Black holes and Rindler superspace: Classical singularity and quantum unitarity

Chung-Hsien Chou,[1] Chopin Soo,[2] and Hoi-Lai Yu[1]
[1]*Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan*
[2]*Department of Physics, National Cheng Kung University, Tainan, Taiwan*
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Canonical quantization of spherically symmetric initial data appropriate to classical interior black hole solutions in four dimensions is solved exactly without gauge fixing the remaining kinematic Gauss law constraint. The resultant mini-superspace manifold is two dimensional, of signature $(+, -)$, nonsingular, and can be identified with the first Rindler wedge. The associated Wheeler-DeWitt equation with evolution in intrinsic superspace time is a free massive Klein-Gordon equation, and the Hamilton-Jacobi semiclassical limit of plane wave solutions can be matched to the interiors of Schwarzschild black holes. Classical black hole horizons and singularities correspond to the boundaries of the Rindler wedge. Exact wave functions of the Dirac equation in superspace are also considered. Precise correspondence between Schwarzschild black holes and free-particle mechanics in superspace is noted. Despite the presence of classical singularities, Hermiticity of the Dirac Hamiltonian operator, and thus unitarity of the quantum theory, is equivalent to an appropriate boundary condition which must be satisfied by the quantum states. This boundary condition holds for quite generic quantum wave packets of energy eigenstates, but fails for the usual Rindler fermion modes which are eigenstates with zero uncertainty in energy.

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I. INTRODUCTION AND OVERVIEW

Quantization of the spherically symmetric sector of four-dimensional pure general relativity (GR) has been investigated in a number of articles [1–3]. Despite the simplicity of this toy model, it is an extremely interesting testing (toy)ground for many intriguing and challenging quantum gravity issues, both on the technical as well as conceptual fronts. Birkhoff's theorem [4] ensures that the classical solutions of this sector are none other than Schwarzschild black holes, and as such they come with classical horizons and singularities. How do these manifest themselves and what roles do they play in the quantum context? Do they affect the unitarity of the quantum theory, and if so in what explicit manner? And for reparametrization invariant theory, one ought to ask whether the unitarity or nonunitary of quantum evolution is with respect to intrinsic time in superspace [5], or some other appropriate choice of "time." Indeed it is not unreasonable to demand any respectable quantum theory of gravity to provide guidance on these issues.

Recent investigations motivated by loop quantum gravity have yielded as insights and possible resolutions an upper bound on the curvature, and reformulation of the Wheeler-DeWitt constraint as a difference time evolution equation wherein classical black hole singularities are not obstacles [2,3]. Here we adopt a more conservative approach based upon continuum physics and exact canonical quantization. The Wheeler-DeWitt equation [5,6] for spherically symmetric mini-superspace appropriate to the discussion of interior black hole solutions [3] is solved exactly. Among the neat features which emerged are (1) The resultant arena for quantum geometrodynamics is two-dimensional, of signature $(+, -)$, nonsingular, and can be identified with the first Rindler wedge. (2) Classical black hole horizons and singularities correspond, remarkably, to the boundaries of the Rindler wedge. (3) The classical super-Hamiltonian constraint is equivalent to a massive free-particle dispersion relation in flat superspace, and the Wheeler-DeWitt equation is a Klein-Gordon equation with evolution in intrinsic superspace time. (4) The Hamilton-Jacobi semiclassical limit consists of plane wave solutions which can be matched to the interiors of Schwarzschild black holes. (5) Positive-definite "probability" current for wave functions and other considerations motivate the investigation of "fermionic" solutions of the associated Dirac equation. (6) With respect to the natural inner product, Hermiticity of the Dirac Hamiltonian, and thus unitarity of the quantum theory, is equivalent to a condition at the boundary of spacelike hypersurfaces of the Rindler wedge which must be satisfied by the solutions. (7) The boundary condition is demonstrated to be satisfied by rather generic wave packets, but not for the usual energy-eigenstate fermionic Rindler modes. This situation has an analogy in free nonrelativistic Schrödinger quantum mechanics wherein Hermiticity of the momentum operator requires the physical Hilbert space to be made up of states which are suitable wave packets which vanish at spatial infinity, and rule out plane wave states with infinitely sharp momentum. From this perspective, the quantum evolution here is thus physically unitary, and the boundary condition guaranteeing unitarity is much milder than, say, "quantum censorship" requirement of vanishing wave function at the boundary.

II. SPHERICALLY SYMMETRIC INITIAL DATA AND CANONICAL QUANTIZATION

We start by following Ref. [3], wherein modulo gauge invariance, the Lie-algebra-valued connection 1-form for spherically symmetric configurations is

$$A = cT_3 d\alpha + (aT_1 + bT_2)d\theta + (-bT_1 + aT_2)\sin\theta d\phi + T_3\cos\theta d\phi, \tag{1}$$

where a, b, c are constants on spacelike Cauchy surfaces, and T_a are the generators of $SO(3)$. The most general spherically symmetric gauge configurations [7] allow for α dependence as well, resulting in additional contributions to the constraints for initial data relevant to exterior black hole solutions. The description of exterior regions wherein α dependence of a, b, and c gives rise to further contributions to the constraints has also been discussed elsewhere recently [8]. As in Ref. [3], the formalism here works perfectly for the purposes of discussing the classical singularity and properties inside the horizon. The corresponding dreibein are $e_1 = \omega_a d\theta - \omega_b \sin\theta d\phi$, $e_2 = \omega_b d\theta + \omega_a \sin\theta d\phi$, $e_3 = \omega_c d\alpha$, and they are related to the densitized triad by $\tilde{E} = p_c T_3 \sin\theta\frac{\partial}{\partial\alpha} + (p_a T_1 + p_b T_2)\times \sin\theta\frac{\partial}{\partial\theta} + (p_a T_2 - p_b T_1)\frac{\partial}{\partial\phi}$, with $\omega_a = \sqrt{|p_c|}p_a/\sqrt{p_a^2 + p_b^2}$, $\omega_b = \sqrt{|p_c|}p_b/\sqrt{p_a^2 + p_b^2}$, $\omega_c = \mathrm{sgn}(p_c)\sqrt{p_a^2 + p_b^2}/\sqrt{|p_c|}$.

In this formulation, the Ashtekar-Barbero super Hamiltonian, with Immirzi parameter γ, is proportional to $\int d^3x N e^{-1}[\epsilon^{abc}F_{ija}\tilde{E}^i_b\tilde{E}^j_c - 2(1+\gamma^2)K^a_{[i}K^b_{j]}\tilde{E}^i_a\tilde{E}^j_b]$, wherein $e := \sqrt{|\det\tilde{E}|}\,\mathrm{sgn}(\det\tilde{E})$. Using the extrinsic curvature $K = \frac{1}{\gamma}(A - \Gamma)$, where Γ is the torsionless connection compatible with the dreibein, the super-Hamiltonian constraint simplifies to

$$\mathcal{H} \propto 2p_c c(p_a a + p_b b) + (p_a^2 + p_b^2)(a^2 + b^2 + \gamma^2) \approx 0, \tag{2}$$

and the symplectic form for the spherically symmetric sector is [3] $\Omega = \frac{1}{2\gamma G}(2\delta a \wedge \delta p_a + 2\delta b \wedge \delta p_b + \delta c \wedge \delta p_c)$.

To tackle gauge invariance, we define $p_a =: r\cos\Theta$, $p_b =: r\sin\Theta$, $p_c: = :z$. Invoking the canonical quantization rules, $2a/2\gamma G = (\hbar/i)(\partial/\partial p_a) = (\hbar/i)\times [\cos\Theta(\partial/\partial r) - (\sin\Theta/r)(\partial/\partial\Theta)]$, $2b/2\gamma G = (\hbar/i)\times (\partial/\partial p_b) = (\hbar/i)[\sin\Theta(\partial/\partial r) + (\cos\Theta/r)(\partial/\partial\Theta)]$, $\frac{c}{2\gamma G} =: p_z = \frac{\hbar}{i}\frac{\partial}{\partial z}$, the quantum constraint acting on wave functions Φ is thus

$$\left[z\hat{p}_z r\hat{p}_r - \frac{\hbar^2 r^2}{4}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\Theta^2}\right) + \frac{r^2}{4G^2}\right]\Phi = 0. \tag{3}$$

The Gauss law constraint $p_b a - p_a b = 0$ is precisely equivalent to invariance under Θ rotations about the z axis, which is solved exactly, without gauge fixing, by the independence of Φ with respect to Θ. For the case of α-independent variables discussed here, the diffeomorphism supermomentum constraint gives rise to no further restrictions. Although r and z are the resultant superspace variables, it can be readily verified the resultant Wheeler-DeWitt equation,

$$\left[-z\frac{\partial}{\partial z}r\frac{\partial}{\partial r} - \frac{1}{4}r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{r^2}{4\hbar^2 G^2}\right]\Phi(r, z) = 0, \tag{4}$$

is not separable in r and z.

To discuss the complete set of solutions, another set of variables $X_- := \sqrt{z}$, $X_+ := \frac{r^2}{4\hbar^2 G^2\sqrt{z}}$ is crucial. The Wheeler-DeWitt constraint in superspace now reduces to

$$X_+X_-\left[-\frac{\partial}{\partial X_+}\frac{\partial}{\partial X_-} + 1\right]\Phi(X_+, X_-) = 0, \tag{5}$$

which is equivalently a simple Klein-Gordon wave equation in light cone coordinates $X_\pm = \frac{1}{2}(X \pm T)$ with unit "mass." Plane waves $\Phi(X_+, X_-) = \exp(ik_+X_+ + ik_-X_-)$ (with mass shell condition $k_+k_- + 1 = 0$) and their superpositions are all solutions, and the semiclassical limit of the theory will be discussed in the following section.

III. SEMICLASSICAL LIMIT, HAMILTON-JACOBI EQUATIONS, AND SCHWARZSCHILD BLACK HOLES

The semiclassical limit can be addressed by considering usual semiclassical states $Ae^{i(S/\hbar)}$, with A being a slow varying function, and S satisfying the Hamilton-Jacobi equation, $(\partial S/\partial X_+)(\partial S/\partial X_-) + 1 = 0$. Separation of variables with $S(X_+, X_-) = S_+(X_+) + S_-(X_-)$ leads to $dS_\pm(X_\pm)/dX_\pm = p_\pm$ being constants related by $p_+ = -1/p_-$, thus yielding (up to a constant) the Hamilton function $S(X_+, X_-) = p_+X_+ + p_-X_-$. These semiclassical states may thus be identified with plane waves $A'e^{i(k_+X_+ + k_-X_-)}$ wherein $p_\pm = \hbar k_\pm$.

We can furthermore map the equations of motion (EOM) for plane wave solutions to classical black holes by studying the correspondence of the classical initial data to the Hamilton-Jacobi theory. To wit, we note that interior to the horizon $(0 < R < 2GM)$, the Schwarzschild metric with constant-R Cauchy surfaces can be described by the vierbein $e^A = (dR/\sqrt{(2GM/R - 1)}, (R/\sqrt{2})(d\theta - \sin\theta d\phi), (R/\sqrt{2})(d\theta + \sin\theta d\phi), \sqrt{(2GM/R - 1)}dt)$. Computing the Ashtekar potential from $A_a = \gamma\omega^0{}_a - \frac{1}{2}\epsilon_{abc}\omega^{bc}$, where ω_{AB} is the torsionless spin connection, and comparing directly with Eq. (1) and the corresponding dreibein yield,

$$\omega_a = \omega_b = \frac{R}{\sqrt{2}}, \qquad \omega_c d\alpha = \frac{r}{\sqrt{z}}d\alpha = \sqrt{\left(\frac{2GM}{R} - 1\right)}dt; \tag{6}$$

$$a = b = \frac{\gamma}{\sqrt{2}}\sqrt{\left(\frac{2GM}{R} - 1\right)}, \qquad c d\alpha = -\gamma \frac{GM}{R^2} dt. \quad (7)$$

It follows that $X_-^2 \equiv z = p_c = \omega_a^2 + \omega_b^2 = R^2$. To infer the EOM from the Hamilton-Jacobi theory, we note that on plane wave states, $\Phi \propto e^{(ik_+X_+ + ik_-X_-)}$,

$$\hat{p}_r\Phi = \left(\frac{2X_+\hbar k_+}{r}\right)\Phi; \qquad \hat{p}_z\Phi = \left(\frac{k_-X_- - k_+X_+}{2X_-^2}\right)\hbar\Phi. \quad (8)$$

Direct substitutions into the semiclassical Hamilton-Jacobi limit of Eq. (4) yield, as expected, the "straight line trajectory" EOM,

$$X_+ = \frac{1}{k_+^2}(2GM - X_-), \quad (9)$$

which is independent of γ. Furthermore, $(r/\sqrt{z})d\alpha = \sqrt{(2GM/R - 1)}dt$ gives $d\alpha/dt = k_+/2G\hbar$ on applying the EOM.

IV. RINDLER SUPERSPACE

It should be noted that $X_\pm \geq 0$, thus the associated superspace is not the whole of Minkowski spacetime; rather, it is precisely the first Rindler wedge, which can be parametrized by $(\xi \geq 0, -\infty < \tau < \infty)$, with $X = \xi\cosh\tau$ and $T = \xi\sinh\tau$, and endowed with supermetric $ds^2 = dT^2 - dX^2 = \xi^2 d\tau^2 - d\xi^2$.

Our previous EOM implies that *classical* black hole horizons and singularities occur at $(X_- = R = 2GM, X_+ = (1/k_+^2)[2GM - X_-] = 0)$ and $(X_- = 0, X_+ = (1/k_+^2)[2GM - X_-] = 2GM/k_+^2)$, respectively. Thus as we span over the range of black hole masses M, we observe that the lower boundary $(X_- = 2GM, X_+ = 0)$ and upper boundary $(X_- = 0, 2GM/k_+^2)$ of the Rindler wedge correspond precisely to the classical black hole horizons and singularities.

On this Rindler wedge the Klein-Gordon equation is

$$\left[\frac{1}{\xi^2}\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial}{\partial\xi} + 1\right]\Phi(\xi, \tau) = 0. \quad (10)$$

The orthonormal modes can be expressed explicitly as $\Phi_\omega(\xi, \tau) = \sqrt{(1 - e^{-2\pi\omega})/2}B_\omega(\xi, \tau)$, wherein the "Minkowski Bessel modes" [9] are given by $B_\omega = \frac{1}{\pi}e^{(\pi\omega/2)}K_{i\omega}(\xi)e^{-i\omega\tau}$. These Minkowski Bessel modes can in fact be understood as the "rapidity Fourier transform" of plane wave solutions $e^{i(kX-\omega T)}$ [9], i.e., $B_\omega(\xi, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\xi\sinh(\eta-\tau))}e^{-i\omega\eta}d\eta$, where $k_\pm = k \mp \omega$, and in terms of η (the rapidity variable), $k = \sinh\eta$, $\omega = \cosh\eta$, thus satisfying the mass shell condition $\omega^2 = k^2 + 1$ for our particular case of unit mass. As discussed, the plane wave solutions which correspond to Schwarzschild black holes are given by the inverse rapidity Fourier transforms of B_ω. A conserved current with respect to the Klein-Gordon inner product can be constructed [5]; however, it is well known in Klein-Gordon theory positive-definite current density for generic superpositions cannot be guaranteed.

While "bosonic" scalar solutions of Eq. (10) are the first to come to mind, the possibility of fermionic solutions of the Wheeler-DeWitt constraint should not be ignored. To wit, we investigate the Dirac equation which is *first-order in intrinsic superspace time* on the Rindler wedge,

$$(i\not{D} - m)\Psi = \left[\frac{\gamma^0}{\xi}i\partial_\tau + \frac{i\gamma^1}{2\xi} + i\gamma^1\partial_\xi - m\right]\Psi = 0, \quad (11)$$

with Ω being the spin connection compatible with Rindler space vielbein $\mathrm{E}^{0,1}{}_\mu dx^\mu = (\xi d\tau, d\xi)$. Moreover, the Lorentz-invariant current density $\tilde{J}^\mu \equiv \det(\mathrm{E})\bar{\Psi}\gamma^\mu\Psi$ obeys $\partial_\mu\tilde{J}^\mu = 0$ and $\tilde{J}^0 = \det(\mathrm{E})\mathrm{E}^0{}_A\bar{\Psi}\gamma^A\Psi = \Psi^\dagger\Psi$ is also positive definite. Orthonormal modes with respect to the inner product $\langle\Psi'|\Psi\rangle = \int_0^\infty \Psi'^\dagger\Psi d\xi$ can be expressed as [10,11] $\Psi_\nu(\tau, \xi) = N_\nu e^{-i\nu\tau}K_{i\nu-(\gamma^0\gamma^1/2)}(\xi)\chi$. It is possible to choose $\chi = \chi^+ + \chi^-$, with constant orthonormal $\pm$ eigenspinors, $\chi^\pm$, of $\gamma^0\gamma^1$ with the resultant normalization constant $N_\nu = \sqrt{\cosh(\pi\nu)/\pi}$.

V. QUANTUM UNITARITY DESPITE THE PRESENCE OF APPARENT CLASSICAL SINGULARITIES

With respect to the inner product discussed above, the Dirac Hamiltonian operator on the Rindler wedge, $H_D = -i\gamma^0\gamma^1\xi\partial_\xi - i\gamma^0\gamma^1/2 + m\gamma^0$, is Hermitian, i.e., $\langle H_D\Psi'|\Psi\rangle = \langle\Psi'|H_D\Psi\rangle$ if and only if for all $-\infty < \tau < \infty$, $\lim_{\xi\to 0}\xi\Psi'^\dagger(\tau, \xi)\gamma^0\gamma^1\Psi(\tau, \xi) = 0$ [12]. In an investigation of the Unruh effect, Oriti [11] conjectured that the fermionic solutions should therefore obey $\lim_{\xi\to 0}\xi^{1/2}\Psi(\xi) = 0$. In our present context of quantum gravity, it is intriguing this last equation is a boundary condition which guarantees Hermiticity of the Hamiltonian and thus *unitarity of the quantum evolution with respect to intrinsic superspace time* despite the threat of apparent classical black hole singularities.

The asymptotic behavior of $K_\alpha(\xi)$ is quite simple:

$$\lim_{\xi\to 0}\xi^{1/2}K_{i\nu\pm(1/2)}(\xi) = \sqrt{2}\Gamma\left(\frac{1}{2} \pm i\nu\right)\left(\frac{\xi}{2}\right)^{\mp i\nu}, \quad (12)$$

where $\pm\vartheta(\nu) = \arg[\Gamma(\frac{1}{2} \pm i\nu)]$. Since $\{\Psi_\nu(\tau, \xi)\}$ forms an orthonormal basis, the boundary condition for unitarity implies the restriction [on $f(\nu)$] on a generic state $\Psi(\tau, \xi)$ which is

$$0 = \lim_{\xi\to 0}\xi^{1/2}\Psi(\tau,\xi) = \lim_{\xi\to 0}\sqrt{\xi}\int_{-\infty}^{\infty} d\nu f(\nu)\Psi_\nu(\tau,\xi)$$
$$= \sqrt{\frac{2}{\pi}}\lim_{\xi\to 0}\int_{-\infty}^{\infty} d\nu f(\nu)e^{-i\nu\tau}\bigg[e^{i\vartheta(\nu)}\left(\frac{\xi}{2}\right)^{-i\nu}\chi^{+} + e^{-i\vartheta(\nu)}\left(\frac{\xi}{2}\right)^{i\nu}\chi^{-}\bigg]. \tag{13}$$

This results in two conditions:

$$\lim_{X_\pm\to 0}\int_{-\infty}^{\infty} d\nu f(\nu)e^{\pm i\vartheta(\nu)}e^{\mp i\nu \ln X_\pm} = 0, \tag{14}$$

wherein $X_\pm = \frac{\xi}{2}e^{\pm\tau}$ has been used, and for $-\infty < \tau < \infty$, the limit $\xi \to 0$ implies both $X_\pm \to 0$. Thus the boundary condition is equivalent to requiring the Fourier transform of $f(\nu)e^{\pm i\vartheta(\nu)}$ to vanish at $\pm\infty$. Since $e^{i\vartheta(\nu)}$ is oscillatory (rather than sharply peaked), the condition can be satisfied by a rather generic wave packet with $f(\nu)$ whose Fourier transform vanishes at $\pm\infty$ [explicitly, for instance, by $f(\nu)$ being Gaussian]. From this perspective, the quantum evolution here is thus physically unitary. However, it should be pointed out that infinitely sharp energy eigenstates, i.e., $f(\nu_0) \propto \delta(\nu - \nu_0)$ which correspond to the usual Rindler modes Ψ_{ν_0}, *fail* to satisfy the boundary condition. An analogous situation happens in free nonrelativistic quantum mechanics wherein Hermiticity of the momentum operator requires a physical Hilbert space of suitable wave packets which vanish at spatial infinity, and rule out plane wave eigenstates with infinitely sharp momentum.

It can also be verified that a semiclassical black hole state is a wave packet (in ν) which satisfies the boundary condition explicitly. We note that the energy eigenstates $\Psi_\nu(\tau,\xi)$ satisfy the associated equation,

$$0 = (i\not{D} + m)(i\not{D} - m)\Psi_\nu$$
$$= \bigg[-\frac{1}{\xi^2}e^{-((\gamma^0\gamma^1\tau)/2)}\partial_\tau^2 e^{((\gamma^0\gamma^1\tau)/2)} + \partial_\xi^2 + \frac{1}{\xi}\partial_\xi - m^2\bigg]\Psi_\nu. \tag{15}$$

This means that if Ψ is a solution of the Dirac equation, the fermionic Lorentz-boosted solution $\Phi^F(\tau,\xi) = e^{((\gamma^0\gamma^1\tau)/2)}\Psi(\tau,\xi)$ will also solve the Klein-Gordon equation (10). Spinorial Minkowski plane wave modes can be written (in terms of the rapidity η) as $P^F_\eta(X,T) = (1/2\sqrt{2\pi})(e^{\eta/2}\chi^- + ie^{-\eta/2}\chi^+)e^{i(kX-\omega T)}$, with orthonormalization $\int_{-\infty}^{\infty}(P^F_{\eta'})^\dagger P^F_\eta dX = \delta(\eta - \eta')$. The identity $K_{i\nu\mp(1/2)}e^{-(i\nu\mp(1/2))\tau} = \frac{1}{2}e^{i(\pi/2)(i\nu\mp(1/2))}\int_{-\infty}^{\infty} e^{-(i\nu\mp(1/2))\eta}\times e^{i(kX-\omega T)}d\eta$ implies that the Rindler energy eigenstates, $\Psi_\nu(\tau,\xi) = N_\nu e^{-i\nu\tau}[K_{i\nu-(1/2)}(\xi)\chi^+ + K_{i\nu+(1/2)}(\xi)\chi^-]$, and Minkowski plane wave states are thus related by $e^{((\gamma^0\gamma^1\tau)/2)}\Psi_\nu(\tau,\xi) = (N_\nu/2)e^{-(\pi\nu/2)}\int_{-\infty}^{\infty}P^F_\eta(T,X)e^{-i\nu\eta}\times d\eta$. Thus the semiclassical fermionic plane wave black hole solution which solves the Dirac equation (11) is just the inverse Fourier transform

$$e^{-((\gamma^0\gamma^1\tau)/2)}P^F_\eta(T,X) = 2\pi\int_{-\infty}^{\infty}\left(\frac{2}{N_\nu}e^{\pi\nu/2}\right)\Psi_\nu(\tau,\xi)e^{i\nu\eta}d\nu, \tag{16}$$

which is a *wave packet* of Rindler modes Ψ_ν with $f_\eta(\nu) = (4\pi/N_\nu)e^{(i\eta+(\pi/2))\nu}$.

VI. SUMMARY AND FURTHER REMARKS

The limitations of classical GR are saliently exposed by the occurrence of singularities in the theory. While singular potentials do not necessarily pose problems to quantum theory (in quantum mechanics situations with singular potentials are tractable and even exactly solvable), classical curvature singularities need not be manifested in the quantum context as singular potentials. The quantum theory of spherically symmetric 4-dimensional GR is a simple model whose classical sector is made up of only black hole solutions to Einstein's theory. In this sense it is precisely a "quantum theory of Schwarzschild black holes," just as nonrelativistic free-particle Schrödinger quantum mechanics is a quantum theory of free Newtonian particle mechanics. Amusingly, this analogy is true even in the details, for Schwarzschild solutions are mapped to straight line trajectories of free motion in flat superspace!

The arena for quantum gravity is not spacetime but superspace [5,6]. Here we discover that the corresponding mini-superspace is free of singularity and has a surprisingly simple structure. We complete the Wheeler-DeWitt canonical quantization program for the case at hand by using a rather conservative approach with continuum variables, obtain the complete solution to the quantum evolution with respect to intrinsic superspace time, and discuss unitarity of the evolution and the Hermiticity of the Hamiltonian in a clear and precise manner. The contention that in all likelihood the mathematics of our formalism is too special and restrictive to be applicable or generalizable to the full theory is a point of view we can empathize with, but the other side of the coin that black holes can be so elegantly described (as free particles in Rindler superspace) suggests that the full theory may in fact contain simplifications which have not yet been fully exploited, and that our results and techniques may serve as launching points in searching out the related exact states in the full theory and in understanding their physical behavior.

The semiclassical limit from the quantum theory here (in a sense a quantum derivation of Birkhoff's theorem) is in complete agreement with Birkhoff's classical result of Schwarzschild solutions for Einstein's field equations. All black holes (plane wave solutions in our quantum context) are semiclassical regardless of the mass, however small. Although this may be nonintuitive, the derived classical limit is in fact reasonable, for all Schwarzschild metrics regardless of mass are solutions of Einstein's equations corresponding to stationary points of the gravitational action. For these 4-manifolds, black holes correspond to (intrinsic) timelike straight line trajectories in superspace, and for these trajectories intrinsic time τ and R radial coordinate time of interior Schwarzschild solutions are monotonically related.

Classical singularities are precisely where the classical theory breaks down. This means we cannot interpret the physics there by drawing the correspondence between even semiclassical, quantum theory, and the classical variables. The relevant question is whether the quantum theory and the behavior of the quantum states remain well defined. For the case at hand, quantum evolution takes place in a non-singular Rindler wedge, and the semiclassical plane wave states can be matched to Scwharzschild geometries, except at $R = 0$. Our formalism also shows that the coordinates $X_{\pm}$ are well defined even when a, b, c diverge, and apparent classical black hole ‘‘singularities’’ are matched to finite values on the X_+ axis. Moreover, the classical EOM (9) also remains valid and can, except for $R = 0$, be interpreted in terms of Schwarzschild geometry. What we lose is just the interpretation, only at $R = 0$, of the wave function as describing Schwarzschild black holes. This is neither a heavy nor unreasonable price, for the result of infinite curvature at $R = 0$ also does not make sense even classically.

While a ‘‘physical observer’’ in a semiclassical Schwarzschild background will have to face the issue of encountering the ‘‘classical singularity’’ in a finite proper time, this situation lies outside the scope of our work as we are only describing the quantum evolution of pure spherically symmetric gravity without matter, and for which the evolution is such that, for the plane wave states, 3-geometries are stacked up in intrinsic time to produce interior Schwarzschild 4-geometry. The demonstration of a completely quantum description (albeit for spherically symmetric sector) in the context of exact canonical quantum gravity leading to the, from the physical point of view, interesting classical limit of black hole physics while remaining at the same time a unitary description with quantum evolution in intrinsic superspace time even for situations with interior black hole geometry and classical singularity is noteworthy. Although not pursued here, it is possible to study the extension of the Rindler superspace to the whole of Minkowski spacetime and investigate ‘‘singularity traversal’’ and the continuation of the trajectories and their correspondence to, in general, complex 4-manifolds.

Solutions of the Dirac equation associated with the Wheeler-DeWitt constraint were also investigated. In flat Rindler spacetime, a solution of the former is also a solution of the latter. This is not necessarily true if superspace is not flat in the full theory [5]. But the existence of fermionic solutions, and ‘‘factorizations’’ of the Wheeler-DeWitt constraint, or its generalizations with and without supersymmetry, should not be ignored. The classical super-Hamiltonian constraint may be playing the analogous role of a ‘‘dispersion relation’’ which can be realized by more than one type of ‘‘particle’’ in the quantum context. In the spherically symmetric sector our Klein-Gordon and Dirac equations are really the resultant quantum constraints of a massive free-particle dispersion relation which is equivalent to the classical super-Hamiltonian constraint of GR. For simplicity, we have adopted a finite-dimensional representation of the Lorentz group of superspace when the Dirac equation and its solutions were discussed explicitly. Since superspace comes with hyperbolic signature, the Lorentz group is noncompact and thus nonunitary for finite-dimensional representations. This has the drawback of the lack of unitary equivalence between states related by local Lorentz transformations in superspace. In quantum field theory, the issue is resolved by assuming that these wave functions are operators acting upon physical states which belong to unitary infinite-dimensional representations. In the present context such a route would result in ‘‘third quantization.’’ Barring this, the solutions here should already be considered as physical states of the theory; however, one is not forbidden to consider these solutions of the Dirac equation with infinite-dimensional unitary representations.

There is a general belief that black holes are rudimentary objects in GR [13]. Intriguingly, this is bolstered by the explicit correspondence that is established here between the description of Schwarzschild black holes and that of elementary free-particle mechanics in superspace. Thus the conjecture that black holes are elementary objects in classical and quantum gravity, and also in particle physics, may in fact be more accurate and noteworthy than expected.

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[12] The boundary of spacelike hypersurfaces for $-\infty < \tau < \infty$ are at the origin with $\xi = 0$, and at $\xi = \infty$ where the fall-off behavior of $K_{i\nu \pm 1/2}$ results in no further restrictions. The hypersurfaces $\xi = 0$, $\tau = \pm\infty$ are lightlike in superspace rather than spacelike.

[13] "Black holes as elementary particles" was conjectured many years ago by G. 't Hooft, for example; see G. Arcioni, Sebastian de Haro, and L. Susskind, arXiv:physics/0611143.