Constants of geodesic motion in higher-dimensional black-hole spacetimes

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In [Phys. Rev. Lett. **98**, 061102 (2007)], we announced the complete integrability of geodesic motion in the general higher-dimensional rotating black-hole spacetimes. In the present paper we prove all the necessary steps leading to this conclusion. In particular, we demonstrate the independence of the constants of motion and the fact that they Poisson commute. The relation to a different set of constants of motion constructed in [J. High Energy Phys. 02 (2007) 004] is also briefly discussed.

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I. INTRODUCTION

Spacetimes of higher dimensions ($D > 4$) have become much studied as a result of their appearance in theories of unification, such as string/M theory. Of such spacetimes, one important class is a sequence of black-hole metrics of greater and greater generality in higher dimensions that have been discovered over the years.

The first such higher-dimensional black-hole spacetime was the metric for a nonrotating black hole in $D > 4$ (the generalization of the 1916 Schwarzschild metric in four dimensions [1]), found in 1963 by Tangherlini [2]. Next was the metric for a rotating black hole in higher dimensions (the generalization of the 1963 Kerr metric in four dimensions [3]), discovered in 1986 by Myers and Perry [4] in the case with zero cosmological constant. Then in 1998, Hawking, Hunter, and Taylor-Robinson [5] found the general $D = 5$ version of the $D = 4$ rotating black hole with a cosmological constant [often called the Kerr-(anti)de Sitter metric] that had been found in 1968 by Carter [6,7]. In 2004, Gibbons, Lü, Page, and Pope [8,9] discovered the general Kerr-de Sitter metrics in all higher dimensions, and in 2006 Chen, Lü, and Pope [10] put these into a simple form similar to Carter's and were able to add a NUT [11] parameter (though not charge) to get the general Kerr-NUT-(A)dS metrics for all D.

It is important to study the properties of these higher-dimensional black-hole spacetimes, and one key property is the nature of geodesic motion in them. In [12,13], we exhibited D constants of geodesic motion and announced that they are all independent (making the geodesic motion integrable) and that the Poisson brackets of any pair of them vanish (making the integrable geodesic motion completely integrable). In this paper we shall prove these assertions.

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After introducing the metric and its basic symmetries, we recapitulate our construction of constants of geodesic motion and show how these constants can be generated from a generating function. The two main proofs demonstrating the independence and the Poisson commutativity of these constants follow. The canonical formalism used in the text is reviewed in the Appendix. We type tensors in boldface with components in normal letters. The spacetime indices are denoted by Latin letters from the beginning of the alphabet, $a, b, c = 1, \ldots, D$, and we use the Einstein summation convention for them. For a rank-2 tensor $\boldsymbol{B}$, the symbol B stands for the matrix of its components $B^a{}_b$. Where it cannot lead to a confusion a dot indicates contraction, i.e., $\boldsymbol{a} \cdot \boldsymbol{b} = a^e b_e$. We assume automatic lowering and raising of indices using the metric. $\boldsymbol{\partial}_{x^a}$ stands for the coordinate vector associated with the coordinate x^a.

II. HIGHER-DIMENSIONAL BLACK-HOLE SPACETIMES

The general Kerr-NUT-(anti)de Sitter spacetime discovered by Chen, Lü, and Pope may, after a suitable Wick rotation of the radial coordinate, be written [10]

$$\boldsymbol{g} = \sum_{\mu=1}^{n} \left[\frac{\boldsymbol{d}x_\mu^2}{Q_\mu} + Q_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} \boldsymbol{d}\psi_k \right)^2 \right] - \frac{\varepsilon c}{A^{(n)}} \left(\sum_{k=0}^{n} A^{(k)} \boldsymbol{d}\psi_k \right)^2, \tag{1}$$

with $n = \lfloor D/2 \rfloor$ and $\varepsilon = D - 2n$. Here, $Q_\mu = X_\mu / U_\mu$,

$$\begin{aligned} U_\mu &= \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\nu^2 - x_\mu^2), \\ X_\mu &= \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} + \frac{\varepsilon c}{x_\mu^2}, \\ A_\mu^{(k)} &= \sum_{\substack{\nu_1<\ldots<\nu_k \\ \nu_i\neq\mu}} x_{\nu_1}^2 \ldots x_{\nu_k}^2, \\ A^{(k)} &= \sum_{\nu_1<\ldots<\nu_k} x_{\nu_1}^2 \ldots x_{\nu_k}^2. \end{aligned} \tag{2}$$

The coordinates x_μ ($\mu = 1, \ldots, n$) correspond to radial and latitude directions, ψ_k ($k = 0, \ldots, n-1+\varepsilon$) to temporal and azimuthal directions. The parameter c_n is proportional to the cosmological constant, and the remaining constants c_k, c, and b_μ are related to the rotation parameters, the mass, and the NUT parameters. Hamamoto, Houri, Oota, and Yasui [14] derived explicit formulas for the curvature and demonstrated that in all dimensions this metric obeys the Einstein equations,

$$R_{ab} = (-1)^n (D-1) c_n g_{ab}. \tag{3}$$

Besides the obvious spacetime isometries generated by the $D-n$ Killing vectors $\boldsymbol{\partial}_{\psi_k}$, the spacetime possesses a whole set of hidden symmetries [12,13], which can be generated from the principal (rank-2 closed) conformal Killing-Yano tensor discovered by Kubizňák and Frolov [15]. These hidden symmetries play the crucial role for the integrability of the geodesic motion.

The metric (1) can be diagonalized. Let us introduce the orthonormal basis one-forms:

$$\boldsymbol{e}^\mu = Q_\mu^{-1/2} \boldsymbol{d}x_\mu, \qquad \boldsymbol{e}^{\hat{\mu}} = \boldsymbol{e}^{n+\mu} = Q_\mu^{1/2} \sum_{k=0}^{n-1} A_\mu^{(k)} \boldsymbol{d}\psi_k,$$

$$\boldsymbol{e}^{2n+1} = (-c/A^{(n)})^{1/2} \sum_{k=0}^{n} A^{(k)} \boldsymbol{d}\psi_k. \tag{4}$$

Then we have

$$\boldsymbol{g} = \sum_{a=1}^{D} \boldsymbol{e}^a \boldsymbol{e}^a = \sum_{\mu=1}^{n} (\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \boldsymbol{e}^{\hat{\mu}} \boldsymbol{e}^{\hat{\mu}}) + \varepsilon \boldsymbol{e}^{2n+1} \boldsymbol{e}^{2n+1}, \tag{5}$$

and the principal conformal Killing-Yano tensor $\boldsymbol{h}$, which obeys the equations

$$(D-1)\nabla_a h_{bc} = g_{ab}\xi_c - g_{ac}\xi_b, \qquad \xi_a = \nabla_c h^c{}_a, \tag{6}$$

takes the extremely simple form

$$\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu \boldsymbol{e}^\mu \wedge \boldsymbol{e}^{\hat{\mu}}. \tag{7}$$

In what follows we shall also use the conformal Killing tensor

$$Q = -hh, \quad \text{i.e.,} \quad Q_{ab} = h_{ac} h_{bd} g^{cd}, \tag{8}$$

which takes the explicit form

$$\boldsymbol{Q} = \sum_{\mu=1}^{n} x_\mu^2 (\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \boldsymbol{e}^{\hat{\mu}} \boldsymbol{e}^{\hat{\mu}}), \tag{9}$$

and satisfies $\nabla_{(a} Q_{bc)} = g_{(ab} Q_{c)}$, where

$$Q_a = \frac{1}{D+2}(2\nabla_c Q^c{}_a + \nabla_a Q^c{}_c). \tag{10}$$

III. CONSTANTS OF MOTION

In [12] we have claimed that in the spacetime (5) there are D independent constants of geodesic motion, given by the following quantities: (a) $n-1$ observables C_j, $j = 1, \ldots, n-1$, given by traces of powers of the projection $\boldsymbol{F}$ of the principal conformal Killing-Yano tensors $\boldsymbol{h}$ [cf. Eqs. (15) and (16) below]

$$C_j = \mathrm{tr}[(-w^{-1}F^2)^j], \tag{11}$$

(b) $D-n$ observables p_j, $j = 0, \ldots, D-n-1$, given by symmetries of the spacetime

$$p_j = \boldsymbol{u} \cdot \boldsymbol{\partial}_{\psi_j}, \tag{12}$$

and (c) the square w of the (unnormalized) velocity $\boldsymbol{u}$,

$$w = \boldsymbol{u} \cdot \boldsymbol{u} = u^a u_a. \tag{13}$$

Moreover, these quantities commute in the sense of Poisson brackets on the phase space. Here we want to elucidate and prove these properties in more detail.

We understand all mentioned quantities as observables (i.e., functions) on the phase space $\mathbf{\Gamma} = \mathbf{T}^* M$. It is well known that the cotangent space $\mathbf{T}^* M$ has a natural phase-space structure (cf. the Appendix or [16]). Since we investigate the relativistic theory and M is a spacetime manifold describing also the physical temporal direction, the phase space $\mathbf{\Gamma} = \mathbf{T}^* M$ is an unphysical phase space which is, however, well suited for an investigation of the geodesic motion. Doing canonical mechanics on it allows us to solve the geodesic motion in an external time which can be identified at the end with the affine parameter of the studied geodesic.

We denote the momentum variable on the cotangent space as $\boldsymbol{u}$. Indeed, since the geodesic motion is governed by the Lagrange function $L = \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} = \frac{1}{2} u^a u^b g_{ab}$, the canonical momentum can be (up to a position of the tensor index) identified with the (unnormalized) velocity $\boldsymbol{u}$. The Hamiltonian then is

$$H = \tfrac{1}{2} w = \tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} = \tfrac{1}{2} u_a u_b g^{ab}. \tag{14}$$

We easily realize that p_j defined in (12) are the special components of momentum and that they are constants of motion since $\boldsymbol{\partial}_{\psi_j}$ are Killing vectors. The quantities C_j, Eq. (11), are constants of motion because the tensor $\boldsymbol{F}$, defined as

$$F_{ab} = (h_{ab} u_c + h_{bc} u_a + h_{ca} u_b) u^c, \tag{15}$$

and the square w of the velocity $\boldsymbol{u}$, are covariantly conserved along the geodesic. Indeed, thanks to (6), for $u^c \nabla_c u^a = 0$ we have $u^c \nabla_c F_{ab} = 0$.

Next we express the constants of motion C_j in terms of the quantities related to the principal conformal Killing-Yano tensor $\boldsymbol{h}$. The components of the tensor (15) can be rewritten as

$$w^{-1}F = PhP, \tag{16}$$

where $\boldsymbol{P}$ is the projector orthogonal to the velocity $\boldsymbol{u}$, $P = I - p$, i.e., $P^a_b = \delta^a_b - p^a_b$. Here we also introduced the projector $\boldsymbol{p}$,

$$p^a_b = w^{-1}u^a u_b, \tag{17}$$

onto the direction $\boldsymbol{u}$. Using the cyclic property of the trace we thus have

$$C_j = (-1)^j w^j \operatorname{tr}[(hP)^{2j}]. \tag{18}$$

The trace of the matrix product could be viewed diagrammatically as a loop formed by joined vertices (each with two "legs") corresponding to matrices in the product. In our case the loop is formed by alternating h and P vertices. Substituting $P = I - p$, we get a sum over all possible loops in which P is replaced either by I or by $-p$. In the case of the identity I, the corresponding vertex is effectively eliminated, and in the case of the one-dimensional projector $\boldsymbol{p} = w^{-1}\boldsymbol{uu}$, the loop splits into disconnected pieces. Namely, we can use the identity

$$\operatorname{tr}(h^{k_1}ph^{k_2}p\cdots h^{k_c}p) = \operatorname{tr}(h^{k_1}p)\operatorname{tr}(h^{k_2}p)\cdots\operatorname{tr}(h^{k_c}p). \tag{19}$$

The trace in (18) thus leads to

$$\operatorname{tr}[(hP)^{2j}] = \operatorname{tr}(h^{2j}) + \sum_{c=1}^{2j}\sum_{\substack{k_1\leq\ldots\leq k_c\\ k_1+\ldots+k_c=2j}}(-1)^c N^{2j}_{k_1\ldots k_j}\prod_{i=1}^{c}\operatorname{tr}(h^{k_i}p). \tag{20}$$

The sum over c is the sum over the number of "splits" of the loop, the indices k_i are the "lengths" of the split pieces, and the combinatorial factor $N^{2j}_{k_1\ldots k_c}$ gives the number of ways in which the loop of the length $2j$ can be split to c pieces of lengths $k_1, \ldots, k_c$. From the fact that the tensor $\boldsymbol{h}$ is antisymmetric, it follows that traces of odd powers of h (optionally multiplied by a projector) are zero. Setting $k_i = 2l_i$ and introducing the rank-2 conformal Killing tensor $\boldsymbol{Q}$ from (8), Eq. (20) thus reduces to

$$\operatorname{tr}[(-hPhP)^j] = \operatorname{tr}(Q^j) + \sum_{c=1}^{j}\sum_{\substack{l_1\leq\ldots\leq l_c\\ l_1+\ldots+l_c=j}}(-1)^c 2N^j_{l_1\ldots l_j}\prod_{i=1}^{c}\operatorname{tr}(Q^{l_i}p), \tag{21}$$

where we used $N^{2j}_{2l_1\ldots 2l_c} = 2N^j_{l_1\ldots l_c}$ which follows from the definition of the N's. If we define the quantities

$$w_j = w\operatorname{tr}(Q^j p) = u_{a_0}Q^{a_0}_{a_1}Q^{a_1}_{a_2}\cdots Q^{a_{j-1}}_{a_j}u^{a_j}, \tag{22}$$

we finally obtain

$$C_j = w^j\operatorname{tr}[(-hPhP)^j] = w^j\operatorname{tr}(Q^j) + 2\sum_{c=1}^{j}\sum_{\substack{l_1\leq\ldots\leq l_c\\ l_1+\ldots+l_c=j}}(-1)^c N^j_{l_1\ldots l_j}w^{j-c}\prod_{i=1}^{c}w_{l_i}, \tag{23}$$

which is Eq. (17) of [12].

Let us note that, by the same argument as that leading to Eq. (20), we can derive the relation for the trace of a power of QP,

$$\operatorname{tr}[(QP)^j] = \operatorname{tr}(Q^j) + \sum_{c=1}^{j}\sum_{\substack{l_1\leq\ldots\leq l_c\\ l_1+\ldots+l_c=j}}(-1)^c N^j_{l_1\ldots l_j}\prod_{i=1}^{c}\operatorname{tr}(Q^{l_i}p). \tag{24}$$

Comparing with Eq. (21), we see that we have proved the relation (16) of [12]:

$$\operatorname{tr}[(-hPhP)^j] + \operatorname{tr}[Q^j] = 2\operatorname{tr}[(QP)^j]. \tag{25}$$

The relation (23) and an algorithm for computing the coefficients $N^j_{l_1\ldots l_c}$ can be derived also in a different way. It was mentioned in [13] that the constants C_j can be generated from the generating function $Z(\beta) = \log W(\beta)$:

$$Z(\beta) = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{2j}\frac{\beta^j}{w^j}C_j = -\sum_{j=1}^{\infty}\frac{1}{2j}\beta^j\operatorname{tr}[(hP)^{2j}] = \operatorname{tr}\log(I-\sqrt{\beta}hP) = \log\det(I-\sqrt{\beta}hP). \tag{26}$$

The third equality follows from the antisymmetry of h. Using properties of the determinant, the antisymmetry of h, $I = P + p$, and the fact that the projector p is one dimensional, we can split $Z(\beta)$ into two pieces [cf. Eq. (2.7) and (2.8) of [13]]:

$$\begin{aligned} Z(\beta) &= \log W_0(\beta) + \log\Sigma(\beta),\\ W_0(\beta) &= \det(I-\sqrt{\beta}h) = \det{}^{1/2}(I+\beta Q),\\ \Sigma(\beta) &= \det(P + (I-\sqrt{\beta}h)^{-1}p)\\ &= \operatorname{tr}((I-\sqrt{\beta}h)^{-1}p) = \operatorname{tr}((I+\beta Q)^{-1}p). \end{aligned} \tag{27}$$

Equation (23) then corresponds to the term proportional to β^j in the power expansion of $Z(\beta)$. The first term of (23) is obtained from $\log W_0(\beta)$, and the sum over all possible splittings of the loop corresponds to the β^j term of $\log\Sigma(\beta)$. Clearly, the jth derivative of $\log\Sigma(\beta)$ (evaluated at $\beta = 0$) contains the sum over all possible products of lth derivatives $\Sigma^{(l)}(0)$ which are proportional to w_l defined in (22). The factors $N^j_{l_1\ldots l_2}$ can thus be obtained by the explicit computation of the derivatives of the generating function $\log\Sigma(\beta)$:

$$C_j = w^j \operatorname{tr}(Q^j) - \frac{2(-w)^j}{(j-1)!} \times \frac{d^j}{d\beta^j} \log\left(1 + \sum_{k=1}^{j} (-1)^k \frac{w_k}{w} \beta^k\right)\bigg|_{\beta=0}. \quad (28)$$

Using software for algebraic manipulation we easily get the first five constants (sufficient for the integrability of geodesic motion up through $D = 13$):

$$\begin{aligned} C_1 &= w \operatorname{tr} Q - 2w_1, \\ C_2 &= w^2 \operatorname{tr} Q^2 - 4ww_2 + 2w_1^2, \\ C_3 &= w^3 \operatorname{tr} Q^3 - 6w^2 w_3 + 6ww_1 w_2 - 2w_1^3, \\ C_4 &= w^4 \operatorname{tr} Q^4 - 8w^3 w_4 + w^2(4w_2^2 + 8w_1 w_3) - 8ww_1^2 w_2 \\ &\quad + 2w_1^4, \\ C_5 &= w^5 \operatorname{tr} Q^5 - 10w^4 w_5 + w^3(10w_2 w_3 + 10w_1 w_4) \\ &\quad - w^2(10w_1 w_2^2 + 10w_1^2 w_3) + 10ww_1^3 w_2 - 2w_1^5. \end{aligned} \quad (29)$$

Taking into account the facts that the eigenvalues of the principal conformal Killing-Yano tensor $\boldsymbol{h}$ are given by the coordinates x_μ, cf. Eq. (7), respectively, that the eigenvalues of $\boldsymbol{Q}$ are x_μ^2, see Eq. (9), we can write down an explicit form for $\operatorname{tr} Q^j$ and w_j:

$$\operatorname{tr} Q^j = 2 \sum_{\mu=1}^{n} x_\mu^{2j}, \quad (30)$$

$$w_j = \sum_{\mu=1}^{n} x_\mu^{2j} (u_\mu^2 + u_{\hat\mu}^2). \quad (31)$$

Let us also point out that on the level of the generating functions the relation (25) corresponds to

$$\det(I - \beta hPhP)\det(I + \beta Q) = \det{}^2(I + \beta QP). \quad (32)$$

It was realized in [13] that the generating function $W(\beta) = \exp Z(\beta) = W_0(\beta)\Sigma(\beta)$ actually generates another set of conserved quantities c_j by

$$W(\beta) = \frac{1}{w} \sum_{j=0}^{\infty} c_j \beta^j, \quad (33)$$

which are quadratic in the velocity $\boldsymbol{u}$. [That they are quadratic can be seen from the fact that W_0 does not depend on the velocity, from Eq. (31), and from $w\Sigma(\beta) = \sum_{j=0}^{\infty} (-1)^j w_j \beta^j$.] The relation between $W(\beta)$ and $Z(\beta)$ implies that

$$C_j = -\frac{2(-w)^j}{(j-1)!} \frac{d^j}{d\beta^j} \log\left(w + \sum_{k=1}^{j} c_k \beta^k\right)\bigg|_{\beta=0}, \quad (34)$$

and, in particular,

$$\begin{aligned} C_1 &= 2c_1, \\ C_2 &= -4wc_2 + 2c_1^2, \\ C_3 &= 6w^2 c_3 - 6wc_1 c_2 + 2c_1^3, \\ C_4 &= -8w^3 c_4 + 8w^2 c_1 c_3 + 4w^2 c_2^2 - 8wc_1^2 c_2 + 2c_1^4, \\ C_5 &= 10w^4 c_5 - 10w^3 c_1 c_4 - 10w^3 c_2 c_3 + 10w^2 c_1^2 c_3 \\ &\quad + 10w^2 c_1 c_2^2 - 10wc_1^3 c_2 + 2c_1^5, \end{aligned} \quad (35)$$

which are the inverse of the relations (3.19) of [13].

IV. INDEPENDENCE OF CONSTANTS OF MOTION

Now we can demonstrate that the quantities w, p_j, and C_j are independent at a generic point of the phase space $\boldsymbol{\Gamma} = \mathbf{T}^* M$. This means that their gradients on the phase space are linearly independent. To prove this, it is sufficient to show that these gradients are independent in the vertical direction of the cotangent bundle $\mathbf{T}^* M$, i.e., that the derivatives of these quantities with respect to the momentum $\boldsymbol{u}$ are linearly independent. To achieve this we will study the wedge product of the ''vertical'' derivatives.

Let us, instead of w and C_j, consider the equivalent set of observables ($j = 1, \ldots, n-1$)

$$\begin{aligned} 2\tilde C_j &= -\frac{1}{2j} w^{1-j} C_j = -\frac{1}{2j} w \operatorname{tr} Q^j + w_j + \ldots, \\ 2\tilde C_0 &= w, \end{aligned} \quad (36)$$

where dots in the first expression denote terms which contain w_k with $k < j$, cf. Eqs. (23) and (28).

We are interested in the quantity[1]

$$\boldsymbol{J} = \boldsymbol{\partial} \tilde C_0 \wedge \cdots \wedge \boldsymbol{\partial} \tilde C_{n-1} \wedge \boldsymbol{\partial} p_0 \wedge \cdots \wedge \boldsymbol{\partial} p_{D-n-1}. \quad (37)$$

Because of (12) and (36), we have $\boldsymbol{\partial} p_j = \boldsymbol{\partial}_{\psi_j}$ and

$$\boldsymbol{\partial} \tilde C_j = -\frac{1}{2j} (\operatorname{tr} Q^j) \boldsymbol{u} + \boldsymbol{Q}^j \cdot \boldsymbol{u} + \cdots, \quad (38)$$

where dots denote linear combinations of $\boldsymbol{Q}^k \cdot \boldsymbol{u}$ with $k < j$; $\boldsymbol{Q}^l \cdot \boldsymbol{u}$ represents the vector with components $Q^a_{a_1} Q^{a_1}_{a_2} \cdots Q^{a_{l-1}}_{a_l} u^{a_l}$. From the antisymmetry of the wedge product it follows that

$$\boldsymbol{J} = \boldsymbol{u} \wedge (\boldsymbol{Q} \cdot \boldsymbol{u}) \wedge \cdots \wedge (\boldsymbol{Q}^{n-1} \cdot \boldsymbol{u}) \wedge \boldsymbol{\partial}_{\psi_0} \wedge \cdots \wedge \boldsymbol{\partial}_{\psi_{D-n-1}}. \quad (39)$$

(Matrix) powers $\boldsymbol{Q}^j$ of the conformal Killing tensor can be written as

[1] The derivative $\boldsymbol{\partial} f$ is the *vector* field on spacetime M with components $\partial f / \partial u_a$, cf. the Appendix (it could be written more explicitly as $\boldsymbol{\partial} f / \boldsymbol{\partial u}$). The wedge product is, strictly speaking, defined for (antisymmetric) *forms*. However, we can easily define the wedge product also for the vectors or lower the vector indices with the help of the metric to get 1-forms.

$$\boldsymbol{Q}^j = \sum_{\mu=1}^{n} x_\mu^{2j}\boldsymbol{e}_\mu \boldsymbol{e}^\mu + \sum_{\mu=1}^{n} x_\mu^{2j}\boldsymbol{e}_{\hat{\mu}}\boldsymbol{e}^{\hat{\mu}}. \tag{40}$$

The second term acts on the subspace of the vectors spanned on $\boldsymbol{\partial}_{\psi_j}$. Thus, thanks to the $\boldsymbol{\partial}_{\psi_0} \wedge \cdots \wedge \boldsymbol{\partial}_{\psi_{D-n-1}}$ term in the wedge product, this part can be ignored in (39). Taking into account that $\boldsymbol{e}_\mu \boldsymbol{e}^\mu = \boldsymbol{\partial}_{x_\mu}\mathbf{d}x_\mu$ and $u^\mu = \boldsymbol{d}x_\mu \cdot \boldsymbol{u}$, the substitution of (40) into (39) leads to

$$\boldsymbol{J} = u^1 \ldots u^n U \boldsymbol{\partial}_{x_1} \wedge \cdots \wedge \boldsymbol{\partial}_{x_n} \wedge \boldsymbol{\partial}_{\psi_0} \wedge \cdots \wedge \boldsymbol{\partial}_{\psi_{D-n-1}}, \tag{41}$$

where

$$U = \sum_{\substack{\text{permutations } \sigma \\ \text{of } [0,\ldots,n-1]}} \operatorname{sign}\sigma\, x_1^{2\sigma_1} \ldots x_n^{2\sigma_n} = \prod_{\substack{\mu,\nu=1\ldots n \\ \nu<\mu}} (x_\mu^2 - x_\nu^2). \tag{42}$$

In a generic point of the phase space we have $u^j \neq 0$ and $x_\mu^2 \neq x_\nu^2$ (for $\mu \neq \nu$) and therefore $\boldsymbol{J} \neq 0$ there, thus showing that the constants of motion are independent.

V. POISSON BRACKETS

Finally we show that the observables w, C_j, and p_j Poisson commute on the phase space.

The Poisson bracket of two functions on the phase space $\boldsymbol{\Gamma} = \mathbf{T}^* M$ can be written as

$$\{A, B\} = \boldsymbol{\nabla} A \cdot \boldsymbol{\partial} B - \boldsymbol{\partial} A \cdot \boldsymbol{\nabla} B, \tag{43}$$

where $\boldsymbol{\nabla} F$ represents an arbitrary (torsion-free) covariant derivative which ignores the dependence of F on the momentum $\boldsymbol{u}$, and $\boldsymbol{\partial} B$ is the derivative of B with respect to the momentum $\boldsymbol{u}$, cf. the Appendix. $\boldsymbol{\nabla} F$ is a 1-form and $\boldsymbol{\partial} F$ a vector field on the spacetime M, and the dot indicates the contraction in spacetime tensor indices. We use naturally the covariant derivative $\boldsymbol{\nabla}$ generated by the metric on M.

Clearly, the commutation of any observable with the Hamiltonian $\frac{1}{2}w$ of the geodesic motion is equivalent to the conservation of the observable, cf. Eq. (A13), so we have

$$\{w, p_j\} = 0, \qquad \{w, C_j\} = 0. \tag{44}$$

The Poisson bracket between observables $p_j = \boldsymbol{u} \cdot \boldsymbol{\partial}_{\psi_j}$ reduces to Lie brackets of the Killing vector fields $\boldsymbol{\partial}_{\psi_j}$, which vanish because $\boldsymbol{\partial}_{\psi_j}$ are coordinate vector fields:

$$\begin{aligned}\{p_i, p_j\} &= \boldsymbol{\partial}_{\psi_j} \cdot (\boldsymbol{\nabla}\boldsymbol{\partial}_{\psi_i}) \cdot \boldsymbol{u} - \boldsymbol{\partial}_{\psi_i} \cdot (\boldsymbol{\nabla}\boldsymbol{\partial}_{\psi_j}) \cdot \boldsymbol{u} \\ &= [\boldsymbol{\partial}_{\psi_j}, \boldsymbol{\partial}_{\psi_i}] \cdot \boldsymbol{u} = 0. \end{aligned} \tag{45}$$

The Poisson bracket of any observable with the observable $p = \boldsymbol{l} \cdot \boldsymbol{u}$ linear in momentum leads to the Lie derivative along the vector field $\boldsymbol{l}$, see Eq. (A12):

$$\{C_i, p_j\} = \mathcal{L}_{\boldsymbol{\partial}_{\psi_j}} C_i = 0. \tag{46}$$

Here, the Lie derivative $\mathcal{L}_{\boldsymbol{\partial}_{\psi_j}} C_i$ ignores the dependence of C_i on the momentum $\boldsymbol{u}$, cf. the Appendix. It vanishes because $\boldsymbol{\partial}_{\psi_j}$ is a Killing vector and the definition of C_i respects the symmetry of the spacetime (it does not depend explicitly on ψ_j).

Finally, it remains to evaluate the brackets $\{C_i, C_j\}$. To simplify the following computation, we will study rescaled observables[2]

$$\tilde{C}_j = (-1)^j w^j C_j = \operatorname{tr}[(h\tilde{P})^{2j}], \tag{47}$$

cf. Eq. (18), and we denote

$$\tilde{\boldsymbol{P}} = w\boldsymbol{P} = w\boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}. \tag{48}$$

Using the cyclic property of the trace, the derivative of $\tilde{C}_j$ in the spacetime direction is

$$\nabla_a \tilde{C}_j = 2j \operatorname{tr}[(\nabla_a h)\tilde{P}(h\tilde{P})^{2j-1}]. \tag{49}$$

Here $\nabla_a h$ is the matrix of components $\nabla_a h^b{}_c$ of the covariant derivative $\boldsymbol{\nabla h}$. Substituting for $\nabla_a h^b{}_c$ from Eq. (6) and using the antisymmetry of h, we obtain

$$\begin{aligned}\frac{D-1}{2j}\nabla_e \tilde{C}_j &= \xi_{a_0}\tilde{P}^{a_0}_{b_1} h^{b_1}{}_{a_1}\tilde{P}^{a_1}_{b_2} \ldots h^{b_{2j-2}}{}_{a_{2j-1}}\tilde{P}^{a_{2j-1}}_{e} \\ &\quad - g_{ea_{2j}}\tilde{P}^{a_{2j}}_{b_{2j-1}} h^{b_{2j-1}}{}_{a_{2j-1}} \ldots h^{b_1}{}_{a_1}\tilde{P}^{a_1}_{b_0}\xi^{b_0} \\ &= 2\xi_{a_0}\tilde{P}^{a_0}_{b_1} h^{b_1}{}_{a_1}\tilde{P}^{a_1}_{b_2} \ldots h^{b_{2j-2}}{}_{a_{2j-1}}\tilde{P}^{a_{2j-1}}_{e}. \end{aligned} \tag{50}$$

For the derivative with respect of the momentum $\boldsymbol{u}$, we get

$$\begin{aligned}\frac{1}{4j}\partial^e \tilde{C}_j &= u^e (h^{d_1}{}_{c_1}\tilde{P}^{c_1}_{d_2} h^{d_2}{}_{c_2}\tilde{P}^{c_2}{}_{d_3} \ldots \tilde{P}^{c_{2j-1}}_{d_{2j}} h^{d_{2j}}{}_{d_1}) \\ &\quad + h^e{}_{c_1}\tilde{P}^{c_1}_{d_2} h^{d_2}{}_{c_2}\tilde{P}^{c_2}_{d_3} \ldots \tilde{P}^{c_{2j-1}}_{d_{2j}} h^{d_{2j}}{}_{c_{2j}} u^{c_{2j}}. \end{aligned} \tag{51}$$

Substituting (50) and (51) into expression (43) for $\{\tilde{C}_i, \tilde{C}_j\}$ and using $\tilde{P}^a_b u^b = 0$, we find

$$\frac{D-1}{16ij}\{\tilde{C}_i, \tilde{C}_j\} = \xi_{a_0}\tilde{P}^{a_0}_{b_1} h^{b_1}{}_{a_1} \ldots \tilde{P}^{a_{2i-1}}_{b_{2i-1}} h^{b_{2i-1}}{}_{c_1} \ldots \tilde{P}^{c_{2j-1}}_{d_{2j}} h^{d_{2j}}{}_{c_{2j}} u^{c_{2j}} - \xi_{a_0}\tilde{P}^{a_0}_{b_1} h^{b_1}{}_{a_1} \ldots \tilde{P}^{a_{2j-1}}_{b_{2j-1}} h^{b_{2j-1}}{}_{c_1} \ldots \tilde{P}^{c_{2i-1}}_{d_{2i}} h^{d_{2i}}{}_{c_{2i}} u^{c_{2i}} = 0. \tag{52}$$

[2]The scaling (47) differs from (36) used in the previous section.

We thus proved that the conserved quantities w, p_j, and C_j Poisson commute with each other. Since the generating function $Z(\beta)$ is given by power series in β with coefficients given (up to constant factors) by the constants C_j, then also this function [and similarly $W(\beta) = \exp Z(\beta)$] Poisson commute with w and p_j, as well as with itself for different choices of β:

$$\{Z(\beta_1), Z(\beta_2)\} = 0, \qquad \{W(\beta_1), W(\beta_2)\} = 0. \tag{53}$$

The same is true also for the quantities c_j generated from $W(\beta)$ introduced in [13]. Therefore the constants of motion all Poisson commute (are in involution), so the geodesic motion is completely integrable [16,17].

VI. SUMMARY

We have explicitly proved the complete integrability of geodesic motion in the general higher-dimensional rotating black-hole spacetimes [10]. The "nontrivial" constants of motion are associated with the Killing tensors which we generated from the principal conformal Killing-Yano tensor. Observables c_j are quadratic in momenta and correspond to rank-2 Killing tensors, whereas constants C_j are of higher order in momenta and correspond to Killing tensors of increasing rank.

The complete integrability of the geodesic motion is related to the issue of separability of the Hamilton-Jacobi equation recently accomplished by Frolov, Krtouš, and Kubizňák [18]. The relation between integrability and separability on a general level has been studied in the series of papers by Benenti and Francaviglia (see, e.g., [19]) where it was demonstrated that the separability is possible only if all the constants of motion, corresponding to Killing vectors and rank-2 Killing tensors, Poisson commute.

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APPENDIX A: COVARIANT CANONICAL FORMALISM ON THE COTANGENT BUNDLE

It is textbook knowledge [16] that the cotangent bundle $\mathbf{T}^*M$ has the natural structure of a phase space, i.e., it possesses a symplectic form $\mathbf{\Omega}$ which defines the Poisson bracket $\{,\}$. For a base manifold M which is equipped with an additional geometric structure, it can be useful to express phase-space quantities and operations with the help of quantities and operations on the base manifold. In this Appendix we shortly review such a procedure.[3]

We call functions on the phase space $\mathbf{\Gamma} = \mathbf{T}^*M$ observables, and we write $F(x, \boldsymbol{u})$ to emphasize the dependence of F on the configuration variable $x \in M$ and on the momentum $\boldsymbol{u} \in \mathbf{T}^*_x M$.

For the base manifold M with a (torsion-free) covariant derivative $\boldsymbol{\nabla}$ (in our case the spacetime manifold with the metric connection), it is possible to introduce the covariant derivative of an observable $F(x, \boldsymbol{u})$ *in the horizontal (configurational) direction* of the phase space $\mathbf{\Gamma}$. For any base manifold vector $\boldsymbol{l} \in \mathbf{T}M$, we define

$$\boldsymbol{l}^e \boldsymbol{\nabla}_e F(x, \boldsymbol{u}) = \frac{d}{d\alpha} F(x(\alpha), \boldsymbol{u}(\alpha)) \bigg|_{\alpha=0}, \tag{A1}$$

where $x(\alpha)$ is a curve starting from x with tangent vector $\boldsymbol{l}$, and $\boldsymbol{u}(\alpha)$ is parallel transport of $\boldsymbol{u}$ along $x(\alpha)$.

For a 1-form $\boldsymbol{p} \in \mathbf{T}^*_x M$ we can also introduce the derivative of an observable $F(x, \boldsymbol{u})$ *in the vertical (momentum) direction*

$$\boldsymbol{p}_e \boldsymbol{\partial}^e F(x, \boldsymbol{u}) = \frac{d}{d\alpha} F(x, \boldsymbol{u} + \alpha \boldsymbol{p}) \bigg|_{\alpha=0}. \tag{A2}$$

Thanks to the linearity of $\mathbf{T}^*_x M$, this derivative is independent of any additional geometrical structure.

Derivatives $\boldsymbol{l}^e \boldsymbol{\nabla}_e F$ and $\boldsymbol{p}_e \boldsymbol{\partial}^e F$ are derivative operators on $\mathbf{\Gamma}$ and as such they define vector fields on $\mathbf{\Gamma}$, which we denote[4] $\boldsymbol{l}^e \boldsymbol{\nabla}_e$ and $\boldsymbol{p}_e \boldsymbol{\partial}^e$. These derivatives and vector fields depend ultralocally on the base manifold quantities $\boldsymbol{l}$ and $\boldsymbol{p}$, respectively, and we can thus introduce differentials $\boldsymbol{\nabla}_e F$ and $\boldsymbol{\partial}^e F$ and mixed tensor quantities $\frac{\boldsymbol{\nabla}_e}{\boldsymbol{\partial} x}$ and $\frac{\boldsymbol{\partial}^e}{\boldsymbol{\partial} \boldsymbol{u}}$ by tearing off $\boldsymbol{l}$ and $\boldsymbol{p}$, respectively, and by tearing off the function F.

Clearly, $\boldsymbol{\nabla}_e F$ is the covariant derivative of the observable $F(x, \boldsymbol{u})$ which "ignores" the momentum $\boldsymbol{u}$ leaving it covariantly constant. On the other side, the derivative $\boldsymbol{\partial}^e F$ ignores the configuration variable x.

For an observable $F(x, \boldsymbol{u})$ given by a contraction of a spacetime tensor field $\boldsymbol{f}(x)$ with several momenta $\boldsymbol{u}$,

$$F(x, \boldsymbol{u}) = \boldsymbol{f}^{abc\ldots}(x) \boldsymbol{u}_a \boldsymbol{u}_b \boldsymbol{u}_c \ldots, \tag{A3}$$

[3]Similarly to the main text we type tensors in bold. Optionally, we write here the tensors with *abstract indices* [20,21] which help to indicate tensorial operations as, for example, contraction. However, the abstract indices do not refer to any particular choice of coordinates. We use small Latin letters for base manifold indices (for tensors from $\mathbf{T}M$), but we do not introduce indices for the phase-space tensors (tensors from $\mathbf{T\Gamma}$). We assume implicitly the tensor product, i.e., $\boldsymbol{ab} = \boldsymbol{a} \otimes \boldsymbol{b}$.

[4]We could be more explicit and write them as $\boldsymbol{l}^e \frac{\boldsymbol{\nabla}_e}{\boldsymbol{\partial} x}$ and $\boldsymbol{p}_e \frac{\boldsymbol{\partial}^e}{\boldsymbol{\partial} \boldsymbol{u}}$. Similarly we could write $\frac{\boldsymbol{\nabla}_e F}{\boldsymbol{\partial} x}$ and $\frac{\boldsymbol{\partial}^e F}{\boldsymbol{\partial} \boldsymbol{u}}$ for quantities $\boldsymbol{\nabla}_e F$ and $\boldsymbol{\partial}^e F$ introduced below. However, we use such an explicit notation only for the mixed tensor fields $\frac{\boldsymbol{\nabla}_e}{\boldsymbol{\partial} x}$ and $\frac{\boldsymbol{\partial}^e}{\boldsymbol{\partial} \boldsymbol{u}}$ (see below) where the notation $\boldsymbol{\nabla}_e$ and $\boldsymbol{\partial}^e$ would be too brief.

the covariant derivative reduces to the standard base manifold covariant derivative

$$\nabla_e F(x, \boldsymbol{u}) = \nabla_e \boldsymbol{f}^{abc\ldots}(x)\,\boldsymbol{u}_a\boldsymbol{u}_b\boldsymbol{u}_c\ldots. \tag{A4}$$

The momentum derivative leaves $\boldsymbol{f}$ intact

$$\boldsymbol{\partial}^e F(x,\boldsymbol{u}) = \boldsymbol{f}^{ebc\ldots}(x)\,\boldsymbol{u}_b\boldsymbol{u}_c\cdots + \boldsymbol{f}^{aec\ldots}(x)\,\boldsymbol{u}_a\boldsymbol{u}_c\cdots + \boldsymbol{f}^{abe\ldots}(x)\,\boldsymbol{u}_a\boldsymbol{u}_b\cdots + \ldots. \tag{A5}$$

A general phase-space observable can then be written as a (infinite) sum of terms of this type.

The mixed tensor $\frac{\nabla_e}{\partial x}$ is a vector field on the phase space (from $\mathbf{T\Gamma}$) and a 1-form on the base manifold (from $\mathbf{T}^*M$). It is actually the horizontal lift from $\mathbf{T}M$ to $\mathbf{T\Gamma}$ corresponding to the covariant derivative ∇. The mixed tensor $\frac{\partial^e}{\partial \boldsymbol{u}}$ is a vector field on the phase space (from $\mathbf{T\Gamma}$) and a vector field on the base manifold (from $\mathbf{T}M$). It gives a natural identification of the cotangent fiber $\mathbf{T}^*_x M$ with its vertical tangent space $\mathbf{TT}^*_x M$.

The inverse symplectic form $\boldsymbol{\Omega}^{-1}$ and the Poisson bracket can be written as

$$\boldsymbol{\Omega}^{-1} = \frac{\nabla_e}{\partial x}\frac{\partial^e}{\partial \boldsymbol{u}} - \frac{\partial^e}{\partial \boldsymbol{u}}\frac{\nabla_e}{\partial x} \tag{A6}$$

and

$$\{A, B\} = \nabla_e A\,\boldsymbol{\partial}^e B - \boldsymbol{\partial}^e A\,\nabla_e B. \tag{A7}$$

They do not depend on a choice of the covariant derivative. Indeed, if we choose another torsion-free covariant derivative $\tilde{\nabla}$ on M, which can be done by specifying the ‘‘difference’’ tensor $\boldsymbol{\Gamma}^b_{ac}$,

$$\tilde{\nabla}_a \boldsymbol{a}^b = \nabla_a \boldsymbol{a}^b + \boldsymbol{\Gamma}^b_{ac}\boldsymbol{a}^c, \tag{A8}$$

the induced covariant derivative of the phase-space observables transforms as

$$\tilde{\nabla}_a F(x,\boldsymbol{u}) = \nabla_a F(x,\boldsymbol{u}) + \boldsymbol{u}_e\boldsymbol{\Gamma}^e_{ac}(x)\,\boldsymbol{\partial}^c F(x,\boldsymbol{u}). \tag{A9}$$

Substituting this into (A7) and using the symmetry $\boldsymbol{\Gamma}^b_{ac} = \boldsymbol{\Gamma}^b_{ca}$, we find that

$$\{A, B\} = \tilde{\nabla}_e A\,\boldsymbol{\partial}^e B - \boldsymbol{\partial}^e A\,\tilde{\nabla}_e B, \tag{A10}$$

i.e., the Poisson bracket is independent of the choice of the covariant derivative. The argument for the symplectic structure is similar.

The Poisson bracket of an observable of type (A3) with an observable p linear in momenta $\boldsymbol{u}$,

$$p(x,\boldsymbol{u}) = \boldsymbol{l}^c(x)\,\boldsymbol{u}_c, \tag{A11}$$

leads, with help of (A4) and (A5), to the Lie derivative:

$$\begin{aligned}\{F, p\} &= \boldsymbol{l}^e\nabla_e F - \boldsymbol{\partial}^e F(\nabla_e \boldsymbol{l}^c)\boldsymbol{u}_c \\ &= (\boldsymbol{l}^e\nabla_e \boldsymbol{f}^{ab\ldots} - \boldsymbol{f}^{eb\ldots}\nabla_e \boldsymbol{l}^a \\ &\quad - \boldsymbol{f}^{ae\ldots}\nabla_e \boldsymbol{l}^b - \ldots)\,\boldsymbol{u}_a\boldsymbol{u}_b\cdots \\ &= (\mathcal{L}_{\boldsymbol{l}}\boldsymbol{f}^{ab\ldots})\,\boldsymbol{u}_a\boldsymbol{u}_b\cdots \equiv \mathcal{L}_{\boldsymbol{l}}F.\end{aligned} \tag{A12}$$

Here $\mathcal{L}_{\boldsymbol{l}}\boldsymbol{f}$ is the standard Lie derivative on M along the vector field $\boldsymbol{l}$. The last equality then defines the generalized Lie derivative $\mathcal{L}_{\boldsymbol{l}}F$ of the phase-space observable F along the base manifold vector field $\boldsymbol{l}$ which effectively ignores the dependence of F on the momentum $\boldsymbol{u}$. It can be extended to general phase-space observables by linearity. It can be also defined similarly to (A1) with $\boldsymbol{u}(\alpha) = \phi_\alpha \boldsymbol{u}$ given by a flow ϕ_α induced by the vector field $\boldsymbol{l}$ acting on $\boldsymbol{u}$. $\mathcal{L}_{\boldsymbol{l}}F$ can be also viewed as the derivative of the observable F along the vector field $\mathcal{L}_{\boldsymbol{l}}$ on $\mathbf{T}^*M$ which is called the complete lift of the vector field $\boldsymbol{l}$ on M.

Clearly, the Poisson bracket with the Hamiltonian (14) leads to the covariant derivative along the $\boldsymbol{u}$ direction:

$$\{F, H\} = \boldsymbol{u}^e\nabla_e F. \tag{A13}$$

Despite the fact that we do not need them in the main text, let us introduce for completeness the mixed tensor fields $\boldsymbol{D}^e x$ and $\nabla \boldsymbol{u}_e$ dual to $\frac{\nabla_e}{\partial x}$ and $\frac{\partial^e}{\partial \boldsymbol{u}}$ defined by

$$\begin{aligned}\frac{\nabla_b}{\partial x}\cdot \boldsymbol{D}^a x &= \boldsymbol{\delta}^a_b, \qquad & \frac{\partial^a}{\partial \boldsymbol{u}}\cdot \nabla \boldsymbol{u}_b &= \boldsymbol{\delta}^a_b,\\ \frac{\nabla_a}{\partial x}\cdot \nabla \boldsymbol{u}_b &= 0, \qquad & \frac{\partial^a}{\partial \boldsymbol{u}}\cdot \boldsymbol{D}^b x &= 0.\end{aligned} \tag{A14}$$

Here the dot ‘‘ · ’’ indicates the contraction of the phase-space tensor indices.

$\boldsymbol{D}^e x$ is a vector field on the base manifold M and a 1-form on the phase manifold $\boldsymbol{\Gamma}$. It is actually the differential of the bundle projection x: $\mathbf{T}^*M \to M$. $\nabla \boldsymbol{u}_e$ is a 1-form both on the base manifold M and phase space $\boldsymbol{\Gamma}$.

These phase space ‘‘forms’’ satisfy the completeness relation,

$$\frac{\nabla_e}{\partial x}\boldsymbol{D}^e x + \frac{\partial^e}{\partial \boldsymbol{u}}\nabla \boldsymbol{u}_e = \boldsymbol{\delta}, \tag{A15}$$

with $\boldsymbol{\delta}$ being the identity tensor on $\mathbf{T\Gamma}$. The symplectic structure $\boldsymbol{\Omega}$ can be written as

$$\boldsymbol{\Omega} = \boldsymbol{D}^e x \nabla \boldsymbol{u}_e - \nabla \boldsymbol{u}_e \boldsymbol{D}^e x. \tag{A16}$$

Finally, if we choose the coordinate derivative $\eth$ associated with a coordinate system x^a on M,

$$\eth \boldsymbol{\partial}_{x^a} = 0, \qquad \eth \boldsymbol{d}x^a = 0, \tag{A17}$$

instead of the covariant derivative ∇, the relations (A6), (A7), and (A16) reduce to the standard relations in terms of the canonical coordinates x^a, u_b on $\boldsymbol{\Gamma}$, namely,

$$\boldsymbol{\Omega} = \boldsymbol{d}x^e \boldsymbol{d}u_e - \boldsymbol{d}u_e \boldsymbol{d}x^e, \qquad \boldsymbol{\Omega}^{-1} = \boldsymbol{\partial}_{x^e}\boldsymbol{\partial}_{u_e} - \boldsymbol{\partial}_{u_e}\boldsymbol{\partial}_{x^e}, \tag{A18}$$

and

$$\{A, B\} = \frac{\partial A}{\partial x^e}\frac{\partial B}{\partial u_e} - \frac{\partial A}{\partial u_e}\frac{\partial B}{\partial x^e}. \tag{A19}$$

All coordinate vectors and 1-forms in (A18) live on the phase space $\mathbf{\Gamma}$.

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