Plausible "faster-than-light" displacements in a two-sheeted spacetime

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(Received 27 June 2007; published 8 October 2007)

In this paper, we explore the implications of a two-point discretization of an extra dimension in a five-dimensional quantum setup. We adopt a pragmatic attitude by considering the dynamics of spin-half particles through the simplest possible extension of the existing Dirac and Pauli equations. It is shown that the benefit of this approach is to predict new physical phenomena while maintaining the number of constitutive hypotheses at minimum. As the most striking feature of the model, we demonstrate the possibility of fermionic matter oscillations between the two four-dimensional sections and hyperfast displacements in case of asymmetric warping (without conflicting special relativity). This result, similar to previous reported ones in braneworld theories, is completely original as it is derived by using quantum mechanics only without recourse to general relativity and bulk geodesics calculation. The model allows causal contact between normally disconnected regions. If it proves to be physically founded, its practical aspects could have deep implications for the search of extra dimensions.

DOI: 10.1103/PhysRevD.76.085005 PACS numbers: 11.10.Kk, 04.62.+v, 11.25.Wx

I. INTRODUCTION

The idea that our observable universe could be a part of a more extended N-dimensional spacetime ($N > 4$) has a long tradition. It traces back to the seminal work of Kaluza in 1921, who extended general relativity in five dimensions in order to treat electromagnetism and gravitation on an equal footing [1]. Unfortunately, the model and its subsequent extensions reveal unsuccessful in their aim of describing the physical reality such that the multidimensional approach was abandoned for a while.

Since that time, much work has been carried out and modern theoretical physics has led us to consider as more probable the existence of a multidimensional universe. Recently, the idea was reintroduced in the context of superstrings and braneworld theories. This renewed interest can be explained by the fact that multidimensional universes may adequately describe known forces and particles and also explain the hierarchy between the gravitational and electroweak scales [2,3]. These scenarios have been extended much in recent years and there is now some approaches suggesting that many connected parallel branes could exist in an extended bulk (thus creating a so-called "manyfold"). As summarized in Ref. [4], the existence of such a multisheeted spacetime could shed some light on a number of puzzling cosmological problems including the nature of dark matter structures (which would be identical to normal matter but located in a distinct sheet), as well as their invisibility (the gauge fields and most notably electromagnetism would be confined within the branes such that the structures belonging to distinct branes would be mutually invisible).

Following this line of thought, other recent developments have tried to improve the original Kaluza's idea of unification by considering discrete extra dimensions instead of continuous ones. In these models, each discrete point in the fifth dimension is endowed with its own metric field such that the resulting universe appears like a multisheeted spacetime composed by a collection of parallel 4D sheets in interaction. Although such latticized models are recognized to suffer from drawbacks (potentially curable), they exhibit promising phenomenological properties, in particular, in cosmology [5,6].

Discrete extra dimensions have also been studied in the spirit of noncommutative geometry. One of the most promising approaches is that of Connes and Lott [7], and Viet and Wali [8]. Their formulation considers a product manifold comprising a continuous four-dimensional part times a discrete one, typically $M_4 \times Z_2$. The nontrivial feature of their theory is the coupling of the two spacetime sheets which results in the Higgs field. In several aspects, this construction appears as a reminiscence of the five-dimensional Kaluza-Klein model with the fifth dimension restricted to only two points.

In recent papers, present authors have developed a phenomenological model of a five-dimensional two-sheeted spacetime and studied the quantum behavior of massive particles in such a universe. Different mathematical approaches were used involving either a noncommutative product manifold [9] or a two-point discretization of the fifth dimension [10]. In the latter approach, the compact extra dimension was treated through a naive discretization scheme involving a finite difference analysis. The procedure was inspired from "multigravity" theories where massive gravitons result as a consequence of a latticized extra dimension [5]. Extending such a procedure to the case of the five-dimensional Dirac equation was demon-

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strated to provide several original results like fermionic matter oscillations between the two four-dimensional sections [9–12]. Therefore, despite all roughness of the approach, the prediction of possible new quantum phenomena made the theory interesting to study.

In the present paper, we propose to extend further the discussion of Refs. [9–12] by considering an asymmetrically warped background instead of a flat one. Since each four-dimensional section is now endowed with its own warp factor, different physical length scales can be defined on the sheets. We show that, under some circumstances, a massive particle can oscillate between the two four-dimensional sections and exhibit hyperfast velocities from the perspective of a four-dimensional observer thanks to the different length scales in the two sheets. This result is similar to that obtained by Chung and Freese in Ref. [13] except that it is here derived by using quantum mechanics exclusively without recourse to geodesic calculation.

The paper is structured as follows. In Sec. II, we derive the two-sheeted Dirac equation for the chosen asymmetrically warped background. The solutions of the Dirac equation are given and compared with the usual ones which are recovered at the decoupling limit (infinite distance between the sheets). Then, in the third part, we demonstrate the oscillatory behavior in the case of a positive energy particle and the possibility of hyperfast motions. In the fourth section, the nonrelativistic limit of the Dirac equation is derived and the quantum dynamics is studied at low energies. It is demonstrated that the oscillatory behavior and the hyperfast velocities still survive in the nonrelativistic limit, thus enabling a possible experimental confirmation of the model.

II. TWO-SHEETED DIRAC EQUATION IN A WARPED BACKGROUND

A. Mathematical framework

In this part, we shall derive the two-sheeted Dirac equation. We start by considering the usual 5D covariant Dirac equation:

$$(i\not{D} - m)\psi = i\gamma^A(x,\lambda)\not{D}_A\psi - m\psi = 0 \tag{1}$$

with $A \in \{0,1,2,3,4\}$ and where $\gamma^A(x,\lambda)$ are the curvature dependent Dirac matrices. $\not{D}_A$ is the covariant derivative given by

$$\not{D}_A = \partial_A + \Gamma_A(x,\lambda) \tag{2}$$

with $\Gamma_A(x,\lambda)$ the spin connections and $\partial_A = \partial/\partial x^A$. In Eq. (1) and (2), x refers to the four-dimensional coordinates and λ to the transverse extra dimension (a subscript or an exponent equals to 4 corresponds to λ). Note that, throughout this paper, we will work in units where $c = 1$, $\hbar = 1$. Now, we assume that the five-dimensional metric takes the following specific form:

$$\begin{aligned} ds^2 &= g_{AB}dx^A dx^B \\ &= dt^2 - R^2(\lambda)[dx^2 + dy^2 + dz^2] + d\lambda^2, \end{aligned} \tag{3}$$

where $R(\lambda)$ is the warp factor. Note that we consider that the warping only involves the space interval of the line element and not the temporal part. This is similar to the choice made by Chung and Freese in Ref. [13] and differs from the classical braneworld approach where the metric takes a canonical form. Our choice of metric corresponds to an asymmetric warping as defined in Ref. [13]. Since the results of this paper are not affected by the exact form of $R(\lambda)$, we are keeping this general form throughout this paper. Note that a signature $(+,-,-,-,+)$ is chosen instead of the usual $(+,-,-,-,-)$ one. It is straightforward to show that a timelike extra dimension is necessary to ensure energy conservation in the present model [a spacelike extra dimension leads to a non-Hermitian Hamiltonian as it is obvious from Eqs. (101) and (102) hereafter].

The five-dimensional Dirac matrices in curved space take the form

$$\gamma^A(x,\lambda) = e_a^A(x,\lambda)\gamma^a, \tag{4}$$

where e_a^A define the vielbein according to

$$g^{AB} = e_a^A(x)e_b^B(x)\eta^{ab} \tag{5}$$

with g_{AB} the 5D metric and η_{ab} the five-dimensional metric tensor of the Minkowski spacetime. The vielbein is given by

$$e_a^A(x) = \mathrm{diag}\left(1, \frac{1}{R}, \frac{1}{R}, \frac{1}{R}, 1\right), \tag{6}$$

which leads to

$$\gamma^0(x,\lambda) = \gamma^0, \qquad \gamma^i(x,\lambda) = \frac{1}{R}\gamma^i, \qquad \gamma^4(x,\lambda) = \gamma^5, \tag{7}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with the usual Dirac matrices γ^μ in flat space, such that one verifies $\{\gamma^a,\gamma^b\} = 2\eta^{ab}$. The spin connection must satisfy the expression [14]:

$$\Gamma_A(x,\lambda) = \tfrac{1}{4}\gamma_B[\partial_A\gamma^B(x,\lambda) + \Gamma^B_{CA}\gamma^C(x,\lambda)], \tag{8}$$

where Γ^B_{CA} are the Christoffel symbols for the metric field defined above. From (8), the following covariant derivative terms can be found:

$$\not{D}_0 = \partial_0, \qquad \not{D}_4 = \partial_4, \qquad \not{D}_i = \partial_i - \tfrac{1}{2}\dot{R}\gamma^i\gamma^5, \tag{9}$$

where a dot implies a derivation along the fifth dimension and the gamma matrices in Eq. (9) are those of flat space. The five-dimensional Dirac equation for the metric field (3) finally becomes

$$\left\{i\gamma^0\partial_0 + \frac{i}{R}\gamma^i\partial_i + i\gamma^5\partial_4 + i\frac{3}{2}\frac{\dot{R}}{R}\gamma^5 - m\right\}\psi = 0. \quad (10)$$

Let us now discretize the fifth dimension. The way to proceed is inspired by those described in Refs. [5,6]. However, instead of considering a one-dimensional lattice, we focus on a more restricted compact extra dimension containing two points located at coordinates $\lambda \in \{-\delta/2, +\delta/2\}$ (see Ref. [10] where such an approach had been developed). At each site, there is thus a four-dimensional submanifold $X_\pm$ where the particle wave function takes the local form $\psi(x, \lambda = \pm\delta/2) = \psi_\pm(x)$. In the proposed geometrical framework the derivative ∂_4 along the discrete dimension simply reduces to a finite difference [10] (see Appendix A):

$$\partial_4\psi|_\pm \to g(\psi_\pm - \psi_\mp) \quad \text{and} \quad \dot{R}_\pm = g(R_\pm - R_\mp), \quad (11)$$

where g is the inverse of the distance δ between the four-dimensional sections and $R_- = R(\lambda = -\delta/2)$, $R_+ = R(\lambda = +\delta/2)$ are the "projected" warp factors on the two sheets.

Using those expressions, the five-dimensional Dirac equation breaks down into a set of two coupled four-dimensional-like Dirac equations:

$$i\gamma^0\partial_0\psi_+ + \frac{i}{R_+}\gamma^i\partial_i\psi_+ + i\gamma^5 g(\psi_+ - \psi_-) + i\gamma^5\frac{3}{2R_+}g(R_+ - R_-)\psi_+ - m\psi_+ = 0 \quad (12)$$

$$i\gamma^0\partial_0\psi_- + \frac{i}{R_-}\gamma^i\partial_i\psi_- + i\gamma^5 g(\psi_- - \psi_+) + i\gamma^5\frac{3}{2R_-}g(R_- - R_+)\psi_- - m\psi_- = 0. \quad (13)$$

This equation can be rewritten in a compact matrix form such that

$$(i\not{D} - m)\psi = 0 \quad (14)$$

with

$$\not{D} = \Gamma^0\partial_0 + \frac{1}{R}\Gamma^\eta\partial_\eta + g\Gamma^5 + g\Delta \quad (15)$$

provided that

$$\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}, \qquad \Gamma^0 = \begin{pmatrix}\gamma^0 & 0 \\ 0 & \gamma^0\end{pmatrix},$$
$$\frac{1}{R} = \begin{pmatrix}1/R_+ & 0 \\ 0 & 1/R_-\end{pmatrix}, \qquad \Gamma^\eta = \begin{pmatrix}\gamma^\eta & 0 \\ 0 & \gamma^\eta\end{pmatrix}, \quad (16)$$

$$\Gamma^5 = \begin{pmatrix}\gamma^5 & -\gamma^5 \\ -\gamma^5 & \gamma^5\end{pmatrix},$$
$$\Delta = \frac{3}{2}\begin{pmatrix}\gamma^5(R_+ - R_-)/R_+ & 0 \\ 0 & \gamma^5(R_- - R_+)/R_-\end{pmatrix}. \quad (17)$$

Equation (14) is the two-sheeted Dirac equation for the metric field (3) which will be studied in this paper. Note that it takes a form which reminds the usual Dirac equation except for the last two terms $ig\Gamma_5$ and $ig\Delta$ which generate the coupling between the sheets. It can be noted that no interaction occurs anymore at the limit where the sheets are infinitely separated, i.e. $g \to 0$.

B. Asymmetrical two-sheeted Klein-Gordon equation

To solve Eq. (14), it is mandatory to introduce the auxiliary field ϕ given by

$$\psi = (i\not{D} + m)\phi = \left(i\Gamma^0\partial_0 + i\frac{1}{R}\Gamma^i\partial_i + m + ig\Gamma^5 + ig\Delta\right)\phi. \quad (18)$$

Then, it can be shown that ϕ is a solution of

$$(i\not{D} - m)(i\not{D} + m)\phi = \left\{-\partial_0^2 + \frac{1}{R^2}\partial_i^2 - m^2 - g\mathrm{H}^i\partial_i - g^2\Gamma\right\}\phi = 0 \quad (19)$$

with

$$\mathrm{H}^i = \begin{pmatrix}0 & -\gamma^i\gamma^5(1/R_+ - 1/R_-) \\ -\gamma^i\gamma^5(1/R_- - 1/R_+) & 0\end{pmatrix},$$
$$\Gamma = \begin{pmatrix}\Sigma_+ & -\Pi \\ -\Pi & \Sigma_-\end{pmatrix}, \quad (20)$$

where we have set

$$\Sigma_+ = \frac{29R_+^2 - 30R_+R_- + 9R_-^2}{4R_+^2},$$
$$\Sigma_- = \frac{29R_-^2 - 30R_+R_- + 9R_+^2}{4R_-^2}, \quad (21)$$
$$\Pi = 2 - \frac{3}{2}\frac{(R_+ - R_-)^2}{R_+R_-}.$$

Following the usual procedure, we can solve this two-sheeted Klein-Gordon equation by using an ansatz of the form $\phi = \phi_0 e^{-i(E_p t - p\cdot x)}$. Introducing this ansatz into Eq. (19) leads to the system

$$\begin{pmatrix}E_p^2 - \frac{p^2}{R_+^2} - m^2 - g^2\Sigma_+ & g^2\Pi - ig(1/R_+ - 1/R_-)\gamma^5\gamma^i p_i \\ g^2\Pi - ig(1/R_- - 1/R_+)\gamma^5\gamma^i p_i & E_p^2 - \frac{p^2}{R_-^2} - m^2 - g^2\Sigma_-\end{pmatrix}\phi_0 = 0. \quad (22)$$

C. Free-field solutions of the two-sheeted Klein-Gordon and Dirac equations

Let us determine the solutions of the two-sheeted Klein-Gordon and Dirac equations. To simplify notations in the forthcoming results, it is convenient to write

$$\chi = \frac{R_+^2 + R_-^2}{R_+^2 R_-^2} \quad \text{and} \quad \Xi = \alpha^2 p^4 + \beta p^2 + \gamma \tag{23}$$

with

$$\alpha = \frac{R_+^2 - R_-^2}{R_+^2 R_-^2},$$
$$\beta = 2g^2\left(\frac{1}{R_-} - \frac{1}{R_+}\right) \times \left[2\left(\frac{1}{R_-} - \frac{1}{R_+}\right) - \left(\frac{1}{R_-} + \frac{1}{R_+}\right)(\Sigma_+ - \Sigma_-)\right], \tag{24}$$

$$\gamma = g^4[4\Pi^2 + (\Sigma_+ - \Sigma_-)^2], \tag{25}$$

$$\kappa = g^2\Pi, \qquad \tau = ig\left(\frac{1}{R_-} - \frac{1}{R_+}\right). \tag{26}$$

From Eq. (22), the energy eigenvalues E and eigenvectors ϕ_0 are easily found:

$$E = E_p = \left\{m^2 + p^2\frac{\chi}{2} + g^2\frac{(\Sigma_+ + \Sigma_-)}{2} + \frac{\sqrt{\Xi}}{2}\right\}^{1/2}$$
$$\text{with } \phi_0 = \begin{pmatrix} \eta\phi_\lambda \\ \vartheta_\lambda\phi_\lambda \end{pmatrix} \tag{27}$$

and

$$E = -E_p = -\left\{m^2 + p^2\frac{\chi}{2} + g^2\frac{(\Sigma_+ + \Sigma_-)}{2} + \frac{\sqrt{\Xi}}{2}\right\}^{1/2}$$
$$\text{with } \phi_0 = \begin{pmatrix} \eta\varphi_\lambda \\ \vartheta_\lambda^*\varphi_\lambda \end{pmatrix} \tag{28}$$

with $\eta = \frac{1}{2}[\alpha p^2 - g^2(\Sigma_+ - \Sigma_-) - \sqrt{\Xi}]$ and

$$E = \tilde{E}_p = \left\{m^2 + p^2\frac{\chi}{2} + g^2\frac{(\Sigma_+ + \Sigma_-)}{2} - \frac{\sqrt{\Xi}}{2}\right\}^{1/2}$$
$$\text{with } \phi_0 = \begin{pmatrix} \tilde{\eta}\phi_\lambda \\ \vartheta_\lambda\phi_\lambda \end{pmatrix} \tag{29}$$

and

$$E = -\tilde{E}_p = -\left\{m^2 + p^2\frac{\chi}{2} + g^2\frac{(\Sigma_+ + \Sigma_-)}{2} - \frac{\sqrt{\Xi}}{2}\right\}^{1/2}$$
$$\text{with } \phi_0 = \begin{pmatrix} \tilde{\eta}\varphi_\lambda \\ \vartheta_\lambda^*\varphi_\lambda \end{pmatrix} \tag{30}$$

with $\tilde{\eta} = \frac{1}{2}[\alpha p^2 - g^2(\Sigma_+ - \Sigma_-) + \sqrt{\Xi}]$. $\lambda = \pm\frac{1}{2}$ and refers throughout the paper to the helicity states of the particle. We also have

$$\vartheta_\lambda = \kappa + 2\tau\lambda p \quad \text{with } \phi_\lambda = \begin{pmatrix} \chi_\lambda \\ 0 \end{pmatrix} \tag{31}$$

and

$$\vartheta_\lambda^* = \kappa - 2\tau\lambda p \quad \text{with } \varphi_\lambda = \begin{pmatrix} 0 \\ \chi_\lambda \end{pmatrix}, \tag{32}$$

where χ_λ is the spinor such that, if one considers the impulsion $\mathbf{p}$, it verifies the usual equation:

$$\sigma^i p_i \chi_\lambda = 2\lambda p \chi_\lambda. \tag{33}$$

One notes that $\{\vartheta_\lambda, \phi_\lambda\}$ and $\{\vartheta_\lambda^*, \varphi_\lambda\}$ are the eigensolutions of the operator T given by

$$T = \kappa - \tau\gamma^5\gamma^i p_i = \begin{pmatrix} \kappa + \tau\sigma^i p_i & 0 \\ 0 & \kappa - \tau\sigma^i p_i \end{pmatrix}, \tag{34}$$

and which constitutes the nondiagonal terms of the matrix in Eq. (22).

Using Eq. (18), it can be shown that

$$\psi = \begin{pmatrix} E + m & -\frac{\sigma^i p_i}{R_+} + ig\Theta_+ & 0 & -ig \\ \frac{\sigma^i p_i}{R_+} + ig\Theta_+ & -E + m & -ig & 0 \\ 0 & -ig & E + m & -\frac{\sigma^i p_i}{R_-} + ig\Theta_- \\ -ig & 0 & \frac{\sigma^i p_i}{R_-} + ig\Theta_- & -E + m \end{pmatrix}\phi_0 \tag{35}$$

with

$$\Theta_\pm = 1 + \frac{3}{2}\frac{R_\pm - R_\mp}{R_\pm}. \tag{36}$$

For the positive energies, Eqs. (27) and (29) suggest to look for solutions of the form:

$$\phi_0 = \eta\begin{bmatrix}\chi_\lambda\\0\\0\\0\end{bmatrix} + \vartheta_\lambda\begin{bmatrix}0\\0\\\chi_\lambda\\0\end{bmatrix} \quad \text{or} \quad \phi_0 = \tilde{\eta}\begin{bmatrix}\chi_\lambda\\0\\0\\0\end{bmatrix} + \vartheta_\lambda\begin{bmatrix}0\\0\\\chi_\lambda\\0\end{bmatrix}. \tag{37}$$

Similarly for negative energies, Eqs. (28) and (30) suggest to look for solutions of the form

$$\phi_0 = \eta\begin{bmatrix}0\\\chi_\lambda\\0\\0\end{bmatrix} + \vartheta^*_\lambda\begin{bmatrix}0\\0\\0\\\chi_\lambda\end{bmatrix} \quad \text{or} \quad \phi_0 = \tilde{\eta}\begin{bmatrix}0\\\chi_\lambda\\0\\0\end{bmatrix} + \vartheta^*\begin{bmatrix}0\\0\\0\\\chi_\lambda\end{bmatrix}. \tag{38}$$

Inserting the ansatz (37) and (38) into Eq. (35) leads (after normalization) to the following solutions for the Dirac equation:

For $E = \pm E_p$

$$u_\lambda(p) = \frac{1}{\sqrt{C}}\begin{Bmatrix}\eta(E_p+m)\chi_\lambda\\(\eta(\frac{2\lambda p}{R_+}+ig\Theta_+)-ig\vartheta_\lambda)\chi_\lambda\\\vartheta_\lambda(E_p+m)\chi_\lambda\\(\vartheta_\lambda(\frac{2\lambda p}{R_-}+ig\Theta_-)-ig\eta)\chi_\lambda\end{Bmatrix}, \qquad v_\lambda(p) = \frac{1}{\sqrt{C}}\begin{Bmatrix}(\eta(\frac{2\lambda p}{R_+}-ig\Theta_+)+ig\vartheta^*_\lambda)\chi_\lambda\\\eta(E_p+m)\chi_\lambda\\(\vartheta^*_\lambda(\frac{2\lambda p}{R_-}-ig\Theta_-)+ig\eta)\chi_\lambda\\\vartheta^*_\lambda(E_p+m)\chi_\lambda\end{Bmatrix}, \tag{39}$$

where $u_\lambda(p)$ and $v_\lambda(p)$ refer, respectively, to the positive and negative energy solutions.

For $E = \pm\tilde{E}_p$

$$\tilde{u}_\lambda(p) = \frac{1}{\sqrt{\tilde{C}}}\begin{Bmatrix}\tilde{\eta}(\tilde{E}_p+m)\chi_\lambda\\(\tilde{\eta}(\frac{2\lambda p}{R_+}+ig\Theta_+)-ig\vartheta_\lambda)\chi_\lambda\\\vartheta_\lambda(\tilde{E}_p+m)\chi_\lambda\\(\vartheta_\lambda(\frac{2\lambda p}{R_-}+ig\Theta_-)-ig\tilde{\eta})\chi_\lambda\end{Bmatrix}, \qquad \tilde{v}_\lambda(p) = \frac{1}{\sqrt{\tilde{C}}}\begin{Bmatrix}(\tilde{\eta}(\frac{2\lambda p}{R_+}-ig\Theta_+)+ig\vartheta^*_\lambda)\chi_\lambda\\\tilde{\eta}(\tilde{E}_p+m)\chi_\lambda\\(\vartheta^*_\lambda(\frac{2\lambda p}{R_-}-ig\Theta_-)+ig\tilde{\eta})\chi_\lambda\\\vartheta^*_\lambda(\tilde{E}_p+m)\chi_\lambda\end{Bmatrix}, \tag{40}$$

where, as previously, $\tilde{u}_\lambda(p)$ and $\tilde{v}_\lambda(p)$ refer, respectively, to the positive and negative energy solutions. In this model the status of the negative energy is the same as that in the classical Dirac equation. The normalization constants C and $\tilde{C}$ are easily calculated:

$$C = \eta^2\left\{(E_p+m)^2 + \frac{p^2}{R_+^2} + g^2(1+\Theta_+^2)\right\} + |\theta_\lambda|^2\left\{(E_p+m)^2 + \frac{p^2}{R_-^2} + g^2(1+\Theta_-^2)\right\} - 2\eta|\theta_\lambda|^2 \tag{41}$$

and

$$\tilde{C} = \tilde{\eta}^2\left\{(\tilde{E}_p+m)^2 + \frac{p^2}{R_+^2} + g^2(1+\Theta_+^2)\right\} + |\theta_\lambda|^2\left\{(\tilde{E}_p+m)^2 + \frac{p^2}{R_-^2} + g^2(1+\Theta_-^2)\right\} - 2\tilde{\eta}|\theta_\lambda|^2, \tag{42}$$

where $|\vartheta_\lambda|^2 = \kappa^2 + \xi^2 p^2$ with $\xi = g(\frac{1}{R_-} - \frac{1}{R_+})$.

It can be easily checked that at the limit where $R_\pm \to 1$, i.e. when both sheets have identical warp factors, the above solutions conform to that of the paper [10].

Reciprocally, at the decoupling limit (i.e. when $g \to 0$), the solutions (39) and (40) become "2 times" the classical Dirac ones expressed using the coordinates x. A careful calculation shows that the spinors of positive energy become

$$u_\lambda = \frac{1}{\sqrt{2E_p(E_p+m)}}\begin{bmatrix}(E_p+m)\chi_\lambda\\\frac{2\lambda p}{R_+}\chi_\lambda\\0\\0\end{bmatrix} \quad \text{and} \quad \tilde{u}_\lambda = \frac{1}{\sqrt{2\tilde{E}_p(\tilde{E}_p+m)}}\begin{bmatrix}0\\0\\(\tilde{E}_p+m)\chi_\lambda\\\frac{2\lambda p}{R_-}\end{bmatrix}, \tag{43}$$

while the energies become

$$E_p = \sqrt{m^2 + p^2 \frac{1}{R_+^2}} \quad \text{and} \quad \tilde{E}_p = \sqrt{m^2 + p^2 \frac{1}{R_-^2}}. \tag{44}$$

These solutions describe two four-dimensional spinors living in distinct sheets provided that the physical momenta p_+ and p_- are $p_+ = p/R_+$ and $p_- = p/R_-$ on the (+) and (−) sheet, respectively.

This choice implies that the physical length coordinates are $x_+ = R_+ x$ and $x_- = R_- x$ on the (+) and (−) sheet, respectively. Note that, under such a coordinate rescaling, both action and phase are conserved since

$$\mathbf{p} \cdot \mathbf{x} = \mathbf{p}_+ \cdot (R_+ \mathbf{x}) = \mathbf{p}_- \cdot (R_- \mathbf{x}) = \mathbf{p}_+ \cdot \mathbf{x}_+ = \mathbf{p}_- \cdot \mathbf{x}_-. \tag{45}$$

More clarifications on the coordinate rescaling will be given later in Sec. IV. Throughout this paper, x and p will refer to "global" coordinates and momenta while $x_\pm$ and $p_\pm$ will refer to "physical" or "ordinary" coordinates and momenta.

III. FERMIONIC OSCILLATIONS BETWEEN THE SHEETS AND HYPERFAST DISPLACEMENTS

A. Particle motion between the two four-dimensional sections

The predictions of the model can be illustrated by studying the following state corresponding to an unpolarized particle of positive energy:

$$\psi = \frac{1}{2\sqrt{V}}(u_{1/2} e^{ip\cdot x} e^{-iE_p t} - \tilde{u}_{1/2} e^{ip\cdot x} e^{-i\tilde{E}_p t}) + \frac{1}{2\sqrt{V}}(u_{-1/2} e^{ip\cdot x} e^{-iE_p t} - \tilde{u}_{-1/2} e^{ip\cdot x} e^{-i\tilde{E}_p t}). \tag{46}$$

By virtue of the two-sheeted structure of spacetime, ψ is an 8-component spinor whose first fourth components are located on the (+) sheet and the last fourth on the (−) sheet.

The probability P_+ (respectively P_-) to find the particle in the (+) sheet [respectively (−) sheet] is simply given by the integration of $|\psi_+|^2$ (respectively $|\psi_-|^2$), the square norm of the first (respectively last) fourth components of ψ, over the space coordinates on the volume V. We get

$$P_+ = A - 2B \cos(E_p - \tilde{E}_p)t \tag{47}$$

and

$$P_- = 1 - P_+ \tag{48}$$

with

$$\begin{aligned} A = {} & \frac{1}{2C}\left\{\eta^2\left[(E_p + m)^2 + \frac{p^2}{R_+^2} + g^2\Theta_+^2\right] + g^2|\vartheta|^2 + 2g\eta\left[\xi\frac{p^2}{R_+} - g\kappa\Theta_+\right]\right\} \\ & + \frac{1}{2\tilde{C}}\left\{\tilde{\eta}^2\left[(\tilde{E}_p + m)^2 + \frac{p^2}{R_+^2} + g^2\Theta_+^2\right] + g^2|\vartheta|^2 + 2g\tilde{\eta}\left[\xi\frac{p^2}{R_+} - g\kappa\Theta_+\right]\right\} \end{aligned} \tag{49}$$

and

$$B = \frac{1}{2\sqrt{C\tilde{C}}}\left\{\eta\tilde{\eta}\left[(E_p + m)(\tilde{E}_p + m) + \frac{p^2}{R_+^2} + g^2\Theta_+^2\right] + g^2|\vartheta|^2 + 2g[\eta + \tilde{\eta}]\left[\xi\frac{p^2}{R_+} - g\kappa\Theta_+\right]\right\}. \tag{50}$$

The presence of the cosine term in Eq. (47) shows that the particle oscillates between the two sheets with a frequency ν proportional to the difference of the two energy eigenvalues, i.e.,

$$2\pi\nu = \Delta E = |E_p - \tilde{E}_p|. \tag{51}$$

If one assumes that the mass term prevails over the other contributions, then the frequency is given by (at first order)

$$2\pi\nu \cong \frac{\sqrt{\Xi}}{2m} \tag{52}$$

and it can be easily checked that the oscillation frequency increases with the coupling strength between the two sheets and it is enhanced for low mass particles. At the limit of very low impulsion and for identical warp factors ($R_+ = R_-$), we get $\nu = g^2/(\pi m)$ which is the result found in Ref. [10] in a flat background.

Equation (47) tells us that, when the particle oscillates, it can disappear periodically from the perspective of any four-dimensional observer. This result is just the discrete counterpart of motions through the bulk predicted in some braneworld theories. A common feature of all those models (be they involving continuous or discrete extra dimension [9–12,15,16]) is the apparent violation of the energy conservation from a four-dimensional point of view. It is obvious that such a violation is only an artifact of low dimensionality.

B. Asymmetrical warp factors and velocities

In order to illustrate the incidence of the different physical length scales on the two sheets ($x_+ \neq x_-$ in the more general case for a same value of x), it is convenient to rewrite the Eq. (14) by multiplying it by Γ^0. Then, the two-sheeted Dirac equation can be recast as

$$i\frac{\partial}{\partial t}\psi = H\psi, \tag{53}$$

where H is the two-sheeted Hamiltonian. In this form, the global velocity operator V can be trivially calculated:

$$(V)_i = \left(\frac{\partial H}{\partial \mathbf{p}}\right)_i = \begin{pmatrix} \frac{1}{R_+}\gamma^0\gamma^i & 0 \\ 0 & \frac{1}{R_-}\gamma^0\gamma^i \end{pmatrix} \quad (54)$$

with $\mathbf{p}$ the global momentum operator. The global velocity can then be expressed:

$$v = \langle\psi|V|\psi\rangle = \frac{1}{R_+}\langle\psi_+|\gamma^0\gamma^i e_i|\psi_+\rangle + \frac{1}{R_-}\langle\psi_-|\gamma^0\gamma^i e_i|\psi_-\rangle. \quad (55)$$

It is instructive to consider the particle velocity measured by physical observers of the two sheets. For an observer of the (+) sheet, any particle confined to this sheet moves according to the velocity operator V_+ (notice the disappearance of the warp factor terms):

$$(V_+)_i = \left(\frac{\partial H}{\partial \mathbf{p}_+}\right)_i = \begin{pmatrix} \gamma^0\gamma^i & 0 \\ 0 & 0 \end{pmatrix}. \quad (56)$$

Similarly any particle located in the (−) sheet will be observed by local observers as moving according to the velocity operator:

$$(V_-)_i = \left(\frac{\partial H}{\partial \mathbf{p}_-}\right)_i = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^0\gamma^i \end{pmatrix}. \quad (57)$$

Therefore, the global velocity operator of a particle unrestricted in its motion can be written

$$(V)_i = \frac{1}{R_+}(V_+)_i + \frac{1}{R_-}(V_-)_i \quad (58)$$

such that in the most general case, the global velocity reads

$$\mathbf{v} = \mathbf{v}_+/R_+ + \mathbf{v}_-/R_- \quad (59)$$

with

$$\mathbf{v}_+ = \langle\psi|\mathbf{V}_+|\psi\rangle = \langle\psi_+|\gamma^0\gamma^i e_i|\psi_+\rangle \quad (60)$$

and similarly for $\mathbf{v}_-$.

We see that the particle velocity can be expressed as a combination of the physical velocities of the particle, had it remained confined in the (+) or the (−) sheet. As the particle oscillates from one sheet to the other one with a time period T, the effective distance x that the particle is able to travel during one period is given by (assuming a rectilinear motion)

$$x = \int_0^T v dt. \quad (61)$$

Let us define a starting point $x_+ = 0$ (or $x_- = 0$) at $t = 0$ in the (+) sheet [or respectively (−) sheet]. Therefore, an oscillating particle detected at time $t = T$ in the (+) sheet [or respectively (−) sheet] is observed at a physical distance x_+ (or respectively x_-) from its starting point:

$$x_+ = R_+ x = \int_0^T \left(v_+ + \frac{R_+}{R_-}v_-\right)dt \quad \text{and/or}$$
$$x_- = R_- x = \int_0^T \left(v_- + \frac{R_-}{R_+}v_+\right)dt. \quad (62)$$

As a consequence of the oscillatory motion, the particle is observed at time $t = T$ and at a distance x_+ (or respectively x_-) as if it had traveled at a fictitious mean velocity $\bar{v}_+$ (or $\bar{v}_-$) from the perspective of an observer of the (+) sheet [or respectively (−) sheet]:

$$\bar{v}_+ = \frac{1}{T}x_+ = R_+\frac{1}{T}\int_0^T v dt \quad (63)$$

and/or

$$\bar{v}_- = \frac{1}{T}x_- = R_-\frac{1}{T}\int_0^T v dt \quad (64)$$

with obviously $\bar{v}_+ \neq v_+$ and $\bar{v}_- \neq v_-$.

Therefore, in the most general case, the fictitious velocity ($\bar{v}_\pm$) of an oscillating particle differs from what it would have, had it remained confined in only one sheet (i.e. $v_\pm$). To clarify this issue, let us consider the case of an unpolarized particle initially mainly localized in the (+) sheet and described by the wave function

$$\psi = \frac{1}{2\sqrt{V}}(u_{1/2}e^{ip\cdot x}e^{-iE_pt} - \tilde{u}_{1/2}e^{ip\cdot x}e^{-i\tilde{E}_pt}) + \frac{1}{2\sqrt{V}}(u_{-1/2}e^{ip\cdot x}e^{-iE_pt} - \tilde{u}_{-1/2}e^{ip\cdot x}e^{-i\tilde{E}_pt}). \quad (65)$$

Considering that the momentum $\mathbf{p}$ is along the Oz axis, and according to the convention used for the spinor determination, the particle velocity becomes

$$\mathbf{v} = ve_3 = \langle\psi|V_3|\psi\rangle = \int \psi^\dagger V_3\psi d^3x. \quad (66)$$

We thus have

$$v = \tfrac{1}{4}\{a - 2b\cos(E_p - \tilde{E}_p + \varphi)t\} \quad (67)$$

with

$$a = u_{1/2}^\dagger V_3 u_{1/2} + u_{-1/2}^\dagger V_3 u_{-1/2} + \tilde{u}_{1/2}^\dagger V_3\tilde{u}_{1/2} + \tilde{u}_{-1/2}^\dagger V_3\tilde{u}_{-1/2} \quad (68)$$

$$be^{i\varphi} = u_{1/2}^\dagger V_3\tilde{u}_{1/2} + u_{-1/2}^\dagger V_3\tilde{u}_{-1/2} \quad \text{and}$$
$$be^{-i\varphi} = \tilde{u}_{1/2}^\dagger V_3 u_{1/2} + \tilde{u}_{-1/2}^\dagger V_3 u_{-1/2}. \quad (69)$$

As a consequence, the mean fictitious velocity as seen by an observer of the (+) sheet becomes

$$\bar{v}_+ = R_+\frac{1}{T}\int_0^T v dt = \frac{1}{4}R_+ a. \quad (70)$$

After tedious calculations, the fictitious particle velocity from the perspective of an observer located in the (+)

sheet reads

$$\bar{v}_+ = R_+ \frac{p}{C}(E_p + m)\left[\frac{\eta^2}{R_+^2} + \frac{|\vartheta_{1/2}|^2}{R_-^2} - \eta\xi^2\right] + R_+ \frac{p}{\tilde{C}}(\tilde{E}_p + m)\left[\frac{\tilde{\eta}^2}{R_+^2} + \frac{|\vartheta_{1/2}|^2}{R_-^2} - \tilde{\eta}\xi^2\right]. \quad (71)$$

For illustrative purposes, it is convenient to simplify this expression by considering its nonrelativistic limit. Then assuming that $g \ll m$, p and $p \ll m$, one gets the more compact form

$$\bar{v}_+ \cong \left\{\frac{\chi}{2m}R_+ + O[g]^2\right\}p + O[p]^2 = \frac{1}{2}\left(1 + \frac{R_+^2}{R_-^2}\right)\frac{p_+}{m} \quad (72)$$

with χ given by Eq. (23).

On the other hand, at the decoupling limit $g \rightarrow 0$ and assuming completely localized particles in the (+) sheet the velocity can be calculated to be [from Eq. (44)]

$$v_+ = \frac{1}{E_p}\frac{p}{R_+} = \frac{p_+}{E_p}. \quad (73)$$

At low impulsion, this expression conforms with the usual one as expected, i.e. $v_+ = p_+/m$. As a consequence, the expression (72) can be conveniently rewritten as

$$\bar{v}_+ = \frac{1}{2}\left(1 + \frac{R_+^2}{R_-^2}\right)v_+. \quad (74)$$

We see confirmed that an oscillating particle can travel between two locations with a fictitious (but effective) speed $\bar{v}_+$ which differs (in the general case where $R_+ \neq R_-$) from what it would have, had it remained in only one sheet [i.e. v_+ in the (+) sheet]. If $R_- \ll R_+$, the apparent particle velocity $\bar{v}_+$ can become huge even if the particle velocity v_+ in the (+) sheet remains moderated.

A numerical example can be given for illustrative purposes. Let us assume a warp factor ratio $R_+/R_- = 250$ and a particle initially located in the (+) sheet with a nonrelativistic velocity $v_+ = 20\ \text{km}\cdot\text{s}^{-1}$. Note that since we have $v_+/v_- = R_-/R_+$ in the nonrelativistic limit [from Eq. (73)], we see immediately that if a particle initially in the (+) sheet reaches the (−) sheet and stays there, its physical velocity as measured by an observed of the (−) sheet will not be $20\ \text{km}\cdot\text{s}^{-1}$ but $5000\ \text{km}\cdot\text{s}^{-1}$. Although this is an important velocity increase, this value is still a nonrelativistic one. At this stage, it is important to stress that the equations of the physical velocities v_+ [see Eq. (73)] and v_- imply that $v_\pm < 1$ whatever R_+, R_-, p, m. Therefore, even if the particle moves faster in the (−) sheet, it can never exceed the light velocity. The laws of special relativity are safe in the present approach. In addition, from Eq. (45) we note that $x_+/x_- = R_+/R_-$. In the present example, this means that the distances are 250 times shorter in the (−) sheet than in the (+) sheet. Let us consider a particle initially localized in the (+) sheet and transferred into the (−) sheet. The particle covers a distance x_- during a time t and goes back in the (+) sheet. For an observer of the (+) sheet, the detected particle has apparently moved a distance x_+ during a time t, such that its apparent (but fictitious) velocity $\overline{v}_+ = x_+/t$ or $\overline{v}_+ = v_+(R_+/R_-)^2$. The latter expression is the consequence of the shortened distance and increased proper velocity in (−) sheet. The reason why this expression differs from Eq. (74) is the result of the oscillating behavior of the particle. Indeed, on the average, the oscillating particle spends as much time in sheet (−) as in sheet (+). The fictitious velocity $\bar{v}_+$ in Eq. (74) is then an average velocity between the real velocity in (+) sheet and the fictitious velocity, had it remained in the (−) sheet only.

For a proper velocity v_+ of $20\ \text{km}\cdot\text{s}^{-1}$ in the (+) sheet, Eq. (74) shows that the fictitious velocity $\bar{v}_+$ [calculated by an observer located in the (+) sheet] is about 2 times the light velocity. Although the proper velocity of the particle remains nonrelativistic in both sheets, its apparent velocity from the perspective of an observer of the (+) sheet exceeds now the light speed.

In Ref. [13], it was suggested that the homogeneity of the universe by the time of nucleosynthesis might be explained by particle motions in an asymmetric two-branes system (one of which being a "hidden" brane). According to the authors, an impulse originating on one brane of the system can take a shortcut through the other brane and affect our brane at a point outside the conventional causal horizon. The present paper tries to go further on those aspects by using a very simple although realistic quantum mechanical model. Although our approach is radically different from that of Ref. [13], our results share obvious similarities (e.g. the particles confined in the other sheet are invisible to us, the oscillating particles can reach distant point outside their "naive" horizon ...). Hence, it is believed that there may generically exist a noninflationary solution to the horizon problem in theories with extra dimensions (be they continuous or discontinuous).

IV. TWO-SHEETED ELECTROMAGNETIC FIELD AND PAULI EQUATION

A. Introduction of the electromagnetic field

To account for the discrete structure of the bulk, the usual $U(1)$ gauge field must be substituted by an extended $U(1) \otimes U(1)$ gauge field such that

$$G = \begin{bmatrix} \mathbf{1}_{4\times4}\exp(-iq\Lambda_+) & 0 \\ 0 & \mathbf{1}_{4\times4}\exp(-iq\Lambda_-) \end{bmatrix}. \quad (75)$$

We look for an appropriate gauge such that $\not{D}_A \rightarrow \not{D} + \not{A}$ with the following rule of transformation:

$$\not{A}' = G\not{A}G^\dagger + G[\not{D}, G^\dagger]. \quad (76)$$

A convenient choice is (see Refs. [9–12,17])

$$A = \begin{bmatrix} iq\frac{1}{R_+}\gamma^\mu A^+_\mu & \gamma^5\chi \\ \gamma^5\chi^\dagger & iq\frac{1}{R_-}\gamma^\mu A^-_\mu \end{bmatrix}, \tag{77}$$

where γ^μ and γ^5 are the usual Dirac matrices. Choosing $\chi = 0$ whatever the gauge choice can be done only if $\Lambda_+ = \Lambda_- = \Lambda$.

In addition, the above definitions impose the following gauge transformations:

$$A^\pm_\mu = A^\pm_\mu + \partial_\mu \Lambda, \tag{78}$$

On the two sheets live the distinct A_+ and A_- fields. Each spacetime sheet possesses its own current and charge density distribution as sources of the local electromagnetic fields. The off-diagonal term has been set equal to zero. This free choice allows a further simplification of the model. If this term is different from zero, it leads to a coupling between the two photon fields such that each charged particle becomes sensitive to the electromagnetic fields of both sheets irrespective of its localization in the bulk. With the present choice, the electromagnetic field of a sheet couples only with the particles belonging to the same sheet.

B. Derivation of the Pauli equation

Let us derive the nonrelativistic limit of the two-sheeted Dirac equation for the metric field (3). We first introduce the gauge contributions $\not{A}$ of both sheets into the two-sheeted Dirac equation following the standard procedure,

$$(i\not{D}_A - m)\Psi = 0, \tag{79}$$

such that

$$\not{D}_A = \Gamma^0(\partial_0 + iq\hat{A}_0) + \frac{1}{R}\Gamma^\eta(\partial_\eta + iq\hat{A}_\eta) + g\Gamma^5 + g\Delta. \tag{80}$$

One can also write

$$i\partial_0\Psi = -i\Gamma^0\frac{1}{R}\Gamma^\eta(\partial_\eta + iq\hat{A}_\eta)\Psi - ig\Gamma^0\Gamma^5\Psi - ig\Gamma^0\Delta\Psi + m\Gamma^0\Psi + q\hat{A}_0\Psi, \tag{81}$$

where

$$\hat{A}_\mu = \begin{bmatrix} A^+_\mu & 0 \\ 0 & A^-_\mu \end{bmatrix}. \tag{82}$$

When m is large compared with the kinetic energy, the most rapid time dependence is in the factor $\exp(\pm imt)$. For a free positive energy particle, and for small kinetic and electromagnetic energies, we may therefore seek a solution of the form $\Psi = \psi e^{-imt}$ with

$$\psi = \begin{bmatrix} \chi_+ \\ \theta_+ \\ \chi_- \\ \theta_- \end{bmatrix}, \tag{83}$$

where χ_+, θ_+, χ_-, θ_- are two-component spinors. One can then write

$$i\partial_0\chi_+ = qA^+_0\chi_+ - i(1/R_+)\sigma_\eta(\partial_\eta + iqA^+_\eta)\theta_+ - ig(\theta_+ - \theta_-) - ig(3/2)\{(R_+ - R_-)/R_+\}\theta_+ \tag{84}$$

$$i\partial_0\chi_- = qA^-_0\chi_- - i(1/R_-)\sigma_\eta(\partial_\eta + iqA^-_\eta)\theta_- + ig(\theta_+ - \theta_-) - ig(3/2)\{(R_- - R_+)/R_-\}\theta_- \tag{85}$$

$$i\partial_0\theta_+ = -i(1/R_+)\sigma_\eta(\partial_\eta + iqA^+_\eta)\chi_+ + qA^+_0\theta_+ + ig(\chi_+ - \chi_-) - 2m\theta_+ + ig(3/2) \times \{(R_+ - R_-)/R_+\}\chi_+ \tag{86}$$

$$i\partial_0\theta_- = -i(1/R_-)\sigma_\eta(\partial_\eta + iqA^-_\eta)\chi_- + qA^-_0\theta_- - ig(\chi_+ - \chi_-) - 2m\theta_- + ig(3/2) \times \{(R_- - R_+)/R_-\}\chi_-. \tag{87}$$

In addition, when p is much smaller than m, θ_+ and θ_- become tiny in comparison with χ_+ and χ_-. For small electromagnetic and kinetic energies, one then get the following expressions:

$$\theta_+ \approx -i(1/R_+)\frac{1}{2m}\sigma_\eta(\partial_\eta + iqA^+_\eta)\chi_+ + i\frac{g}{2m}(\chi_+ - \chi_-) + i\frac{g}{2m}(3/2)\{(R_+ - R_-)/R_+\}\chi_+ \tag{88}$$

$$\theta_- \approx -i(1/R_-)\frac{1}{2m}\sigma_\eta(\partial_\eta + iqA^-_\eta)\chi_- - i\frac{g}{2m}(\chi_+ - \chi_-) + i\frac{g}{2m}(3/2)\{(R_- - R_+)/R_-\}\chi_-. \tag{89}$$

In order to give these equations a more conventional form, vectors must be used instead of covariant or contravariant terms. The procedure, which is similar to that used in cosmology [18], can be summarized as follows. For a metric field given by $g_{ij} = R^2\delta_{ij}$ $(i, j = 1, 2, 3)$, we can define an orthonormal basis,

$$\mathbf{e}_i = \frac{\mathbf{g}_i}{R} = R\mathbf{g}^i, \tag{90}$$

such that $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = g_{ji}$. What are usually called the components of a vector $\mathbf{a}$ in elementary treatments are neither the covariant components a_i nor the contravariant components a^i, but the ordinary components:

$$\bar{a}_i = \mathbf{a} \cdot \mathbf{e}_i = Ra^i = R^{-1}a_i. \tag{91}$$

Moreover, to be consistent, vectors and operators relative to a specific sheet have to be expressed using the metric of the corresponding sheet, i.e. following Ref. [18]:

$$\nabla_{\pm} = \sum_i \mathbf{e}_i \frac{1}{R_{\pm}} \frac{\partial}{\partial x^i} = \frac{i}{R_{\pm}} \sum_i \mathbf{p}_i = i \sum_i \mathbf{p}_{\pm,i} \quad \text{where } \mathbf{p}_i = -i\mathbf{e}_i \frac{\partial}{\partial x^i} \quad \text{and} \quad \mathbf{p}_{\pm,i} = -i\mathbf{e}_i \frac{\partial}{\partial x^i_{\pm}} = \frac{1}{R_{\pm}} \mathbf{p}_i \tag{92}$$

and similarly, magnetic vector potentials and magnetic fields are given by

$$\bar{A}_{\pm,i} = R_{\pm} A^i_{\pm} = R_{\pm}^{-1} A_{\pm,i} \text{ and similarly for } \bar{B}_{\pm,j}. \tag{93}$$

Note that we have introduced the global vector $\mathbf{p}$ exactly as we did for the Dirac equation [Eq. (45)].

By setting $\Phi_{\pm}$ and $\mathbf{A}_{\pm}$ for the usual electric and magnetic potential, it can be easily shown that the Pauli equation reads

$$H\varphi = i\partial_0 \varphi \quad \text{with } \varphi = \begin{bmatrix} \chi_+ \\ \chi_- \end{bmatrix}, \tag{94}$$

where we have used the fact that

$$\sigma_\eta \sigma_\nu \overline{(\nabla_{\pm} - iq\mathbf{A}_{\pm})}_\eta \overline{(\nabla_{\pm} - iq\mathbf{A}_{\pm})}_\nu = (\nabla_{\pm} - iq\mathbf{A}_{\pm})^2 + q\sigma \cdot \mathbf{B}_{\pm}. \tag{95}$$

The resulting Hamiltonian can be written as the following sum:

$$H = (H_k + H_m + H_p) + (H_c + H_{cm} + H_{cp}), \tag{96}$$

where

$$H_k = -\frac{1}{2m} \begin{bmatrix} (\nabla_+ - iq\mathbf{A}_+)^2 & 0 \\ 0 & (\nabla_- - iq\mathbf{A}_-)^2 \end{bmatrix} \tag{97}$$

$$H_m = -\frac{q}{2m} \begin{bmatrix} \sigma \cdot \mathbf{B}_+ & 0 \\ 0 & \sigma \cdot \mathbf{B}_- \end{bmatrix} \tag{98}$$

$$H_p = \begin{bmatrix} q\Phi_+ & 0 \\ 0 & q\Phi_- \end{bmatrix} \tag{99}$$

$$H_c = \frac{g^2}{m} \begin{bmatrix} 1 + (3/2)(\frac{R_+ - R_-}{R_+}) + (9/8)(\frac{R_+ - R_-}{R_+})^2 & -1 + (3/4)\frac{(R_+ - R_-)^2}{R_+ R_-} \\ -1 + (3/4)\frac{(R_+ - R_-)^2}{R_+ R_-} & 1 + (3/2)(\frac{R_- - R_+}{R_-}) + (9/8)(\frac{R_- - R_+}{R_-})^2 \end{bmatrix} \tag{100}$$

$$H_{cm} = i\frac{gq}{2m} \begin{bmatrix} 0 & \sigma \cdot \{\mathbf{A}_+ - \mathbf{A}_-\} \\ -\sigma \cdot \{\mathbf{A}_+ - \mathbf{A}_-\} & 0 \end{bmatrix} \tag{101}$$

$$H_{cp} = \frac{g}{2m} \begin{bmatrix} 0 & \sigma \cdot \{\nabla_+ - \nabla_-\} \\ -\sigma \cdot \{\nabla_+ - \nabla_-\} & 0 \end{bmatrix}. \tag{102}$$

This two-sheeted Pauli equation is very similar to that derived in Refs [9,10] except for the gradient operator which is distinct in the two sheets as a consequence of the two metric fields. It is worth being noticed that, at the limit of decoupling, we get 2 times the usual Pauli equation as expected. Therefore, at the limit of small g, the difference between the two-sheeted Pauli equation and the usual one are not expected to be significant.

The first three terms of the Hamiltonian correspond to the classical contributions of the "one-sheeted" Pauli's equation. H_k is the kinetic term whereas H_m and H_p relate to the magnetic and Coulomb terms, respectively. The last three terms correspond to new predictions of the model. The term H_c behaves as a constant coupling between the two sheets. This term is responsible for the spontaneous particle oscillations studied previously in the relativistic limit. H_{cp} and H_{cm} introduce the geometrical coupling between the sheets through kinetic and magnetic terms. It is worth stressing that the coupling which involves the kinetic mixing disappears if the two sheets have the same warp factors whereas H_{cm} does not.

C. Hyperfast velocities in the nonrelativistic limit

As previously, three different velocity operators can be defined: $\mathbf{V}_+$ ($\mathbf{V}_-$) which corresponds to the velocity of a confined particle in the (+) [respectively (−)] sheet and the global velocity operator $\mathbf{V}$. These operators read

$$\mathbf{V}_+ = \frac{\partial H}{\partial \mathbf{p}_+} = \begin{bmatrix} \frac{1}{m}(\mathbf{p}_+ - q\mathbf{A}_+) & \frac{ig}{2m}\sigma \\ -\frac{ig}{2m}\sigma & 0 \end{bmatrix} \tag{103}$$

and

$$\mathbf{V}_- = \frac{\partial H}{\partial \mathbf{p}_-} = \begin{bmatrix} 0 & -\frac{ig}{2m}\sigma \\ \frac{ig}{2m}\sigma & \frac{1}{m}(\mathbf{p}_- - q\mathbf{A}_-) \end{bmatrix} \tag{104}$$

and

$$\mathbf{V} = \frac{\partial H}{\partial \mathbf{p}} = \begin{bmatrix} \frac{1}{R_+}\frac{1}{m}(\frac{1}{R_+}\mathbf{p} - q\mathbf{A}_+) & \frac{ig}{2m}\sigma \cdot \{\frac{1}{R_+} - \frac{1}{R_-}\} \\ -\frac{ig}{2m}\sigma \cdot \{\frac{1}{R_+} - \frac{1}{R_-}\} & \frac{1}{R_-}\frac{1}{m}(\frac{1}{R_-}\mathbf{p} - q\mathbf{A}_-) \end{bmatrix} = \frac{1}{R_+}\frac{\partial H}{\partial \mathbf{p}_+} + \frac{1}{R_-}\frac{\partial H}{\partial \mathbf{p}_-} = \frac{1}{R_+}\mathbf{V}_+ + \frac{1}{R_-}\mathbf{V}_-. \tag{105}$$

We point out the existence of a nondiagonal components of the velocity operator. However, since this term is proportional to g we can ignore it for the moment and in the forthcoming calculations to concentrate only on the effect of the diagonal terms in $\mathbf{V}$.

D. Illustrative case

In the following, we neglect the terms g^2 of the Hamiltonian in front of the g terms. In this way we underline the contribution of H_{cm} and H_{cp}. To illustrate the survivance of hyperfast displacement in the nonrelativistic limit, it is convenient to consider the case of a particle initially located in the first sheet and embedded in a region of constant curlless magnetic vector potential. For simplicity reasons, let us consider a neutronlike particle, i.e. a chargeless particle with a magnetic moment. We do not consider the complications arising from the anomalous magnetic moment of this particle. Instead, we assume that it is identical to that of an electron. In the absence of any magnetic field or scalar potential and by neglecting the g^2 terms in front of the g terms, the Hamiltonian reads

$$H = \begin{bmatrix} K_+ & -i\alpha\sigma\cdot\mathbf{P} \\ i\alpha\sigma\cdot\mathbf{P} & K_- \end{bmatrix} \tag{106}$$

with $\alpha = g/(2m)$ and

$$K_\pm = \frac{1}{R_\pm^2}\frac{\mathbf{p}^2}{2m} \tag{107}$$

and

$$\mathbf{P} = e\mathbf{A} + \left\{\frac{1}{R_+} - \frac{1}{R_-}\right\}\mathbf{p}. \tag{108}$$

In the following, one considers that $\mathbf{A}$ and $\mathbf{p}$ are collinear. The eigenvectors of the above Hamiltonian are

$$u_{\pm,\lambda} = \frac{1}{\sqrt{N_\pm}}\begin{bmatrix} (E_\pm - K_-)\chi_\lambda \\ i\alpha\sigma\cdot\mathbf{P}\chi_\lambda \end{bmatrix} \tag{109}$$

with the corresponding eigenvalues

$$E_\pm = \tfrac{1}{2}\left(K_+ + K_- \pm \sqrt{(K_+ - K_-)^2 + 4\alpha^2P^2}\right). \tag{110}$$

χ_λ is a spinor where $\lambda = \pm 1/2$ and stands for both spin states and $N_\pm = (E_\pm - K_-)(2E_\pm - K_+ - K_-)$.

It is not difficult to convince oneself that the particle ability of reaching the second spacetime sheet still survives in the nonrelativistic limit. The wave function assuming that the particle is initially ($t = 0$) located in the (+) sheet with some polarization state $(n_{1/2}^2 - n_{-1/2}^2)/(n_{1/2}^2 + n_{-1/2}^2)$ is

$$\psi = \frac{1}{\sqrt{V}}\{n_{1/2}(au_{+,1/2}e^{-iE_+t} + bu_{-,1/2}e^{-iE_-t}) + n_{-1/2}(au_{+,-1/2}e^{-iE_+t} + bu_{-,-1/2}e^{-iE_-t})\}e^{ip\cdot x} \tag{111}$$

with $n_{1/2}^2 + n_{-1/2}^2 = 1$ and

$$a = \sqrt{\frac{N_+}{N_+ + N_-}} \quad \text{and} \quad b = \sqrt{\frac{N_-}{N_+ + N_-}}. \tag{112}$$

The probability P to find the particle in the second sheet can be trivially calculated,

$$P = \frac{1}{1+\kappa^2}\sin^2\{(1/2)(E_+ - E_-)t\}, \tag{113}$$

with

$$\kappa = \frac{K_+ - K_-}{2\alpha P}. \tag{114}$$

Without going further, it can already be noticed that if $A = 0$,

$$\kappa = \frac{p_+}{2g}\left(1 + \frac{R_+}{R_-}\right), \tag{115}$$

and

$$E_+ - E_- = \frac{p_+^2}{2m}\left|1 - \frac{R_+}{R_-}\right|\sqrt{\left(1 + \frac{R_+}{R_-}\right)^2 + \frac{4g^2}{p_+^2}} \tag{116}$$

with $p_+ = p/R_+$. In the absence of confining effect, the particle oscillates between the two four-dimensional sections provided that $p_+ \neq 0$. However, if p_+ becomes larger than g, P quickly drops to zero and the spontaneous oscillations are strongly suppressed.

Returning back to the general case where $A \neq 0$ and $p_+ \neq 0$, it is interesting to calculate the particle velocity. Assuming $\mathbf{p}$ and $\mathbf{A}$ are oriented along the Oz axis, the global particle velocity v is given by

$$\begin{aligned} v = v_z &= \int \psi^\dagger V_z\psi dV \\ &= \frac{1}{R_+^2}\frac{p}{m}\cos^2\{(1/2)(E_+ - E_-)t\} \\ &\quad + \frac{1}{R_-^2}\frac{p}{m}\left(\frac{\frac{R_-^2}{R_+^2}(K_+ - K_-)^2 + 4\alpha^2P^2}{(K_+ - K_-)^2 + 4\alpha^2P^2}\right) \\ &\quad \times \sin^2\{(1/2)(E_+ - E_-)t\}. \end{aligned} \tag{117}$$

Since the particle oscillates between the two sheets, the velocity exhibits also an oscillatory behavior. The fictitious velocity $\overline{v}_+$ from the point of view of an observer in the sheet (+), after a period $T = \frac{2\pi}{(E_+ - E_-)}$ is

$$\overline{v} = \frac{R_+}{T}\int_0^T v\,dt \tag{118}$$

and therefore

$$\overline{v}_+ = \frac{1}{R_+}\frac{p}{m}\frac{1}{2}\left\{1 + \frac{(K_+ - K_-)^2 + 4\alpha^2P^2\frac{R_+^2}{R_-^2}}{(K_+ - K_-)^2 + 4\alpha^2P^2}\right\}. \tag{119}$$

From Eqs. (107) and (108) and for a large enough A, the previous expression simplifies further:

$$\overline{v}_+ \approx \frac{v_+}{2}\left\{1 + \frac{R_+^2}{R_-^2}\right\} \tag{120}$$

with $v_+ = p_+/m$ the usual particle velocity had it remained confined in the (+) sheet. We recover the relation already derived from the relativistic approach at the limit of low velocities.

Again, we see that, if $R_+ \gg R_-$, the particle moves with an apparent velocity that can exceed the velocity v_+ this particle would have, had it remained in the (+) sheet. The novelty in comparison with the result (74) is that the particle oscillates through the application of a magnetic potential and therefore this result suggests that the model could perhaps be experimentally investigated (this possibility and the properties of H_{cm} have been explored more in Refs. [11,12]).

We stress that the results of this paper have been obtained for a free particle, e.g. by assuming that the magnetic field, the scalar potential, and any other environmental contribution can be neglected (excepted for a constant curlless magnetic vector potential as discussed previously). This condition is obviously a very restrictive one and, in most cases, the suppression of these terms could be hardly achieved. In Ref. [12], it was demonstrated that any diagonal term in the Hamiltonian strongly suppresses the particle oscillations. In fact, it is not difficult to demonstrate that the more energetic the particle is, the more the oscillations are suppressed. Even the gravitational potential, whose contribution modifies the particle energy, will affect the oscillations and restricts the particle motion between the sheets. As demonstrated in Ref. [12], our model suggests that any massive particle could be spontaneously confined within the sheets (through environmental interactions) without requiring any complementary scalar field or repulsive gravity [12]. Since no particle disappearance has been noted to date, it is very likely that the degree of confinement is very strong, and/or that the coupling constant g is very small. Therefore, it is expected that the phenomena described in this paper will be hardly observed, especially if one takes into account the fact that no current experimental setup is suitably designed for searching for these phenomena. Any experiment aiming at demonstrated the behavior predicted in the present paper will require very particular conditions although they might not be completely out of reach of our present technology [11,12].

V. CONCLUSIONS

In this paper, we have studied the quantum dynamics of spin half particles in an asymmetrically two-sheeted spacetime. It was shown that any free particle oscillates between the two sheets as a consequence of the geometrical coupling along the discrete extra dimension. By oscillating, any massive particle is able to travel between distant points which are normally outside its four-dimensional horizon. The reason arises from the differential warping which leads to reduced length scales and increased velocities in one of the sheet.

APPENDIX A

The discrete derivative used in present and previous papers [10–12] is defined as follows.

A compact oriented discretized dimension $Z_n = \{s_i | i \in \{0, \ldots, n-1\}\}$ is considered, with s_i being each of the n sites of Z_n (see Fig. 1). Z_n is invariant under the cyclic group Z/nZ. Z_n is oriented positively with increasing values of site index i. For each site s_i of Z_n, it is possible to define a coordinate $\lambda(s_i) = \lambda_i = \delta i + q$. Note that both δ and q have the dimension of a length. q is an arbitrary constant (the presence of this term will be clarified shortly afterwards) whereas δ is the distance between nearest-neighbor sites.

One then defines a positive restricted algebraic distance $d_a(s_i, s_j)$ on Z_n. That means that one goes from site s_i to site s_j by imposing a positive direction, i.e.,

$$d_a(s_i, s_j) = \begin{cases} \lambda_j - \lambda_i & \text{if } j \geq i \\ \lambda_j - \lambda_i + \delta n & \text{if } j < i. \end{cases} \tag{A1}$$

This imposes

$$d_a(s_{i-1}, s_i) = \delta \quad \text{for } i \in \{1, \ldots, n-1\} \tag{A2}$$

and

$$d_a(s_{n-1}, s_0) = \delta \tag{A3}$$

with respect to the cyclic properties.

If one considers an arbitrary function $\phi(\lambda(s_i)) = \phi(\lambda_i) = \phi_i$, the derivative $\partial_\lambda \phi(\lambda_i)$ at each site s_i is naturally given by

$$\partial_\lambda \phi(\lambda_i) \equiv \frac{\phi(\lambda_i) - \phi(\lambda_{i-1})}{d_a(s_{i-1}, s_i)} \quad \text{for } i \in \{1, \ldots, n-1\} \tag{A4}$$

with of course

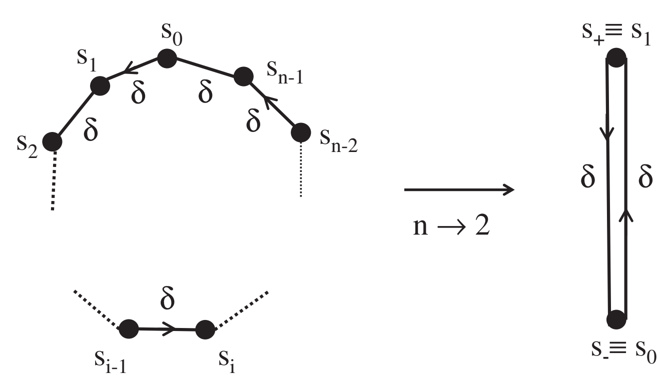

FIG. 1. General polygonal representation of the compact discretized Z_n dimension (left) and of its Z_2 limit (right).

$$\partial_\lambda \phi(\lambda_0) \equiv \frac{\phi(\lambda_0) - \phi(\lambda_{n-1})}{d_a(s_{n-1}, s_0)}. \tag{A5}$$

These expressions can be written in the usual form:

$$\partial_\lambda \phi_i \equiv (1/\delta)(\phi_i - \phi_{i-1}) \quad \text{for } i \in \{1, \ldots, n-1\} \tag{A6}$$

and

$$\partial_\lambda \phi_0 \equiv (1/\delta)(\phi_0 - \phi_{n-1}). \tag{A7}$$

By construction, the derivative is invariant through cyclic permutation and corresponds to the so-called "Z_n-derivative."

The "Z_2-derivative" is simply obtained by setting $n = 2$ in previous expressions. The result reads

$$\partial_\lambda \phi_1 = (1/\delta)(\phi_1 - \phi_0) \quad \text{and} \quad \partial_\lambda \phi_0 = (1/\delta)(\phi_0 - \phi_1). \tag{A8}$$

Considering the following substitution $(\phi_0, \phi_1) \to (\phi_-, \phi_+)$ the derivative can be expressed into the form

$$\partial_\lambda \phi_\pm = \pm(1/\delta)(\phi_+ - \phi_-). \tag{A9}$$

In addition, by setting $q = -\delta/2$, we get $\lambda_\pm = \pm\delta/2$ for the coordinate of each site.

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[14] It is straightforward to show that the expression (8) of the spin connection is equivalent to the more familiar one $\Gamma_A = \frac{1}{8}\omega_{abA}[\gamma^a, \gamma^b]$ where the connection term is given by the usual expression $\omega_{abA} = \eta_{ac}\omega^c_{bA} = \eta_{ac}e^c_B(\frac{\partial}{\partial x^A}e^B_b + \Gamma^B_{CA}e^C_b)$. See for instance T. R. Slatyer and R. R. Volkas, J. High Energy Phys. 04 (2007) 062; or S. L. Parameswarana, S. Randjbar-Daemib, and A. Salvioa, Nucl. Phys. **B767**, 54 (2007).
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