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Conformal Killing-Yano tensors for the Plebański-Demiański family of solutions

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We present explicit expressions for the conformal Killing-Yano tensors for the Plebański-Demiański family of type D solutions in four dimensions. Some physically important special cases are discussed in more detail. In particular, it is demonstrated how the conformal Killing-Yano tensor becomes the Killing-Yano tensor for the solutions without acceleration. A possible generalization into higher dimensions is studied. Whereas the transition from the nonaccelerating to accelerating solutions in four dimensions is achieved by the conformal rescaling of the metric, we show that such a procedure is not sufficiently general in higher dimensions—only the maximally symmetric spacetimes in ''accelerated'' coordinates are obtained.

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I. INTRODUCTION

The complete family of type D spacetimes in four dimensions, including the black hole spacetimes like the Kerr metric, the metrics describing the accelerating sources as the C-metric, or the nonexpanding Kundt's class type D solutions, can be represented by the general seven-parameter metric discovered by Plebański and Demiański [1] (cf. also [2]). Recently, Griffiths and Podolský [3–7] put this metric into a new form which enabled a better physical interpretation of parameters and simplified a procedure how to derive all special cases. Among subclasses of this solution let us mention the six-parameter family of metrics without acceleration derived and studied already by Carter [8] and later by Plebański [9].

It turns out that the elegant form of the Plebański-Demiański metric not only yields the new solutions in 4D (see, e.g. [10,11]), but also inspires for its generalizations into higher dimensions which became popular in connection with string theories and the brane world models with large extra dimensions. Recently, Chen, Lü, and Pope [12] were able to cast the Carter's subclass of nonaccelerating solutions into higher dimensions—thus constructing the general Kerr-NUT-(A)dS metrics in all dimensions.

One of the most remarkable properties of the Carter's subclass of nonaccelerating solutions, which is also inherited by its higher-dimensional generalizations [13,14], is the existence of hidden symmetries associated with the Killing-Yano tensor [15,16]. Indeed, it is this tensor which is responsible for the ''miraculous'' properties of the Kerr metric, including the integrability of geodesic motion or the separability of Hamilton-Jacobi and Klein-Gordon equations [8,17]. Similar results were obtained recently in higher dimensions [18–20].

In four dimensions the integrability conditions for the existence of nondegenerate Killing-Yano tensor restricts the Petrov type of spacetime to type D (see, e.g., [21]). However, Demiański and Francaviglia [22] demonstrated that from the known type D solutions only spacetimes without acceleration of sources actually admit this tensor.

The purpose of the present paper is to show that the general Plebański-Demiański metric admits the conformal generalization of the Killing-Yano tensor. We also explicitly demonstrate how in the absence of acceleration this tensor becomes the known Killing-Yano tensor of the Carter's metric. The particular forms of this tensor for the physically important cases are presented.

We also study a generalization of the Plebański-Demiański class into higher dimensions. Namely, we try to ''accelerate'' the higher-dimensional Kerr-NUT-(A)dS metric in the same way as it can be achieved in four dimensions—by a conformal rescaling of the metric accompanied with a modification of the metric functions. We demonstrate that this ansatz does not work in odd dimensions and in even dimension it leads only to the trivial case of maximally symmetric spacetimes. However, it allows us to identify the conformal Killing-Yano tensor in higher-dimensional flat and (A)dS spacetimes related to the ''accelerated'' coordinates.

II. CONFORMAL KILLING-YANO TENSORS

In this section we shall briefly describe the conformal Killing-Yano (CKY) tensors and their basic properties. The CKY tensors were first proposed by Kashiwada and Tachibana [23,24] as a generalization of the Killing-Yano (KY) tensors [25]. Since then both of these tensors found wide applications in physics related to hidden (super)symmetries, conserved quantities, symmetry operators, or separation of variables (see, e.g., [26–39]).

The conformal Killing-Yano (CKY) tensor $\boldsymbol{k}$ of rank-p in D dimensions is a p-form, the covariant derivative of which has vanishing harmonic part, i.e., it can be split into the antisymmetric and divergence parts:

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$$\nabla_\mu k_{\alpha_1\ldots\alpha_p} = \nabla_{[\mu} k_{\alpha_1\ldots\alpha_p]} + \frac{p}{D-p+1} g_{\mu[\alpha_1} \nabla_{|\kappa|} k^\kappa{}_{\alpha_2\ldots\alpha_p]}. \tag{1}$$

This defining equation is invariant under the Hodge duality; the antisymmetric part transforms into the divergence part and vice versa. This implies that the dual $*\boldsymbol{k}$ is the CKY tensor whenever $\boldsymbol{k}$ is the CKY tensor.

Two special subclasses of CKY tensors are of particular interest: (a) Killing-Yano tensors [25] with zero divergence part in (1) and (b) closed conformal Killing-Yano tensors with vanishing antisymmetric part in (1). These subclasses transform into each other under the Hodge duality.

In what follows, we shall deal mainly with the rank-2 CKY tensors, which are the only nontrivial in four dimensions, and which obey the equations

$$\nabla_\mu k_{\alpha\beta} = \nabla_{[\mu} k_{\alpha\beta]} + 2 g_{\mu[\alpha} \xi_{\beta]}, \tag{2}$$

where we have denoted $\boldsymbol{\xi}$ the divergence of $\boldsymbol{k}$:

$$\xi_\alpha = \frac{1}{D-1} \nabla_\kappa k^\kappa{}_\alpha. \tag{3}$$

Equivalently, we could use the alternative definition (see, e.g., [33,38])

$$\nabla_{(\mu} k_{\alpha)\beta} = g_{\mu\alpha}\xi_\beta - \xi_{(\mu} g_{\alpha)\beta}. \tag{4}$$

It was demonstrated by Jezierski and Łukasik [34] that in an Einstein space of arbitrary dimension the vector $\boldsymbol{\xi}$, given by (3), either vanishes, which means that the CKY tensor $\boldsymbol{k}$ is in fact a KY tensor, or it is a Killing vector. (We shall see below on a particular example of the Plebański-Demiański metric that this may hold more generally, in the presence of the electromagnetic field.) It may be also possible to construct other isometries using the (conformal) KY tensor. For example, for the general (higher-dimensional) Kerr-NUT-(A)dS spacetimes it was demonstrated that all the isometries follow from the existence of the principal KY tensor [40].

Whereas the opposite is not generally true (see, e.g., [41] for a discussion concerning the nonconformal case), the (conformal) KY tensor implies the existence of the (conformal) Killing tensor given by[1]:

$$Q_{\alpha\beta} = k_{\alpha\kappa} k_\beta{}^\kappa. \tag{5}$$

This tensor satisfies

$$\nabla_{(\alpha} Q_{\beta\gamma)} = g_{(\alpha\beta} Q_{\gamma)}, \tag{6}$$

where

$$Q_\alpha = \frac{1}{D+2} (2\nabla_\kappa Q^\kappa{}_\alpha + \nabla_\alpha Q^\kappa{}_\kappa) \tag{7}$$

for the conformal Killing tensor whereas it vanishes for the Killing tensor.

III. PLEBAŃSKI-DEMIAŃSKI METRIC

The original form of the Plebański-Demiański metric [1] is given by

$$\boldsymbol{g} = \Omega^2 \Big[-\frac{Q(\boldsymbol{d\tau} - p^2 \boldsymbol{d\sigma})^2}{r^2+p^2} + \frac{P(\boldsymbol{d\tau} + r^2 \boldsymbol{d\sigma})^2}{r^2+p^2} + \frac{r^2+p^2}{P} \boldsymbol{d}p^2 + \frac{r^2+p^2}{Q} \boldsymbol{d}r^2 \Big]. \tag{8}$$

This metric obeys the Einstein-Maxwell equations with the electric and magnetic charges e and g and the cosmological constant Λ when functions $P = P(p)$ and $Q = Q(r)$ take the particular form

$$\begin{aligned} Q &= k + e^2 + g^2 - 2mr + \epsilon r^2 - 2nr^3 - (k + \Lambda/3) r^4, \\ P &= k + 2np - \epsilon p^2 + 2mp^3 - (k + e^2 + g^2 + \Lambda/3) p^4, \end{aligned} \tag{9}$$

the conformal factor is

$$\Omega^{-1} = 1 - pr, \tag{10}$$

and the vector potential reads

$$\boldsymbol{A} = -\frac{1}{r^2+p^2} [e r (\boldsymbol{d\tau} - p^2 \boldsymbol{d\sigma}) + g p (\boldsymbol{d\tau} + r^2 \boldsymbol{d\sigma})]. \tag{11}$$

Our claim is that the general Plebański-Demiański metric (8) admits the conformal Killing-Yano tensor:

$$\boldsymbol{k} = \Omega^3 [p \boldsymbol{dr} \wedge (\boldsymbol{d\tau} - p^2 \boldsymbol{d\sigma}) + r \boldsymbol{dp} \wedge (\boldsymbol{d\tau} + r^2 \boldsymbol{d\sigma})]. \tag{12}$$

Using the GRTensor, one can easily check that the Eqs. (2), or (4), are satisfied. An independent proof is given in the Sec. V.

The dual $\boldsymbol{h} = *\boldsymbol{k}$ is also a conformal Killing-Yano tensor and it reads

$$\boldsymbol{h} = \Omega^3 [r \boldsymbol{dr} \wedge (p^2 \boldsymbol{d\sigma} - \boldsymbol{d\tau}) + p \boldsymbol{dp} \wedge (r^2 \boldsymbol{d\sigma} + \boldsymbol{d\tau})], \tag{13}$$

which is equivalent to

$$\boldsymbol{h} = \Omega^3 \boldsymbol{db}, \tag{14}$$

where

$$2\boldsymbol{b} = (p^2 - r^2) \boldsymbol{d\tau} + p^2 r^2 \boldsymbol{d\sigma}. \tag{15}$$

It is interesting to mention that $\boldsymbol{k}$ and $\boldsymbol{h}$ are the CKY tensors for the metric (8) with an arbitrary conformal factor Ω and arbitrary functions $P(p)$, $Q(r)$, i.e., irrespectively of

[1]The (conformal) Killing tensor implies the conserved quantity for the (null) geodesic motion which is of second order in momenta (see, e.g., [42]). Since (with an arbitrary prefactor) the metric is a (conformal) Killing tensor, any (conformal) Killing tensor is defined up to a term proportional (with an arbitrary prefactor) to the metric.

the fact if the metric (8) solves the Einstein equations or not. We shall return to this remark in Sec. V where we shall also see that in the absence of acceleration of sources $\boldsymbol{k}$ becomes the KY tensor, whereas $\boldsymbol{h}$ becomes the closed CKY tensor.

Although we deal with a more general space than required by the theorem in [34], for Ω given by (10) and for arbitrary functions $P(p)$ and $Q(r)$ both isometries of the spacetime still follow from $\boldsymbol{h}$ and $\boldsymbol{k}$ by Eq. (3):

$$\boldsymbol{\xi}_{(h)} = \boldsymbol{\partial}_\tau, \qquad \boldsymbol{\xi}_{(k)} = \boldsymbol{\partial}_\sigma. \tag{16}$$

The conformal Killing tensor given by (5) which corresponds to $\boldsymbol{k}$ reads

$$\boldsymbol{Q}_{(k)} = \Omega^4\Big[\frac{Qp^2(\boldsymbol{d}\tau - p^2\boldsymbol{d}\sigma)^2}{r^2+p^2} + \frac{Pr^2(\boldsymbol{d}\tau + r^2\boldsymbol{d}\sigma)^2}{r^2+p^2} + \frac{r^2(r^2+p^2)}{P}\boldsymbol{d}p^2 - \frac{p^2(r^2+p^2)}{Q}\boldsymbol{d}r^2\Big]. \tag{17}$$

It inherits the ''universality'' of $\boldsymbol{k}$, i.e., it is a conformal Killing tensor of the metric (8) with an arbitrary Ω, and arbitrary $Q(r)$ and $P(p)$. In the absence of acceleration $\boldsymbol{Q}_{(k)}$ becomes a Killing tensor which generates the Carter's constant for a geodesic motion [17]. The conformal Killing tensor associated with $\boldsymbol{h}$ is

$$\boldsymbol{Q}_{(h)} = \Omega^4\Big[\frac{Qr^2(\boldsymbol{d}\tau - p^2\boldsymbol{d}\sigma)^2}{r^2+p^2} + \frac{Pp^2(\boldsymbol{d}\tau + r^2\boldsymbol{d}\sigma)^2}{r^2+p^2} + \frac{p^2(r^2+p^2)}{P}\boldsymbol{d}p^2 - \frac{r^2(r^2+p^2)}{Q}\boldsymbol{d}r^2\Big]. \tag{18}$$

Both tensors are related as

$$\boldsymbol{Q}_{(h)} = \boldsymbol{Q}_{(k)} + \Omega^2(p^2 - r^2)\boldsymbol{g}. \tag{19}$$

Following [3] one can easily perform the transformations of coordinates and parameters to obtain the complete family of type D spacetimes and the corresponding forms of CKY tensors. In the next two sections we shall consider two special cases. First we deal with the generalized black holes and then we demonstrate what happens when the acceleration of sources is removed.

IV. GENERALIZED BLACK HOLES

Following [3] let us introduce two new continuous parameters α (the acceleration) and ω (the ''twist'') by the rescaling

$$p \to \sqrt{\alpha\omega}\,p, \qquad r \to \sqrt{\frac{\alpha}{\omega}}\,r, \qquad \sigma \to \sqrt{\frac{\omega}{\alpha^3}}\,\sigma, \qquad \tau \to \sqrt{\frac{\omega}{\alpha}}\,\tau, \tag{20}$$

and relabel the other parameters as

$$m \to \Big(\frac{\alpha}{\omega}\Big)^{3/2} m, \qquad n \to \Big(\frac{\alpha}{\omega}\Big)^{3/2} n, \qquad e \to \frac{\alpha}{\omega}\,e, \qquad g \to \frac{\alpha}{\omega}\,g, \qquad \epsilon \to \frac{\alpha}{\omega}\,\epsilon, \qquad k \to \alpha^2 k. \tag{21}$$

Then the metric and the vector potential take the form

$$\boldsymbol{g} = \Omega^2\Big[-\frac{Q(\boldsymbol{d}\tau - \omega p^2\boldsymbol{d}\sigma)^2}{r^2+\omega^2p^2} + \frac{P(\omega\boldsymbol{d}\tau + r^2\boldsymbol{d}\sigma)^2}{r^2+\omega^2p^2} + \frac{r^2+\omega^2p^2}{P}\boldsymbol{d}p^2 + \frac{r^2+\omega^2p^2}{Q}\boldsymbol{d}r^2\Big], \tag{22}$$

$$\boldsymbol{A} = -\frac{1}{r^2+\omega^2p^2}[er(\boldsymbol{d}\tau - \omega p^2\boldsymbol{d}\sigma) + gp(\omega\boldsymbol{d}\tau + r^2\boldsymbol{d}\sigma)], \tag{23}$$

with

$$\Omega^{-1} = 1 - \alpha pr, \tag{24}$$

and

$$\begin{aligned} Q &= \omega^2k + e^2 + g^2 - 2mr + \epsilon r^2 - \frac{2\alpha n}{\omega}r^3 - \Big(\alpha^2k + \frac{\Lambda}{3}\Big)r^4, \\ P &= k + \frac{2n}{\omega}p - \epsilon p^2 + 2\alpha m p^3 - \Big(\alpha^2(\omega^2k + e^2 + g^2) + \omega^2\frac{\Lambda}{3}\Big)p^4. \end{aligned} \tag{25}$$

The CKY tensors are (up to trivial constant factors)

$$\Omega^{-3}\boldsymbol{k} = \omega p\boldsymbol{d}r \wedge (\boldsymbol{d}\tau - \omega p^2\boldsymbol{d}\sigma) + r\boldsymbol{d}p \wedge (\omega\boldsymbol{d}\tau + r^2\boldsymbol{d}\sigma), \tag{26}$$

and, $\boldsymbol{h} = \Omega^3\boldsymbol{d}\boldsymbol{b}$, with Ω given in (24) and

$$2\boldsymbol{b} = (\omega^2p^2 - r^2)\boldsymbol{d}\tau + \omega p^2r^2\boldsymbol{d}\sigma. \tag{27}$$

Let us consider two special cases. First, we relabel $\omega = a$, perform an additional coordinate transformation

$$p \to \cos\theta, \qquad \tau \to \tau - a\phi, \qquad \sigma \to -\phi, \tag{28}$$

and set

$$k = 1, \qquad n = -\alpha am, \qquad \epsilon = 1 - \alpha^2(a^2 + e^2 + g^2) - \frac{\Lambda}{3}a^2. \tag{29}$$

(One parameter—NUT charge—was set to zero and the scaling freedom was used to eliminate the other two.) We obtained a six-parameter solution which describes the accelerating rotating charged black hole with the cosmological constant:

$$\boldsymbol{g} = \Omega^2\Big\{-\frac{Q}{\Delta}[\boldsymbol{d}\tau - a\sin^2\theta\boldsymbol{d}\phi]^2 + \frac{\Delta}{Q}\boldsymbol{d}r^2 + \frac{P}{\Delta}[a\boldsymbol{d}\tau - (r^2+a^2)\boldsymbol{d}\phi]^2 + \frac{\Delta}{P}\sin^2\theta\boldsymbol{d}\theta^2\Big\}, \quad (30)$$

where

$$\begin{aligned}\Omega^{-1} &= 1 - \alpha r\cos\theta,\\ \Delta &= r^2 + a^2\cos^2\theta,\\ Q &= (a^2+e^2+g^2-2mr+r^2)(1-\alpha^2r^2) - \frac{\Lambda}{3}(a^2+r^2)r^2,\\ \frac{P}{\sin^2\theta} &= 1 - 2\alpha m\cos\theta + \Big[\alpha^2(a^2+e^2+g^2) + \frac{\Lambda a^2}{3}\Big]\cos^2\theta.\end{aligned} \quad (31)$$

In the brackets in (30) we can easily recognize the familiar form of the Kerr solution. The conformal factor and the modification of metric functions correspond to the acceleration and the cosmological constant. The CKY tensor $\boldsymbol{k}$ takes the form

$$\Omega^{-3}\boldsymbol{k} = a\cos\theta\boldsymbol{d}r\wedge[\boldsymbol{d}\tau - a\sin^2\theta\boldsymbol{d}\phi] - r\sin\theta\boldsymbol{d}\theta\wedge[a\boldsymbol{d}\tau - (r^2+a^2)\boldsymbol{d}\phi], \quad (32)$$

where Ω is given in (31). Except the conformal factor we recovered the Killing-Yano tensor for the Kerr metric derived by Penrose and Floyd [15].

The second interesting example is obtained if instead of (28) and (29) we perform

$$p \to \frac{l + a\cos\theta}{\omega}, \qquad \tau \to \tau - \frac{(l+a)^2}{a}\phi, \qquad \sigma \to -\frac{\omega}{a}\phi, \quad (33)$$

set the acceleration $\alpha = 0$, and adjust

$$\epsilon = 1-(a^2/3+2l^2)\Lambda, \qquad n = l + \Lambda l(a^2-4l^2)/3, \qquad \omega^2 k = (1-l^2\Lambda)(a^2-l^2). \quad (34)$$

Then we have a nonaccelerated rotating charged black hole with the NUT parameter and cosmological constant:

$$\boldsymbol{g} = -\frac{Q}{\Delta}\Big[\boldsymbol{d}\tau - \Big(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}\Big)\boldsymbol{d}\phi\Big]^2 + \frac{\Delta}{Q}\boldsymbol{d}r^2 + \frac{P}{\Delta}[a\boldsymbol{d}\tau - (r^2+(a+l)^2)\boldsymbol{d}\phi]^2 + \frac{\Delta}{P}\sin^2\theta\boldsymbol{d}\theta^2, \quad (35)$$

where

$$\begin{aligned}\Delta &= r^2 + (l + a\cos\theta)^2,\\ \frac{P}{\sin^2\theta} &= 1 + \frac{4\Lambda}{3}al\cos\theta + \frac{\Lambda}{3}a^2\cos^2\theta,\\ Q &= a^2 - l^2 + e^2 + g^2 - 2mr + r^2 - \frac{\Lambda}{3}(3(a^2-l^2)l^2 + (a^2+6l^2)r^2 + r^4).\end{aligned} \quad (36)$$

The CKY tensor $\boldsymbol{k}$ becomes the KY tensor (see also the next section) and takes the form

$$\boldsymbol{k} = (l + a\cos\theta)\boldsymbol{d}r\wedge\{\boldsymbol{d}\tau + \boldsymbol{d}\phi[2l(\cos\theta-1) - a\sin^2\theta]\} - r\sin\theta\boldsymbol{d}\theta\wedge\{a\boldsymbol{d}\tau - \boldsymbol{d}\phi[(l+a)^2 + r^2]\}. \quad (37)$$

The dual CKY tensor becomes closed, $\boldsymbol{h} = \boldsymbol{d}\boldsymbol{b}$, with

$$2\boldsymbol{b} = ((l+a\cos\theta)^2 - r^2)(a\boldsymbol{d}\tau - (l+a)^2\boldsymbol{d}\phi) - r^2(l+a\cos\theta)^2\boldsymbol{d}\phi. \quad (38)$$

In particular, in vacuum ($e = g = \Lambda = 0$) we recover the KY tensor for the Kerr metric ($l = 0$), respectively, for the NUT solution ($a = 0$) studied recently in [34], respectively [35].

V. CARTER'S METRIC

Let us take the Plebański-Demiański metric in the form (22) and set the acceleration $\alpha = 0$, and $\omega = 1$. Then the conformal factor becomes $\Omega = 1$ and we recover the Carter's family of nonaccelerating solutions [8] in the form used in [9]:

$$\boldsymbol{g} = -\frac{Q(\boldsymbol{d}\tau - p^2\boldsymbol{d}\sigma)^2}{r^2+p^2} + \frac{P(\boldsymbol{d}\tau + r^2\boldsymbol{d}\sigma)^2}{r^2+p^2} + \frac{r^2+p^2}{P}\boldsymbol{d}p^2 + \frac{r^2+p^2}{Q}\boldsymbol{d}r^2, \quad (39)$$

where

$$Q = k + e^2 + g^2 - 2mr + \epsilon r^2 - \frac{\Lambda}{3}r^4, \qquad P = k + 2np - \epsilon p^2 - \frac{\Lambda}{3}p^4, \quad (40)$$

and the vector potential is given again by (11).

We also get

$$\boldsymbol{k} = p\boldsymbol{d}r\wedge(\boldsymbol{d}\tau - p^2\boldsymbol{d}\sigma) + r\boldsymbol{d}p\wedge(\boldsymbol{d}\tau + r^2\boldsymbol{d}\sigma), \quad (41)$$

which is the Killing-Yano tensor given by Carter in [16]. Its dual,

$$\boldsymbol{h} = *\boldsymbol{k} = \boldsymbol{d}\boldsymbol{b}, \quad (42)$$

with $\boldsymbol{b}$ given by (15), becomes the closed CKY tensor. Again, these properties are independent of the particular form of $P(p)$ and $Q(r)$. The conformal Killing tensor (17) becomes the Killing tensor

$$\boldsymbol{K} = \frac{Qp^2(\boldsymbol{d}\tau - p^2\boldsymbol{d}\sigma)^2}{r^2+p^2} + \frac{Pr^2(\boldsymbol{d}\tau + r^2\boldsymbol{d}\sigma)^2}{r^2+p^2} + \frac{r^2(r^2+p^2)}{P}\boldsymbol{d}p^2 - \frac{p^2(r^2+p^2)}{Q}\boldsymbol{d}r^2. \tag{43}$$

Both isometries of spacetime may be derived from the existence of $\boldsymbol{k}$, but in a different manner than before. We have $\boldsymbol{\xi}_{(h)} = \boldsymbol{\partial}_\tau$ whereas $\boldsymbol{\xi}_{(k)} = 0$ since $\boldsymbol{k}$ is now a KY tensor. Nevertheless, the second isometry is given by

$$\partial^\alpha_\sigma = K^\alpha{}_\beta \xi^\beta_{(h)}. \tag{44}$$

Let us observe that the full Plebański-Demiański metric with acceleration is related to the Carter's metric only by a conformal rescaling and a modification of the metric functions $P(p)$ and $Q(r)$. It allows us to use the theorem—proved recently by Jezierski and Łukasik [34] which says that whenever $\boldsymbol{k}$ is the CKY tensor for the metric $\boldsymbol{g}$ then $\Omega^3\boldsymbol{k}$ is the CKY tensor for the conformally rescaled metric $\Omega^2\boldsymbol{g}$. This would justify the transition from the known KY tensor (41) to the CKY tensor (12), up to the fact, that in the transition from (8) to (39) we also need to change functions $P(p)$ and $Q(r)$. Fortunately, as mentioned above, the "universality" of $\boldsymbol{k}$, i.e., the property that (41) remains KY tensor for the metric (39) with arbitrary function $P(p)$ and $Q(r)$, can be demonstrated. Indeed, the only nontrivial components of the covariant derivative $\boldsymbol{\nabla k}$, namely

$$\nabla_p k_{\sigma r} = \nabla_r k_{p\sigma} = \nabla_\sigma k_{rp} = r^2 + p^2, \tag{45}$$

are completely independent of the form of $Q(r)$ and $P(p)$. (Using these derivatives we easily find that $\boldsymbol{k}$ is the Killing-Yano tensor satisfying $\nabla_{(\alpha}k_{\beta)\gamma} = 0$.) Therefore one can start with the metric $\boldsymbol{g}$ (39), with the KY tensor $\boldsymbol{k}$ (41), and with arbitrary functions $P(p)$ and $Q(r)$ so that, after performing the conformal scaling $\boldsymbol{g} \to \Omega^2\boldsymbol{g}$ we obtain the metric (8). The theorem ensures that $\Omega^3\boldsymbol{k}$ is the universal CKY tensor of the new metric, and, in particular, of the Plebański-Demiański solution where Ω is given by (10) and functions $P(p)$ and $Q(r)$ by (9).

VI. SOME REMARKS ON HIGHER DIMENSIONS

As we mentioned in the introduction, the Carter's metric (39) (with charges set to zero) has a form of the Chen-Lü-Pope metric [12] describing the higher-dimensional generally rotating NUT–(A)dS black hole. Indeed, in an even dimension[2] $D = 2n$ the metric reads

$$\boldsymbol{g} = \sum_{\mu=1}^{n}\left[\frac{U_\mu}{X_\mu}\boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu}\left(\sum_{k=0}^{n-1}A^{(k)}_\mu \boldsymbol{d}\psi_k\right)^2\right] \tag{46}$$

with quantities U_μ and $A^{(k)}_\mu$ given by

$$U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(x_\nu^2 - x_\mu^2), \qquad A^{(k)}_\mu = \sum_{\substack{\nu_1,\ldots,\nu_k=1\\ \nu_1<\ldots<\nu_k,\nu_i\neq\mu}}^{n} x^2_{\nu_1}\ldots x^2_{\nu_k} \tag{47}$$

and with the metric functions

$$X_\mu = b_\mu x_\mu + \sum_{k=0}^{n} c_k x_\mu^{2k}. \tag{48}$$

Here, x_μ, $\mu = 1,\ldots,n$, correspond to radial[3] and latitudinal directions, while ψ_j, $j = 0,\ldots,n-1$, to temporal and longitudinal directions. The metric can be rewritten in the diagonal form using a properly chosen orthonormal frame $\boldsymbol{e}^a$, $a = 1,\ldots,D$ [40]. The curvature tensors have been computed explicitly in [43] and it turns out that the Ricci tensor $\mathbf{Ric}$ is diagonal in the frame $\boldsymbol{e}^a$ as well.

It has been found in [14] and discussed in [40] that the metric (46) possesses the Killing-Yano tensor $\boldsymbol{k}$ and dual closed conformal Killing-Yano tensor $\boldsymbol{h}$

$$\boldsymbol{k} = *\boldsymbol{h}, \qquad \boldsymbol{h} = \frac{1}{2}\sum_{\mu=1}^{n}\left[\boldsymbol{d}x_\mu^2 \wedge \sum_{j=0}^{n-1}A^{(j)}_\mu \boldsymbol{d}\psi_j\right] \tag{49}$$

independently of the specific form of the metric functions $X_\mu(x_\mu)$.

For $D = 4$ we recover the Carter's metric without electromagnetic field,[4] $e = g = 0$, by the identification

$$\begin{aligned} &\psi_0 = t, && \psi_1 = -\sigma, && \\ &x_1 = ir, && X_1 = Q, && b_1 = 2im, \\ &x_2 = p, && X_2 = P, && b_2 = 2n, \\ &c_0 = k, && c_1 = -\epsilon, && c_2 = -\Lambda/3. \end{aligned} \tag{50}$$

It is natural to ask if it is possible to generalize the four-dimensional accelerated Plebański-Demiański metric into higher dimensions. An obvious procedure to follow would be to start with the metric (46), rescale it (in analogy with the four-dimensional case) by a conformal factor Ω^2,

$$\tilde{\boldsymbol{g}} = \Omega^2\boldsymbol{g}, \tag{51}$$

and adjust the metric functions $X_\mu(x_\mu)$ in such a way that the scaled metric would satisfy the Einstein equations. Because of the same argument which we used in four dimensions such a metric would possess a conformal Killing-Yano tensor $\tilde{\boldsymbol{h}} = \Omega^3\boldsymbol{h}$.

[2]In the odd dimensional case there is an additional coordinate ψ_n and additional (nondiagonal) terms in the metric [12]. Since our procedure of a generalization of Carter's class to higher dimensions does not work in odd dimensions, we do not discuss this case in more detail.

[3]The radial coordinate (and some related quantities) are rescaled by the imaginary unit i in order to put the metric to a more symmetric form (cf., e.g., [12]).

[4]For a discussion of "charging" the higher-dimensional black hole in a way analogous to the four-dimensional Carter's metric, see [44].

The Ricci tensor $\widetilde{\mathbf{Ric}}$ of the rescaled metric $\tilde{\boldsymbol{g}}$ is related to the Ricci tensor of the unscaled metric $\boldsymbol{g}$ by a well-known expression (see, e.g., the appendix in [45]), which can be written as

$$\widetilde{\mathbf{Ric}} = \mathbf{Ric} + (D-2)\Omega\boldsymbol{\nabla}\boldsymbol{\nabla}\Omega^{-1} + \boldsymbol{g}(\Omega\boldsymbol{\nabla}^2\Omega^{-1} - (D-1)\Omega^2(\boldsymbol{\nabla}\Omega^{-1})^2). \quad (52)$$

Here the "square" of 1-forms is defined using the inverse unscaled metric $\boldsymbol{g}^{-1}$. We require $\widetilde{\mathbf{Ric}} = -\lambda\tilde{\boldsymbol{g}}$ with λ proportional to the cosmological constant. The Ricci tensor $\widetilde{\mathbf{Ric}}$ thus must be diagonal in the frame $\boldsymbol{e}^a$. The conditions on off-diagonal terms give the equations for the conformal factor Ω.

In a generic odd dimension these conditions are too strong—they admit only a constant conformal factor Ω. In even dimensions the conditions on off-diagonal terms lead to equations

$$\Omega^{-1}{}_{,\mu\nu} = \frac{x_\nu\Omega^{-1}{}_{,\mu}}{x_\nu^2 - x_\mu^2} + \frac{x_\mu\Omega^{-1}{}_{,\nu}}{x_\mu^2 - x_\nu^2}, \qquad 0 = \frac{x_\mu\Omega^{-1}{}_{,\mu}}{x_\nu^2 - x_\mu^2} + \frac{x_\nu\Omega^{-1}{}_{,\nu}}{x_\mu^2 - x_\nu^2}. \quad (53)$$

It gives the conformal factor depending on two constants c and a,

$$\Omega^{-1} = c + ax_1 \ldots x_n, \quad (54)$$

which is obviously a generalization of the four-dimensional factor (24) (with $c = 1$ and $a = i\alpha$).

Unfortunately, the conditions for diagonal terms of the Ricci tensor are in even dimensions $D > 4$ rather restrictive. Analyzing first the condition for the scalar curvature and then checking all diagonal terms one finds that either[5]

$$\Omega^{-1} = x_1 \ldots x_2, \qquad X_\mu = \bar{b}_\mu x_\mu^{2n-1} + \sum_{k=0}^{n} c_k x_\mu^{2k}, \quad (55)$$

with $\lambda = (D-1)c_0$, or

$$\Omega^{-1} = 1 + ax_1 \ldots x_2, \qquad X_\mu = \sum_{k=0}^{n} c_k x_\mu^{2k}, \quad (56)$$

with $\lambda = (D-1)((-1)^{n-1}c_n + a^2c_0)$. The first case is not a new solution: the substitution

$$x_\mu = 1/\bar{x}_\mu, \qquad \psi_j = \bar{\psi}_{n-1-j}, \qquad X_\mu = \bar{x}_\mu^{-n+1}\bar{X}_\mu \quad (57)$$

transforms the rescaled metric $\tilde{\boldsymbol{g}}$ back to the form (46) in "barred" coordinates. In the second case the metric functions X_μ depend on a smaller number of parameters and one has to expect that the metric describes only a subclass of the "accelerated black hole solutions." It is actually the trivial subclass—it was shown in [43] that the metric (46) with X_μ given by (56) represents the maximally symmetric spacetime; therefore the scaled metric $\tilde{\boldsymbol{g}}$, being the Einstein space conformally related to the maximally symmetric spacetime, must describe also the maximally symmetric spacetime. In analogy with the four-dimensional case we expect that the metric (51) with metric functions (56) describes the Minkowski or (anti-)de Sitter space in some kind of "accelerated" coordinates. However, such an interpretation needs further detailed study.

We conclude that we did not find a reasonable nontrivial generalization of the accelerated Plebański-Demiański metric to higher dimensions. However, the physically trivial case (56) allows us to write down the conformal Killing-Yano tensor for the maximally symmetric spacetimes related to the "accelerating" "rotating" coordinates x_μ, ψ_j.

[5]The trivial global scaling was eliminated by setting $a = 1$ in (55) and $c = 1$ in (56).

VII. SUMMARY

We have explicitly demonstrated that the complete family of type D spacetimes which can be derived from the Plebański-Demiański metric possesses the conformal generalization of the Killing-Yano tensor. In the absence of the acceleration of sources, i.e., for a special subclass of solutions described by Carter and Plebański, this tensor becomes the known Killing-Yano tensor. Several examples were discussed in more detail and specific forms of these tensors were given.

The Plebański-Demiański metric also motivates for a generalization into higher dimensions. Its nonaccelerated subclass, the Carter's metric, has been already generalized into higher dimensions by Chen, Lü, and Pope [12]. However, it seems that further generalizations, although almost obvious at first sight, cannot be easily obtained. For example, the attempts to "naturally" charge these solutions failed so far (see, e.g., [44,46]). In the present paper we have demonstrated that also the generalization to accelerated solutions is not straightforward. In particular, we have shown that the direct analogue of the Plebański-Demiański complete family (with acceleration) in higher dimensions cannot be obtained in a manner similar to the four-dimensional case, that is, by a conformal scaling of the Chen-Lü-Pope metric, possibly with the "natural" change of the metric functions. The question about the existence of the C-metric in higher dimensions therefore still remains open.

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