligned}\mathcal{L} = & i\psi_\sigma^\dagger \partial_t \psi_\sigma - \frac{1}{2} |\nabla \psi_\sigma|^2 + i\phi^* \partial_t \phi - \frac{1}{4} |\nabla \phi|^2 \\ & + g\psi_1^\dagger \psi_1^\dagger \phi + g\psi_1 \psi_1 \phi^*.\end{aligned}\quad (56)$$

The canonical dimensions of the fermion field ψ_σ and the boson field ϕ are both $d/2$. Therefore the coupling constant g is relevant at weak coupling below $d = 4$. There are three other relevant terms one can add to the Lagrangian density (56): $\mu_\sigma \psi_\sigma^\dagger \psi_\sigma$ and $-(g^2/c_0)\phi^* \phi$. μ_σ is a chemical potential for each spin component of fermions and here we consider the system at zero density $\mu_\sigma = 0$. Furthermore we assume that the system is fine-tuned so that the coefficient in front of $\phi^* \phi$ satisfies

$$\frac{1}{c_0} = \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^2}. \quad (57)$$

[(c_0)⁻¹ is zero in dimensional regularization.] This condition is equivalent to the fine-tuning to the infinite scattering length. We denote the propagators of ψ and ϕ by $G(p)$ and $D(p)$, respectively.

The renormalization of the theory can be performed in the standard way considering $\epsilon = 4 - d$ to be a small parameter for perturbation. There is a one-loop self-energy diagram for ϕ which is logarithmically divergent (Fig. 1). Integrating out modes in the momentum shell $e^{-s}\Lambda < k < \Lambda$, we obtain

$$\begin{aligned}\Sigma(p) &= \frac{g^2}{c_0} + ig^2 \int \frac{dk}{(2\pi)^{d+1}} G\left(\frac{p}{2} + k\right) G\left(\frac{p}{2} - k\right) \\ &= -\frac{g^2}{8\pi^2} \left(p_0 - \frac{p^2}{4}\right) \ln \frac{\Lambda}{e^{-s}\Lambda},\end{aligned}\quad (58)$$

which corresponds to the wave-function renormalization of ϕ

$$Z_\phi = 1 - \frac{g^2}{8\pi^2} s. \quad (59)$$

The anomalous dimension of ϕ is found by the standard formula

$$\gamma_\phi = -\frac{1}{2} \frac{\partial \ln Z_\phi}{\partial s} = \frac{g^2}{16\pi^2}. \quad (60)$$

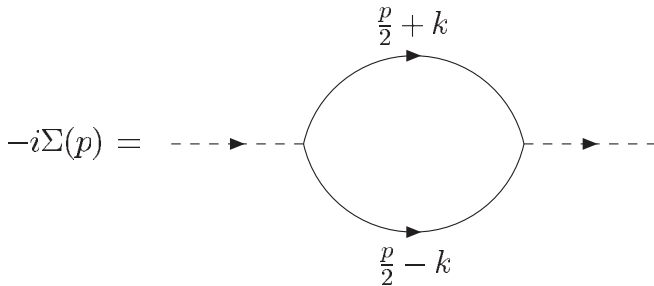


FIG. 1. One-loop self-energy diagram to renormalize the wave function of ϕ .

There is no divergent one-particle irreducible diagram that renormalizes the $\psi_1 \psi_1 \phi^*$ coupling. As a result, the β function that governs the running of g ,

$$\frac{\partial g}{\partial s} = \beta(g), \quad (61)$$

is determined by the dimension of g :

$$\beta(g) = \left(2 - \frac{d}{2} - \gamma_\phi\right)g = \frac{\epsilon}{2}g - \frac{g^3}{16\pi^2}. \quad (62)$$

There is a fixed point located at

$$g^2 = 8\pi^2\epsilon. \quad (63)$$

At this fixed point, the theory is a nonrelativistic CFT describing fermions at unitarity.

2. Scaling dimensions of operators

Since the one-fermion operator ψ is not renormalized, its scaling dimension is

$$\Delta_\psi = \frac{d}{2}. \quad (64)$$

Using the operator-state correspondence, we find that there is a one-fermion state in a harmonic potential with energy $(d/2)\omega$, which is obvious.

The two-fermion operator ϕ , on the other hand, has a scaling dimension different from its canonical dimension $d/2$. At the fixed point,

$$\Delta_\phi = \frac{d}{2} + \gamma_\phi = 2. \quad (65)$$

This is, of course, consistent with Eq. (27). Since there is no other contribution to Δ_ϕ , Eq. (65) is exact to all order in ϵ . According to the operator-state correspondence, this implies the existence of a two-fermion state with zero orbital angular momentum and energy 2ω . The wave function of this state is given by Eq. (55).

Let us look at three-fermion operators. The simplest operator is $\phi\psi_1$. This operator has zero orbital angular momentum $l = 0$. The diagram that contributes to its anomalous dimension to order ϵ is depicted in Fig. 2. It is evaluated as

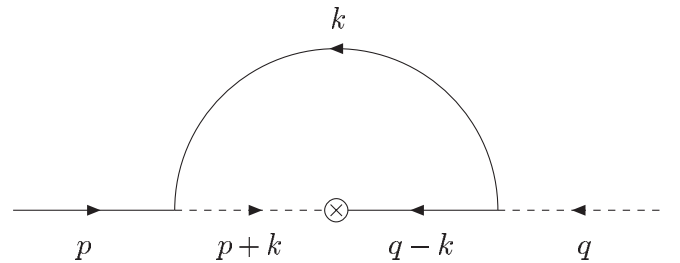


FIG. 2. One-loop diagram to renormalize the three-fermion operator $\phi\psi_1$.

$$-ig^2 \int \frac{dk}{(2\pi)^{d+1}} G(k)D(k)G(-k) = -\frac{g^2}{6\pi^2} \ln \frac{\Lambda}{e^{-s}\Lambda}. \quad (66)$$

Therefore the renormalized operator differs from the bare operator by a rescaling factor: $(\phi\psi_1)_{\text{ren}} = Z_{\phi\psi_1}^{-1} \phi\psi_1$, where

$$Z_{\phi\psi_1} = 1 - \frac{g^2}{6\pi^2} s. \quad (67)$$

At the fixed point, $-\partial \ln Z_{\phi\psi_1} / \partial s = 4\epsilon/3$ is the anomalous dimension of the composite operator $\phi\psi_1$ (more precisely, the nontrivial part of the anomalous dimension because there is a trivial part equal to γ_ϕ). We thus find the scaling dimension

$$\Delta_{\phi\psi_1} = \Delta_\phi + \Delta_{\psi_1} + \frac{4}{3}\epsilon = 4 + \frac{5}{6}\epsilon. \quad (68)$$

According to the operator-state correspondence, $\phi\psi_1$ corresponds to a three-fermion state with $l = 0$ and energy equal to

$$E_3^{(0)} = [4 + \frac{5}{6}\epsilon + O(\epsilon^2)]\omega. \quad (69)$$

This state continues to the first excited state of three fermions in a harmonic potential at $d = 3$. Even within the leading correction in ϵ , the result extrapolated to $\epsilon = 1$, $E_3^{(0)} \approx 4.83\omega$, is not too far from the true result of 4.66622ω at $d = 3$.

The three-fermion ground state in a harmonic potential at $d = 3$ has $l = 1$. There are two lowest $l = 1$ operators near four spatial dimensions; $\phi\nabla\psi_1$ and $(\nabla\phi)\psi_1$. Let us consider the renormalization of a general operator $a\phi\nabla\psi_1 + b(\nabla\phi)\psi_1$. Inserting this operator into Fig. 2, we find

$$\begin{aligned} & -ig^2 \int \frac{dk}{(2\pi)^{d+1}} [a(-\mathbf{k} + \mathbf{q}) + b(\mathbf{k} + \mathbf{p})] \\ & \times G(k)D(k+p)G(-k+q) \\ & = -\frac{g^2}{6\pi^2} \left[\frac{a+5b}{6} \mathbf{p} + \frac{5a+7b}{12} \mathbf{q} \right] \ln \frac{\Lambda}{e^{-s}\Lambda}. \end{aligned} \quad (70)$$

In order to have well-defined anomalous dimensions, this should be proportional to $a\mathbf{p} + b\mathbf{q}$. Thus we have two solutions $(a, b) \propto (1, 1)$ and $(a, b) \propto (2, -1)$, which have anomalous dimensions $4\epsilon/3$ and $-\epsilon/3$, respectively. The first possibility corresponds to the operator $\nabla(\phi\psi_1)$, that is obviously not a primary operator. Its scaling dimension is trivially equal to $\Delta_{\phi\psi_1} + 1$ (corresponding to an excitation in the center of mass motion). The other operator $2\phi\nabla\psi_1 - (\nabla\phi)\psi_1$ has the nontrivial scaling dimension

$$\Delta_{2\phi\nabla\psi_1 - (\nabla\phi)\psi_1} = \Delta_\phi + \Delta_{\psi_1} + 1 - \frac{1}{3}\epsilon = 5 - \frac{5}{6}\epsilon. \quad (71)$$

The operator-state correspondence tells us that the three-fermion state with $l = 1$ in a harmonic potential has the energy

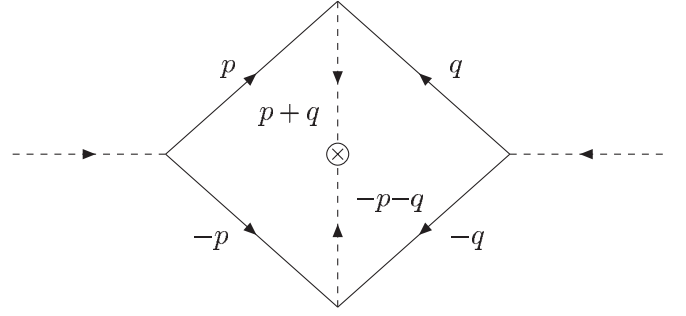


FIG. 3. One-loop diagram to renormalize the four-fermion operator ϕ^2 .

$$E_3^{(1)} = [5 - \frac{5}{6}\epsilon + O(\epsilon^2)]\omega. \quad (72)$$

The extrapolation to $\epsilon = 1$ gives $E_3^{(1)} \approx 4.17\omega$, which is not too far from the true ground state energy 4.27272ω at $d = 3$.

We now turn to the four-fermion state with $l = 0$ represented by the operator ϕ^2 . The first nontrivial correction to its scaling dimension is of order ϵ^2 given by the diagram depicted in Fig. 3. The two-loop integral can be performed analytically and we find

$$\Delta_{\phi^2} = 4 + 8\epsilon^2 \ln \frac{27}{16}. \quad (73)$$

Thus the ground state of four fermions in a harmonic potential has the energy

$$E_4^{(0)} = \left[4 + 8\epsilon^2 \ln \frac{27}{16} + O(\epsilon^3) \right] \omega. \quad (74)$$

The correction, although it is of order ϵ^2 , has the large coefficient. Therefore, in order to extrapolate to $\epsilon = 1$, we shall not use Eq. (73) directly but will combine it with a result near two spatial dimensions.

For a general even number of fermions $N = 2n$, the operator ϕ^n corresponds to the ground state in a harmonic potential with $l = 0$. Its scaling dimension is given by

$$\Delta_{\phi^n} = N + N(N-2)\epsilon^2 \ln \frac{27}{16} + O(\epsilon^3). \quad (75)$$

When $N = 2n + 1$ is odd, the operators $\phi^n\psi_1$ and $2\phi^n\nabla\psi_1 - \phi^{n-1}(\nabla\phi)\psi_1$ correspond to states with orbital angular momentum $l = 0$ and $l = 1$, respectively. Their scaling dimensions are

$$\Delta_{\phi^n\psi_1} = N + 1 + \frac{4N-7}{6}\epsilon + O(\epsilon^2) \quad (76)$$

and

$$\Delta_{2\phi^n\nabla\psi_1 - \phi^{n-1}(\nabla\phi)\psi_1} = N + 2 + \frac{2N-21}{18}\epsilon + O(\epsilon^2). \quad (77)$$

According to the operator-state correspondence, the energy

of N -fermion state in a harmonic potential is simply given by $E_N^{(l)} = \Delta_O \omega$.

The leading-order results [$E_N^{(0)} = N\omega$ for even N and $E_N^{(0)} = (N+1)\omega$ and $E_N^{(1)} = (N+2)\omega$ for odd N] can be easily understood by recalling that, in the limit of $d \rightarrow 4$ from below, fermion pairs at unitarity form pointlike bosons and they do not interact with each other or with extra fermions [10,12]. So the ground state for $N = 2n$ fermions consists of n free composite bosons, each of which has the lowest energy 2ω in a harmonic potential at $d = 4$. When $N = 2n + 1$, the ground state has $l = 0$ and consists of n composite bosons and one extra fermion in the lowest energy states. In order to have an $l = 1$ state, one of the $n + 1$ particles has to be excited to the first excited state, which costs additional 1ω . At $d = 4$, we observe the odd-even staggering in the ground state energy as $E_N^{(0)} - (E_{N-1}^{(0)} + E_{N+1}^{(0)})/2 = 1\omega$ for odd N .

B. Near two spatial dimensions

1. Fixed point

The other field-theoretical representation of fermions at unitarity is provided by the Lagrangian density

$$\mathcal{L} = i\psi_\sigma^\dagger \partial_t \psi_\sigma - \frac{1}{2} |\nabla \psi_\sigma|^2 + \bar{g}^2 \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\uparrow \psi_\downarrow. \quad (78)$$

Therefore the renormalized operator is $(\psi_\uparrow \psi_\downarrow \nabla \psi_\uparrow)_{\text{ren}} = (Z_{\psi_\uparrow \psi_\downarrow \nabla \psi_\uparrow})^{-1} \psi_\uparrow \psi_\downarrow \nabla \psi_\uparrow$, where

$$Z_{\psi_\uparrow \psi_\downarrow \nabla \psi_\uparrow} = 1 + \frac{3\bar{g}^2}{4\pi} s. \quad (82)$$

At the fixed point, the anomalous dimension becomes $\gamma_{\psi_\uparrow \psi_\downarrow \nabla \psi_\uparrow} = -\partial \ln Z_{\psi_\uparrow \psi_\downarrow \nabla \psi_\uparrow} / \partial s = -3\bar{\epsilon}/2$. So the scaling dimension of the operator $\psi_\uparrow \psi_\downarrow \nabla \psi_\uparrow$ is

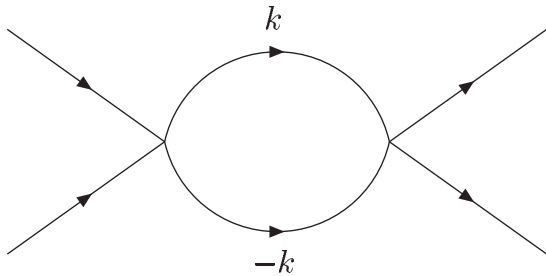


FIG. 4. One-loop diagram to renormalize the four-point vertex coupling \bar{g}^2 .

Again we consider the system at zero density $\mu_\sigma = 0$. When $\bar{\epsilon} = d - 2$ is small, the coupling \bar{g}^2 is renormalized by the logarithmically divergent one-loop diagram (Fig. 4). The renormalization group equation for \bar{g}^2 is

$$\frac{\partial \bar{g}^2}{\partial s} = -\bar{\epsilon} \bar{g}^2 + \frac{\bar{g}^4}{2\pi}. \quad (79)$$

The fixed point is located at

$$\bar{g}^2 = 2\pi \bar{\epsilon}. \quad (80)$$

At this fixed point, the theory is a nonrelativistic CFT describing fermions at unitarity.

2. Scaling dimensions of operators

The scaling dimensions of the one-fermion operator ψ and the two-fermion operator $\psi_\uparrow \psi_\downarrow$ (its renormalization is given by Fig. 5) are given by the same formulas as around four dimensions; $\Delta_\psi = d/2$ and $\Delta_{\psi_\uparrow \psi_\downarrow} = 2$. Here we concentrate our attention on three-fermion operators. The simplest operator is $\psi_\uparrow \psi_\downarrow \nabla \psi_\uparrow$, which has the orbital angular momentum $l = 1$. By considering the diagram where one more fermion line is attached to Fig. 5, the renormalization of this operator is given by

$$-i\bar{g}^2 \int \frac{dk}{(2\pi)^{d+1}} \left(\mathbf{p}_3 - \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} - \mathbf{k} \right) G\left(\frac{\mathbf{p}_1 + \mathbf{p}_2}{2} + \mathbf{k}\right) G\left(\frac{\mathbf{p}_1 + \mathbf{p}_2}{2} - \mathbf{k}\right) - i\bar{g}^2 \int \frac{dk}{(2\pi)^{d+1}} \left(\frac{\mathbf{p}_2 + \mathbf{p}_3}{2} + \mathbf{k} - \mathbf{p}_1 \right) \times G\left(\frac{\mathbf{p}_2 + \mathbf{p}_3}{2} + \mathbf{k}\right) G\left(\frac{\mathbf{p}_2 + \mathbf{p}_3}{2} - \mathbf{k}\right) = \frac{3\bar{g}^2}{4\pi} (\mathbf{p}_2 - \mathbf{p}_1) \ln \frac{\Lambda}{e^{-s}\Lambda}. \quad (81)$$

$$\Delta_{\psi_\uparrow \psi_\downarrow \nabla \psi_\uparrow} = \frac{3d}{2} + 1 + \gamma_{\psi_\uparrow \psi_\downarrow \nabla \psi_\uparrow} = 4. \quad (83)$$

According to the operator-state correspondence, the ground state energy of three fermions in a harmonic potential is given by

$$E_3^{(1)} = [4 + O(\bar{\epsilon}^2)]\omega. \quad (84)$$

For three-fermion operators with $l = 0$, the calculation is somewhat more involved, because there are three lowest

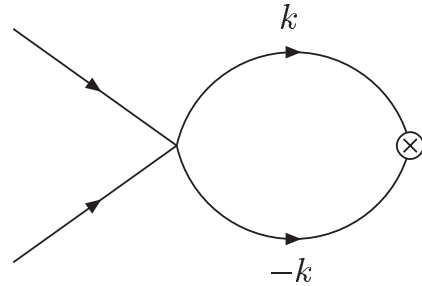


FIG. 5. One-loop diagram to renormalize the two-fermion operator $\psi_\uparrow \psi_\downarrow$.

operators that can mix with each other: $\psi_1\psi_1\nabla^2\psi_1$, $\psi_1\nabla\psi_1\cdot\nabla\psi_1$, and $\psi_1\psi_1\partial_i\psi_1$. The linear combinations with well-defined anomalous dimensions are

- (i) $\nabla\cdot(\psi_1\psi_1\nabla\psi_1)$ with $\gamma = -3\bar{\epsilon}/2$,
- (ii) $\psi_1\psi_1\partial_i\psi_1$ with $\gamma = -3\epsilon/2$,
- (iii) $\psi_1\psi_1(i\partial_i + \frac{1}{2}\nabla^2)\psi_1$ with $\gamma = -\bar{\epsilon}$.

The first operator is not a primary operator. Its scaling dimension is trivially equal to $\Delta_{\psi_1\psi_1\nabla\psi_1} + 1$ (corresponding to an excitation of the center of mass motion). The third operator annihilates the vacuum and thus does not correspond to any eigenstate of the system in a harmonic potential. The second operator is, therefore, the one that corresponds to the lowest energy eigenstate of three fermions in a harmonic potential with $l = 0$. The energy of this state is

$$E_3^{(0)} = \left[\frac{3d}{2} + 2 - \frac{3}{2}\bar{\epsilon} \right] \omega = [5 + O(\bar{\epsilon}^2)] \omega. \quad (85)$$

We can develop the same analysis for operators having more than three fermion numbers. The lowest four-fermion operator with $l = 0$ is $\psi_1\psi_1\nabla\psi_1\cdot\nabla\psi_1$. Its anomalous dimension is computed to be $-3\bar{\epsilon}$, which corresponds to the ground state energy in a harmonic potential

$$E_4^{(0)} = [6 - \bar{\epsilon} + O(\bar{\epsilon}^2)] \omega. \quad (86)$$

For five fermions, the operator $\psi_1\psi_1(\nabla\psi_1\cdot\nabla\psi_1)\nabla\psi_1$ with the anomalous dimension $-7\bar{\epsilon}/2$ corresponds to the ground state in a harmonic potential with $l = 1$. Its energy is given by

$$E_5^{(1)} = [8 - \bar{\epsilon} + O(\bar{\epsilon}^2)] \omega. \quad (87)$$

We can find two nontrivial operators with $l = 0$ corresponding to the lowest two energy eigenstates of five fermions in a harmonic potential. The operators $a\psi_1\psi_1(\nabla\psi_1\cdot\nabla\psi_1)\nabla^2\psi_1 + b\psi_1\nabla_i\psi_1(\nabla\psi_1\cdot\nabla\psi_1)\nabla_i\psi_1 + c\psi_1\psi_1((\nabla_i\nabla\psi_1)\cdot\nabla\psi_1)\nabla_i\psi_1 - d\psi_1\psi_1(\nabla\psi_1\cdot\nabla\psi_1)i\partial_i\psi_1$ with

$$(a, b, c, d) \propto (\pm 19\sqrt{3} - 5\sqrt{35}, \mp 16\sqrt{3}, -6\sqrt{35} \mp 6\sqrt{3}, 16\sqrt{35}) \quad (88)$$

have well-defined anomalous dimensions $-(51 \pm \sqrt{105})\bar{\epsilon}/16$. Therefore, there are five-fermion states with $l = 0$ and energies equal to

$$E_5^{(0)} = \left[9 - \frac{11 \pm \sqrt{105}}{16} \bar{\epsilon} + O(\bar{\epsilon}^2) \right] \omega. \quad (89)$$

Finally, the lowest six-fermion operator with $l = 0$ is $\psi_1\psi_1(\nabla\psi_1\cdot\nabla\psi_1)(\nabla\psi_1\cdot\nabla\psi_1)$. Its anomalous dimension is computed to be $-5\bar{\epsilon}$, which corresponds to the ground state energy in a harmonic potential

$$E_6^{(0)} = [10 - 2\bar{\epsilon} + O(\bar{\epsilon}^2)] \omega. \quad (90)$$

We note that the leading-order results for $E_N^{(l)}$ can be easily understood by recalling that, in the limit of $d \rightarrow 2$ from above, fermions at unitarity become noninteracting [10,12]. So the energy eigenvalue of each N -fermion state is just a sum of single particle energies in a harmonic potential at $d = 2$. Clearly, the ground state energy shows the shell structure at $d = 2$.

C. Interpolations to $d = 3$ and discussion

We determined the exact scaling dimensions of the one-fermion operator ψ and the two-fermion operator $\phi = \psi_1\psi_1$ in arbitrary spatial dimension d . For the scaling dimensions of N -fermion operators with $N \geq 3$, a few lowest-order terms in the expansions over $\bar{\epsilon} = d - 2$ and $\epsilon = 4 - \epsilon$ were computed as summarized in Table I. According to the operator-state correspondence, we find that the ground state of three fermions in a harmonic potential has the orbital angular momentum $l = 1$ near $d = 2$, while $l = 0$ near $d = 4$. So there must be at least one level crossing between the states with $l = 0$ and $l = 1$ as d increases. Using the ϵ expansions, the spatial dimension at which this level crossing occurs can be estimated to be $d \approx 3.4$, which means that the three-fermion ground state at $d = 3$ has $l = 1$. The same level crossing has to occur for the five-fermion case about $d \approx 3.64$, which implies the five-fermion ground state with $l = 1$ at $d = 3$. On the other hand, the ground state of four or six fermions in a harmonic potential has zero orbital angular momentum near $d = 2$ and $d = 4$. Thus the level crossing

TABLE I. Scaling dimensions of operators corresponding to N -fermion states with orbital angular momentum l in a harmonic potential. Known results for the energy eigenvalues $E_N^{(l)}$ at $d = 3$ are also shown in units of $\hbar\omega$ [13–15].

	$\bar{\epsilon} = d - 2$ expansion	$\epsilon = 4 - d$ expansion	Known value at $d = 3$
$N = 2$ ($l = 0$)	2	2	2
$N = 3$ ($l = 0$)	$5 + O(\bar{\epsilon}^2)$	$4 + \frac{5}{6}\epsilon + O(\epsilon^2)$	4.666 22
$N = 3$ ($l = 1$)	$4 + O(\bar{\epsilon}^2)$	$5 - \frac{5}{6}\epsilon + O(\epsilon^2)$	4.272 72
$N = 4$ ($l = 0$)	$6 - \bar{\epsilon} + O(\bar{\epsilon}^2)$	$4 + 8\epsilon^2 \ln_{16}^{27} + O(\epsilon^3)$	$5.1 \pm 0.1 / 5.07 \pm 0.01$
$N = 5$ ($l = 0$)	$9 - \frac{11+\sqrt{105}}{16}\bar{\epsilon} + O(\bar{\epsilon}^2)$	$6 + \frac{13}{6}\epsilon + O(\epsilon^2)$	-
$N = 5$ ($l = 1$)	$8 - \bar{\epsilon} + O(\bar{\epsilon}^2)$	$7 - \frac{11}{18}\epsilon + O(\epsilon^2)$	7.6 ± 0.1
$N = 6$ ($l = 0$)	$10 - 2\bar{\epsilon} + O(\bar{\epsilon}^2)$	$6 + 24\epsilon^2 \ln_{16}^{27} + O(\epsilon^3)$	$8.7 \pm 0.1 / 8.67 \pm 0.03$

with higher orbital angular momentum states is unlikely and we expect $l = 0$ for the ground state at $d = 3$.

In order to make quantitative discussions, we can use the Padé approximants to interpolate the two expansions around $d = 2$ and $d = 4$. For each operator, we approximate its scaling dimension as a function of $d = 2 + \bar{\epsilon}$ by a ratio of two polynomials;

$$[X/Y] = \frac{a_0 + a_1 \bar{\epsilon} + \cdots + a_X \bar{\epsilon}^X}{1 + b_1 \bar{\epsilon} + \cdots + b_Y \bar{\epsilon}^Y}. \quad (91)$$

We demand that the series expansions of (91) around $d = 2$ and $d = 4$ match the computed results. $X + Y$ is fixed by the number of known terms in the two expansions, while there is a freedom in distributing the sum between X and Y .

The different four Padé approximants for the scaling dimension of each three-fermion operator are plotted as functions of d in Fig. 6. We find the behaviors of the Padé approximants are quite consistent with the exact results both for $l = 0$ (left panel) and $l = 1$ (right panel). For three-fermions with $l = 0$, the interpolated results at $d = 3$ are

$$\begin{aligned} [3/0] &= 4.71, & [2/1] &= 4.7, \\ [1/2] &= 4.72, & [0/3] &= 4.72. \end{aligned} \quad (92)$$

We see that all Padé approximants give very close results in a small interval $E_3^{(0)} \approx 4.71 \pm 0.01$. The harmonic oscillator frequency ω was set to 1 again. This is close to the exact result 4.666 22 at $d = 3$ [13], while the numbers obtained by the Padé interpolations are slight overestimates of the exact value.

Similarly, for three fermions with $l = 1$, the interpolated results are given by

$$\begin{aligned} [3/0] &= 4.29, & [2/1] &= 4.3, \\ [1/2] &= 4.32, & [0/3] &= 4.29, \end{aligned} \quad (93)$$

which span a small interval $E_3^{(1)} \approx 4.30 \pm 0.02$. Again the result is very close to, but slightly larger than the exact value 4.272 72 at $d = 3$ [13]. One may expect that these agreements will be further improved once additional terms in the expansions around $d = 2$ and $d = 4$ are included.

For four fermions with $l = 0$, the Padé interpolations to $d = 3$ give

$$\begin{aligned} [4/0] &= 5.55, & [3/1] &= 4.94, & [2/2] &= 4.94, \\ [1/3] &= 4.90, & [0/4] &= 6.17. \end{aligned} \quad (94)$$

Some of these estimates are not too far from the numerical result 5.1 ± 0.1 [14] and 5.07 ± 0.01 [15] at $d = 3$, but the results span a larger interval $E_4^{(0)} \approx 5.53 \pm 0.64$. This may be because the coefficient of the next-to-leading-order correction around $d = 4$ is sizable compared to the leading term. If one excluded the two extremely asymmetric cases $[4/0]$ and $[0/4]$ where all terms in the Padé approximant come to the numerator or denominator, one would have a rather small interval about $E_4^{(0)} \approx 4.92 \pm 0.02$.

For five fermions with $l = 0$, the interpolated results at $d = 3$ are given by

$$\begin{aligned} [3/0] &= 7.71, & [2/1] &= 7.64, \\ [1/2] &= 7.66, & [0/3] &= 7.82, \end{aligned} \quad (95)$$

which are in an interval $E_5^{(0)} \approx 7.73 \pm 0.09$. On the other hand, for five fermions with $l = 1$, the interpolated results are

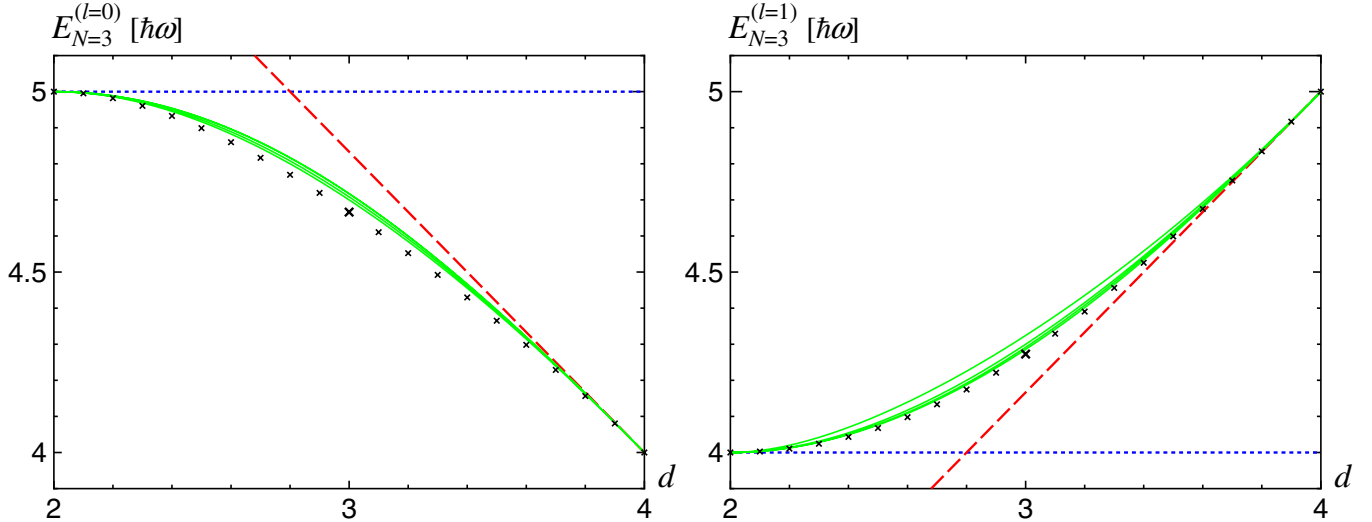


FIG. 6 (color online). Energy of three fermions in a harmonic potential with $l = 0$ (left panel) and $l = 1$ (right panel) as functions of spatial dimension d . The four solid curves, although hard to distinguish, represent different Padé approximants interpolating the expansions around $d = 2$ and $d = 4$. The dashed (dotted) lines are extrapolations from the $\epsilon = 4 - d$ ($\bar{\epsilon} = d - 2$) expansions. The symbols (\times) indicate the exact values for each d [9].

$$\begin{aligned} [3/0] &= 7.10, & [2/1] &= 7.16, \\ [1/2] &= 7.19, & [0/3] &= 7.09, \end{aligned} \quad (96)$$

which are in an interval $E_5^{(1)} \approx 7.14 \pm 0.05$. We find $E_5^{(1)} < E_5^{(0)}$ at $d = 3$, which means $l = 1$ for the five-fermion ground state in a harmonic potential. However, its energy eigenvalue is a substantial underestimate of the numerical result 7.6 ± 0.1 at $d = 3$ [14].

For six fermions with $l = 0$, the Padé interpolations to $d = 3$ give

$$\begin{aligned} [4/0] &= 10.1, & [3/1] &= 7.92, & [2/2] &= 7.92, \\ [1/3] &= 7.80, & [0/4] &= 16.4. \end{aligned} \quad (97)$$

The results now span a considerably larger interval $E_6^{(0)} \approx 12.1 \pm 4.3$, probably because of the huge next-to-leading-order coefficient around $d = 4$. The large error signals the worse convergence of the series expansions as the number of fermions increases. If the two extremely asymmetric cases $[4/0]$ and $[0/4]$ were excluded, one would have $E_6^{(0)} \approx 7.86 \pm 0.06$. For comparison, the numerical result is 8.7 ± 0.1 [14] and 8.67 ± 0.03 [15] at $d = 3$.

As the number of fermions goes to infinity, one can expect the series expansions over $\bar{\epsilon} = d - 2$ and $\epsilon = 4 - \epsilon$ for the ground state energy break down. Indeed, the energy of N fermions at unitarity in a harmonic potential scales with different powers of N in different spatial dimensions as $E_N \sim N^{(d+1)/d}$ for sufficiently large N . Therefore it is not surprising that our extrapolations to $d = 3$ do not work well for five and six fermions. It is possible that the situation is improved once we know the next terms in the expansions around $d = 2$ and $d = 4$.

Here we comment on the convergence of the $\bar{\epsilon}$ and ϵ expansions. Since the exact integral equation to determine the energy eigenvalues of three fermions in a harmonic potential is known [9], one can estimate the radii of convergence of the expansions around $d = 2$ and $d = 4$ by studying their asymptotic behaviors. It turns out that the expansions for the three-fermion state with $l = 0$ are convergent when $|\bar{\epsilon}| \lesssim 1.0$ or $|\epsilon| \lesssim 0.48$, while those with $l = 1$ are convergent when $|\bar{\epsilon}| \lesssim 1.0$ or $|\epsilon| \lesssim 1.4$. On

this basis, we speculate that the expansions over $\bar{\epsilon} = d - 2$ and $\epsilon = 4 - d$ have nonzero radii of convergence for systems with a finite number of particles. The full details will be reported elsewhere [16].

IV. EXAMPLE 2: ANYONS

Anyons in two spatial dimensions present another example of a nonrelativistic CFT. In this section, we compute the scaling dimensions of some operators near the bosonic limit and the fermionic limit, where perturbative expansions in terms of statistical parameter θ are available. Our analytical results, as we will see, are consistent with results obtained by the conventional Rayleigh-Schrödinger perturbation theory in a harmonic potential or numerical simulations.

The field-theoretical representation of anyons is provided by the following Lagrangian density where a nonrelativistic field φ is minimally coupled to a Chern-Simons gauge field $a_\mu = (a_0, \mathbf{a})$:

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\theta} \partial_t \mathbf{a} \times \mathbf{a} - \frac{1}{2\theta} a_0 \nabla \times \mathbf{a} - \frac{1}{2\xi} (\nabla \cdot \mathbf{a})^2 \\ &\quad + i\varphi^* (\partial_t + ia_0) \varphi - \frac{1}{2} |(\nabla - i\mathbf{a})\varphi|^2 - \frac{v}{4} (\varphi^* \varphi)^2. \end{aligned} \quad (98)$$

φ is either a bosonic or fermionic field. We denote the propagator of φ by $G(p)$. In the Coulomb gauge $\xi = 0$, the only nonvanishing components of the a_μ propagator are

$$D_{i0}(p) = -D_{0i}(p) = -2i\theta \frac{\epsilon_{ij} p_j}{p^2}. \quad (99)$$

We define the three-point vertex $\Gamma_0 = -1$, $\Gamma_i(p, p') = (p_i + p'_i)/2m$ and the four-point vertex $\Gamma_{ij} = -\delta_{ij}/m$. The contact interaction coupling v has to be fine-tuned so that the system is scale invariant. We start with the case where φ is bosonic.

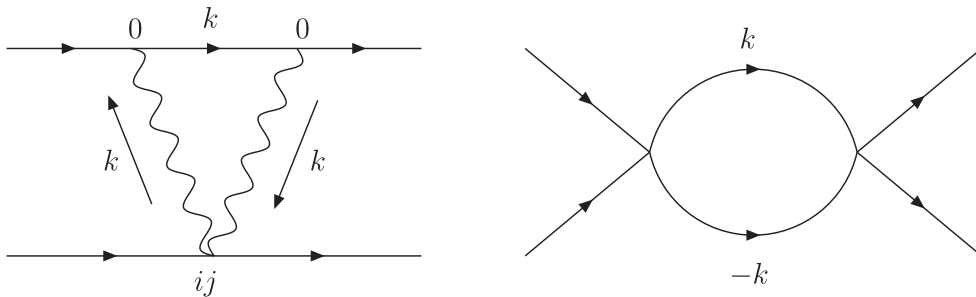


FIG. 7. One-loop diagrams to renormalize the contact interaction coupling v .

A. Near the bosonic limit

1. Fixed points

There are two one-loop diagrams which are logarithmically divergent and renormalize the coupling v (Fig. 7). Integrating out modes in the momentum shell $e^{-s}\Lambda < k < \Lambda$, the first diagram is evaluated as

$$\begin{aligned} & -4 \int \frac{dk}{(2\pi)^3} G(k) \Gamma_0 D_{0i}(k) \Gamma_{ij} D_{j0}(k) \Gamma_0 \\ & = -i \frac{4\theta^2}{\pi} \ln \frac{\Lambda}{e^{-s}\Lambda}, \end{aligned} \quad (100)$$

while the second one as

$$\frac{v^2}{2} \int \frac{dk}{(2\pi)^3} G(k) G(-k) = i \frac{v^2}{4\pi} \ln \frac{\Lambda}{e^{-s}\Lambda}. \quad (101)$$

Therefore the renormalization group equation for v is

$$\frac{\partial v}{\partial s} = \frac{4\theta^2}{\pi} - \frac{v^2}{4\pi}. \quad (102)$$

We find two fixed points located at [17]

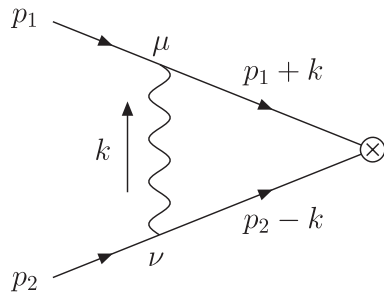
$$v = \pm 4|\theta|. \quad (103)$$

At these fixed points, the theory is a nonrelativistic CFT. The repulsive (upper sign) or attractive (lower sign) contact interaction corresponds to a different boundary condition imposed on the s -wave two-body wave function at origin $\sim r^{\pm|\theta|/\pi}$ [18,19].

2. Scaling dimensions of operators

Since the one-body operator φ is not renormalized, its scaling dimension is $\Delta_\varphi = 1$, independent of θ . Using the operator-state correspondence, we find that there is a one-anyon state in a harmonic potential with energy 1ω , which is obvious.

The two-body operator φ^2 is renormalized by the diagrams depicted in Fig. 8, which are potentially logarithmically divergent and contribute to the anomalous dimension to order θ . Since the first diagram turns out to be finite, we can concentrate on the second diagram which is given by



$$i \frac{v}{2} \int \frac{dk}{(2\pi)^3} G(k) G(-k) = -\frac{v}{4\pi} \ln \frac{\Lambda}{e^{-s}\Lambda}. \quad (104)$$

Therefore the renormalized operator differs from the bare operator by a rescaling factor: $(\varphi^2)_{\text{ren}} = Z_{\varphi^2}^{-1} \varphi^2$, where

$$Z_{\varphi^2} = 1 - \frac{v}{4\pi} s. \quad (105)$$

At the fixed point, $\gamma_{\varphi^2} = -\partial \ln Z_{\varphi^2} / \partial s = \pm |\theta|/\pi$ is the anomalous dimension of the composite operator φ^2 . We thus find the scaling dimension

$$\Delta_{\varphi^2} = 2\Delta_\varphi + \gamma_{\varphi^2} = 2 \pm \frac{|\theta|}{\pi}. \quad (106)$$

According to the operator-state correspondence, φ^2 corresponds to a two-anyon state in a harmonic potential with energy equal to

$$E_2 = \left[2 \pm \frac{|\theta|}{\pi} \right] \omega. \quad (107)$$

It is straightforward to generalize our analysis to the lowest N -anyon operator φ^N . Its scaling dimension is given by

$$\Delta_{\varphi^N} = N \pm \frac{N(N-1)}{2} \frac{|\theta|}{\pi}, \quad (108)$$

and therefore, the corresponding N -anyon state in a harmonic potential has the energy $E_N = \Delta_{\varphi^N} \omega$. This result coincides with the exact energy eigenvalues of N anyons in a harmonic potential [20–22].

B. Near the fermionic limit

Scaling dimensions of operators

If φ is a fermionic field, the contact interaction term $(\varphi^* \varphi)^2$ vanishes and the left diagram in Fig. 7 turns out to be finite. Therefore, the system is automatically scale invariant. We denote the statistical parameter in this case by $\theta' = \theta - \pi$.

Let us first look at the lowest two-body operators $\varphi \nabla_i \varphi$. Inserting this operator into the left diagram of Fig. 8, its

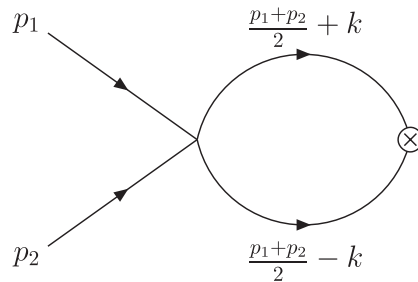


FIG. 8. One-loop diagrams to renormalize the two-anyon operators.

renormalization is given by

$$i \int \frac{dk}{(2\pi)^3} (\mathbf{p}_2 - \mathbf{k})_i G(p_1 + k) \Gamma_\mu(p_1, p_1 + k) D_{\mu\nu}(k) \Gamma_\nu(p_2, p_2 - k) G(p_2 - k) \\ - i \int \frac{dk}{(2\pi)^3} (\mathbf{p}_1 + \mathbf{k})_i G(p_1 + k) \Gamma_\mu(p_1, p_1 + k) D_{\mu\nu}(k) \Gamma_\nu(p_2, p_2 - k) G(p_2 - k) = i \frac{\theta'}{\pi} \epsilon_{ij} (\mathbf{p}_2 - \mathbf{p}_1)_j \ln \frac{\Lambda}{e^{-s}\Lambda}. \quad (109)$$

We thus find the linear combinations $\varphi \nabla_x \varphi \mp i \varphi \nabla_y \varphi$ with well-defined anomalous dimensions $\gamma = \pm \theta'/\pi$. Therefore, the scaling dimensions of such operators are

$$\Delta_{\varphi \nabla_x \varphi \mp i \varphi \nabla_y \varphi} = 3 \pm \frac{\theta'}{\pi}. \quad (110)$$

The operator-state correspondence tells us that the two-anyon states in a harmonic potential have the energies

$$E_2 = \left[3 \pm \frac{\theta'}{\pi} \right] \omega. \quad (111)$$

This result coincides with the exact energy eigenvalues of two anyons in a harmonic potential.

Similarly, we can find the lowest three-body operator $\varphi \nabla_x \varphi \nabla_y \varphi$ has a vanishing anomalous dimension to order θ' . Therefore, the ground state energy of three anyons in a harmonic potential near the fermionic limit is given by

$$E_3 = [5 + O(\theta^2)] \omega. \quad (112)$$

The same result has been derived using the conventional Rayleigh-Schrödinger perturbation theory up to order θ'^2 [20].

We now turn to the four-anyon case. There are four lowest operators that can mix with each other: $\varphi \nabla_x \varphi \nabla_y \varphi \nabla_{xx} \varphi$, $\varphi \nabla_x \varphi \nabla_y \varphi \nabla_{xy} \varphi$, $\varphi \nabla_x \varphi \nabla_y \varphi \nabla_{yy} \varphi$, and $\varphi \nabla_x \varphi \nabla_y \varphi \partial_t \varphi$. The linear combinations with well-defined anomalous dimensions are

- (i) $\varphi \nabla_x \varphi \nabla_y \varphi \partial_t \varphi$ with $\gamma = 0$,
- (ii) $\varphi \nabla_x \varphi \nabla_y \varphi \nabla^2 \varphi$ with $\gamma = 0$,
- (iii) $\varphi \nabla_x \varphi \nabla_y \varphi (\nabla_x - i \nabla_y)^2 \varphi$ with $\gamma = \frac{5}{2} \frac{\theta'}{\pi}$,
- (iv) $\varphi \nabla_x \varphi \nabla_y \varphi (\nabla_x + i \nabla_y)^2 \varphi$ with $\gamma = -\frac{5}{2} \frac{\theta'}{\pi}$.

The combination of the first two operators $\varphi \nabla_x \varphi \nabla_y \varphi (i \partial_t + \frac{1}{2} \nabla^2) \varphi$ annihilates the vacuum and thus does not correspond to any eigenstate of the system in a harmonic potential. The other three operators, therefore, correspond to the energy eigenstates of four anyons in a harmonic potential. According to the operator-state correspondence, the energies of these three states are given by

$$E_4 = [8 + O(\theta^2)] \omega \quad \text{and} \\ E_4 = \left[8 \pm \frac{5}{2} \frac{\theta'}{\pi} + O(\theta^2) \right] \omega. \quad (113)$$

To our knowledge, these analytical results have not been derived so far. Our numbers are consistent with slopes observed in the numerical simulation [23].

V. CONCLUSION

In this paper we study Schrödinger algebra and its representation in terms of operators. We show that irreducible representations are built upon primary operators. We also point out a correspondence between primary operators and eigenstates in a harmonic potential. We illustrate this connection by computing the energy eigenvalues of up to six fermions at unitarity in a harmonic potential using expansions over $4 - d$ and $d - 2$, as well as the energy eigenvalues of up to four anyons in a harmonic potential using expansions over θ and $\theta - \pi$.

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APPENDIX A: COMMUTATORS IN THE SCHRÖDINGER ALGEBRA

The Schrödinger algebra is formed from the operators $N, D, M_{ij}, K_i, P_i, C, H$. The commutators of N and M_{ij} with other operators are

$$[N, D] = [N, M_{ij}] = [N, K_i] = [N, P_i] = [N, C] \\ = [N, H] = 0, \quad (A1)$$

$$[M_{ij}, M_{kl}] = i(\delta_{ik} M_{jk} - \delta_{jk} M_{il} + \delta_{il} M_{kj} - \delta_{jl} M_{ki}), \quad (A2)$$

$$[M_{ij}, K_k] = i(\delta_{ik} K_j - \delta_{jk} K_i), \\ [M_{ij}, P_k] = i(\delta_{ik} P_j - \delta_{jk} P_i), \quad (A3)$$

TABLE II. Part of the Schrödinger algebra. Given are the values of $[A, B]$.

	B	P_j	K_j	D	C	H
A						
P_i		0	$-i\delta_{ij}N$	$-iP_i$	$-iK_i$	0
K_i		$i\delta_{ij}N$	0	iK_i	0	iP_i
D		iP_j	$-iK_j$	0	$-2iC$	$2iH$
C		iK_j	0	$2iC$	0	iD
H		0	$-iP_j$	$-2iH$	$-iD$	0

$$[M_{ij}, C] = [M_{ij}, D] = [M_{ij}, H] = 0. \quad (\text{A4})$$

The rest of the algebra is summarized in Table II.

APPENDIX B: THE STRESS TENSOR

We will find the stress tensor Π_{ij} which appears in the momentum conservation equation:

$$\partial_i j_i + \partial_j \Pi_{ij} = 0. \quad (\text{B1})$$

From the evolution equation for the field operators ψ and ψ^\dagger it follows that

$$\begin{aligned} \partial_i(j(x)) &= \frac{1}{4}(\psi^\dagger \nabla^2 \nabla \psi + \nabla^2 \nabla \psi^\dagger \psi - \nabla^2 \psi^\dagger \nabla \psi \\ &\quad - \nabla \psi^\dagger \nabla^2 \psi)(x) - \int dy \nabla_x V(x-y) :n(x)n(y):, \end{aligned} \quad (\text{B2})$$

where $:\cdots:$ denotes normal ordering [i.e., $:n(x)n(y): = \psi_\alpha^\dagger(x)\psi_\beta^\dagger(y)\psi_\beta(y)\psi_\alpha(x)$]. It is not obvious that the right-hand side of Eq. (B2) can be written as the derivative of a stress tensor. To do that, let us introduce the Laplace transform $\alpha(Q)$ of the function $rV(r)$:

$$4\pi rV(r) = \int_0^\infty dQ \alpha(Q) e^{-Qr}. \quad (\text{B3})$$

In other words, we write the potential $V(r)$ as a superposition of Yukawa potentials,

$$V(r) = \int_0^\infty dQ \alpha(Q) \frac{e^{-Qr}}{4\pi r}. \quad (\text{B4})$$

We also introduce, for each value of Q , an auxiliary field $\sigma_Q(x)$,

$$\sigma_Q(x) = \int dy \frac{e^{-Q|x-y|}}{4\pi|x-y|} n(y). \quad (\text{B5})$$

It satisfies the equation

$$(-\nabla^2 + Q^2)\sigma_Q(x) = n(x). \quad (\text{B6})$$

The stress tensor can now be introduced:

$$\begin{aligned} \Pi_{ij} &= \frac{1}{2}(\partial_i \psi^\dagger \partial_j \psi + \partial_j \psi^\dagger \partial_i \psi) - \frac{1}{4} \delta_{ij} \nabla^2 n \\ &\quad + \int_0^\infty dQ \alpha(Q) \left\{ -\partial_i \sigma_Q \partial_j \sigma_Q \right. \\ &\quad \left. + \frac{\delta_{ij}}{2} [(\nabla \sigma_Q)^2 + Q^2 \sigma_Q^2] \right\}. \end{aligned} \quad (\text{B7})$$

By using Eq. (B6) it is straightforward to verify that Eq. (B1) is satisfied.

Notice that Π_{ij} is not unique. For example, one can replace

$$\Pi_{ij} \rightarrow \Pi_{ij} + (\partial_i \partial_j - \delta_{ij} \nabla^2) \Phi \quad (\text{B8})$$

with any Φ without destroying the momentum conservation.

Let us now show that if $V(r)$ is a short-range potential with infinite scattering length, then

$$\int d\mathbf{x} \Pi_{ii}(x) = 2H. \quad (\text{B9})$$

By using Eq. (B6) and the following property of the Yukawa potential,

$$\int d\mathbf{x} \frac{e^{-Q|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \frac{e^{-Q|\mathbf{x}-\mathbf{z}|}}{4\pi|\mathbf{x}-\mathbf{z}|} = \frac{e^{-Q|\mathbf{y}-\mathbf{z}|}}{8\pi Q} \quad (\text{B10})$$

(which can be shown, e.g., by using the Fourier transforms), we find

$$\begin{aligned} \int d\mathbf{x} \Pi_{ii}(x) &= 2T + V \\ &\quad + \int dQ d\mathbf{x} dy \alpha(Q) Q \frac{e^{-Q|\mathbf{x}-\mathbf{y}|}}{8\pi} :n(x)n(y):, \end{aligned} \quad (\text{B11})$$

where T is the kinetic energy and V is the potential energy. Since $V(r)$ is a short-range potential giving an infinite scattering length, the low-energy physics does not change when one rescales the potential as

$$V(r) \rightarrow \lambda^2 V(\lambda r). \quad (\text{B12})$$

In particular the Hamiltonian is unchanged under the transformation (B12). Setting $\lambda = 1 + \epsilon$, $\epsilon \ll 1$ and expanding H to the linear order in ϵ , we find

$$\int dQ d\mathbf{x} dy \alpha(Q) Q \frac{e^{-Q|\mathbf{x}-\mathbf{y}|}}{8\pi} :n(x)n(y): = V. \quad (\text{B13})$$

Therefore, we obtain Eq. (B9).

One can use this relationship to prove that in the normal phase (above the critical temperature), the bulk viscosity of a Fermi gas at unitarity is identically zero. Indeed, the bulk viscosity is given by Kubo's formula:

$$\zeta = \lim_{\omega \rightarrow 0} \frac{1}{9\omega} \int_0^\infty dt \int d\mathbf{x} e^{i\omega t} \langle [\Pi_{ii}(t, \mathbf{x}), \Pi_{jj}(0, \mathbf{0})] \rangle, \quad (\text{B14})$$

but the integral over \mathbf{x} can be taken according to Eq. (B9). Moreover, as $\langle [H, \mathcal{O}] \rangle = 0$ in thermal equilibrium for any operator \mathcal{O} , the bulk viscosity is zero. This result was derived previously using a different approach [24].

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