Exact solutions of Einstein-Yang-Mills theory with higher-derivative coupling

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We construct an exact classical solution of an Einstein-Yang-Mills system in ten space-time dimensions whose action contains a term quartic in Yang-Mills field strength. The solution provides compactification proposed by Cremmer and Scherk; the ten-dimensional space-time with a cosmological constant is compactified to the direct product of four-dimensional Minkowski space-time and a six-dimensional sphere S^6 whose radius is determined by coupling constants. We also construct a solution which compactifies the ten-dimensional space-time to $\mathrm{AdS}_4 \times S^6$ when the cosmological constant is absent.

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Unification of fundamental forces with space-time and matter often requires higher-dimensional space-time rather than our four-dimensional Universe. The early Kaluza-Klein theory unifies gravity and the electromagnetic interaction by considering five-dimensional space-time with one direction compactified into a circle S^1 [1]. This old idea has been revisited several times. After supergravity was discovered many people tried to unify all forces and matter in higher-dimensional space-time with extra-dimensional space compactified on various manifolds [2]. String theory was proposed as the most attractive candidate of unification, but it is defined only in ten-dimensional space-time. In order to realize the four-dimensional Universe one has to find a suitable six-dimensional internal space. Many candidates of such spaces were proposed; Calabi-Yau manifolds and orbifolds. Internal manifolds can be deformed while satisfying equations of motion. In such cases there appear moduli of space-time, which leads to unwanted massless particles in the four-dimensional world. Recently a new mechanism has been suggested to fix these moduli by turning on the Ramond-Ramond flux on the internal space [3,4]. This flux compactification has been extensively studied in recent years.

We would like to reexamine the compactification scenario with fixed moduli proposed by Cremmer and Scherk a long time ago [5] (see also [6,7]) in a theory with a cosmological constant. By placing solitons on a compact internal space they showed that decompactifying limit with a large radius of the internal space is disfavored and the radius is fixed to a certain value determined by coupling constants. They considered the 't Hooft-Polyakov monopole [8] on S^2 and the Yang-Mills instanton [9] on S^4, both of which can satisfy the first order (self-dual) equations, with proper values of the coupling constants. Since string theory is defined in ten dimensions, it is natural to consider this scenario with stable Bogomol'nyi-Prasad-Sommerfield (BPS) solitons on a six-dimensional internal space like S^6.

Higher-dimensional generalization of self-dual equations was suggested by Tchrakian some years ago [10]. The eight dimensional case is known as octonionic instantons [11]. Though several works have been done for generalized self-dual equations [12,13], a six-dimensional case has not been discussed because of the lack of conformal property. Recently we have found a new solution to the generalized self-dual equations in an SO(6) pure Yang-Mills theory on a six-dimensional sphere S^6 [14] with an additional fourth order term in field strengths, which we call a four-derivative term.

In this article we propose to use this solution in the context of the compactification of the Cremmer-Scherk type. In our model ten-dimensional space-time with (without) a cosmological constant is compactified to the product of a four-dimensional Minkowski space M_4 (anti-de Sitter space AdS_4) and a six-dimensional sphere S^6. Here the dimensionality of the internal space, six, is required by the four-derivative term. In the presence of gravity the four-derivative coupling constant α can differ from the constant $\beta = eR_0^2/3$ in the generalized self-dual equations where e and R_0 are the gauge coupling constant and the radius of S^6. When the relation $\alpha = \beta$ holds the generalized self-dual equations become the Bogomol'nyi equations so that solutions are BPS. We find for both $M_4 \times S^6$ and $\mathrm{AdS}_4 \times S^6$ that certain relations exist between the radius of S^6, the gauge coupling, the four-derivative coupling α, and the gravitational coupling constants. When the four-derivative coupling constant α vanishes in the case of $M_4 \times S^6$, these relations reduce to those of the original work by

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Cremmer and Scherk. The advantage of our model over the Cremmer-Scherk model is that the Yang-Mills soliton in our model satisfies the self-dual equations (the Bogomol'nyi equations for $\alpha = \beta$). This ensures the stability of configuration at least for the sector of Yang-Mills fields.

Let space-time be a ten-dimensional manifold. We consider an Einstein-Yang-Mills theory. Our action contains as dynamical variables the Yang-Mills (gauge) fields $A^{[ab]}_{\hat{\mu}}$ and a graviton field or the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$. Indices with a hat "^" will refer to a ten-dimensional space-time $(X^0, X^1, \cdots, X^9)$. Latin indices $(a, b, \cdots)$ run from 1 to 6 and refer to an internal space. The Clifford algebra associated with the orthogonal group SO(6) is useful and we represent generators of the Lie algebra so(6) as their elements. The Clifford algebra is defined by gamma matrices $\{\Gamma_a\}$ which satisfy the anticommutation relations $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$. These matrices can be realized as 8×8 matrices with complex coefficients. The generators of so(6) are represented by $\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]$. We often abbreviate the Yang-Mills fields as $A_{\hat{\mu}} = \frac{1}{2}A^{[ab]}_{\hat{\mu}}\Gamma_{ab}$ and we also use notations with differential forms. Thus the gauge fields are expressed as $A = A_{\hat{\mu}}dX^{\hat{\mu}}$. In this notation, the corresponding field strength F is written as $F = dA + eA \wedge A$, where e is a gauge coupling. Covariant derivative $\mathcal{D}_{\hat{\mu}}$ on an adjoint representation $Y = \frac{1}{2}Y^{[ab]}\Gamma_{ab}$ is defined as $\mathcal{D}_{\hat{\mu}}Y = \partial_{\hat{\mu}}Y + e(A_{\hat{\mu}}Y - YA_{\hat{\mu}})$, where Y is a scalar multiplet. The total action is

$$\begin{aligned}\mathcal{S}_{\text{total}} &= \mathcal{S}_E + \mathcal{S}_{YMT},\\ \mathcal{S}_E &= \frac{1}{16\pi G}\int dv \mathcal{R},\\ \mathcal{S}_{YMT} &= \frac{1}{16}\int \mathrm{Tr}\{-F \wedge *F + \alpha^2(F \wedge F) \wedge *(F \wedge F)\\ &\quad - V_0 dv\}.\end{aligned} \tag{1}$$

Here $\mathcal{S}_E$ is the Einstein-Hilbert action. The Yang-Mills action $\mathcal{S}_{YMT}$ contains terms both in quadratic and quartic in field strength F. Such a quartic term has been studied by Tchrakian [10] and so we call it the Tchrakian term. The 10-form dv is an invariant volume form with respect to the metric $\hat{g}$ and is written as $dv = \sqrt{-\hat{g}}d^{10}X$ in a local patch. The scalar curvature is denoted by $\mathcal{R}$. The asterisk "$*$" denotes the Hodge dual operator. This operator defines an inner product over differential forms, and for a given form ω, $\omega \wedge *\omega$ is proportional to the invariant volume form dv.[1] The parameters of this action are the gravitational constant G, the gauge coupling e, the four-derivative coupling α, and the cosmological constant V_0.

[1]The Hodge dual operator acting on a differential form on a space with Minkowski signature satisfies the following relation: $(F_{\mu\nu}dx^{\mu\nu}) \wedge *(F_{\rho\sigma}dx^{\rho\sigma}) = -F_{\mu\nu}F^{\mu\nu}dv$.

We show the explicit form of the Yang-Mills part with components of A and F,

$$\begin{aligned}\mathcal{S}_{YMT} = -\int dv\Big\{&\frac{1}{8}F^{[ab]}_{\hat{\mu}\hat{\nu}}F^{\hat{\mu}\hat{\nu},[ab]}\\ &+\frac{\alpha^2}{8}\tilde{T}^{[abcd]}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}T^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma},[ab][cd]}\\ &+\frac{\alpha^2}{3\cdot 16}S_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}S^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \frac{1}{2}V_0\Big\},\end{aligned} \tag{2}$$

$$\begin{aligned}F &= \frac{1}{4}F^{[ab]}_{\hat{\mu}\hat{\nu}}dX^{\hat{\mu}} \wedge dX^{\hat{\nu}}\Gamma_{ab},\\ S_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} &= F^{[ab]}_{\hat{\mu}\hat{\nu}}F^{[ab]}_{\hat{\rho}\hat{\sigma}} + F^{[ab]}_{\hat{\mu}\hat{\rho}}F^{[ab]}_{\hat{\sigma}\hat{\nu}} + F^{[ab]}_{\hat{\mu}\hat{\sigma}}F^{[ab]}_{\hat{\nu}\hat{\rho}},\end{aligned} \tag{3}$$

$$\begin{aligned}T^{[ab][cd]}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} &= (F^{[ab]}_{\hat{\mu}\hat{\nu}}F^{[cd]}_{\hat{\rho}\hat{\sigma}} + F^{[ab]}_{\hat{\mu}\hat{\rho}}F^{[cd]}_{\hat{\sigma}\hat{\nu}} + F^{[ab]}_{\hat{\mu}\hat{\sigma}}F^{[cd]}_{\hat{\nu}\hat{\rho}}),\\ \tilde{T}^{[abcd]}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} &= \frac{1}{6}(T^{[ab][cd]}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + T^{[ac][db]}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + T^{[ad][bc]}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + T^{[cd][ab]}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\\ &\quad + T^{[db][ac]}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + T^{[bc][ad]}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}).\end{aligned} \tag{4}$$

The Euler-Lagrange equations from these actions read the usual Einstein equation and the equations for the Yang-Mills fields:

$$\begin{aligned}\mathcal{R}_{\hat{\mu}\hat{\nu}} - \tfrac{1}{2}\hat{g}_{\hat{\mu}\hat{\nu}}\mathcal{R} &= 8\pi G\mathcal{T}_{\hat{\mu}\hat{\nu}},\\ \mathcal{D}_{\hat{\mu}}[\sqrt{-g}F^{\hat{\mu}\hat{\nu}} - 2\alpha^2\sqrt{-g}F^{[\hat{\mu}\hat{\nu}}F^{\hat{\rho}\hat{\sigma}]}F_{\hat{\rho}\hat{\sigma}}] &= 0.\end{aligned} \tag{5}$$

Here the energy-momentum tensor $\mathcal{T}_{\hat{\mu}\hat{\nu}}$ is obtained by the variation of the Yang-Mills part with respect to the metric:

$$\begin{aligned}\mathcal{T}_{\hat{\mu}\hat{\nu}} &= \frac{1}{2}F_{\hat{\mu}}{}^{\hat{\rho},[ab]}F^{[ab]}_{\hat{\nu}\hat{\rho}} + \alpha^2\tilde{T}^{[abcd]}_{\hat{\mu}\hat{\rho}\hat{\sigma}\hat{\tau}}T_{\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}\hat{\tau},[ab][cd]}\\ &\quad + \frac{\alpha^2}{3\cdot 2}S_{\hat{\mu}\hat{\rho}\hat{\sigma}\hat{\tau}}S_{\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}\hat{\tau}} - \frac{1}{2}g_{\hat{\mu}\hat{\nu}}\chi,\\ \chi &= \frac{1}{4}F^{[ab]}_{\hat{\mu}\hat{\nu}}F^{\hat{\mu}\hat{\nu},[ab]} + \frac{\alpha^2}{4}\tilde{T}^{[abcd]}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}T^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma},[ab][cd]}\\ &\quad + \frac{\alpha^2}{3\cdot 8}S_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}S^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + V_0.\end{aligned} \tag{6}$$

To solve these equations, we make an ansatz which is the same as that of Cremmer-Scherk. Our ansatz for the metric is the following:

$$\begin{aligned}ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + \frac{\delta_{IJ}}{(1 + y^2/4R_0^2)^2}dy^I dy^J\\ &= \hat{g}_{\hat{\mu}\hat{\nu}}dX^{\hat{\mu}}dX^{\hat{\nu}},\\ y^2 &= \sum_{a=1}^{6}(y^I)^2,\end{aligned} \tag{7}$$

where the coordinates X are the total space-time coordinates. The metric $\eta_{\mu\nu} = \mathrm{diag}(- + ++)$ is the Lorentz metric on the four-dimensional Minkowski space. Greek indices without a hat "^", for instance μ, will refer to the first four coordinates. Capital indices $(I, J, \cdots)$ run from one to six and refer to the compact space. The six-dimensional space is taken as a sphere with a radius R_0. The Riemann tensor, Ricci tensor, and scalar curvature are

$$R^I{}_{JKL} = \frac{1}{R_0^2}(\delta^I_K g_{JL} - \delta^I_L g_{JK}),$$
$$\mathcal{R}_{IJ} = \frac{5}{R_0^2} g_{IJ}, \qquad \mathcal{R} = \frac{30}{R_0^2}. \tag{8}$$

The other components of the curvature tensor vanish. In this space, the Einstein equations in (5) reduce to simple equations,

$$-\frac{1}{2}\eta_{\mu\nu}\frac{30}{R_0^2} = 8\pi G\mathcal{T}_{\mu\nu}, \qquad 0 = \mathcal{T}_{\mu I},$$
$$-\frac{1}{2}\frac{20}{R_0^2} g_{IJ} = 8\pi G\mathcal{T}_{IJ}. \tag{9}$$

We now make ansatz for the gauge fields. We assume that the fields A do not depend on the four-dimensional coordinates, $\partial_\mu A = 0$, and the four-dimensional components vanish $A_\mu = 0$. In other words $A = A_I(y)dy^I$, $F = \frac{1}{2}F_{IJ}dy^I \wedge dy^J$. With this ansatz, the four-dimensional part of the energy-momentum tensor becomes $-\frac{1}{2}\eta_{\mu\nu}\chi$, and the equation reduces to $30/R_0^2 = 8\pi G\chi$. This equation requires that the χ is a constant. Suppose that the field strength fulfils the generalized self-dual condition

$$F = i\beta\gamma_7 *_6 (F \wedge F), \tag{10}$$

where β is a real parameter, $\gamma_7 = -i\Gamma_1 \cdots \Gamma_6$ and "$*_6$" means the Hodge dual on the six-dimensional sphere. Then the second part of the equations of motion (5) is fulfilled automatically by the relation $\mathcal{D}F = 0$, where the exterior covariant derivative is defined as $\mathcal{D}F = dF + e(A \wedge F - F \wedge A)$. In fact we have an explicit solution to the self-dual equation:

$$A = \frac{1}{4eR_0^2} y^a e^b \Gamma_{ab}, \qquad F = \frac{1}{4eR_0^2} e^a \wedge e^b \Gamma_{ab},$$
$$\beta = \frac{eR_0^2}{3}. \tag{11}$$

Here $e^I = dx^I/(1 + x^2/4R_0^2)^2$ are sechsbein and we identify the internal space index and the sphere index. The energy-momentum tensor $\mathcal{T}_{IJ}$ and χ of this configuration becomes

$$\mathcal{T}_{IJ} = -\frac{1}{2}\left\{(1-\zeta)\frac{5}{4e^2R_0^4} + V_0\right\}g_{IJ}, \qquad \zeta \equiv \frac{\alpha^2}{\beta^2},$$
$$\chi = (1+\zeta)\frac{15}{4e^2R_0^4} + V_0. \tag{12}$$

With this ansatz we obtain algebraic equations from the Einstein equations:

$$\frac{30}{R_0^2} = 8\pi G\left(\frac{1}{2}(1+\zeta)\frac{15}{2e^2R_0^4} + V_0\right),$$
$$\frac{10}{R_0^2} = 8\pi G\left((1-\zeta)\frac{5}{8e^2R_0^4} + \frac{V_0}{2}\right), \tag{13}$$

From these we finally obtain

$$\frac{1}{\pi G} = \frac{1}{e^2R_0^2}(2 + 4\zeta), \qquad V_0 = \frac{15}{4e^2R_0^4}(1 + 3\zeta). \tag{14}$$

When the four-derivative coupling vanishes, $\alpha = 0$ and therefore $\zeta = 0$, these relations reduce to those of Cremmer and Scherk's [5].[2] When the relation $\alpha = \beta$ holds ($\zeta = 1$) our solution saturates the Bogomol'nyi bound and becomes a BPS state. The energy density is given by an integral over S^6 as follows:

$$\begin{aligned} E^{S^6}_{YMT} &= \frac{1}{16}\int_{S^6} \mathrm{Tr}\{-F \wedge *_6 F + \alpha^2 (F \wedge F) \wedge *_6 (F \wedge F)\} \\ &= \frac{1}{16}\int_{S^6} \mathrm{Tr}(iF \mp \alpha\gamma_7 *_6 (F \wedge F)) \wedge *_6 (iF \mp \alpha\gamma_7 *_6 (F\wedge F)) \pm \frac{i}{8}\alpha \int_{S^6} \mathrm{Tr}\gamma_7 F \wedge F \wedge F \\ &\geq \pm\frac{i}{8}\alpha\int_S^6 \mathrm{Tr}\gamma_7 F \wedge F \wedge F \\ &= \mp\alpha\int_{S^6} \epsilon_{abcdef} F^{[ab]} \wedge F^{[cd]} \wedge F^{[ef]} \equiv \pm\alpha\mathcal{Q}, \end{aligned} \tag{15}$$

where the field strength F has only components along S^6. For $\alpha = \beta$ the Bogomol'nyi bound is saturated so that the energy attains the local minimum. We can also consider a system coupled with scalar fields. Suppose that scalar fields Q^m transform as a representation of SO(6). The index m labels the representation space. An action $\mathcal{S}_Q$ of the scalar fields Q with a Higgs potential given by

[2] We need to redefine e the half when we compare to the result of [5].

$$\mathcal{S}_Q = \frac{1}{2}\left\{\int dv D_{\hat{\mu}} Q^m D^{\hat{\mu}} Q^m + V(Q^2)\right\},$$
$$D_{\hat{\mu}} Q^m = \partial_{\hat{\mu}} Q^m - \frac{1}{2} ie A_{\hat{\mu}}^{[ab]} R(\Gamma_{ab})_{mm'} Q^{m'} \tag{16}$$

is added to $\mathcal{S}_{\text{total}}$. The equations of motion are modified. In general, our solution mentioned above does not satisfy the modified equations anymore. However, for the scalars which fulfil the covariantly constant condition $D_{\hat{\mu}} Q^m = 0$ and attain the absolute minimum $V(Q) = 0$, the configurations of A and g in Eqs. (7) and (11) are still solutions for the modified equations. Thus we can argue the Higgs mechanism around our solutions.

Next we suppose that the four-dimensional part is an anti-de Sitter space AdS_4 of radius R_A. Our ansatz for the metric is the following:

$$ds^2 = \eta_{\mu\nu}(x) dx^\mu dx^\nu + g_{IJ}(y) dy^I dy^J = \hat{g}_{\hat{\mu}\hat{\nu}} dX^{\hat{\mu}} dX^{\hat{\nu}}, \tag{17}$$

$$g_{IJ}(y) dy^I dy^J = \frac{\delta_{IJ}}{(1 + y^2/4R_0^2)^2} dy^I dy^J,$$
$$y^2 = \sum_{a=1}^{6} (y^I)^2,$$
$$\eta_{\mu\nu}(x) dx^\mu dx^\nu = \frac{R_A^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2),$$
$$d\Omega^2 = \frac{|dz|^2}{(1 + |z|^2/4)^2}, \tag{18}$$

where z parametrizes a whole complex plane. The metric $\eta_{\mu\nu}(x)$ is a maximally symmetric metric on the four-dimensional anti-de Sitter space. The Riemann tensor and the Ricci tensor are

$$R^\mu{}_{\nu\rho\sigma} = -\frac{1}{R_A^2}(\delta^\mu_\rho \eta_{\nu\sigma} - \delta^\mu_\sigma \eta_{\nu\rho}),$$
$$\mathcal{R}_{\mu\nu} = -\frac{3}{R_A^2}\eta_{\mu\nu},$$
$$R^I{}_{JKL} = \frac{1}{R_0^2}(\delta^I_K g_{JL} - \delta^I_L g_{JK}),$$
$$\mathcal{R}_{IJ} = \frac{5}{R_0^2} g_{IJ}. \tag{19}$$

The total scalar curvature is the sum of those of two parts: $\mathcal{R} = -\frac{12}{R_A^2} + \frac{30}{R_0^2}$. In this space, the Einstein equations are

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\mathcal{R} = 8\pi G \mathcal{T}_{\mu\nu},$$
$$\mathcal{R}_{IJ} - \tfrac{1}{2} g_{IJ}\mathcal{R} = 8\pi G \mathcal{T}_{IJ}. \tag{20}$$

The ansatz for the gauge fields is the same as that in the previous case. With the ansatz, we obtain algebraic equations from the Einstein equations as

$$\frac{3}{R_A^2} - \frac{15}{R_0^2} = -4\pi G\left\{(1+\zeta)\frac{15}{4e^2 R_0^4} + V_0\right\},$$
$$\frac{6}{R_A^2} - \frac{10}{R_0^2} = -4\pi G\left\{(1-\zeta)\frac{5}{4e^2 R_0^4} + V_0\right\}. \tag{21}$$

We are interested in a relation to string theory and therefore we consider the case with the vanishing cosmological constant, $V_0 = 0$. In this case, the radii (R_A, R_0) are written by the couplings,

$$R_0^2 = (5 + 7\zeta)\frac{\pi G}{4e^2}, \qquad R_A^2 = \frac{5 + 7\zeta}{5 + 15\zeta} R_0^2. \tag{22}$$

Thus the additional higher-derivative coupling term of the Tchrakian type does not affect critically the equations of motion. When $\zeta = 1$ our solution becomes a solution of the Bogomol'nyi equation again.

The solutions presented in this article are new solutions in the system with a Tchrakian term. The origin of this term has not been clear so far but it seems rather universal in order to construct solitons with codimensions higher than four: for instance it has played a crucial role to construct a finite energy monopole (with codimension five) in a six-dimensional space-time [13]. Though the parameter $\zeta(= \alpha^2/\beta^2)$ is a free parameter, we expect that the system goes to $\zeta = 1$ because it becomes BPS. There are several discussions on the instability of higher-dimensional Yang-Mills theories [15]. To compute the mass spectra of the fluctuations around our solutions is left for the future. When the scalar fields Q^m are nontrivially coupled, the system may allow BPS composite solitons which are made of solitons with different codimensions, as in the case of usual self-dual Yang-Mills equations coupled to Higgs fields [16].

Finally our solution of $AdS_4 \times S^6$ may have a relation with D2-branes, and we hope that there exists some impact on AdS/conformal field theory (CFT) duality [17].

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[1] G. Nordström, Phys. Z. **15**, 504 (1914); T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl. **1921**, 966 (1921); O. Klein, Z. Phys. **37**, 895 (1926); Surv. High Energy Phys. **5**, 241 (1986).

[2] M. J. Duff, B. E. W. Nilsson, and C. N. Pope, Phys. Rep. **130**, 1 (1986).

[3] K. Dasgupta, G. Rajesh, and S. Sethi, J. High Energy Phys. 08 (1999) 023.

[4] S. Kachru, R. Kallosh, A. Linde, and S. P. Trivedi, Phys. Rev. D **68**, 046005 (2003).

[5] E. Cremmer and J. Scherk, Nucl. Phys. **B108**, 409 (1976); **B118**, 61 (1977).

[6] Z. Horvath, L. Palla, E. Cremmer, and J. Scherk, Nucl. Phys. **B127**, 57 (1977).

[7] R. Kerner and D. H. Tchrakian, Phys. Lett. B **215**, 87 (1988).

[8] G. 't Hooft, Nucl. Phys. **B79**, 276 (1974); A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].

[9] A. A. Belavin, A. M. Polyakov, A. S. Schwarts, and Yu. S. Tyupkin, Phys. Lett. B **59**, 85 (1975).

[10] D. H. Tchrakian, J. Math. Phys. (N.Y.) **21**, 166 (1980).

[11] B. Grossman, T. W. Kephart, and J. D. Stasheff, Commun. Math. Phys. **96**, 431 (1984); **100**, 311(E) (1985); R. V. Buniy and T. W. Kephart, Phys. Lett. B **548**, 97 (2002).

[12] D. H. Tchrakian, Phys. Lett. B **150**, 360 (1985); D. O'Se and D. H. Tchrakian, Lett. Math. Phys. **13**, 211 (1987); Z. Ma and D. H. Tchrakian, J. Math. Phys. (N.Y.) **31**, 1506 (1990).

[13] H. Kihara, Y. Hosotani, and M. Nitta, Phys. Rev. D **71**, 041701 (2005); E. Radu and D. H. Tchrakian, Phys. Rev. D **71**, 125013 (2005).

[14] H. Kihara and M. Nitta, arXiv:hep-th/0703166.

[15] S. Randjbar-Daemi, A. Salam, and J. A. Strathdee, Phys. Lett. B **124**, 345 (1983); **144**, 455 (1984); O. DeWolfe, D. Z. Freedman, S. S. Gubser, G. T. Horowitz, and I. Mitra, Phys. Rev. D **65**, 064033 (2002); A. E. Mosaffa, S. Randjbar-Daemi, and M. M. Sheikh-Jabbari, arXiv:hep-th/0612181.

[16] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, J. Phys. A **39**, R315 (2006).

[17] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998); O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).