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Determinant of the SU(N) caloron with nontrivial holonomy

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The 1-loop quantum weight of the SU(N) Kraan-van Baal–Lee-Lu caloron with nontrivial holonomy is calculated. The caloron is the most general self-dual solution with unit topological charge in the 4d Yang-Mills theory with one compactified dimension (finite temperature).

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I. INTRODUCTION

The finite-temperature field theory is defined by considering the Euclidean space-time which is compactified in the "time" direction whose inverse circumference is a temperature T, with the usual periodic boundary conditions for boson fields and antiperiodic conditions for fermion fields. In particular, the gauge field is periodic in time, so the theory is no longer invariant under arbitrary gauge transformations. Only time-periodic gauge transformations are allowed and hence the number of gauge invariants increases. The new invariant is the holonomy or the eigenvalues of the Polyakov line that winds along the compact time direction [1]:

$$L = \mathrm{P}\exp\left(\int_0^{1/T} dtA_4\right)\bigg|_{|\vec{x}|\to\infty}. \qquad (1)$$

This invariant together with the topological charge and the magnetic charge can be used for the classification of the field configurations [2]; its zero vacuum average is one of the common criteria of confinement.

A generalization of the usual Belavin-Polyakov-Schwartz-Tyupkin (BPST) instantons [3] for arbitrary temperatures and holonomies is the Kraan-van Baal-Lee-Lu (KvBLL) caloron with nontrivial holonomy [4–6]. It is a self-dual electrically neutral configuration with unit topological charge and arbitrary holonomy. This solution was constructed by Kraan and van Baal [4] and Lee and Lu [6] for the SU(2) gauge group, and in [5] for the general SU(N) case; it has been named the KvBLL caloron (recently, the exact solutions of higher topological charge were constructed and discussed [7,8]). There are plenty of lattice studies supporting the presence of these solutions [9]; see also [10] for a very brief review. In a recent paper [11] the caloron ensemble was studied analytically; although some contributions were neglected there, the results are in very good agreement with phenomenology.

The holonomy is called "trivial" if the Polyakov loop (1) acquires values belonging to the group center $Z(N)$. For this case the KvBLL caloron reduces to the periodic Harrington-Shepard [12] caloron known before. The latter is purely an SU(2) configuration and its quantum weight was studied in detail by Gross, Pisarski, and Yaffe [2].

The KvBLL caloron in the theory with the SU(N) gauge group on the space $R^3 \times S^1$ can be interpreted as a composite of N distinct fundamental monopoles (dyons) [13,14] (see Fig. 1). As was proven in [5,15], the exact KvBLL gauge field reduces to a superposition of BPS [16] dyons, when the separation ϱ_i between dyons is large (in units of inverse temperature). On the contrary, the KvBLL caloron reduces to the usual BPST instanton, when the distances ϱ_l between all the dyons become small compared to the inverse nontriviality of holonomy.

We refer the reader to the papers [5,15] for the detailed discussion and construction of the caloron solutions, to the original works [4] for the SU(2) case, and to further works on higher topological charge solutions [7,8].

This paper is in the series of papers [15,17–20] where we calculate the functional determinant for KvBLL calorons with nontrivial holonomy [4,6] in the finite-temperature Yang-Mills theory.

Here we calculate the 1-loop gluonic and ghost functional determinants for the case of an arbitrary SU(N) gauge group. The calculation is performed in the limit of far-separated dyon constituents and up to an overall numerical constant. The constant for the gluonic determinant remains known only for the SU(2) case [17,21].

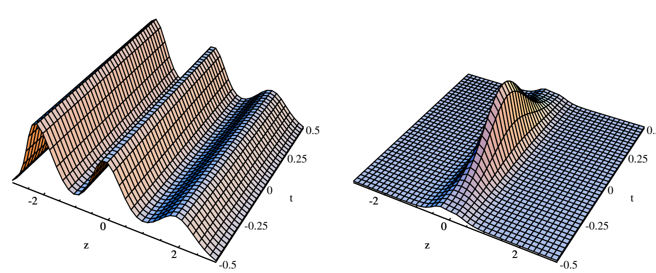

FIG. 1 (color online). The action density of the $SU(3)$ KvBLL caloron as a function of z, t at fixed $x = y = 0$; eigenvalues of A_4 at spatial infinity are $\mu_1 = -0.307T$, $\mu_2 = -0.013T$, $\mu_3 = 0.32T$. The action density is periodic in the t direction. At large dyon separation the density becomes static (left panel, $\varrho_{1,2} = 1/T$, $\varrho_3 = 2/T$). As the separation decreases the action density becomes more like a 4d lump (right panel, $\varrho_{1,2} = 1/(3T)$, $\varrho_3 = 2/(3T)$). The axes are in units of inverse temperature $1/T$.

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We find new 3-particle interactions arising between constituent dyons due to the gluonic determinant. These terms were not present in the fermionic (fundamental-representation) determinant and also vanished in the SU(2) gluonic determinant.

Similar to our previous results, the determinant is infrared divergent, the leading divergence is proportional to the volume or the system, and there are, of course, several subleading divergent terms. It is not surprising and was known long ago [2,22] that nontrivial holonomy increases the effective action by a factor proportional to the volume. Nevertheless, that does not make the studies of nontrivial holonomy unphysical, since in the ensemble of many calorons the moduli space integrals can compensate the above divergences.

Since there are subleading divergences coming from the Coulomb tail of dyon fields, it is natural that the result would also depend on the position of the large ball, with which we make the infrared cutoff. We will display this dependence, but we note that it is unphysical unless the box in not a real border of plasma region. One could also expect that the quantum corrections will dump the Coulomb tails of dyons.

We present the relevant notations in Sec. II and illustrate the notations by the old results. The method of computation is described in Sec. III and the actual computations are carried out in subsequent sections and appendixes. The final result is presented in Sec. VI.

We do not draw here any physical conclusions on the behavior of the whole caloron ensemble since that is now a separate business [11,23]. Our results could be useful for extending Ref. [23] to the $SU(N>2)$ case and including the corrections due to the nonzero modes in the work [11].

II. NOTATIONS AND REVIEW

Consider the SU(N) Yang-Mills theory and a caloron solution with the asymptotics [24]

$$A_\mu \overset{\vec{x}\to\infty}{\to} 2\pi\delta_{\mu 4}\mathrm{diag}(\mu_1, \ldots, \mu_N).$$

For the SU(2) case the standard choice is $\mu_1 = -\omega$, $\mu_2 = \omega$ where $0 \le \omega < \frac{1}{2}$. As usual, we set the temperature $T = 1$ throughout the computation, and restore the temperature dependence only in the final result.

The caloron can be viewed as composed of dyons (BPS monopoles with A_4 playing the role of a Higgs field), the inverse dyon size ν being defined as

$$\nu_l = \mu_{l+1} - \mu_l; \qquad \nu_N = \mu_1 - \mu_N + 1. \tag{2}$$

Traditionally the first $N-1$ dyons are called the "M dyons" and the Nth dyon is called the "L dyon," because an additional gauge transformation is needed for it to have correct asymptotics.

We also introduce a notation

$$\mathrm{v}_{mn} \equiv 2\pi(\mu_m - \mu_n) \bmod 2\pi, \tag{3}$$

which coincides with $\mathrm{v} = \mathrm{v}_{21}$ and $\bar{\mathrm{v}} = \mathrm{v}_{12}$ used previously in the SU(2) calculations.

The positions of dyon centers are denoted by y_i. The distance from the ith dyon center to a point x is denoted as $\vec{x} - \vec{y}_i = r_i$; for the SU(2) case the standard notation is $r_1 = s$; $r_2 = r$ [17]. The distance between dyon cores is denoted by $r_{ij} = |\vec{r}_j - \vec{r}_i|$.

It is convenient to use a so-called "algebraic gauge," in which the asymptotic gauge field is vanishing at the expense of introducing twisted boundary conditions for field fluctuations. The twist $a(\vec{x}, 1/T) = e^{-i\tau}a(\vec{x}, 0)$ is hence related to the holonomy as $\tau = 2\pi\mathrm{diag}(\mu_1, \ldots, \mu_N)$. The holonomy and, correspondingly, the twist could also be multiplied by elements of the center of the gauge group $e^{2\pi i(k/N)}$. It does not affect the adjoint gauge field and determinant but it affects fundamental determinants [15]:

$$\log \mathrm{Det}(-\nabla_N^2) = \sum_n \Big(\frac{\pi}{4}P''(\tau_n)r_{n,n-1} + \frac{1}{2}P(\tau_n)V^{(3)} - \frac{\nu_n \log\nu_n}{6} - \frac{\log r_{n,n-1}}{12\pi r_{n,n-1}}\Big) + c_N + \frac{1}{6}\log\mu + \mathcal{O}(1/r) \tag{4}$$

where

$$c_N = -\frac{13}{72} - \frac{\pi^2}{216} + \frac{\log\pi}{6} - \frac{\zeta'(2)}{\pi^2}. \tag{5}$$

P is a periodical function with a period 2π such that

$$P(\mathrm{v}) = \frac{q^2(2\pi - q)^2}{12\pi^2}; \qquad q = \mathrm{v} \bmod 2\pi. \tag{6}$$

The determinant in the adjoint representation of SU(2) reads [17,20]

$$\log \mathrm{Det}(-D_2^2) = VP(\mathrm{v}) + 2\pi P''(\mathrm{v})r_{12} + \frac{3\pi - 4\mathrm{v}}{3\pi}\log\mathrm{v} + \frac{3\pi - 4\bar{\mathrm{v}}}{3\pi}\log\bar{\mathrm{v}} + \frac{2}{3}\log\mu + \frac{5}{3}\log(2\pi) + c_2 + \frac{1}{r_{12}}\Big[\frac{1}{\mathrm{v}} + \frac{1}{\bar{\mathrm{v}}} + \frac{23\pi}{54} - \frac{8\gamma_E}{3\pi} - \frac{74}{9\pi} - \frac{4}{3\pi}\log\Big(\frac{\mathrm{v}\bar{\mathrm{v}}r_{12}^2}{\pi^2}\Big)\Big] + \mathcal{O}\Big(\frac{1}{r_{12}^2}\Big). \tag{7}$$

Now we proceed to the calculation of the SU(N) adjoint-representation determinant.

III. METHOD OF COMPUTATION

For self-dual fields the gluonic and ghost determinants over nonzero modes for the background gauge fixing are related [25] to the adjoint scalar determinant in the same background: $\mathrm{Det}'(W_{\mu\nu}) = \mathrm{Det}(-D^2)^4$, where $W_{\mu\nu}$ is the quadratic form for spin-1, adjoint-representation quantum fluctuations and D^2 is the covariant Laplace operator for spin-0, adjoint-representation ghost fields. So the total contribution to the effective action of gluon and ghost determinants is $2\log\mathrm{Det}(-D^2)$ which corresponds to two physical degrees of freedom.

We calculate the quantum determinant by integrating its variation with respect to parameters $\mathcal{P}$ of the solution, following [15,17,18,26]. In this case the problem reduces to a four dimensional integral of the gauge field variation multiplied by a vacuum current, which can be expressed through the Green function known implicitly for any self-dual configuration,

$$\frac{\partial \log \mathrm{Det}(-D^2[A])}{\partial \mathcal{P}} = -\int d^4x\,\mathrm{tr}(\partial_{\mathcal{P}} A_\mu J_\mu), \qquad (8)$$

where J_μ is the vacuum current in the external background, determined by the Green function:

$$J_\mu \equiv \overrightarrow{D}_\mu \mathcal{G} + \mathcal{G}\overleftarrow{D}_\mu. \qquad (9)$$

Here $\mathcal{G}$ is the periodical Green function of the covariant Laplace operator in the adjoint representation

$$-D_x^2 G(x,y) = \delta^{(4)}(x-y), \qquad (10)$$

$$\mathcal{G}(x,y) = \sum_{n=-\infty}^{+\infty} G(x_4,\vec{x}; y_4+n,\vec{y}). \qquad (11)$$

The Green functions in the self-dual backgrounds are known explicitly [27,28] if the gauge field is expressed in terms of the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [29]: $A_\mu = v^\dagger \partial_\mu v$. These look quite simple for the fundamental representation [30]

$$G^{\mathrm{fund}}(x,y) = \frac{v^\dagger(x)v(y)}{4\pi^2(x-y)^2}, \qquad (12)$$

but become more complicated for the adjoint representation [31–33]

$$\begin{aligned} G^{ab}(x,y) = {} & \frac{\frac{1}{2}\,\mathrm{tr}\, t^a \langle v(x)|v(y)\rangle t^b \langle v(y)|v(x)\rangle}{4\pi^2(x-y)^2} \\ & + \frac{1}{4\pi^2}\int_{-1/2}^{1/2} dz_1 dz_2 dz_3 dz_4 M(z_1,z_2,z_3,z_4) \\ & \times \frac{1}{2}\,\mathrm{tr}(\mathcal{V}^\dagger(x,z_1)\mathcal{V}(x,z_2)t^a) \\ & \times \mathrm{tr}(\mathcal{V}^\dagger(y,z_4)\mathcal{V}(y,z_3)t^b), \end{aligned} \qquad (13)$$

where t^a are Hermitian fundamental-representation generators of SU(N) normalized to $\mathrm{tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$; $\mathcal{V}(x,z)$ is one of the components of v [see Eq. (B21)]; and M is a piecewise rational function [34]. Fortunately we do not need an explicit form of this function for the SU(N) caloron since in the large separation limit the contribution of the last term (or M-term) is exponentially small away from the dyons. Near the dyons, the field is essentially reduced to the SU(2), so one can use there the results of [17,32].

In what follows it will be convenient to split the periodic propagator into three parts and consider them separately:

$$\begin{aligned} \mathcal{G}(x,y) &= \mathcal{G}^{\mathrm{r}}(x,y) + \mathcal{G}^{\mathrm{s}}(x,y) + \mathcal{G}^{\mathrm{m}}(x,y), \\ \mathcal{G}^{\mathrm{s}}(x,y) &\equiv G(x,y), \\ \mathcal{G}^{\mathrm{r}}(x,y) + \mathcal{G}^{\mathrm{m}}(x,y) &\equiv \sum_{n\neq 0} G(x_4,\vec{x}; y_4+n,\vec{y}). \end{aligned} \qquad (14)$$

Here $\mathcal{G}^{\mathrm{m}}(x,y)$ corresponds to the part of the propagator arising from the M-term. The vacuum current (9) will also be split into three parts, "singular," "regular," and "M," in accordance with (14)

$$J_\mu = J_\mu^{\mathrm{r}} + J_\mu^{\mathrm{s}} + J_\mu^{\mathrm{m}}. \qquad (15)$$

As was proposed in [17] we divide the space into regions surrounding the dyons and the remaining space (outer region). Near each of the dyons the gauge field becomes essentially the SU(2) dyon configuration plus an additional constant-field background. In this region we can use the results of [17]. In the outer region, far from the exponential cores of the dyons, the vacuum current considerably simplifies and we only have to perform integration in (8).

In the following two sections we give results for these two domains, and in Sec. we combine them together and integrate over the space.

IV. CORE DOMAIN

In this section we write a contribution to the variation of the total determinant arising from the core region of a dyon. We take a ball of radius R around the dyon. In that region the field is approximately the one of the SU(2) dyon, embedded along one of the simple roots, *plus* an extra constant A_4 field [5]. More precisely, in the fundamental representation the gauge field near the lth dyon is a zero $N\times N$ matrix with only a 2×2 block at the lth position filled by the BPS dyon gauge field, plus a constant diagonal $N\times N$ matrix [15],

$$\begin{aligned} A_\mu^{l^{\mathrm{th}}\ \mathrm{block}\ 2\times 2} &= A_\mu^{\mathrm{dyon}}(\nu_l, \vec{x}-\vec{y}_l) \\ &\quad + 2\pi i\left(\frac{\mu_l+\mu_{l+1}}{2}\right)\delta_{\mu 4} 1_{2\times 2}, \\ A_\mu^{\mathrm{outside}\ l^{\mathrm{th}}\ \mathrm{block}\ 2\times 2} &= 2\pi i\,\mathrm{diag}(\mu_1,\ldots,\mu_N)\delta_{\mu 4}. \end{aligned} \qquad (16)$$

Under the action of the SU(2) subgroup, the adjoint representation of SU(N) splits into one triplet, $2(N-2)$ doublets, and $(N-2)^2$ singlets. The determinant of the arbitrary SU(2) configuration embedded into SU(N) is then expressed as a sum of the SU(2) adjoint-representation determinant plus $2(N-2)$ fundamental-representation determinants [35].

As is seen from Eq. (16), the SU(2) dyon field is accomplished by the constant diagonal matrix. This matrix can be killed by a gauge transformation, which is not periodical, and thus can change the determinant. It is equivalent to the additional twist of boundary conditions for the $2(N-2)$ fundamental-representation determinants.

As a demonstration let us consider the SU(3) case. The fundamental gauge field reads

$$A_\mu = \begin{pmatrix} A^{\rm dyon}_{\mu 2\times 2} & 0 \\ & 0 \\ 0 \quad 0 & 0 \end{pmatrix} + 2\pi i \delta_{\mu 4} \begin{pmatrix} \frac{\mu_1+\mu_2}{2} & 0 & 0 \\ 0 & \frac{\mu_1+\mu_2}{2} & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}. \tag{17}$$

In the adjoint representation of SU(3) in an appropriate basis it becomes

$$\hat{A}_\mu = \begin{pmatrix} \hat{A}^{\rm dyon}_{\mu 3\times 3} & 0 & 0 & 0 \\ 0 & -A^{\rm dyon}_{\mu 2\times 2} - i\pi\delta_{\mu 4}(\nu_3-\nu_2) & 0 & 0 \\ 0 & 0 & A^{\rm dyon}_{\mu 2\times 2} + i\pi\delta_{\mu 4}(\nu_3-\nu_2) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

So, there is one block 3×3 giving an adjoint-representation dyon field and two 2×2 blocks giving a fundamental-representation dyon accomplished by a unit matrix. As it was shown in [15] this extra unit matrix only changes an IR divergent part of the dyon determinant (the one, depending on radius of the ball). These divergences cancel with the terms in the outer-region determinant depending on the radius of the holes R, as is shown in Appendix A. So we can freely drop them.

Summing up one adjoint [(A4)] and $2(N-2)$ fundamental dyon determinants [(A3)] for all N dyons, we obtain the following contribution to the derivative of the caloron determinant from the considered domain:

$$\partial_{\mathcal{P}} \sum_n \left(-\frac{(6+N)\nu_n \log(\nu_n)}{3} + \log(\nu_n) \right) + \text{IR}, \tag{19}$$

where "IR" denotes the IR divergent terms.

V. OUTER DOMAIN

We proceed to consider the far domain, i.e. the region of space outside the dyons' cores. The caloron field becomes diagonal with $\mathcal{O}(e^{-\nu_i r_i})$ precision, and this significantly simplifies the results. For instance, the 4th component of the fundamental caloron gauge field reads

$$A_4^{mn} = i\delta^{mn}\left(2\pi\mu_m + \frac{1}{2r_m} - \frac{1}{2r_{m-1}}\right). \tag{20}$$

In what follows, we will consider the derivative of the determinant with respect to μ_m. It turns out that in this domain only A_4 depends on μ_m nontrivially [5]. Thus we need only the 4th component of the vacuum current as follows from (8). As we know from the SU(2) case, this component of the current is especially simple [17]:

$$J_4^{\rm SU(2)} = \frac{i}{2} T_3 P'\left(v + \frac{1}{r_1} - \frac{1}{r_2}\right). \tag{21}$$

The natural generalization of this expression is

$$J_4^{\rm SU(N)} = \operatorname*{diag}_{n,m=1}^{N} \left[\frac{i}{2} P'\left(2\pi(\mu_m - \mu_n) + \frac{1}{2r_m} - \frac{1}{2r_{m-1}} - \frac{1}{2r_n} + \frac{1}{2r_{n-1}} \right) \right]. \tag{22}$$

The expression in brackets is simply the eigenvalue of the gauge field (20) in the adjoint representation. This formula is definitely right for large r_m, where the field becomes almost constant [2,36] and generalizes the SU(2) expression. Moreover, we check it by a direct computation in Appendix B. We conclude that

$$-\text{tr}(\partial_{\mathcal{P}} A_\mu J_\mu) = \frac{1}{2}\sum_{n,m} \partial_{\mathcal{P}} P\left(\text{v}_{mn} + \frac{1}{2r_m} - \frac{1}{2r_{m-1}} - \frac{1}{2r_n} + \frac{1}{2r_{n-1}} \right) \tag{23}$$

where

$$\text{v}_{mn} \equiv 2\pi(\mu_m - \mu_n). \tag{24}$$

The variation over $\mathcal{P}$ can be integrated up to a constant, and the integral over space will be performed in the next section.

Integration

In order to get a variation of the determinant we have to integrate in Eq. (23) over the space with N spherical holes of radius R. The following integrals will be very helpful:

$$\int \left(\frac{1}{2r_m} - \frac{1}{2r_{m-1}} - \frac{1}{2r_n} + \frac{1}{2r_{n-1}} \right)^2 d^3x \simeq \pi(r_{m,n} + r_{m-1,n-1} - r_{m,n-1} - r_{m-1,n} + r_{m,m-1} + r_{n,n-1}), \tag{25}$$

$$\int\left(\frac{1}{2r_m}-\frac{1}{2r_{m-1}}-\frac{1}{2r_n}+\frac{1}{2r_{n-1}}\right)^3 d^3x \simeq -3\pi\log\left(\frac{r_{m,n-1}}{r_{n,m-1}}\frac{r_{m,n,m-1}}{r_{n,m,n-1}}\frac{r_{n,m-1,n-1}}{r_{m,n-1,m-1}}\right) \tag{26}$$

where $r_{nm}=|\vec{y}_n-\vec{y}_m|$ is the distance between dyons and

$$2r_{lmn}\equiv r_{lm}+r_{mn}+r_{nl} \tag{27}$$

is the perimeter of the triangle, formed by lth, mth, nth dyons. The $\simeq$ sign means that we drop all the terms dependent on the radius of the holes R since they cancel precisely with the dyons' IR divergences as discussed in Appendix A. To derive the last equation we used

$$\int\frac{1}{r_nr_mr_l}d^3x\simeq -4\pi\log r_{nml}+C.$$

It is important to point out that Eq. (26) is *not* applicable for the case $m=n\pm 1$, since it diverges. The reason is that the divergences near dyon cores are not balanced anymore. Nevertheless it is straightforward to verify that if one replaces a zero r_{nn} under the logarithm in Eq. (26) by some fixed ϵ, then it is still valid up to a constant, which cancels in the final result.

So we can integrate in Eq. (23)

$$\int\frac{1}{2}\sum_{m,n}P\left(\mathrm{v}_{mn}+\frac{1}{2r_m}-\frac{1}{2r_{m-1}}-\frac{1}{2r_n}+\frac{1}{2r_{n-1}}\right)d^3x\simeq\sum_{m,n}\frac{\pi}{4}P''(\mathrm{v}_{mn})(r_{m,n}+r_{m-1,n-1}-r_{m,n-1}-r_{m-1,n}+r_{m,m-1}+r_{n,n-1})-\sum_{m,n}\frac{[\mathrm{v}_{mn}]-\pi}{2\pi}\log\left(\frac{r_{m,n-1}}{r_{n,m-1}}\frac{r_{m,n,m-1}}{r_{n,m,n-1}}\frac{r_{n,m-1,n-1}}{r_{m,n-1,m-1}}\right)+\sum_{n,m}\frac{1}{2}P(\mathrm{v}_{mn})V^{(3)}. \tag{28}$$

We denote $[\mathrm{v}_{mn}]=\mathrm{v}_{mn}\bmod 2\pi$. To simplify the above expression we use the identity

$$\sum_{m,n}\frac{[\mathrm{v}_{mn}]-\pi}{4\pi}\log\frac{r_{m,n,m-1}r_{n,m-1,n-1}}{r_{n,m,n-1}r_{m,n-1,m-1}}=\sum_{m,n}\nu_n\log r_{m,n,m-1}-\sum_n\log r_{n,n-1}. \tag{29}$$

Then Eq. (28) becomes

$$\begin{aligned}\log\mathrm{Det}(-D_N^2)^{\mathrm{far}}&=\int\frac{1}{2}\sum_{m,n}P\left(\mathrm{v}_{mn}+\frac{1}{2r_m}-\frac{1}{2r_{m-1}}-\frac{1}{2r_n}+\frac{1}{2r_{n-1}}\right)d^3x\\&\simeq-\sum_{m,n}\frac{[\mathrm{v}_{mn}]-\pi}{2\pi}\log\left(\frac{r_{m,n-1}}{r_{n,m-1}}\right)-\sum_{m,n}2\nu_n\log r_{m,n,m-1}+2\sum_n\log r_{n,n-1}\\&\quad+\sum_{m,n}\frac{\pi}{4}P''(\mathrm{v}_{mn})(r_{m,n}+r_{m-1,n-1}-r_{m,n-1}-r_{m-1,n}+r_{m,m-1}+r_{n,n-1})+\sum_{m,n}\frac{1}{2}P(\mathrm{v}_{mn})V^{(3)}.\end{aligned} \tag{30}$$

The "R-terms" are exactly the ones of Eq. (A6) but with R appearing as a lower limit of integration; this provides their cancellation when we add the core contribution. The second equality in (30) is valid when the variation does not involve changing of the far region. Note that the $\frac{\log r}{r}$ correction comes only from the far region, so we can calculate it. It comes from the next P'''' term in the Taylor series; this term obviously involves 4-center Coulomb integrals:

$$\int\frac{d^3x}{r_1r_2r_3r_4}, \tag{31}$$

taken over R^3 with holes around centers. Since this integral converges both in the IR and UV (near the holes), it can involve only logarithms of some dimensionless combinations of distances between these four points, divided by the distance. In the approximation that the dyons are spread homogeneously, such terms would be of order of unity, and we neglect them. The only *large* logarithms come from the case where three of the four points coincide; in this case the integral diverges as the logarithm near the ith dyon:

$$\int_{\mathrm{R}^4\backslash B_R}\frac{1}{r_i^3r_j}=4\pi\frac{\log(r_{ij}/R)}{r_{ij}}+\mathcal{O}(1/r_{ij}). \tag{32}$$

So for the correction to $\log\mathrm{Det}(-D^2)$, one sums all the contributions of the form (32). Note that $P^{\mathrm{IV}}=\frac{2}{\pi}$ is a constant, so some terms cancel in the sum. The result for $N>2$ is

$$\log\mathrm{Det}(-D^2)_{\mathrm{correction}}=-\frac{6+N}{6\pi}\sum_{n>m}^{N}\frac{\log r_{nm}}{r_{nm}}. \tag{33}$$

For $N=2$ the coefficient is doubled and becomes $-\frac{8}{3\pi}$, since there are more coincident points. This matches our SU(2) result [Eq. (60) in [17]].

VI. THE RESULT

From Eqs. (19) and (30), we can conclude that for large dyons' separations, $\varrho_m \ll 1/\nu_m + 1/\nu_{m-1}$, the SU(N) caloron determinant is the sum of these expressions plus some integration constant and $\frac{\log r}{r}$ improvement (33). Restoring the temperature dependence we obtain

$$\log \mathrm{Det}(-D_N^2) = -\sum_{m,n} \frac{[\mathrm{v}_{mn}] - \pi}{2\pi} \log\left(\frac{r_{m,n-1}}{r_{n,m-1}}\right) - \sum_{m,n} 2\nu_n \log r_{m,n,m-1} + 2\sum_n \log r_{n,n-1} + \sum_{m,n} \frac{\pi}{4} P''(\mathrm{v}_{mn}) T (r_{m,n} + r_{m-1,n-1} - r_{m,n-1} - r_{m-1,n} + r_{m,m-1} + r_{n,n-1}) + \sum_{m,n} \frac{1}{2} P(\mathrm{v}_{mn}) T^3 V^{(3)} - \sum_n \frac{(6+N)\nu_n \log \nu_n}{3} + \sum_n \log \nu_n - \frac{6+N}{6\pi} \frac{\log r_{nm} T}{r_{nm} T} + c_N. \quad (34)$$

Note that the coefficient $-\frac{6+N}{6\pi}$ should be doubled for the $N = 2$ case.

The contribution to the effective action from nonzero modes of gluons and ghosts would be

$$\delta S_{\mathrm{eff}} = -\log \frac{\mathrm{Det}(-D^2)}{(\mathrm{Det}' W_{\mu\nu})^{1/2}} = \log \mathrm{Det}(-D^2). \quad (35)$$

The constant c_N will, of course, contain a standard UV divergence, $c_N = c + \frac{N}{3} \log \mu_{\mathrm{PV}}$, coming from the instanton determinant [37], where μ_{PV} is a Pauli-Villars mass. This divergence, together with $(\frac{\mu_{\mathrm{PV}}}{g\sqrt{2\pi}})^{4N}$ coming from zero modes, gives the standard Yang-Mills β function and is commonly incorporated into the running coupling:

$$\mu_{\mathrm{PV}}^{(11/3)N} e^{-[8\pi^2/g^2(\mu_{\mathrm{PV}})]} = \Lambda^{(11/3)N} \quad (36)$$

where Λ is the scale parameter obtained here through the "transmutation of dimensions."

Now let us combine with the result for SU(N) caloron zero modes [38,39] and the classical action $8\pi^2/g^2(\mu_{\mathrm{PV}})$. The caloron measure is [39]

$$\int_{\mathcal{G}} \omega \simeq 2^{6N} \pi^{4N} \left[1 + \sum_m \frac{1}{4\pi\varrho_m}\left(\frac{1}{\nu_{m-1}} + \frac{1}{\nu_m}\right)\right] \times \prod_n \nu_n d^3\varrho_1 \ldots d^3\varrho_{N-1} d^4\xi. \quad (37)$$

So the total contribution of one caloron to the effective action becomes

$$e^{-S_{\mathrm{eff}}} \approx \left(\frac{\Lambda e^{\gamma_E}}{4\pi T}\right)^{(11/3)N} C_N \int (\mathrm{Det}(-D_N^2))^{-1} \left(\frac{8\pi^2}{g^2(\mu_{\mathrm{PV}})}\right)^{2N} \left[1 + \sum_m \frac{1}{4\pi\varrho_m}\left(\frac{1}{\nu_{m-1}} + \frac{1}{\nu_m}\right)\right] \prod_n \nu_n d^3\varrho_1 \ldots d^3\varrho_{N-1} d^4\xi. \quad (38)$$

We have collected the factor $4\pi e^{-\gamma_E} T/\Lambda$ because it is the natural argument of the running coupling constant at nonzero temperatures [36,40]. When we have done so in the SU(2) case [17], we got a constant numerically very close to 1, so we expect the constant C to be of order of unity.

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APPENDIX A: CANCELLATION OF IR DIVERGENCES OF DYONS

Consider the field near the dyon constituent of the SU(N) caloron. In the fundamental representation it is given by Eq. (16). In the adjoint representation this field looks like one block with the SU(2) BPS dyon in the SU(2)-adjoint representation (3×3) and $2(N-2)$ blocks with the SU(2) BPS dyon in the SU(2)-fundamental representation (2×2) *plus* a constant part, specified below:

$$-iA_{\mathrm{const}}^{\mathrm{adj}} = \pi \mathrm{diag}(0, 0, 0, -(\mu_1 + \mu_2 - 2\mu_3), -(\mu_1 + \mu_2 - 2\mu_3), +(\ldots), +(\ldots), \ldots, +(\mu_1 + \mu_2 - 2\mu_N), + (\mu_1 + \mu_2 - 2\mu_N), \mu_3 - \mu_4, \mu_4 - \mu_3, \ldots (\text{all pairs without } \mu_1, \mu_2) \ldots, 0, 0, \ldots). \quad (\mathrm{A1})$$

There are $N-2$ zeroes in the end, corresponding to elements of the Cartan subalgebra other than T^3. Let us check the size of the matrix (A1):

$$3 + 4(N-2) + (N-2)(N-3) + (N-2) = N^2 - 1,$$

which is as it should be for the adjoint representation of SU(N).

This constant background is exactly equivalent to twisting the boundary conditions. So we have to sum the logarithms of the determinants for one adjoint-representation SU(2) dyon, $2(N-2)$ differently twisted fundamental-representation SU(2) dyons, and a determinant for $(N-2)(N-3)$ different constant A_4 field eigenvalues. It would be interesting to check that this is asymptotically the same as the adjoint determinant for the far region that we will calculate below.

In [15] we proved that the twisting of boundary conditions for the fundamental-representation determinant results in the shifting of the argument of P, where P is the standard perturbative potential

$$P(\mathrm{v}) = \frac{1}{3(2\pi)^2}\mathrm{v}^2(2\pi - \mathrm{v})^2\Big|_{\mathrm{mod}2\pi}. \tag{A2}$$

Now we can just write the result (for the first dyon, for simplicity), as a sum of the formula taken from [17] for the triplet [SU(2) adjoint representation]:

$$\partial_{\mathcal{P}} \log \mathrm{Det}(-D^2)_{\text{near dyon}} = \partial_{\mathcal{P}}\left(\tilde{c}_{\text{dyon}}\nu_1 - \frac{8}{3}\nu_1\log(\nu_1) + \log\nu_m + \int^R P\left(2\pi\nu_1 - \frac{1}{r}\right)4\pi r^2 dr\right) \tag{A3}$$

with $2(N-2)$ formulas from [15] for the SU(2) dyon with twisted boundary conditions ["twist" is a corresponding matrix element of Eq. (A1)] [41],

$$\partial_{\mathcal{P}} \log \mathrm{Det}(-\nabla^2)_{\text{near dyon}} = \sum_{i=1}^{2(N-2)} \partial_{\mathcal{P}}\Big\{\hat{c}_{\text{dyon}}\nu_1 - \frac{\log(\nu_1)}{6}\nu_1 + \int^R \frac{1}{2}\Big[P\Big(2\pi(\mu_1 - (\mu_1+\mu_2)/2) + \frac{1}{2r} - i(A^{\text{adj}}_{\text{const}})_{2i+1}\Big) + P\Big(2\pi(\mu_2 - (\mu_1+\mu_2)/2) - \frac{1}{2r} - i(A^{\text{adj}}_{\text{const}})_{2i+2}\Big)\Big]4\pi r^2 dr\Big\}, \tag{A4}$$

and with a constant-field determinant for the remaining matrix elements of $-D^2$ (i.e. twists without background field),

$$\partial_{\mathcal{P}} \log \mathrm{Det}(-D^2)^{\text{const}}_{\text{near dyon}} = \sum_{i=1}^{(N-2)(N-3)} \partial_{\mathcal{P}} \int_0^R \frac{1}{2}P(-i(A^{\text{adj}}_{\text{const}})_{2+4(N-2)+i})4\pi r^2 dr.$$

Totally, we get

$$\partial_{\mathcal{P}} \log \mathrm{Det}(-D^2)_{\text{near dyon}} = \partial_{\mathcal{P}}\left(c_{\text{dyon}}\nu_m - \left(\frac{8}{3} + \frac{N-2}{3}\right)\log(\nu_m)\nu_m + \log\nu_m\right) + (R\text{-terms}). \tag{A5}$$

And it is easy to check explicitly that the "R-terms" exactly match the asymptotics of far-from-dyons domain (see the Sec. V),

$$(R\text{-terms}) = \lim_{r_{i\neq 1}\to\infty}\int^R \sum_{i>j}\delta P\left[2\pi(\mu_i - \mu_j) + \frac{1}{2r_i} - \frac{1}{2r_{i-1}} - \left(\frac{1}{2r_j} - \frac{1}{2r_{j-1}}\right)\right]4\pi r_1^2 dr_1. \tag{A6}$$

So we conclude that the R-terms are trivial and exactly match the contributions from the outer region, as it should be, of course, since the result cannot depend on the radius of the auxiliary balls that we have chosen.

APPENDIX B: CALCULATION OF THE CURRENTS FOR THE OUTER DOMAIN

1. Singular current

The contribution of the singular part of the propagator to the variation of the determinant is 4 times the fundamental-representation result [2,25], if we write this variation in terms of the fundamental representation. So we just take our old result from [15] (that formula was not written there explicitly; it was in our intermediate computations). For the component J^{si} it is quite natural to introduce bipolar spatial coordinates with unit basis vectors $\hat{r}_i = \frac{r_i}{|r|}$, $\hat{s}_i = \frac{r_{i-1}}{|r|}$, $n_\phi = \frac{\hat{r}\times\hat{s}}{|\hat{r}\times\hat{s}|}$. In these coordinates the current (already multiplied by 4) is

$$J^{s\,i}{}_4 = -\frac{i(r_i^3 - s_i^3)}{12\pi^2 r_i^3 s_i^3}, \tag{B1}$$

$$J^{s\,i}{}_{\phi} = -\frac{i(r_i + s_i)\sqrt{-(\varrho_i - r_i - s_i)(d + r_i - s_i)(\varrho_i - r_i + s_i)(\varrho_i + r_i + s_i)}}{4\pi^2 r_i^2 s_i^2(\varrho_i + r_i + s_i)^2}, \tag{B2}$$

$$J^{s\,i}{}_{\hat{r}_i} = 0; \qquad J^{s\,i}{}_{\hat{s}_i} = 0. \tag{B3}$$

We also remind the reader about the notations: $r_i = x - y_i$ is a vector from the ith dyon center to the current point, $s_i \equiv r_{i-1}$; and $\varrho_i = |y_i - y_{i-1}|$ is a distance between these two dyons. Also, the standard "circular rule" $r_{N+1} \equiv r_1$ is implied.

2. *M*-term current

Let us prove that the contribution to the current from the M-term of the adjoint propagator is zero with exponential precision (i.e. it decays exponentially out of the dyon cores). As was shown in [17], when making the propagator periodic the M-term simplifies to

$$\begin{aligned} \mathcal{G}^{\mathrm{m}ab}(x, y) \equiv \frac{1}{8\pi^2}\int_{-1/2}^{1/2} dz dz' M(z, z') \\ \times \operatorname{tr}(v^{2\dagger}(x, z)v^2(x, z)\tau^a) \\ \times \operatorname{tr}(v^{2\dagger}(y, z')v^2(y, z')\tau^b), \end{aligned} \tag{B4}$$

since the property $v(y^n, z) = e^{2\pi i n z}v(y, z)$ used to derive that result still holds for the SU(N) ADHMN [28,29] construction. Here $M(z_1, z_2, z, z) = \delta(z_1 - z_2)M(z_1, z)$.

First of all, we note that only the lower components of v are left and only the Cartan (diagonal) components are nonzero: From Eq. (B34) we see that for each m the function $s_m(z)$, and hence $v_m^2(z)$, is peaked near $z = \mu_m$ and exponentially decays away from this point. So $v_m^{2\dagger}(x, z)v_n^2(x, z) \sim \delta_{mn}$ with exponential precision. This leads us to the conclusion that

$$\begin{aligned} \mathcal{G}^{\mathrm{m}ab}(x, y) \propto \delta^{a\in\mathrm{Cartan}}\delta^{b\in\mathrm{Cartan}}, \\ \mathcal{G}^{\mathrm{m}ab}(x, y) = \mathcal{G}^{\mathrm{m}ab}(y, x). \end{aligned} \tag{B5}$$

The second equation means that the terms with derivatives in the expression for the current (9) cancel each other. It follows from the first one that the adjoint action of A on $\mathcal{G}^m$ gives zero since both A and $\mathcal{G}^m$ lie approximately in the Cartan subalgebra. Therefore we conclude that

$$J_\mu^{\mathrm{m}} \simeq 0. \tag{B6}$$

3. Regular current

The adjoint-representation regular current is

$$J^{ab} = D_x^{ac}\mathcal{G}_{cb}(x, y) + \mathcal{G}_{ac}(x, y)D_y^{cb} \tag{B7}$$

where $a, b, c = 1, \ldots, N^2 - 1$ and we take the regular part of the propagator:

$$\begin{aligned} (\mathcal{G}^{\mathrm{r}})^{ab}(x, y) \equiv \sum_{n\neq 0}\frac{4}{8\pi^2(x - y_n)^2} \\ \times \operatorname{tr}[t^a\langle v(x)|v(y_n)\rangle t^b\langle v(y_n)|v(x)\rangle], \\ y_n^i = y^i - \delta^{i4}n. \end{aligned} \tag{B8}$$

It is possible to rewrite these formulas in the fundamental notations and evaluate them explicitly. Some details of the calculation together with a short review of ADHM construction are presented below. We denote the adjoint indices by $a, b, c = 1, \ldots, N^2 - 1$ and fundamental indices by $i, j, k, l, m, n = 1, \ldots, N$.

First we represent the covariant derivatives in the fundamental representation. With the help of the identities

$$\begin{aligned} \overrightarrow{D}_\mu^{ad}\operatorname{tr}(t^d A t^b B) = \operatorname{tr}[t^a(\overrightarrow{D}_\mu A)t^b B - t^a A t^b(B\overleftarrow{D}_\mu)], \\ \operatorname{tr}(t^a A t^d B)\overleftarrow{D}_\mu^{db} = \operatorname{tr}[t^a(A\overleftarrow{D}_\mu)t^b B - t^a A t^b(\overrightarrow{D}_\mu B)], \end{aligned} \tag{B9}$$

one gets for J^{ab} the obvious four terms plus $\delta_{\mu 4}(2\operatorname{tr}[t^a v_x^\dagger v_y t^b v_y^\dagger v_x]/\pi^2 n^3)$ from the derivative acting on the denominator.

All the terms in the adjoint current have the form $\operatorname{tr}[t^a B t^b C]$. The variation of the determinant has the form $-\delta A^c T^c_{ab} J^{ab}$. To write it in the fundamental representation we use the identities

$$A^c T^c_{ab} = 2\operatorname{tr}(t^b[t^a, A]) \tag{B10}$$

where $A = A^i t^i$ and

$$t^a_{ij}t^a_{kl} = 1/2(\delta_{il}\delta_{jk} - 1/N\delta_{ij}\delta_{kl}). \tag{B11}$$

The Hermitian generators t^a are normalized as $\operatorname{tr}(t^a t^b) = 1/2$ [for SU(2) these are $\tau^a/2$]. We get

$$\delta A^c T^c_{ab}\operatorname{tr}(t^a B t^b C) = \tfrac{1}{2}\operatorname{tr}(B)\operatorname{tr}(\delta A C) - \tfrac{1}{2}\operatorname{tr}(C)\operatorname{tr}(\delta A B). \tag{B12}$$

So in terms of the *fundamental* indices ($i, j = 1, \ldots, N$) we get for the current J^{ij} (that is to be coupled to A in the *fundamental* representation to get the variation of the determinant)

$$\begin{aligned} (J_\mu^{\mathrm{r}})^{ij}(x) = \sum_{n\neq 0,\{B,C\}}\frac{1}{4\pi^2(x - y_n)^2}(\operatorname{tr}(B)C^{ij} - \operatorname{tr}(C)B^{ij}) \\ - \delta_{\mu 4}\sum_{n\neq 0}\frac{1}{\pi^2(x - y_n)^3}(\operatorname{tr}(E)F^{ij} - \operatorname{tr}(F)E^{ij}). \end{aligned} \tag{B13}$$

Here

$$E = v_x^\dagger v_{y_n} \equiv b; \qquad F = v_{y_n}^\dagger v_x \equiv b^\dagger \tag{B14}$$

and we put $y = x$ (so that now $y_n^i = x^i - \delta^{i4}n$) according to Eq. (9). The set $\{(B, C)\}$ consists of four pairs taken from Eqs. (B7) and (B9):

$$\{(B, C)\} = \{(D_x v_x^\dagger v_{y_n}, v_{y_n}^\dagger v_x), (v_x^\dagger v_{y_n}, D_x v_{y_n}^\dagger v_x), (v_x^\dagger v_{y_n} D_y, v_{y_n}^\dagger v_x), (v_x^\dagger v_{y_n}, v_{y_n}^\dagger v_x D_y)\}. \tag{B15}$$

Since the current and the field are approximately diagonal in the fundamental representation, we consider the diagonal components $(J_\mu^{\rm r})^i \equiv (J_\mu^{\rm r})^{ii}$. In these notations the contribution to Eq. (B13) can be rewritten as

$$(J^{\rm r}{}_\mu)^i(x) = \sum_{j=1}^N \sum_{n\neq 0}\bigg(\sum_{\{(B,C)\}} \frac{1}{4\pi^2(x-y_n)^2}(B_jC_i - C_jB_i) - \frac{\delta_{\mu 4}}{\pi^2(x-y_n)^3}(E_jF_i - F_jE_i)\bigg). \tag{B16}$$

From Eqs. (B15) and (B16) we get 8 + 2 terms in the resulting contribution to the current. Now in order to calculate explicitly these B, C, E, F we need ADHMN construction. A brief review and a calculation follow.

4. Expressions of the ADHMN construction

The basic object in the ADHMN construction [28,29] is the $(2 + N) \times 2$ matrix Δ linear in the space-time variable x and depending on an additional compact variable z belonging to the unit circle:

$$\Delta_\beta^K(z, x) = \begin{cases} \lambda_\beta^m(z), & K = m, \quad 1 \le m \le N, \\ (B(z) - x_\mu \sigma_\mu)_\beta^\alpha, & K = N + \alpha, \quad 1 \le \alpha \le 2, \end{cases} \tag{B17}$$

where $\alpha, \beta = 1, 2$ and $m = 1, \dots, N$; $\sigma_\mu = (i\vec{\sigma}, 1_2)$. As usual, the superscripts number rows of a matrix and the subscripts number columns. The functions $\lambda_\beta^m(z)$ forming a $N \times 2$ matrix carry information about color orientations of the constituent dyons, encoded in the N two-spinors ζ:

$$\lambda_\beta^m(z) = \delta(z - \mu_m)\zeta_\beta^m. \tag{B18}$$

The quantities ζ_β^m transform as contravariant spinors of the gauge group SU(N) but as covariant spinors of the spatial SU(2) group. The 2×2 matrix B is a differential operator in z and depends on the positions of the dyons in the 3d space $\vec{y}_m$ and the overall position in time $\xi_4 = x_4$:

$$B_\beta^\alpha(z) = \frac{\delta_\beta^\alpha \partial_z}{2\pi i} + \frac{\hat{A}_\beta^\alpha(z)}{2\pi i} \tag{B19}$$

with

$$\hat{A}(z) = A_\mu \sigma_\mu, \qquad \vec{A}(z) = 2\pi i \vec{y}_m(z), \qquad A_4 = 2\pi i \xi_4, \tag{B20}$$

where inside the interval $\mu_m \le z \le \mu_{m+1}$, $\vec{y}(z) = \vec{y}_m$ is the position of the mth dyon with inverse size $\nu_m \equiv \mu_{m+1} - \mu_m$.

One has to find N quantities $v_n^K(x)$, $n = 1, \dots, N$,

$$v_n^K(x) = \begin{cases} v_n^{1m}(x), & K = m, \quad 1 \le m \le N, \\ v_n^{2\alpha}(z, x), & K = N + \alpha, \quad 1 \le \alpha \le 2, \end{cases} \tag{B21}$$

which are normalized independent solutions of the differential equation

$$\lambda_m^{\dagger\alpha}(z) v_n^{1m} + [B^\dagger(z) - x_\mu \sigma_\mu^\dagger]_\beta^\alpha v_n^{2\beta}(z, x) = 0, \qquad v_l^{\dagger 1m} v_n^{1l} + \int_{-1/2}^{1/2} dz v_\alpha^{\dagger 2m} v_n^{2\alpha} = \delta_n^m, \tag{B22}$$

or, in shorthand notation,

$$\Delta^\dagger v = 0, \qquad v^\dagger v = 1_N. \tag{B23}$$

Note that only the lower component v^2 depends on z.

Expressing v as

$$v(x) = \begin{pmatrix} -1_n \\ u(x) \end{pmatrix} \phi^{-1/2}, \qquad u(x) = (B^\dagger - x^\dagger)^{-1} \lambda^\dagger, \tag{B24}$$

let us find $u(z, x)$—the main object of ADHM construction. It is the solution to the equation

$$(B^\dagger - x^\dagger)u = \lambda^\dagger, \qquad B^\dagger - x^\dagger = \frac{\partial_z}{2\pi i} - r^\dagger(z). \tag{B25}$$

Define the Green functions as

$$f = (\Delta^\dagger \Delta)^{-1}, \qquad G = ((B - x)^\dagger (B - x))^{-1}, \qquad \phi_{ij}(x) = \delta_{ij} + \lambda_i^\alpha G^{\alpha\beta} \lambda^{\dagger\alpha}{}_j \tag{B26}$$

where $(i = 1, \dots, N)$, $(\alpha, \beta = 1, 2)$. One can note that

$$f = (G^{-1} + \lambda^\dagger \lambda)^{-1} = G - G\lambda_i^\dagger \phi_{ij}^{-1} \lambda G. \tag{B27}$$

Acting on (B27) with $\lambda^\dagger$ on the right yields

$$G\lambda_j^\dagger = f\lambda_i^\dagger \phi_{ij}. \tag{B28}$$

The Green function is expressed as follows:

$$f(z, z') = s_m(z) f_{mn} s_n^\dagger(z') + 2\pi s(z, z') \delta_{[z][z']}, \tag{B29}$$

$$f_{mn} = F_{mn}^{-1}. \tag{B30}$$

The functions appearing in Eq. (B29) are

$$s_m(z) = e^{2\pi i x_0(z-\mu_m)} \frac{\sinh[2\pi r_m(\mu_{m+1} - z)]}{\sinh(2\pi r_m \nu_m)} \delta_{m[z]} + e^{2\pi i x_0(z-\mu_m)} \frac{\sinh[2\pi r_{m-1}(z - \mu_{m-1})]}{\sinh(2\pi r_{m-1}\nu_{m-1})} \delta_{m,[z]+1},$$
$$s(z,z') = e^{2\pi i x_0(z-z')} \frac{\sinh(2\pi r_{[z]}(\min\{z,z'\} - \mu_{[z]}))\sinh(2\pi r_{[z]}(\mu_{[z]+1} - \max\{z,z'\}))}{r_{[z]}\sinh(2\pi r_{[z]}\nu_{[z]})},$$

$$u_i = (B-x)f\lambda_j^\dagger \phi_{ji} = \left(\frac{\partial_z}{2\pi i} - r_\mu\sigma_\mu\right) s_k^f(z) f_{kj}\zeta_j^\dagger \phi_{ji}, \tag{B31}$$

and the convenient notation $r_i = x - y_i$ is a vector from the ith dyon to the current point. First we note that F_{ij} and ϕ_{ij} are diagonal matrices with exponential precision,

$$f_{ij} \simeq 2\pi\delta_{ij}(r_i + r_{i-1} + \varrho_i)^{-1}, \tag{B32}$$

$$\phi_{ij} \simeq \delta_{ij}\frac{r_i + r_{i-1} + \varrho_i}{r_i + r_{i-1} - \varrho_i}. \tag{B33}$$

To pass to the periodical gauge we multiply $v(x)$ from the right by $g_{ij} = \delta_{ij}e^{2\pi i\mu_i x_0}$. Totally within the exponential precision we get for v

$$v_i(x,z) = \begin{pmatrix} -\delta_{ij}\phi_{ii}^{-1/2}e^{2\pi i\mu_i x_0} \\ (B-x)s_i(z)f_{ii}\zeta_i^\dagger \phi_{ii}^{1/2}e^{2\pi i\mu_i x_0}\end{pmatrix} \text{no index summations.} \tag{B34}$$

Consider the covariant derivative of $v_i(x)$ in the periodical gauge (integration over z is assumed):

$$\begin{aligned} D_\mu\langle v(x)| &= \partial_\mu\langle v| - \partial_\mu\langle v|v\rangle\langle v| = \partial_\mu\langle v|(1 - |v\rangle\langle v|) = \partial_\mu\langle v|\Delta f\Delta^\dagger = -\langle v|\partial_\mu\Delta f\Delta^\dagger = -\langle v|\mathcal{B}\sigma_\mu f\Delta^\dagger \\ &= (v^{2\dagger}\sigma_\mu f\lambda^\dagger, v^{2\dagger}\sigma_\mu f(B-x)^\dagger) = f_{ii}\zeta_i\phi_{ii}^{1/2}e^{-2\pi i\mu_i x_0}(B-x)^\dagger s_i^\dagger(z)\sigma_\mu(f\lambda^\dagger, f(B-x)^\dagger) \\ &= f_{ii}\zeta_i\phi_{ii}^{1/2}e^{-2\pi i\mu_i x_0}(B-x)^\dagger s_i^\dagger(z)\sigma_\mu(s_i(z)f_{ii}\zeta_i^\dagger, (s_i(z)f_{ii}s_i^\dagger(z') + 2\pi s(z,z'))(B-x)^\dagger), \end{aligned} \tag{B35}$$

$$|v(x)\rangle D_\mu = -(D_\mu\langle v(x)|)^\dagger. \tag{B36}$$

5. Formula for the regular current

Let us denote $D = (B - x)$. We will, in a moment, express Eq. (B15) through c and b, defined as

$$\begin{aligned} c &= D_\mu v_x^\dagger v_y \\ &= -\phi n f_{ii}^2\zeta_i(Ds(z))^\dagger\sigma_\mu(s(z)fs^\dagger(z') + 2\pi s(z,z')) \\ &\quad\times e^{-2\pi i n(z'-\mu_i)}Ds(z')\zeta_i^\dagger e^{-2\pi i\mu_i n} \\ &= -\phi n f_{ii}^2\zeta_i(\tilde{D}\tilde{s}(z))^\dagger\sigma_\mu(\tilde{s}(z)f\tilde{s}^\dagger(z') + 2\pi\tilde{s}(z,z')) \\ &\quad\times\tilde{D}\tilde{s}(z')\zeta_i^\dagger e^{-2\pi i n z'}. \end{aligned} \tag{B37}$$

Here ˜ means that the time dependence is separated (so that $\tilde{s}$ is time independent) and integration over z and z' is assumed. To derive this we used

$$D_x^\dagger v_y^2 = (D_y^\dagger - n)v_y^2 = -\lambda v_y^1 - n v_y^2 \tag{B38}$$

following from Eq. (B23), and noticed that the first term cancels with the scalar product of upper components. For b we easily get

$$\begin{aligned} b &= v_x^\dagger v_y \\ &= \exp(-2\pi i\mu_i n)/\phi_i + \phi_i f_{ii}^2\zeta_i((Ds)^\dagger e^{-2\pi i n z}Ds)\zeta_i^\dagger. \end{aligned} \tag{B39}$$

We also need the following formulas:

$$v_x^\dagger v_y D_\mu^{(y)} = -c^\dagger_{n\to -n}, \tag{B40}$$

$$v_y^\dagger v_x = b^\dagger, \tag{B41}$$

$$b_{n\to -n} = b^\dagger, \tag{B42}$$

$$v_y^\dagger v_x D_\mu^{(y)} = -D_\mu^{(y)}v_y^\dagger v_x + 2A_\mu v_y^\dagger v_x = -c_{n\to -n} + 2Ab^\dagger, \tag{B43}$$

$$D_\mu^{(x)}v_y^\dagger v_x = -v_y^\dagger v_x D_\mu^{(x)} + 2Av_y^\dagger v_x = c^\dagger + 2Ab^\dagger. \tag{B44}$$

In the total sum, changing $n \to -n$ does not affect the expression since we divide by n^2, so we can make $n \to -n$ in the whole expression by conjugating b and then drop $n \to -n$.

Then the set of $\{(B, C)\}$ in Eq. (B15) becomes

$$\{(B, C)\} = \{(c, b^\dagger), (b, c^\dagger + 2Ab^\dagger), (-c^\dagger_{n\to -n}, b^\dagger), (b, -c_{n\to -n} + 2Ab^\dagger)\}. \tag{B45}$$

Totally, we get for the current (B16) (the index μ is hidden in c)

$$(J^r{}_\mu)^i(x) = \sum_{n\neq 0; j=1,\ldots,N} \frac{2c_j b_i^\dagger - 2c_j^\dagger b_i - 4A_j b_i b_j^\dagger}{4\pi n^2} - \delta_{\mu 4}\frac{b_j b_i^\dagger}{\pi^2 n^3} - (i \leftrightarrow j). \tag{B46}$$

There are only terms with $(i) \neq (j)$ in the current that we are calculating.

Now, by these explicit formulas, the current can be evaluated by performing integrals over z, z' and summing over n. The reference formulas for summation can be found e.g. in [17]. We used MATHEMATICA for these calculations. Below, the result is presented.

Just for illustration and a consistency check, we write first for the SU(2) case. For SU(2) the ADHMN data is taken to be $\mu_1 = -\omega$, $\mu_2 = \omega$, $\nu_1 = 2\omega$, $\nu_2 = 1 - 2\omega$. So we get for the current (here we write the diagonal matrix, which couples to the gauge field in the fundamental representation)

$$(J_4^r)^1 = -(J_4^r)^2 = \left(iP'\left(2\pi(\mu_1 - \mu_2) + \frac{1}{r} - \frac{1}{s}\right) + \frac{i}{12\pi^2}\left(\frac{1}{s^3} - \frac{1}{r^3}\right)\right). \tag{B47}$$

As usual, the current is expressed as a derivative of the perturbative potential P. The second term in this expression cancels exactly with the contribution of the singular current. Totally, adding the singular current, we get for the "far region" contribution to the variation

$$\delta \log \mathrm{Det}(-D^2)_{\mathrm{far}} = \int_{\mathrm{far}} \delta(A_{41}^{\mathrm{fund}} - A_{42}^{\mathrm{fund}}) \times P'\left(2\pi(\mu_1 - \mu_2) + \frac{1}{r} - \frac{1}{s}\right) = \int_{\mathrm{far}} \delta P\left(2\pi(\mu_1 - \mu_2) + \frac{1}{r} - \frac{1}{s}\right). \tag{B48}$$

We recall the convenient notation: $r_i = x - y_i$ is a vector from the ith dyon to the current point, $s_i \equiv r_{i-1}$, and a standard "circular rule" $r_{N+1} \equiv r_1$. For SU(2) we set $r = r_1$, $s = s_1 = r_2$.

For the SU(N) case we get the total result:

$$\delta \log \mathrm{Det}(-D^2)_{\mathrm{far}} = \int_{\mathrm{far}} \sum_{i,j=1}^{N} \delta\left(2\pi\mu_i + \frac{1}{2r_i} - \frac{1}{2s_i}\right) P'\left[2\pi(\mu_i - \mu_j) + \frac{1}{2r_i} - \frac{1}{2s_i} - \left(\frac{1}{2r_j} - \frac{1}{2s_j}\right)\right] \mathrm{sgn}(\mu_i - \mu_j) = \int_{\mathrm{far}} \sum_{i>j} \delta P\left[2\pi(\mu_i - \mu_j) + \frac{1}{2r_i} - \frac{1}{2s_i} - \left(\frac{1}{2r_j} - \frac{1}{2s_j}\right)\right]. \tag{B49}$$

Note that the spatial part of the regular current cancels exactly with the singular part of the current. This proves Eq. (23).

[1] A. M. Polyakov, Phys. Lett. **72B**, 477 (1978).
[2] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
[3] A. Belavin, A. Polyakov, A. Schwartz, and Yu. Tyupkin, Phys. Lett. **59**, 85 (1975).
[4] T. C. Kraan and P. van Baal, Phys. Lett. B **428**, 268 (1998); Nucl. Phys. **B533**, 627 (1998).
[5] T. C. Kraan and P. van Baal, Phys. Lett. B **435**, 389 (1998).
[6] K. Lee and C. Lu, Phys. Rev. D **58**, 025011 (1998).
[7] F. Bruckmann and P. van Baal, Nucl. Phys. **B645**, 105 (2002).
[8] F. Bruckmann, D. Nogradi, and P. van Baal, Nucl. Phys. **B698**, 233 (2004).
[9] E. M. Ilgenfritz, B. V. Martemyanov, M. Muller-Preussker, S. Shcheredin, and A. I. Veselov, Phys. Rev. D **66**, 074503 (2002); E. M. Ilgenfritz, B. V. Martemyanov, M. Muller-Preussker, S. Shcheredin, and A. I. Veselov, Nucl. Phys. B, Proc. Suppl. **119**, 754 (2003); E. M. Ilgenfritz, B. V. Martemyanov, M. Muller-Preussker, and A. I. Veselov, Phys. Rev. D **69**, 114505 (2004).
[10] P. van Baal, AIP Conf. Proc. **892**, 241 (2007).
[11] D. Diakonov and V. Petrov, Phys. Rev. D **76**, 056001 (2007).
[12] B. J. Harrington and H. K. Shepard, Phys. Rev. D **17**, 2122 (1978); **18**, 2990 (1978).
[13] K. Lee and P. Yi, Phys. Rev. D **56**, 3711 (1997).
[14] K. Lee, E. J. Weinberg, and P. Yi, Phys. Lett. B **376**, 97 (1996); Phys. Rev. D **54**, 6351 (1996); E. J. Weinberg, arXiv:hep-th/9908095.
[15] N. Gromov and S. Slizovskiy, Phys. Rev. D **73**, 025022 (2006).
[16] E. B. Bogomolnyi, Yad. Fiz. **24**, 861 (1976) [Sov. J. Nucl. Phys. **24**, 449 (1976)]; M. K. Prasad and C. M. Sommerfeld, Phys. Rev. Lett. **35**, 760 (1975).

[17] D. Diakonov, N. Gromov, V. Petrov, and S. Slizovskiy, Phys. Rev. D **70**, 036003 (2004).
[18] N. Gromov and S. Slizovskiy, Phys. Rev. D **71**, 125019 (2005).
[19] N. Gromov, in Proceedings of the 58th Scottish Universities Summer School in Physics (SUSSP58): A NATO Advanced Study Institute and EU Hadron Physics 13 Summer Institute, St. Andrews, Scotland, 2004.
[20] N. Gromov, arXiv:hep-th/0701192.
[21] In the previous paper [15] we have proved that the corresponding constant is zero for the fundamental-representation determinant, but for the adjoint representation the constant is not expected to vanish.
[22] N. Weiss, Phys. Rev. D **24**, 475 (1981); **25**, 2667 (1982).
[23] P. Gerhold, E.M. Ilgenfritz, and M. Muller-Preussker, Nucl. Phys. **B760**, 1 (2007); Nucl. Phys. **B774**, 268 (2007).
[24] We use notations consistent with [5].
[25] L.S. Brown and D.B. Creamer, Phys. Rev. D **18**, 3695 (1978).
[26] K. Zarembo, Nucl. Phys. **B463**, 73 (1996).
[27] N.H. Christ, E.J. Weinberg, and N.K. Stanton, Phys. Rev. D **18**, 2013 (1978); E. Corrigan, P. Goddard, and S. Templeton, Nucl. Phys. **B151**, 93 (1979).
[28] W. Nahm, Phys. Lett. **90B**, 413 (1980).
[29] M.F. Atiyah, V.G. Drinfeld, N.J. Hitchin, and Yu.I. Manin, Phys. Lett. **65A**, 185 (1978).
[30] E. Corrigan, D.B. Fairlie, S. Templeton, and P. Goddard, Nucl. Phys. **B140**, 31 (1978).
[31] E. Corrigan, D.B. Fairlie, S. Templeton, and P. Goddard, Nucl. Phys. **B140**, 31 (1978).
[32] W. Nahm, Phys. Lett. **90B**, 413 (1980).
[33] I. Jack, Nucl. Phys. **B174**, 526 (1980).
[34] See Appendix B of [17] for its explicit form in the case of SU(2).
[35] See, for example, [42] where this was done for the instanton solution.
[36] D. Diakonov and M. Oswald, Phys. Rev. D **68**, 025012 (2003); **70**, 016006 (2004); **70**, 105016 (2004).
[37] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976).
[38] T.C. Kraan, Commun. Math. Phys. **212**, 503 (2000).
[39] D. Diakonov and N. Gromov, Phys. Rev. D **72**, 025003 (2005).
[40] S. Huang and M. Lissia, Nucl. Phys. **B438**, 54 (1995); K. Kajantie, M. Laine, K. Rummukainen, and M.E. Shaposhnikov, Nucl. Phys. **B503**, 357 (1997); S. Chapman, Phys. Rev. D **50**, 5308 (1994).
[41] Note that to get the normal SU(2) dyon we have to center the interval (μ_1, μ_2) at zero, because in SU(2) we obviously have the zero-trace condition $\mu_1 + \mu_2 = 0$.
[42] C.W. Bernard, N.H. Christ, A.H. Guth, and E.J. Weinberg, Phys. Rev. D **16**, 2967 (1977).