

**Spherical gravitational collapse in  $N$  dimensions**

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We investigate here spherically symmetric gravitational collapse in a space-time with an arbitrary number of dimensions and with a general type I matter field, which is a broad class that includes most of the physically reasonable matter forms. We show that given the initial data for matter in terms of the initial density and pressure profiles at an initial surface  $t = t_i$  from which the collapse evolves, there exist the rest of the initial data functions and classes of solutions of Einstein equations which we construct here, such that the space-time evolution goes to a final state which is either a black hole or a naked singularity, depending on the nature of initial data and evolutions chosen, and subject to validity of the weak energy condition. The results are discussed and analyzed in the light of the cosmic censorship hypothesis in black hole physics. The formalism here combines the earlier results on gravitational collapse in four dimensions in a unified treatment. Also the earlier work is generalized to higher-dimensional space-times to allow a study of the effect of the number of dimensions on the possible final outcome of the collapse in terms of either a black hole or naked singularity. No restriction is adopted on the number of dimensions, and other limiting assumptions such as self-similarity of space-time are avoided, in order to keep the treatment general. Our methodology allows us to consider to an extent the genericity and stability aspects related to the occurrence of naked singularities in gravitational collapse.

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**I. INTRODUCTION**

A considerable amount of work has continued in recent years on the validity or otherwise of the cosmic censorship conjecture (CCC) in black hole physics. The reason for this interest is that CCC is fundamental to many aspects of the basic theory and applications of black holes, and the astrophysical implications resulting from the phenomena of continual gravitational collapse of a massive star which has exhausted its nuclear fuel. As of today, no theoretical proof, or even any satisfactory mathematical formulation of CCC is available, whereas an intriguing finding that has emerged from recent investigations on gravitational collapse scenarios in general relativity is that the final end state of such a collapse could be either a black hole (BH), or a naked singularity (NS). The theoretical and observational properties of these objects could be quite different from each other and so it is of much interest to get an insight into how each of these phases come about as end states of a dynamically developing collapse governed by gravitational dynamics.

When a sufficiently massive star starts collapsing gravitationally on exhausting its nuclear fuel, it would not settle to a stable configuration such as a neutron star. What happens in such a case is that an endless gravitational collapse ensues, where the sole governing force is gravity. According to general relativity, the outcome of such a process would be necessarily a space-time singularity. If an event horizon of gravity forms well in advance before the singularity forms that gives rise to a black hole as the

final state of collapse. In the case otherwise, when the horizon formation is delayed during the collapse, these extreme density and curvature regions may fail to be covered by the horizon, and a visible naked singularity may develop (see, e.g., [1–7] for some recent reviews). The possible physical consequences of the later scenario have drawn some attention recently [1]. Most of the gravitational collapse models studied so far are spherical, and the matter cloud collapses under reasonable physical conditions such as an energy condition, and regularity conditions on the initial data from which the collapse develops.

A noteworthy suggestion that has emerged towards a possible theoretical formulation of CCC is that any naked singularities resulting from matter models which may also develop singularities in special relativity, should not be regarded as physical [8–12]. Clearly, it will require a serious effort to cast this into a mathematical statement and a possible proof for CCC. Also, it may not be easy to discard completely all the matter fields such as dust, perfect fluids, and matter with various other reasonable equations of state, which have been studied and used extensively in relativistic astrophysics for a long time from the perspective of understanding gravitational collapse processes and final states.

Another possibility that indeed appears worth exploring is that we may actually be living in a higher-dimensional (HD) space-time. The recent developments in string theory and other related field theories strongly indicate that gravity is possibly a higher-dimensional interaction, which reduces to the general relativistic description at lower energies. Hence, there is a possibility that while CCC may possibly fail in the four-dimensional manifold of general relativity, it may well be restored in higher dimen-

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sions due to the extra physical effects arising from our transition itself to a higher-dimensional space-time continuum. Such considerations would inspire a study of gravitational collapse in higher-dimensional space-times. From such a perspective, many works have reported results in recent years on spherically symmetric collapse in HD. The recent revival of interest in this problem is partly motivated by the Randall-Sundrum brane-world scenario [13] and different “sectors” of the general problem have been studied by restricting to certain subcases such as those in self-similar space-times, or space-times with a specific number of dimensions [14–22].

It follows that a general investigation of gravitational collapse in a space-time with an arbitrary number of higher dimensions will be of considerable interest. From the perspective of CCC, the effects the number of dimensions may have on the final outcome of gravitational collapse in terms of BH/NS phases will be of much interest.

From such a perspective, we investigate here the issue of BH/NS end-state formations in a higher-dimensional gravitational collapse in some detail, in order to bring out in a transparent manner the effect of dimensions on the final fate of evolution of a matter cloud which collapses from a given regular matter initial data. A spherically symmetric collapse is considered here in  $N \geq 3$  dimensions and the matter content is chosen to be of type I, which obeys the weak energy condition. The collapsing matter field we consider here is a general and broad class, which includes most of the physically reasonable matter fields such as dust, perfect fluids, massless scalar fields and such others and restrictions of any special form are not imposed on the form of matter.

We consider spherically symmetric space-times here, however, when compared to some of the earlier studies of collapse in higher dimensions mentioned above, we deal here with the general case. That is, there is no further restriction imposed on the number of dimensions, the space-time is not assumed to obey various special conditions such as self-similarity or assuming the existence of various Killing fields, which are somewhat restrictive assumptions for collapse models. The results here also generalize earlier results [23–27], related to gravitational collapse final states, and provide a unified treatment.

In order to investigate collapse final states, given the initial data for matter in terms of the initial density and pressure profiles at an initial surface  $t = t_i$  from which the collapse develops, we construct classes of solutions to Einstein equations, such that the space-time evolution goes to a final state which is either a black hole or naked singularity, depending on the nature of the rest of the free initial data functions and possible evolutions, subject to validity of the weak energy condition. The methodology used here allows us to consider in some detail the genericity and stability aspects related to the occurrence of naked singularities in gravitational collapse.

In Sec. II, we describe the basic equations and also the various regularity conditions for gravitational collapse. Section III then considers spherical collapsing clouds and their basic dynamics. The apparent horizon and structure of trapped surface formation is discussed in Sec. IV, which has a direct bearing on the nature of singularity in terms of being either visible or covered. The issues related to equation of state and that of validity of energy conditions in our consideration are discussed in Sec. V, and we also discuss briefly collapse in lower space-time dimensions in Sec. IV. The exterior space-time and matching conditions are given in Sec. VII and the final section summarizes some concluding remarks.

## II. EINSTEIN EQUATIONS, REGULARITY, AND ENERGY CONDITIONS

Let us consider a general  $N$ -dimensional spherically symmetric metric of the form

$$ds^2 = -g_{ab}(x^0, x^1)dx^a dx^b + R^2(x^0, x^1)d\Omega_{N-2}^2, \quad (1)$$

where  $a, b$  run from 0 to 1, and

$$d\Omega_{N-2}^2 = \sum_{i=1}^{N-2} \left[ \prod_{j=1}^{i-1} \sin^2(\theta^j) \right] (d\theta^i)^2 \quad (2)$$

is the metric on  $(N-2)$  sphere with  $\theta^i$  being the spherical coordinates. From this metric, we get the elements of Einstein tensor as [18,19],

$${}^N G_{ab} = \frac{N-2}{2R^2} [g_{ab}\{2R\Box R + (N-3)Q\} - 2RR_{,ab}], \quad (3)$$

$${}^N G_{22} = -\frac{1}{2}[R^2\mathfrak{R} + (N-3)\{2R\Box R + (N-4)Q\}], \quad (4)$$

$${}^N G_{ii}(i > 2) = \left[ \prod_{k=1}^{i-2} \sin^2(\theta^k) \right] {}^N G_{22}, \quad (5)$$

where we have

$$Q = 1 + R^a R_{,a}; \quad \Box R = g^{ab} R_{,ab} \quad (6)$$

and

$$\mathfrak{R} = g^{ab} \mathfrak{R}_{ab}. \quad (7)$$

Here  $\mathfrak{R}_{ab}$  is the Ricci tensor evaluated by the two-metric  $g_{ab}$ , and  $\mathfrak{R}$  is the scalar curvature evaluated by the same. Also, throughout this paper we use the usual convention, that is, a comma (,) in subscript denotes partial differentiation while a semicolon (;) denotes covariant differentiation.

Let us now describe the space-time geometry within the spherically symmetric collapsing cloud by the comoving coordinates  $(t, r, \theta^i)$ , which are specified as below. We take the matter field to be of type I, which is a broad class including most of the physically reasonable matter forms,

including dust, perfect fluids, massless scalar fields, and such others. This is specified by the requirement that the energy-momentum tensor for the matter admits one timelike and  $(N - 1)$  spacelike eigenvectors [28]. We now choose our coordinates  $(t, r, \theta^i)$  to be along these eigenvectors, which makes the coordinate system to be comoving, that is, the coordinate system moves with the matter. We can use the freedom of coordinate transformations of the form  $t' = f(t, r)$  and  $r' = g(t, r)$  to make the  $g_{tr}$  term in metric (1) and the radial velocity of the matter to vanish. In that case the general metric in the comoving coordinates  $(t, r, \theta^i)$  must have three arbitrary functions of  $t$  and  $r$  and this can be written in the form [29],

$$ds^2 = -e^{2\nu(t,r)} dt^2 + e^{2\psi(t,r)} dr^2 + R^2(t, r) d\Omega_{N-2}^2 \quad (8)$$

In this comoving frame the energy-momentum tensor for any matter field which is type I is given in a diagonal form,

$$T_t^t = -\rho(t, r); \quad T_r^r = p_r(t, r); \quad T_{\theta^i}^{\theta^i} = p_\theta(t, r). \quad (9)$$

The quantities  $\rho$ ,  $p_r$ , and  $p_\theta$  are, respectively, the energy density, and radial and tangential pressures, ascribed to the matter field. The matter cloud has a compact support with  $0 < r < r_b$ , where  $r_b$  denotes the boundary of the cloud, outside which it is to be suitably matched through suitable junction conditions with another space-time geometry.

We take the matter field to satisfy the weak energy condition, that is, the energy density measured by any local timelike observer is non-negative. This ensures the physical reasonability for the collapsing matter fields we are considering. Another energy condition frequently used is the dominant energy condition, which demands that for any timelike observer the local energy flow is nonspacelike. We note that these two are frequently regarded as the main and important energy conditions which are physically reasonable [3]. For these energy conditions to be satisfied, we must have for any timelike vector  $V^i$ ,

$$T_{ik} V^i V^k \geq 0 \quad (10)$$

and  $T^{ik} V_k$  nonspacelike. For the energy-momentum tensor (9), these amount, respectively, to the conditions

$$\rho \geq 0; \quad \rho + p_r \geq 0; \quad \rho + p_\theta \geq 0; \quad (11)$$

$$|p_r| \leq \rho; \quad |p_\theta| \leq \rho. \quad (12)$$

Now with the above metric (8), the following quantities can be evaluated:

$$R^a R_{,a} = -\dot{R}^2 e^{-2\nu} + R'^2 e^{-2\psi}, \quad (13)$$

$$\square R = -e^{-2\nu} [\ddot{R} + \dot{R}(-\dot{\nu} + \dot{\psi})] + e^{-2\psi} [R'' + R'(\nu' - \psi')], \quad (14)$$

and

$$R_{,ab} = \left( \ddot{R} - \dot{\nu} \dot{R} - \frac{e^{2\nu} \nu' R'}{e^{2\psi}} \right) \delta_a^0 \delta_b^0 + \left( R'' - \psi' R' - \frac{e^{2\psi} \dot{\psi} \dot{R}}{e^{2\nu}} \right) \delta_a^1 \delta_b^1 + \mathcal{Q}_{ab}, \quad (15)$$

where we have

$$\mathcal{Q}_{ab} = (\dot{R}' - \nu' \dot{R} - \dot{\psi} R') (\delta_a^0 \delta_b^1 + \delta_a^1 \delta_b^0). \quad (16)$$

Using these quantities the Einstein equations  $G_{ik} = T_{ik}$  take the form (in the units  $8\pi G = c = 1$ ),

$$\rho = \frac{(N-2)F'}{2R^{N-2}R'}; \quad p_r = -\frac{(N-2)\dot{F}}{2R^{N-2}\dot{R}}, \quad (17)$$

$$\nu' = \frac{(N-2)(p_\theta - p_r)}{\rho + p_r} \frac{R'}{R} - \frac{p_r'}{\rho + p_r}, \quad (18)$$

$$-2\dot{R}' + R' \frac{\dot{G}}{G} + \dot{R} \frac{H'}{H} = 0, \quad (19)$$

$$G - H = 1 - \frac{F}{R^{N-3}}, \quad (20)$$

where we have defined

$$G(t, r) = e^{-2\psi} (R')^2; \quad H(t, r) = e^{-2\nu} (\dot{R})^2. \quad (21)$$

The function  $F = F(t, r)$  is known as the Misner Sharp mass, which gives the total mass in a shell of comoving radius  $r$ , at an epoch  $t$ . The energy condition  $\rho \geq 0$  implies  $F \geq 0$  and  $F' \geq 0$ . Since the area radius vanishes at the center of the cloud, from Eq. (17) it is evident that in order to preserve the regularity of density and pressures at any nonsingular epoch  $t$ , we must have  $F(t, 0) = 0$ , that is the mass function should vanish at the center of the cloud.

As seen from Eq. (17), there is a density singularity in the space-time at  $R = 0$ , and at  $R' = 0$ . However, the later ones are due to shell crossings [30], which basically indicates the breakdown of the coordinate system we have used. These are not generally regarded as genuine singularities, which can be possibly removed from the space-time to extend the manifold through the same [31]. Hence we shall consider here only the shell-focusing singularity at  $R = 0$ , which is a genuine physical singularity where all matter shells collapse to a zero physical radius. We shall discuss this in some more detail in the next section.

We note that, in general, for a general matter field with nonvanishing pressures as we consider here, there are a variety of dynamical time evolutions possible from the given matter density and pressure profiles as prescribed on an initial surface (which we call here matter initial data), from which the collapse evolves. In particular, even if the cloud commences gravitational collapse at the initial surface  $t = t_i$ , there can be classes of solutions of Einstein equations where the evolution is such that a bounce is possible at a later stage for the cloud. We con-

sider here only a continually collapsing class of models, because our interest is in the physical situation which corresponds to the case when the mass of the star is so high that on exhausting its nuclear fuel, it must undergo a continual gravitational collapse, completing the same in a finite time. Thus the continual collapse condition is included as a part of our work here. In this case, trapped surfaces develop and a space-time singularity necessarily forms as collapse end state and we need to find the conditions when the final singularity is necessarily covered within an event horizon (as hypothesized by the cosmic censorship), or when it will be naked with strong gravity regions being visible to faraway observers. In the case of a bounce or dispersal, no singularity of course needs to form in the space-time, a situation with which we do not concern ourselves here.

We can use the scaling freedom available for the radial coordinate  $r$  to write  $R = r$  at the initial epoch  $t = t_i$ . It is interesting to note that, if we had wished to scale the radial coordinate at the initial epoch as  $R(t_i, r) = r^\beta$ ,  $\beta$  being any constant, then the only possible allowed value for  $\beta$  is unity. Because in the case otherwise, either  $R'$  would blow up at the center (which is not allowed as the Einstein's equations would require the metric functions to be at least  $C^2$ ), or  $R'$  would go to zero at the center causing a shell-crossing singularity (which we would like to avoid, by construction), which violates the regularity of the initial data.

We now introduce a function  $v(t, r)$  as defined by

$$v(t, r) \equiv R/r. \quad (22)$$

We then have  $R(t, r) = rv(t, r)$ , and

$$v(t_i, r) = 1; \quad v[t_s(r), r] = 0; \quad \dot{v} < 0. \quad (23)$$

The time  $t = t_s(r)$ , that is  $v = 0$ , corresponds to the shell-focusing singularity at  $R = 0$ , which is a genuine space-time singularity where all the matter shells collapse to a vanishing physical radius. The condition  $\dot{v} < 0$  here corresponds to a continual collapse of the cloud. The description of shell-focusing singularity at  $R = 0$  in terms of the function  $v(t, r)$  has several advantages. The physical radius goes to the zero value at the shell-focusing singularity, but we also have  $R = 0$  at the regular center of the cloud at  $r = 0$ . This is to be distinguished from the genuine singularity at the collapse end state by the fact, for example, that the density and other physical quantities including the curvature scalars all remain finite at the regular center  $r = 0$ , even though  $R = 0$  holds there. This is achieved, as we point out below, by a suitable behavior of the mass function, which should go to a vanishing value sufficiently fast in the limit of approach to the regular center where (even though  $R$  goes to zero) the density must remain finite. On the other hand, when we use the function  $v(t, r)$ , we note that at  $t = t_i$  we have  $v = 1$  on the entire initial surface, and then as the collapse evolves, the function  $v$  continu-

ously decreases to become zero only at the singularity  $t_s(r)$ , that is,  $v = 0$  uniquely corresponds to the genuine space-time singularity at  $R = 0$ .

From the point of view of dynamic evolution of the initial data prescribed at the initial epoch  $t = t_i$ , there are five arbitrary functions of the comoving shell-radius  $r$  [26], as given by

$$\begin{aligned} v(t_i, r) &= v_0(r), & \psi(t_i, r) &= \psi_0(r), & R(t_i, r) &= r, \\ \rho(t_i, r) &= \rho_0(r), & p_r(t_i, r) &= p_{r_0}(r), & p_\theta(t_i, r) &= p_{\theta_0}(r). \end{aligned} \quad (24)$$

We note that not all the initial data above are mutually independent, because from Eq. (18) we get,

$$v_0(r) = \int_0^r \left( \frac{(N-2)(p_{\theta_0} - p_{r_0})}{r(\rho_0 + p_{r_0})} - \frac{p'_{r_0}}{\rho_0 + p_{r_0}} \right) dr. \quad (25)$$

Thus, apart from the matter initial data describing the initial density and pressure profiles, the rest of the initial data which is free is  $\psi_0(r)$ , which essentially describes the velocities of the collapsing matter shells as we shall discuss later.

To ensure regularity of the initial data, the initial pressures must be taken to have physically reasonable behavior at the center. Considering that the total force at the center of the collapsing cloud should be zero, we have the gradients of initial pressures vanishing at the center. Also, regularity of the initial data requires that we must have,  $p_{r_0}(0) - p_{\theta_0}(0) = 0$ , that is, at the center the difference between the radial and tangential pressure vanishes. The metric functions have to be  $C^2$  differentiable everywhere as per the requirements of the Einstein equations, and as seen from the equation for  $v'$ , the above condition is implied by the requirement that  $v'$  does not blow at the regular center. This means that the matter should behave like a perfect fluid at the center of the cloud with the net force vanishing there. We note that these regularity conditions do not exclude the purely tangential pressure ( $p_r = 0$ ) or purely radial pressure collapse models ( $p_\theta = 0$ ) which we shall refer to later, because in those cases the above implies that  $p_\theta \rightarrow 0$  or  $p_r \rightarrow 0$ , respectively, close to the center, where the matter then closely approximates dust. We note that these regularity conditions give us a sufficient condition for the regularity of the metric function  $v_0(r)$  at any nonsingular initial epoch. It follows from Eq. (25) that at the center of the cloud both  $v_0$  and  $v'_0$  go to zero. Hence  $v_0(r)$  has the form

$$v_0(r) = r^2 g(r), \quad (26)$$

where,  $g(r)$  is an arbitrary function which is at least  $C^1$  for  $r = 0$ , and it is at least a  $C^2$  function for  $r > 0$ , as Einstein equations demand the metric functions to be at least  $C^2$  everywhere. Another regularity condition frequently used in collapse considerations is that there are no trapped



surfaces at the initial surface from which the collapse begins.

We thus see that there are five total field equations with seven unknowns,  $\rho$ ,  $p_r$ ,  $p_\theta$ ,  $\psi$ ,  $\nu$ ,  $R$ , and  $F$ , giving us the freedom of choice of two free functions. Selection of these free functions, subject to the weak energy condition and the given regular initial data for collapse at the initial surface, determines the matter distribution and metric of the space-time, and thus leads to a particular time evolution of the initial matter and velocity distributions. As we shall show, it turns out that given the matter initial profiles in terms of  $\rho_0$ ,  $p_{r_0}$ , and  $p_{\theta_0}$ , there exist the rest of the initial data at  $t = t_i$ , and classes of solutions, which we find by means of explicit construction, which give either a black hole or a naked singularity as the end state of collapse. The outcome depends on the nature of rest of the initial functions, and the classes of dynamical evolutions as allowed by the Einstein equations.

An important point to be noted here is that in the description above we have made no mention so far of the equation of state that the matter must obey. Typically, these are of the form  $p_r = p_r(\rho)$  and  $p_\theta = p_\theta(\rho)$ . If these are specified, then there is no freedom left, and we have seven equations for seven variables. If we are to incorporate this right away, the only way to proceed to find the collapse end state would be to assume a specific equation of state that the matter must satisfy, and then to examine the collapse problem and the nature of the final singularity as resulting from the dynamical evolution as governed by the Einstein equations. There have been many collapse studies in the past using such an approach, e.g. for dust equation of state, perfect fluids, etc. The limitation of such an approach, however, has been that there is very little existing knowledge on what a realistic equation of state should be that the matter has to satisfy at the extreme high densities that a continual collapse realizes in its advanced stages. For example, even for neutron star densities which are relatively low as compared to those of continual collapse, there is a great deal of uncertainty on the equation of state for such neutron matter. As a result, the neutron star mass limits are uncertain to that extent. Thus, specific or special assumptions used on the equation of state may turn out to be physically unrealistic or restrictive and untenable in the final stages of collapse. In fact, diametrically opposite views exist on the possible equation of state in very late stages of collapse. For example, while there are many arguments suggesting that pressures must play an important role in the later stages of collapse, the opposite view is that in such late stages the matter must necessarily be dustlike (see e.g. [32,33]).

Under the situation, the path we take here is that we do not assume any specific or particular equation of state presently, and carry out our further considerations in a general way in terms of the allowed initial matter profiles, and the allowed dynamical evolutions of the Einstein

equations, to determine the black hole and naked singularity end states for collapse. We then discuss subsequently, in Sec. V, the role that the equation of state will play towards further fine tuning the BH/NS outcomes as collapse end states. The advantage such an approach has is, first we write all the collapse equations in generality, and then only different subcases can be examined depending on the corresponding equation of state under consideration. As we shall show, various important subcases such as dust, perfect fluids, and others can be included as special cases of the treatment given here.

In this paper our basic strategy is as described below. We actually do not choose any explicit form of  $F(t, r)$ ; it is allowed to be a completely general class, subject to regularity conditions. Until Sec. V, we do not make any special choices of the two free functions, that we refer to, but deal with and construct certain general classes, as determined by the differentiability conditions on the concerned functions [e.g. as specified by the Eq. (45) onwards] so that basically the singularity curve (which is defined as the time taken for a shell labeled “ $r$ ” to reach the singularity) is expandable up to at least first order, i.e. the singularity curve is regular enough so as to have a well-defined tangent, for these classes of evolutions. Then we show that these classes contain both BH and NS final states, as decided by the sign of the tangent of singularity curve.

In other words, given the matter initial data, we construct the rest of the functions so that in the continual collapse ( $\dot{R} < 0$ ), the final singularity curve is regular (expandable) as above with a well-defined tangent. With that we then show, when the final singularity is visible, or covered in a black hole. The key point is, once we have the matter initial data, we show the existence of classes of the rest of the initial data and evolutions which are solutions to the Einstein equations, by explicit construction as specified here, so that the final state is either a BH or NS depending on the choice made of the rest of the initial data and evolutions. This is subject to energy conditions and other regularity conditions. We are of course not dealing with all possible classes of evolutions from the given matter initial data, which is not our purpose, but we just show the existence of the classes that lead the collapse to above final states.

For a given matter initial data, once we have chosen the rest of the initial data, and the evolutions as solutions to the Einstein equations, that clearly fixes the equation of state for the matter. That can be quite “exotic” at times, depending on the choices made. However, as shown explicitly in Sec. V, the construction given here does include well-known equations of state, such as dust, perfect fluids, etc.

### III. GRAVITATIONAL COLLAPSE OF MATTER CLOUDS

The method we follow is outlined as below. In the case of a black hole developing as collapse end state, the space-

time singularity is necessarily hidden behind the event horizon of gravity, whereas in the case of a naked singularity developing there are families of future-directed non-spacelike trajectories which terminate in the past at the singularity, which can in principle communicate information to faraway observers in the space-time. The existence of such families confirms the NS formation, as opposed to a BH collapse end state. We study the singularity curve produced by the collapsing matter, and it is shown that the tangent to the same at the central singularity at  $r = 0$  is related to the radially outgoing null geodesics from the singularity, if there are any. By determining the nature of the singularity curve and its relation to the initial data and the classes of collapse evolutions, we are able to deduce whether the trapped surface formation in collapse takes place before or after the singularity. It is this causal structure of the trapped region forming during collapse that determines the possible emergence or otherwise of non-spacelike curves from the singularity. This settles the final outcome of collapse in terms of either a BH or NS.

Given the matter initial profiles in terms of the functions  $\rho_0(r)$ ,  $p_{r_o}(r)$ ,  $p_{\theta 0}(r)$  at the initial epoch  $t = t_i$  from which the collapse commences, our purpose now is to construct and examine possible evolutions (classes of solutions to Einstein equations) of such a matter cloud to investigate its final states.

While constructing the classes of solutions which give the collapse evolutions, given the matter initial data at  $t = t_i$ , we preserve as much generality as possible. Hence we allow the mass function  $F(t, r)$  for the collapsing cloud to have a general form as given by

$$F(t, r) = r^{N-1} \mathcal{M}(r, v), \quad (27)$$

where  $\mathcal{M} > 0$  is at least a  $C^1$  function of “ $r$ ” for  $r = 0$ , and at least a  $C^2$  function for  $r > 0$ .

It is to be noted that  $F$  must have this general form that follows from the regularity and finiteness of the density profile at the initial epoch  $t = t_i$ , and at all other later regular epochs before the cloud collapses to the final singularity at  $R = 0$ . This requires, from the Einstein equation (17), that  $F$  must behave as  $r^{(N-1)}$  close to the regular center. Hence we note that since  $\mathcal{M}$  is a general (at least  $C^2$ ) function, the Eq. (27) is not really any ansatz or a special choice, but quite a generic class of the mass profiles for the collapsing cloud, consistent with and allowed by the regularity conditions. We thus make no special choice of  $F$  but allow it to be a general function as given by the above equation. Then Eq. (17) gives

$$\rho(r, v) = \frac{N-2}{2} \left[ \frac{(N-1)\mathcal{M} + r[\mathcal{M}_{,r} + \mathcal{M}_{,v}v']}{v^{N-2}(v + rv')} \right] \quad (28)$$

and

$$p_r(r, v) = -\frac{(N-2)}{2} \left[ \frac{\mathcal{M}_{,v}}{v^{N-2}} \right]. \quad (29)$$

The regular density distribution at the initial epoch is given by

$$\rho_0(r) = \frac{N-2}{2} [(N-1)\mathcal{M}(r, 1) + r\mathcal{M}(r, 1)_{,r}]. \quad (30)$$

It is evident that, in general, as  $v \rightarrow 0$ ,  $\rho \rightarrow \infty$ , and  $p_r \rightarrow \infty$ . That is, both the density and radial pressure blow up at the shell-focusing singularity.

It is seen that given any regular initial density and pressure profiles for the matter cloud from which the collapse develops, there always exist energy profiles or velocity functions for the collapsing matter shells, and classes of dynamical evolutions as determined by the Einstein equations, so that the collapse end state would be either a naked singularity or a black hole, depending on nature of the allowed choice. Thus, given the matter initial data at the initial surface  $t = t_i$ , these evolutions take the collapse to end either as a BH or NS depending on the choice of the class, subject to regularity and energy conditions.

To see this, we need to construct classes of solutions to Einstein equations to this effect. Let us define a suitably differentiable function  $A(r, v)$  as follows:

$$v'(r, v) = A(r, v)_{,v} R'. \quad (31)$$

That is,  $A(r, v)_{,v} \equiv v'/R'$ , and since at  $t = t_i$  we have  $R = r$  which gives  $[A(r, v)_{,v}]_{v=1} = v'_0(r)$ . Our main interest here is in studying the shell-focusing singularity at  $R = 0$  which is the physical singularity where all the matter shells collapse to zero radius. Therefore we assume that there are no shell-crossing singularities in the space-time where  $R' = 0$ , and that the function  $A(r, v)$  is well defined. From Eq. (26), we generalize and choose the form of  $v(t, r)$  as the class given by

$$v(t, r) = r^2 g_1(r, v), \quad (32)$$

where  $g_1(r, v)$  is a suitably differentiable function and  $g_1(r, 1) = g(r)$ . It then follows that  $A(r, v)$  has the form

$$A(r, v) = r g_2(r, v). \quad (33)$$

From the regularity conditions and that the total force at the center of the collapsing cloud should be zero at any nonsingular epoch, it is evident that both  $g_1(r, v)$  and  $g_2(r, v)$  should be well defined at  $r = 0$  and  $v \neq 0$ . In fact in some of the models discussed in Sec. V, these functions are well behaved even at the singularity. In general, it is possible that these functions may in fact blow up at the singularity in some cases. But, as we prove later, even if these functions do blow up, the singularity curve can still be well defined and expandable.

Some comments are in order on our assumption that  $R' > 0$ , that is, we have considered here the situation with no shell-crossing singularities. This is because, it is generally believed (see e.g. [30,31]) that such singularities can be possibly removed from the space-time as they are

typically gravitationally weak, and because space-time extensions have been constructed through the same in certain cases [34]. In contrast, in several physically reasonable collapse models including dust and perfect fluids  $R = 0$  turns out to be a gravitationally strong curvature singularity. Under the situation, we are interested only in examining the nature of the shell-focusing singularities at  $R = 0$ , which are genuine curvature singularities arising at the termination of collapse, where the physical radii for all collapsing shells vanish, and the space-time necessarily terminates without extension.

Specifically,  $R' > 0$  implies that we must have  $v + rv' > 0$ . Since  $v$  is necessarily positive throughout the collapse, it follows that this will be satisfied always whenever  $v'$  is greater than or equal to zero. Even when it is negative, the condition that the magnitude of  $rv'$  should be less than that of  $v$  is sufficient to ensure that there will be no shell crosses. Later in this section we derive an expression for the quantity  $v'$ , in terms of the initial data and the other free evolutions as allowed by the Einstein equations. Hence it follows that we can specifically state the condition for avoidance of shell crossings in terms of the behavior of these functions.

Coming to the dynamical collapse evolutions, using Eq. (31) in Eq. (19) we get, as a class of solutions of Einstein's equations,

$$G(r, v) = b(r)e^{2rA(r, v)}. \quad (34)$$

Here  $b(r)$  is another arbitrary function of the shell radius  $r$ . By the regularity condition on the function  $\dot{v}$  at the center of the cloud we get the form of  $b(r)$  as

$$b(r) = 1 + r^2 b_0(r), \quad (35)$$

where  $b_0(r)$  is the energy distribution function for the shells. Using Eq. (31) in Eq. (18), we get,

$$(N - 2)p_\theta = RA_{,v}(\rho + p_r) + (N - 2)p_r + \frac{Rp'_r}{R'}. \quad (36)$$

In general, both the density and radial pressure blow up at the singularity, so the above equation implies that the tangential pressure would also typically blow up at the singularity. Now using Eqs. (27), (31), and (34) in Eq. (20), we get

$$R^{(N-3)/2} \dot{R} = -e^{\nu(r, v)} \sqrt{(1 + r^2 b_0) R^{(N-3)} e^{2rA(r, v)} - R^{(N-3)} + r^{(N-1)} \mathcal{M}}. \quad (37)$$

The negative sign in the right-hand side of the above equation corresponds to a collapse scenario where we have  $\dot{R} < 0$ . Defining a function  $h(r, v)$  as

$$h(r, v) = \frac{e^{2rA(r, v)} - 1}{r^2} = 2g_2(r, v) + \mathcal{O}(r^2), \quad (38)$$

Eq. (20) becomes

$$v^{(N-3)/2} \dot{v} = -\sqrt{e^{(rA+\nu)} v^{(N-3)} b_0 + e^{2\nu} (v^{(N-3)} h + \mathcal{M})}. \quad (39)$$

Integrating the above equation with respect to  $v$ , we get

$$t(v, r) = \int_v^1 \frac{v^{(N-3)/2} dv}{\sqrt{e^{(rA+\nu)} v^{(N-3)} b_0 + e^{2\nu} (v^{(N-3)} h + \mathcal{M})}}. \quad (40)$$

Note that the variable  $r$  is treated as a constant in the above equation. The above equation gives the time taken for a shell labeled  $r$  to reach a particular epoch  $v$  from the initial epoch  $v = 1$ . Expanding  $t(v, r)$  around the center of the cloud, we get

$$t(v, r) = t(v, 0) + r\mathcal{X}(v) + \mathcal{O}(r^2), \quad (41)$$

where the function  $\mathcal{X}(v)$  is given as

$$\mathcal{X}(v) = -\frac{1}{2} \int_v^1 dv \times \frac{v^{(N-3)/2} [v^{(N-3)} b_1 + h_1 v^{(N-3)} + \mathcal{M}_1(v)]}{[v^{(N-3)} b_{00} + v^{(N-3)} h_0 + \mathcal{M}_0(v)]^{3/2}}, \quad (42)$$

wherein we have defined

$$\begin{aligned} b_{00} &= b_0(0); & \mathcal{M}_0(v) &= \mathcal{M}(0, v); & h_0 &= h(0, v) \\ b_1 &= b'_0(0); & \mathcal{M}_1(v) &= \mathcal{M}_{,r}(0, v); & h_1 &= h_{,r}(0, v). \end{aligned} \quad (43)$$

Hence we see that the time taken for a shell labeled  $r$  to reach the space-time singularity at  $R = 0$  (which is the singularity curve) is given as

$$t_s(r) = \int_0^1 \frac{v^{(N-3)/2} dv}{\sqrt{e^{(rA+\nu)} v^{(N-3)} b_0 + e^{2\nu} (v^{(N-3)} h + \mathcal{M})}}. \quad (44)$$

As we want to consider here continual collapse, we focus only on those classes of solutions where  $t_s(r)$  is finite and sufficiently regular. This means that the cloud collapses in a finite amount of time. In the physical situation of a continual collapse of a massive matter cloud in a finite amount of time the function  $t_s(r)$  has to be of course finite. As for regularity, to check the existence conditions for a well-defined, continuous, and  $C^2$  singularity curve, let us

define a function  $Q(r, v)$  as

$$Q(r, v) = \frac{v^{(N-3)/2}}{\sqrt{e^{(rA+v)} v^{(N-3)} b_0 + e^{2\nu} (v^{(N-3)} h + \mathcal{M})}}. \quad (45)$$

Also we consider the following functions:

$$\phi_1(r) = \int_0^1 Q(r, v)_{,r} dv; \quad \phi_2(r) = \int_0^1 Q(r, v)_{,rr} dv. \quad (46)$$

Let  $\mathcal{A}$  be the rectangular area in the  $(r, v)$  plane defined by the lines

$$r = 0; \quad r = \epsilon; \quad v = 0; \quad v = 1. \quad (47)$$

Now if the following conditions are satisfied [35],

- (1)  $Q(r, v)$  is a continuous function of  $r$  and  $v$  in  $\mathcal{A}$ .
- (2)  $Q(r, v)_{,r}$  and  $Q(r, v)_{,rr}$  are continuous functions of  $r$  and  $v$  in  $\mathcal{A}$ .
- (3) The integrals  $\phi_1(r)$  and  $\phi_2(r)$  converge uniformly in  $\mathcal{A}$ . Then we can write

$$\phi_1(r) = \frac{d}{dr}[t_s(r)]; \quad \phi_2(r) = \frac{d^2}{dr^2}[t_s(r)], \quad (48)$$

and this implies that the singularity curve  $t_s(r)$  would be a well-defined  $C^2$  function near the center.

We would like to emphasize here that regularity of different functions that makes  $Q(r, v)$ , namely  $\nu$ ,  $A$ , and  $M$ , is a sufficient but by no means necessary condition for the existence of a  $C^2$  singularity curve. As we can easily see, even if these functions blow up at  $r = 0$ ,  $v = 0$ , the function  $Q$  may still be well defined. For example, if we suppose these functions tend to infinity at the singularity the function  $Q$  would tend to zero. For any regular collapse, the function  $t_s(r)$  has to be finite and well defined.

Several well-studied collapse models such as dust collapse and others satisfy these or stronger conditions, and so the singularity curve is well defined and expandable. Some examples of such singularity curves are as follows. In the case of an  $N$ -dimensional dust collapse, the expression of the singularity curve would be [36],

$$t_s(r)_{\text{dust}} = \int_0^1 \frac{v^{(N-3)/2} dv}{\sqrt{\mathcal{M}(r) + v^{(N-3)} b_0(r)}}, \quad (49)$$

where  $\mathcal{M}(r)$  and  $b_0(r)$  are well-defined  $C^2$  functions of the comoving coordinate  $r$  and well defined at  $r = 0$ . It can then be easily seen that the singularity curve is differentiable at the center, with

$$\frac{dt_s(r)}{dr} = -\frac{1}{2} \int_0^1 \frac{v^{(N-3)/2} (\mathcal{M}_1 + v^{(N-3)} b_1) dv}{(\mathcal{M}_0 + v^{(N-3)} b_{00})^{3/2}}, \quad (50)$$

where we have defined

$$\begin{aligned} b_{00} &= b_0(0), & \mathcal{M}_0 &= \mathcal{M}(0), \\ b_1 &= b'(0), & \mathcal{M}_1 &= \mathcal{M}'(0). \end{aligned} \quad (51)$$

Another such example is an  $N$ -dimensional Einstein cluster, which describes a nonsteady spherically symmetrical system of noncolliding particles moving in such a way that relative to a suitably moving frame of coordinates, their motion is purely transversal. In this case, the singularity curve is given by [37]

$$t_s(r)_{Ec} = \int_0^1 \frac{v^{(N-3)/2} \sqrt{v^2 + \frac{L(r)^2}{r^2}} dv}{e^\nu \sqrt{b_0 v^{N-1} - \left(\frac{L(r)^2}{r^4}\right) v^{N-3} + \mathcal{M}(r) \left(v^2 + \frac{L(r)^2}{r^2}\right)}}. \quad (52)$$

Here  $L(r)$  is a function of the radial coordinate  $r$  only. In the Newtonian limit, this function corresponds to the angular momentum per unit mass of the system. Therefore the function  $L(r)$  is called as the specific angular momentum, and has the form

$$L(r) \equiv r^2 l(r). \quad (53)$$

Again, since  $\mathcal{M}(r)$  and  $b_0(r)$  and  $L(r)$  are well defined  $C^2$  functions of the coordinate  $r$ , we see that the above singularity curve is well defined and differentiable at  $r = 0$ .

To generalize this and to give another explicit example, it can be shown that given any initial data of the form (24), there always exist classes of dynamical evolutions, which give rise to a well-defined and differentiable singularity curve. Let us, by the freedom of choice of free functions, choose the evolution functions  $\mathcal{M}(r, v)$  and  $A(r, v)$  in the following way:

$$\begin{aligned} \mathcal{M}(r, v) &= m(r) - p_r(r) v^{N-1}, \\ A(r, v)_{,v} &= \nu_0(x)_{,x}, \quad x = rv. \end{aligned} \quad (54)$$

From Einstein equations it is clear that for the above class of evolutions the radial pressure remains static. However, the tangential pressure blows up along with the density at the singularity  $v = 0$  and is given by

$$2p_\theta(r, v) = p_r + p'_r(R/R') + \nu_0(rv)_{,R} [\rho(r, v) + p_r]. \quad (55)$$

One can now easily check that the above class of evolutions admits a well-defined and differentiable singularity curve because both the functions  $\mathcal{M}(r, v)$  and  $A(r, v)$  are well defined and  $C^2$  at  $r = 0$  and  $v = 0$ .

Once we have a singularity curve which is at least  $C^2$ , we can Taylor expand the function near the center as

$$t_s(r) = t_{s_0} + r\mathcal{X}(0) + \mathcal{O}(r^2), \quad (56)$$

where  $t_{s_0}$  is the time when the central singularity at  $R = 0$ ,  $r = 0$ , develops, and is given as



$$t_{s_0} = \int_0^1 \frac{v^{(N-3)/2} dv}{\sqrt{v^{(N-3)}b_{00} + v^{(N-3)}h_0 + \mathcal{M}_0(v)}}. \quad (57)$$

From the above equation it is clear that for  $t_{s_0}$  to be defined,

$$v^{(N-3)}b_{00} + v^{(N-3)}h_0 + \mathcal{M}_0(v) > 0. \quad (58)$$

In other words, a continual collapse in finite time ensures that the above condition holds. Also, from Eqs. (39) and (41) we get for small values of  $r$ , along constant  $v$  surfaces,

$$v^{(N-3)/2}v' = \sqrt{[v^{(N-3)}b_{00} + v^{(N-3)}h_0 + \mathcal{M}_0(v)]\mathcal{X}(v)} + \mathcal{O}(r) \quad (59)$$

It is now clear that the value of  $\mathcal{X}(0)$  depends on the functions  $b_0$ ,  $\mathcal{M}$ , and  $h$ , which in turn depend on the initial data at  $t = t_i$ , the dynamical variable  $v$ , and the evolution function  $A(r, v)$ . Thus, a given set of initial matter distributions and the dynamical profiles including the energy distribution of shells completely determine the tangent at the center to the singularity curve.

#### IV. APPARENT HORIZON AND THE NATURE OF THE SINGULARITY

It is now possible to examine, given the matter initial data at the initial surface  $t = t_i$ , how the final fate of collapse is determined in terms of either a black hole or a naked singularity. If there are families of future-directed nonspacelike trajectories reaching faraway observers in space-time, which terminate in the past at the singularity, then we have a naked singularity forming as the collapse final state and in the case otherwise when no such families exist and event horizon forms sufficiently earlier than the singularity to cover it, we have a black hole. This is decided by the causal behavior of the trapped surfaces developing in the space-time during the collapse evolution, and the apparent horizon, which is the boundary of trapped surface region in the space-time.

In general, the equation of apparent horizon in a spherically symmetric space-time is given as

$$g^{ik}R_{,i}R_{,k} = 0. \quad (60)$$

Thus we see that at the boundary of the trapped region the vector  $R_{,i}$  is null. Substituting (8) into (60) we get,

$$R'^2 e^{-2\psi} - \dot{R}^2 e^{-2\nu} = 0. \quad (61)$$

Using Eq. (20) we can now write the equation of apparent horizon as

$$\frac{F}{R^{N-3}} = 1, \quad (62)$$

which gives the boundary of the trapped surface region of the space-time. If the neighborhood of the center gets trapped prior to the epoch of singularity, then it is covered

and a black hole results, otherwise it could be naked when nonspacelike future-directed trajectories escape from it.

Thus the important point is to determine if there are any future-directed nonspacelike paths emerging from the singularity. To investigate this, and to examine the nature of the central singularity at  $R = 0$ ,  $r = 0$ , let us consider the equation for outgoing radial null geodesics which is given by

$$\frac{dt}{dr} = e^{\psi-\nu}. \quad (63)$$

We want to examine if there would be any families of future-directed null geodesics coming out of the singularity, thus causing a naked singularity phase as collapse end state. The singularity occurs at  $v[t_s(r), r] = 0$ , i.e.  $R[t_s(r), r] = 0$ . Therefore, if there are any future-directed null geodesics terminating in the past at the singularity, we must have  $R \rightarrow 0$  as  $t \rightarrow t_s$  along these curves. Now writing Eq. (63) in terms of variables ( $u = r^\alpha$ ,  $R$ ), we have

$$\frac{dR}{du} = \frac{1}{\alpha} r^{-(\alpha-1)} R' \left[ 1 + \frac{\dot{R}}{R'} e^{\psi-\nu} \right]. \quad (64)$$

In order to get the expression of the tangent to null geodesics emerging in the  $(R, u)$  plane, we choose a particular value of  $\alpha$  such that the geodesic equation is expressed only in terms of known limits. For example, if  $\mathcal{X}(0) \neq 0$ , and the functions  $\mathcal{M}$  and  $h$  are well defined for  $0 \leq r \leq r_b$  and  $0 \leq v \leq 1$  we choose  $\alpha = \frac{N+1}{N-1}$ . Using Eq. (20) and considering  $\dot{R} < 0$ , we then get the null geodesic equation in the form

$$\frac{dR}{du} = \frac{N-1}{N+1} \left( \frac{R}{u} + \frac{v'v^{(N-3)/2}}{\left(\frac{R}{u}\right)^{(N-3)/2}} \right) \left( \frac{1 - \frac{F}{R^{N-3}}}{\sqrt{G}[\sqrt{G} + \sqrt{H}]} \right). \quad (65)$$

If the null geodesics do terminate at the singularity in the past with a definite tangent, then at the singularity the tangent to the geodesics have  $\frac{dR}{du} > 0$  in the  $(u, R)$  plane, and must have a finite value. In the case of a massive singularity in dimensions greater than or equal to four [i.e.  $F(t_s(r), r) > 0$  for  $r \neq 0$ ], all singularities for  $r > 0$  are covered since  $\frac{F}{R^{N-3}} \rightarrow \infty$  and hence  $\frac{dR}{du} \rightarrow -\infty$ . This is when both the pressures  $p_r$  and  $p_\theta$  are positive with the energy condition being satisfied. Therefore in such a case only the central singularity at  $R = 0$ ,  $r = 0$  could be naked.

Hence we need to examine the central singularity at  $r = 0$ ,  $R = 0$  to determine if it is visible or not and to determine if there are any solutions existing to the outgoing null geodesics equation, which terminate in the past at the singularity, going to faraway observers in future. The conditions are to be determined under which this can happen. We also note that, since the singularity curve and the evolution functions are regular, we can calculate the limit of the functions  $H$ ,  $G$ , and  $F/R$  at  $r \rightarrow 0$ ,  $t \rightarrow t_{s_0}$ . From Eq. (34), as  $A(r, v)$  is a well-defined function, we can see

that  $G(t_{s_0}, 0) = 1$ . Also from Eq. (39) we see that at this point  $H \approx r^2/v^{N-3}$ . Calculating this limit on the  $t = t_{s_0}$  plane from Eq. (59), at the point  $(t_{s_0}, 0)$ , we have  $H = 0$ . Hence we see from Eq. (20) that  $F/R^{N-3} = 0$  in this limit.

Let now  $x_0$  be the tangent to the outgoing null geodesics in the  $(R, u)$  plane, at the central singularity, then it is given by

$$x_0 = \lim_{t \rightarrow t_s} \lim_{r \rightarrow 0} \frac{R}{u} = \left. \frac{dR}{du} \right|_{t=t_s; r \rightarrow 0}. \quad (66)$$

To find out whether the null geodesic equation admits any solution of  $x_0$  which is positive and finite at the central singularity, we can use the values of  $H$ ,  $G$ , and  $F/R$  at  $(t_{s_0}, 0)$  in Eq. (65). Also we use Eq. (59) to get the value of  $v'v^{(N-3)/2}$ , on  $v = 0$  surface at  $r = 0$  [that is, on the point  $(t_{s_0}, 0)$ ]. Thus solving Eq. (65), we get

$$x_0^{(N-1)/2} = \frac{N-1}{2} \sqrt{\mathcal{M}_0(0)\mathcal{X}(0)} \quad (67)$$

and the equation of radial null geodesic emerging from the singularity is given by  $R = x_0 u$  in the  $(R, u)$  plane, or in  $(t, r)$  coordinates it is given by

$$t - t_s(0) = x_0 r^{(N+1)/(N-1)}. \quad (68)$$

It follows now that if  $\mathcal{X}(0) > 0$ , then  $x_0 > 0$ , and we get radially outgoing null geodesics coming out from the singularity, giving rise to a naked central singularity. However, if  $\mathcal{X}(0) < 0$  we have a black hole solution, as there will be no such trajectories coming out. If  $\mathcal{X}(0) = 0$  then we will have to take into account the next higher order nonzero term in the singularity curve equation, and a similar analysis has to be carried out by choosing a different value of  $\alpha$ .

To show that the above is a necessary as well as sufficient condition for an outgoing radial null geodesic emerging from the singularity to exist, let us assume that such geodesics do exist and in the  $(R, u)$  plane, its equation is  $R = x_0 u$  and  $x_0 > 0$ . Then at the central singularity ( $R = 0, u = 0$ ), the tangent to such geodesic must be  $x_0$ . Also this tangent must be the root of the equation,

$$\frac{dR}{du} - \frac{N-1}{N+1} \left( \frac{R}{u} + \frac{v'v^{(N-3)/2}}{\left(\frac{R}{u}\right)^{(N-3)/2}} \right) \left( \frac{1 - \frac{F}{R^{N-3}}}{\sqrt{G}[\sqrt{G} + \sqrt{H}]} \right) = 0 \quad (69)$$

at the point  $(R = 0, u = 0)$ . This is possible if and only if

$$x_0 = \left[ \frac{N-1}{2} \sqrt{\mathcal{M}_0(0)\mathcal{X}(0)} \right]^{(N-1)/2} \quad (70)$$

and for the slope to be defined and positive we must have  $\mathcal{X}(0) \geq 0$ .

Now to see that  $\mathcal{X}(0) > 0$  is a sufficient condition for the existence of an outgoing radial null geodesic emerging from the singularity, let us consider the case that the singularity curve has a positive tangent at the central

singularity. Consider now the curve

$$t - t_s(0) = \left[ \frac{N-1}{2} \sqrt{\mathcal{M}_0(0)\mathcal{X}(0)} \right]^{(N-1)/2} r^{(N+1)/(N-1)}. \quad (71)$$

Along this curve  $t \rightarrow t_{s_0}$  as  $r \rightarrow 0$ . And as we have  $\mathcal{X}(0) > 0$ , this curve is outgoing in the sense that  $t$  increases as we increase  $r$  along the curve. Let us now calculate the quantity  $(-g_{tt}dt^2 + g_{rr}dr^2)$  along this curve in the vicinity of the central singularity. Using (34), (39), and (59), we have for this curve at the point  $(t_{s_0}, 0)$ ,

$$\begin{aligned} -e^{2\nu} dt^2 + e^{-2rA} R^2 dr^2 &= \frac{N+1}{N-1} \left[ \frac{N-1}{2} \sqrt{\mathcal{M}_0(0)\mathcal{X}(0)} \right]^{N-1} \\ &\times r^{2/(N-1)} (-dr^2 + dr^2) = 0. \end{aligned} \quad (72)$$

That is, in the vicinity of the central singularity the curve (71) is null. Thus we see that given any positive value of the tangent to the singularity curve at the central singularity, we can always find a null and outgoing curve terminating in the past at the central singularity, making the singularity naked.

We make below some remarks on the nature of the apparent horizon and its relation with the visibility or otherwise of the singularity. To find the equation of apparent horizon near the central singularity, let the time corresponding to a shell labeled by  $r$  entering the apparent horizon, in term of the variable  $v$ , be  $v_{ah}(r)$ . Then from Eq. (62), we can easily see that  $v_{ah}(r)$  is the root of the equation,

$$r^2 \mathcal{M}(r, v) - v^{(N-3)} = 0. \quad (73)$$

Now using Eq. (40), we get the equation for apparent horizon in the  $(t, r)$  plane as

$$\begin{aligned} t_{ah}(r) &= t_s(r) \\ &- \int_0^{v_{ah}(r)} \frac{v^{(N-3)/2} dv}{\sqrt{e^{(rA+\nu)} v^{(N-3)} b_0 + e^{2\nu} (v^{(N-3)} h + \mathcal{M})}}. \end{aligned} \quad (74)$$

It is obvious that the necessary condition for the existence of a locally naked singularity is that the apparent horizon curve must be an increasing function at the central singularity, in the lowest power of  $r$ .

We note that in the above the functions  $h$  and  $\mathcal{M}$  are expanded with respect to  $r$  around  $r = 0$  and the first-order terms are considered. At times, however, these are assumed to be expandable with respect to  $r^2$ , and it is argued that such smooth functions would be physically more relevant. Such an assumption comes from the analyticity with respect to the local Minkowskian coordinates (see e.g. [38]), and it is really the freedom of definition mathematically. We may remark that the formalism as discussed above

would work for such smooth functions also, which is a special case of the above discussion.

We thus see how the initial data in terms of the free functions available determine the BH/NS phases as collapse end states, because  $\mathcal{X}(0)$  is determined by these initial and dynamical profiles as given by Eq. (42). It is clear, therefore, that given any regular initial density and pressure profiles for the matter cloud from which the collapse develops, we can always choose velocity profiles so that the end state of the collapse would be either a naked singularity or a black hole, and vice versa.

We believe numerical work on collapse models may provide further insights into these interesting dynamical phenomena, and especially when collapse is nonspherical, which remains a major open problem to be considered [39–51]. Numerical and some analytical works have been done in recent years on spherical scalar field collapse [52–60] and also on some perfect fluid models [61–69]. While we have worked out explicitly here the emergence of null geodesics from the singularity, thus showing it to be naked (or otherwise) in an analytic manner, the numerical simulations generally discuss the formation or otherwise of trapped surfaces and apparent horizon, and such considerations may possibly break down closer to the epoch of actual singularity formation. In that case, this may not allow for actual detection of BH/NS end states, whereas important insights on critical phenomena and dispersal have already been gained through numerical methods. Probably, a detailed numerical investigation of the structure of null geodesics in collapse models may provide further insights here.

## V. EQUATION OF STATE AND ENERGY CONDITIONS

As stated above, we work here with type I matter fields, which is a rather general form of matter. However, it is important to note that suitable care must be taken in interpreting these results. While we have shown that the initial data and dynamical evolutions chosen do determine the BH/NS end states for collapse, the point is, actually, all these dynamical variables are not explicitly determined by the initial data given at the initial epoch (note that  $v$  plays the role of a time coordinate here). Hence these functions are fully determined only as a result of time development of the system from the initial data provided we have the relation between the density and pressures, that is a given “equation of state.”

In principle, it is possible to choose these functions freely [e.g. the matter and velocity profiles at the initial epoch and the dynamical evolutions such as  $F(v, r)$  and  $\nu(v, r)$ ], only subject to an energy condition and regularity, which then fully determines the collapse evolution. One can then calculate the energy density, and the radial and tangential pressures for the matter. However, in that case, the resultant “equation of state” could be quite strange in

general. If any equation of state of the form  $p_r = f(\rho)$  and/or  $p_\theta = g(\rho)$  is given, then it is clear from Eqs. (17) and (18) that there would be a constraint on the otherwise arbitrary function  $\mathcal{M}$  and  $A$ , specifying the required class, if the solution of the constraint equation exists. It is certainly true that presently we have practically very little idea on what kind of an equation of state the matter should follow, especially at very high densities and closer to the collapse end states, where we are already dealing with ultrahigh energies and pressures. Hence if we allow for the possibility that we could freely choose the property of the matter fields as above, or the equation of state, then our analysis is certainly valid and gives several useful conclusions on possible collapse end states. In such a case, it is also possible that the chosen equation of state will be in general such that the pressures may explicitly depend not only on the energy density, but also on the time coordinate.

All the same, it is important to point out that the analysis as given above in fact does include several well-known equations of state and useful classes of collapse models, also satisfying the energy conditions throughout the collapse, as we demonstrate below.

### A. Dust collapse

The idealized class of dust collapse models where the pressures are taken to be vanishing has been studied extensively so far and has yielded many important insights on collapse evolutions. In this special case, the Einstein equations can be solved completely to get the  $N$ -dimensional generalization of the usual Tolman-Bondi-Lemaitre (TBL) dust collapse metric [70–72] and it is given as

$$ds^2 = dt^2 - \frac{R'^2}{1 + r^2 b_0(r)} dr^2 - R^2(t, r) d\Omega_{N-2}^2. \quad (75)$$

The equations of motion are given by

$$\frac{(N-2)F'}{2R^{(N-3)}R'} = \rho \quad (76)$$

and

$$\dot{R}^2 = \frac{F(r)}{R^{(N-3)}} + f(r). \quad (77)$$

In the case of dust, the mass function must be  $F = F(r)$ , and hence the regularity condition implies that

$$F(r) = r^{(N-1)} \mathcal{M}(r). \quad (78)$$

The energy condition here gives  $0 < r < r_b$ , and so we must have  $\mathcal{M}(r) \geq 0$  and  $3\mathcal{M} + r\mathcal{M}_r \geq 0$ . In this case, as we have already seen in Sec. III, the function  $\mathcal{X}(v)$  is given as

$$\mathcal{X}(v) = -\frac{1}{2} \int_v^1 \frac{v^{(N-3)/2} (\mathcal{M}_1 + v^{(N-3)} b_1) dv}{(\mathcal{M}_0 + v^{(N-3)} b_{00})^{3/2}}, \quad (79)$$

and the time taken for the central shell to reach the singu-

larity is given by

$$t_{s_0} = \int_0^1 \frac{v^{(N-3)/2} dv}{\sqrt{\mathcal{M}_0 + v^{(N-3)} b_{00}}}. \quad (80)$$

It is now seen clearly that any given sets of density and velocity profiles at the initial epoch completely determine the tangent to the singularity curve at the central singularity. Also Eq. (67) becomes

$$x_0^{(N-1)/2} = \frac{N-1}{2} \sqrt{\mathcal{M}_0} \mathcal{X}(0). \quad (81)$$

It therefore follows that, given any specific density profile of the collapsing dust cloud, we can always choose a velocity profile so that the end state of the collapse would be either a naked singularity or a black hole depending on the choice made, such that energy conditions are satisfied throughout the collapse. The converse also holds, namely, one can choose a given velocity profile for the cloud at the initial epoch, and then there are density profiles which will lead the collapse to either of the BH/NS final states, and these conclusions hold irrespective of the number of dimensions of the space-time. Hence we see that our results unify and generalize the earlier results of dust collapse [73–79]. Basically the point that follows here is that, given an initial density profile for the collapsing cloud, the space of velocity profile functions is divided into the regions that lead the collapse either to a black hole or naked singularity evolution, depending on the choice made, and the converse holds similarly.

### B. Collapse with static radial pressure

While the dust equation of state discussed above is fairly standard and extensively used, it is widely believed that pressures could play an important role in gravitational collapse considerations. We discuss below a class of collapse models with nonzero pressures, which is however idealized in the sense that while the tangential pressure can be arbitrary, the radial pressure is taken to be static.

As we already pointed out in Sec. III, if we consider the classes of collapse in which the radial pressure remains static, the constraint equation for  $\mathcal{M}$  has the following solution:

$$\mathcal{M}(r, v) = m(r) - p_r(r) v^{N-1}. \quad (82)$$

In addition to this if there is an equation of state of the form  $p_\theta = f(\rho)$ , then that gives the constraint equation for the function  $A(r, v)$  in the following way:

$$2f(\rho) = p_r + p'_r(R/R') + A(r, v)_v [\rho(r, v) + p_r]. \quad (83)$$

For this class of models the energy conditions are given by

$$(N-1)[m - p_r v^{N-1}] + r[m_{,r} - p_{r,r} v^{N-1} - (N-1)p_r v^{N-2} v'] \geq 0, \quad (84)$$

$$(N-1)[m - p_r v^{N-1}] + r[m_{,r} - p_{r,r} v^{N-1} - (N-1)p_r v^{N-2} v'] + p_r R^2 R' \geq 0, \quad (85)$$

$$\frac{\rho}{2} + \frac{1}{2} \left[ (\rho + p_r)(A_{,v} + 1) + p'_r \frac{R}{R'} \right] \geq 0. \quad (86)$$

As shown in [80] there exist classes of functions  $m$ ,  $p_r$ , and  $A$  such that naked singularity is the end state for the collapse and also the above three energy conditions are satisfied. It follows that, given initial matter profiles, there exist classes of collapse evolutions satisfying the energy conditions as we see above, such that either of the BH/NS end states can result subject to above equation of state.

### C. Isentropic perfect fluid with a linear equation of state

Perfect fluids have been widely used in astrophysical considerations and a linear equation of state is well studied. We discuss below how the formalism outlined here applies to this case to find the BH/NS configurations as a perfect fluid collapse end states.

For an isentropic perfect fluid, whose pressure is a linear function of the density only, the equation of state of the collapsing matter is given by

$$p_r(t, r) = p_\theta(t, r) = k\rho(t, r), \quad (87)$$

where  $k \in [-1, 1]$  is a constant. The case  $k = 0$  gives the dust case we discussed above and  $k = 1$  is the stiff fluid case. Let us at present consider only the case of positive pressures. In that case  $k > 0$  and the energy conditions give

$$M_{,v} < 0. \quad (88)$$

From the above equation of state and the Einstein equations we can immediately see that the function  $\mathcal{M}$  is now the solution of the equation

$$(N-1)k\mathcal{M} + kr\mathcal{M}_{,r} + Q(r, v)\mathcal{M}_{,v} = 0, \quad (89)$$

where

$$Q(r, v) = (k+1)rv' + v. \quad (90)$$

Now the above equation (89) has a general solution of the form [81],

$$\mathcal{F}(X, Y) = 0, \quad (91)$$

where  $X(r, v, \mathcal{M})$  and  $Y(r, v, \mathcal{M})$  are the solutions of the system of equations,

$$-\frac{d\mathcal{M}}{(N-1)k\mathcal{M}} = \frac{dr}{kr} = \frac{dv}{Q}. \quad (92)$$

Thus we can easily see that Eq. (89) admits classes of solutions when  $v' > 0$ . Also solving the equation for the central shell  $r = 0$ , with boundary conditions  $\rho \rightarrow \infty$  as  $v \rightarrow 0$ , we get

$$\mathcal{M}(0, v) = \frac{m_0}{v^{(N-1)k}}. \quad (93)$$



By choosing  $m_0 > 0$ , we can make the central shell to satisfy the energy condition  $\rho(t, 0) > 0$  for all epochs. Then by the continuity of the density function, we can say that there exists an  $\epsilon$  ball around the central shell for which  $v'(t, r) > 0$  and also  $\rho(t, r) > 0$ . But as is known from (59), at the central singularity  $\sqrt{v}v' \approx \mathcal{X}(0)$ , hence this implies that we can have classes of solutions which satisfy the energy conditions and also admit a naked singularity as the collapse end state.

For further discussion on perfect fluid collapse, and the details of black hole and naked singularity formation, we refer to [61–69], and references therein.

It is seen from the above that several well-known classes of collapse models form subcases of the consideration given here. Along with these well-known models, the above analysis would work for any other models with other equations of state, if that permits solutions to the constraint equations on  $\mathcal{M}$  and  $A$ . Hence it follows that the considerations above provide an interesting framework for the study of dynamical collapse, which is one of the most important open problems in gravity physics today.

## VI. THE CASE OF 2 + 1 DIMENSIONAL COLLAPSE

Several studies on gravitational collapse scenarios in  $(2 + 1)$  dimensional space-times, have been carried out by various authors [82–85]. These provide interesting toy models which may provide quite important insights from the perspective of quantum gravity. This is because in  $(2 + 1)$  dimensional space-times there is no gravity outside matter. Also the space-time metric is always conformally flat as the Weyl tensor vanishes identically everywhere. The situation in  $(3 + 1)$  and higher dimensions is far more complicated as compared to this.

To investigate the final outcome in  $(2 + 1)$  dimensional collapse, let us consider the geometry of the trapped surfaces in this case in some detail. From Eq. (62) we see that the equation of apparent horizon in this case is given by

$$F(t, r) = 1. \quad (94)$$

It is interesting to note that the geometry of trapped surfaces here is completely determined by the mass function of the cloud, and is independent of the area radius of the collapsing shells. If the mass function of the collapsing configuration is bounded from above, say, for example, with  $F(t, r) < 1$  for  $t \in [-\infty, t_s(r)]$ , and  $r \in [0, r_b]$  (where  $r_b$  is the boundary of the collapsing cloud), we then see that the trapping does not occur and the complete singularity that forms as collapse end state is necessarily visible to an outside observer. This is strikingly different from four or higher-dimensional cases where a massive singularity is always trapped.

An interesting subcase of this situation is that of  $(2 + 1)$  dimensional dust collapse, when  $F(t, r) = F(r)$  with the mass function having no time dependence. Here we see

that the initial mass of the collapsing cloud completely determines the final outcome in terms of BH or NS. Clouds with small enough mass always form a visible singularity, whereas for larger masses a trapped region is present at all epochs. However, as demanded by the regularity conditions, we must avoid trapped surfaces on the initial surface  $t = t_i$  where the collapse commences. In that case, for the dust collapse case, there are no trapped surfaces developing at all at any other later epochs until the singularity formation, and as a result the  $(2 + 1)$  dust collapse always necessarily produces a visible naked singularity, as opposed to BH/NS phases obtained in usual four-dimensional dust collapse which we discussed earlier.

## VII. EXTERIOR SPACE-TIME AND MATCHING CONDITIONS

To complete a collapse model, we need to match the interior space-time to a suitable exterior space-time. As we are interested here in modeling collapse of astrophysical objects (such as massive stars), we have assumed the matter to have compact support at the initial surface, with the boundary of the cloud being at some  $r = r_b$ . If one assumes the pressures at the boundary of the cloud to be vanishing, then it is always possible to match the interior space-time with an empty Schwarzschild exterior. However, in all cases it may not be possible to make the pressures at the boundary of the cloud to vanish. Hence we outline here the procedure to match the interior with a general class of exterior metrics, which are the generalized Vaidya space-times [86,87], at the boundary hypersurface  $\Sigma$  given by  $r = r_b$ . For the required matching we use the Israel-Darmois conditions [88,89], where we match the first and second fundamental forms (the metric coefficients and the extrinsic curvature, respectively) at the boundary of the cloud.

Whereas the procedures used below are standard, we shall describe the particular case treated here in some detail so as to give the exact picture of the overall collapse scenario emerging. We note a useful fact that since we are matching the second fundamental form  $K_{ij}$ , there is no surface stress energy or surface tension at the boundary (see e.g. [90]). The metric just inside  $\Sigma$  is

$$ds_-^2 = -e^{2\nu(t,r)} dt^2 + e^{2\psi(t,r)} dr^2 + R^2(t, r) d\Omega^2, \quad (95)$$

which describes the geometry of the collapsing cloud. The metric in the exterior of  $\Sigma$  is given by

$$ds_+^2 = -\left(1 - \frac{2M(r_v, V)}{r_v}\right) dV^2 - 2dV dr_v + r_v^2 d\Omega^2, \quad (96)$$

where  $V$  is the retarded (exploding) null coordinate and  $r_v$  is the Vaidya radius. Matching the area radius at the boundary we get

$$R(r_b, t) = r_v(V). \quad (97)$$

Then on the hypersurface  $\Sigma$ , the interior and exterior metrics are given by

$$ds_{\Sigma^-}^2 = -e^{2\nu(t,r_b)} dt^2 + R^2(t, r_b)^2 d\Omega^2 \quad (98)$$

and

$$ds_{\Sigma^+}^2 = -\left(1 - \frac{2M(r_v, V)}{r_v} + 2\frac{dr_v}{dV}\right) dV^2 + r_v^2 d\Omega^2. \quad (99)$$

Matching the first fundamental form gives

$$\left(\frac{dV}{dt}\right)_{\Sigma} = \frac{e^{\nu(t,r_b)}}{\sqrt{1 - \frac{2M(r_v, V)}{r_v} + 2\frac{dr_v}{dV}}}; \quad (r_v)_{\Sigma} = R(t, r_b). \quad (100)$$

Next, to match the second fundamental forms (extrinsic curvatures) for the interior and exterior metrics, we note that the normal to the hypersurface  $\Sigma$ , as calculated from the interior metric, is given as

$$n_-^i = [0, e^{-\psi(r_b, t)}, 0, 0] \quad (101)$$

and the nonvanishing components of the normal as derived from the generalized Vaidya space-time are

$$n_+^V = -\frac{1}{\sqrt{1 - \frac{2M(r_v, V)}{r_v} + 2\frac{dr_v}{dV}}}, \quad (102)$$

$$n_+^{r_v} = \frac{1 - \frac{2M(r_v, V)}{r_v} + \frac{dr_v}{dV}}{\sqrt{1 - \frac{2M(r_v, V)}{r_v} + 2\frac{dr_v}{dV}}}. \quad (103)$$

Here the extrinsic curvature is defined as

$$K_{ab} = \frac{1}{2} \mathcal{L}_{\mathbf{n}} g_{ab}. \quad (104)$$

That is, the second fundamental form is the Lie derivative of the metric with respect to the normal vector  $\mathbf{n}$ . The above equation is equivalent to

$$K_{ab} = \frac{1}{2} [g_{ab,c} n^c + g_{cb} n_{,a}^c + g_{ac} n_{,b}^c]. \quad (105)$$

Now setting  $[K_{\theta\theta}^- - K_{\theta\theta}^+]_{\Sigma} = 0$  on the hypersurface  $\Sigma$  we get

$$RR' e^{-\psi} = r_v \frac{1 - \frac{2M(r_v, V)}{r_v} + \frac{dr_v}{dV}}{\sqrt{1 - \frac{2M(r_v, V)}{r_v} + 2\frac{dr_v}{dV}}}. \quad (106)$$

Simplifying the above equation using Eq. (100) and the Einstein equations, we get

$$F(t, r_b) = 2M(r_v, V). \quad (107)$$

Using the above equation and (100) we now get,

$$\left(\frac{dV}{dt}\right)_{\Sigma} = \frac{e^{\nu}(R'e^{-\psi} + \dot{R}e^{-\nu})}{1 - \frac{F(t, r_b)}{R(t, r_b)}}. \quad (108)$$

Finally, setting  $[K_{\tau\tau}^- - K_{\tau\tau}^+]_{\Sigma} = 0$ , where  $\tau$  is the proper time on  $\Sigma$ , we get

$$M(r_v, V)_{,r_v} = \frac{F}{2R} + \frac{Re^{-\nu}}{\sqrt{G}} \sqrt{H}_{,t} + Re^{2\nu} \nu' e^{-\psi}. \quad (109)$$

Any generalized Vaidya mass function  $M(v, r_v)$ , which satisfies Eq. (109) will then give a unique exterior space-time with required equations of motion given by other matching conditions (97), (107), and (108).

To see that the set of all such functions  $M(v, r_v)$  is nonempty, we have the examples of a charged Vaidya space-time  $M = M(V) + Q(V)/r_v$ , and the anisotropic de Sitter space-time  $M = M(r_v)$  as two different solutions of the Eq. (109) (see for example [87,91]). This gives two unique exterior space-times, both of which are subclasses of the generalized Vaidya metric.

### VIII. CONCLUDING REMARKS

We make here some remarks towards a conclusion and note some unresolved and open issues.

- (1) Towards investigating end states of a continual gravitational collapse, given a general type I matter field and given the matter initial profiles at the initial surface from which the collapse develops, we constructed here classes of solutions to the Einstein equations such that the collapse evolution goes to the formation of either of a black hole or naked singularity end state, depending on the choice of the evolution made and choice of rest of the initial data functions such as velocities of the collapsing shells. This is subject to satisfying the energy condition, and several reasonable and important equations of state are included in the framework here. It also becomes clear that in higher dimensions both black holes and naked singularities can occur as collapse end states.
- (2) We like to note that what we have deduced here is the occurrence of a locally naked singularity only, as opposed to that of a globally naked singularity. That is, we show when the null geodesics escape from the space-time singularity, going out in the future, but we do not address the question of when they go out of the boundary of the matter cloud. It is possible, in principle, that the singularity is only locally naked and trajectories do come out but they all fall back into the singularity again, without going out of the boundary of the star, thus not being globally visible. This issue is still not studied for the general class of models such as the classes we considered here and may be of interest. However, for the case of dust models this has been studied in some detail and it is shown that whenever the singularity is locally naked, one can always choose the classes of the mass and energy functions suitably, as one moves away from the center, in such a manner that the singularity becomes globally visible. The point is, while the local visibility of the central singularity is basically decided by the conditions near the center, the global visibility really depends on the overall behavior of these functions within the matter cloud,

away from the center. This we are still free to choose. In other words, for the dust collapse models once the singularity is locally visible, there are always classes of functions which we can choose so as to make it globally visible.

Another important related point here is, as such there is no scale in the problem, and the size of the collapsing cloud could be quite large. In such a case, even if the singularity is only locally visible, still it can be seen for a long enough time by the observers. Thus, in principle, a locally naked singularity is also as serious a violation of the cosmic censorship as a globally visible singularity, and there may not be a qualitative difference in the two cases in many situations of physical interest.

- (3) Another important and interesting issue frequently mentioned regarding the occurrence of naked singularities in gravitational collapse is their genericity and stability. It is argued that if these are not generic or stable, these need not be taken seriously. This is interesting because in general relativity there is no well-defined notion or criteria available for stability or genericity, which one can then apply and test for a given model to ascertain these. On the other hand, a consideration of this issue is important all the same in that, depending on the collapse situation under discussion, one would like to formulate these notions in some way to examine if the naked singularities developing as collapse end states are “generic” or “stable” in some suitable sense. This would typically involve taking into account the topology and metric of the concerned function spaces which define the given collapse scenario. Without discussing this here in further detail, we note that since we have formulated here the collapse in a general manner, for general physically reasonable matter fields, the classes constructed here may provide a good arena to explore and test these important issues for the cosmic censorship hypothesis.
- (4) A related issue of course is that of nonspherical collapse. Our considerations here are restricted to spherical collapse, and the question remains open regarding the final end state of a nonspherical collapse. So one can ask if the conclusions available in spherically symmetric collapse remain the same and stable under possible nonspherical perturbations. Even though some nonspherical collapse models have been discussed and investigated so far such as the Szekeres quasispherical collapse or some cylindrical collapse models [39–51], these are

somewhat restricted in nature. A strong theorem about the formation of trapped surfaces in cylindrically symmetric space-times is given in [50]. Recently, a numerical construction of naked singular solutions with the cylindrical symmetry was done in [51]. It is not clear as yet if this issue can be approached really in an analytic manner, and possibly detailed numerical simulations of the collapsing stars could be the answer.

- (5) A question that is frequently asked in connection with the occurrence of naked singularities as collapse end states is how to understand this phenomena physically. A naked singularity signifies the escape of light and particle trajectories from the ultradense space-time regions. However, the gravity must become so strong in these regions. In such a case, how can anything escape at all from such a region is the question. Thus, while a black hole which is a region from which not even light would escape, may appear to be the only physically reasonable outcome in such situations, formation of a naked singularity in collapse may appear to be counterintuitive.

The point that comes out from considerations such as ours is that the naked singularities are more an artifact of general relativity, rather than that of a purely Newtonian physics. Even though the matter density grows higher and higher without bound and blows up closer to a space-time singularity, which would denote the growth of attractive forces of gravity, there are other important factors which are purely general relativistic effects which can delay the formation of trapped surfaces governing the trapping of light.

An interesting effect that does this is the space-time shear. It is intriguing to find that the physical agencies such as the space-time shear, and related inhomogeneities in matter density distribution within a dynamically collapsing cloud, could naturally delay the formation of trapped surfaces during gravitational collapse [92,93]. In other words, such physical factors do naturally give rise to naked singularity phases in the collapse where the formation of apparent horizon and the trapped surfaces is delayed. Even though the matter densities are arbitrarily large and growing, the shear could distort the trapped surface geometry in such a manner as to avoid the trapping of light and facilitates the escape of null rays from such ultradense regions.

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