Experimental status of the $\pi\pi$ isoscalar S wave at low energy: $f_0(600)$ pole and scattering length

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The experimental results obtained in the last few years on kaon decays ($K \rightarrow 2\pi$ and, above all, *Ke4* decays) allow a reliable, model-independent determination of low energy $\pi\pi$ scattering in the S0 wave. Using them and, eventually, other sets of data, it is possible to give a precise parametrization of the S0 wave as well as to find the scattering length and effective range parameter. One can also perform an extrapolation to the pole of the "sigma resonance" [$f_0(600)$]. We obtain the results $a_0^{(0)} = 0.233 \pm 0.013 M_\pi^{-1}$, $b_0^{(0)} = 0.285 \pm 0.012 M_\pi^{-3}$, and, for the sigma pole, $M_\sigma = 484 \pm 17$ MeV, $\Gamma_\sigma/2 = 255 \pm 10$ MeV.

I. INTRODUCTION

The lightest scalar and isoscalar mesonic "resonance," sometimes called the sigma resonance, has enjoyed a peculiar status. The Particle Data Tables (PDT) [1] that refer to it as the $f_0(600)$ resonance give for its pole a mass between 400 and 1200 MeV, and a half width between 300 and 500 MeV; and the same values for the "Breit-Wigner" mass and half width: wide ranges, indeed. In fact, in the old days the resonance was many times reported as nonexistent. On the other hand, calculations based on chiral perturbation theory and dispersion relations [2,3] give the following values for the complex pole corresponding to it [2]:[1]

$$M_\sigma = 441 \pm 17 \text{ MeV}, \qquad \Gamma_\sigma/2 = 230 \pm 15 \text{ MeV} \tag{1.1}$$

(the errors here are purely indicative; they are obtained from the spread of the values in the articles in Ref. [2]) and [3]

$$M_\sigma = 441^{+16}_{-8} \text{ MeV}, \qquad \Gamma_\sigma/2 = 279^{+9}_{-12.5} \text{ MeV}. \tag{1.2}$$

Similar results are obtained by Zhou *et al.* [4] who also use chiral perturbation theory and dispersion relations and get

$$M_\sigma = 470 \pm 50 \text{ MeV}, \qquad \Gamma_\sigma/2 = 285 \pm 25 \text{ MeV}.$$

It would be nice to be able to compare these results with *experiment*, without the tremendous uncertainties given in the PDT. Furthermore, it is desirable to use, in the comparison with experiment, only data from the S0 wave at low

and intermediate energies without introducing extraneous information on the S0 wave at high energies, or large amounts of data from other waves. This is what is done in the present paper, where the only theory we use is unitarity and analyticity, the last only what follows from general field theory.[2]

In the present paper it will be shown that, indeed, it is possible to get a determination of M_σ, Γ_σ, from experimental data on the S0 wave alone, and with a precision comparable to the theoretical one in (1.1) or (1.2). In fact, we will show that, *from experiment*, one has the values

$$M_\sigma = 484 \pm 17 \text{ MeV}, \qquad \Gamma_\sigma/2 = 255 \pm 10 \text{ MeV}. \tag{1.3}$$

The number for M_σ is stable; that for Γ_σ less so. Thus, theoretical calculations appear to be reasonably compatible with experiment.

Among the low-energy parameters, the scattering length, $a_0^{(0)}$, and effective range parameter, $b_0^{(0)}$, are particularly important. They are defined as in Ref. [6],

$$\frac{s^{1/2}}{2M_\pi k^{2l+1}} \text{ Re}\hat{f}_l^{(I)}(s) \underset{k\to 0}{\simeq} a_l^{(I)} + b_l^{(I)} k^2 + \cdots;$$

$$\hat{f}_l^{(I)} = \sin\delta_l^{(I)} e^{i\delta_l^{(I)}}, \qquad k = \sqrt{s/4 - M_\pi^2},$$

where $\hat{f}_l^{(I)}$ is the partial wave of definite isospin I and angular momentum l, and $\delta_l^{(I)}$ stands for its corresponding phase shift. The same methods that allow us to extrapolate the experimental S0 amplitude to the sigma pole permit an extrapolation to find $a_0^{(0)}, b_0^{(0)}$. The best values we get are

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[1]The determinations (1.1) and (1.2) are, in fact, based on different methods. We quote them together to give an idea on the spread of *theoretical* calculations that, one way or the other, use chiral perturbation theory to get the sigma pole.

[2]In contrast, the authors of Ref. [3] also use debatable theoretical information (see Ref. [5]) for example, on the (unmeasurable) scalar form factor of the pion, or on Regge theory. While in the articles in Ref. [2] some theoretical input is used to control the left-hand cut.

$$a_0^{(0)} = 0.233 \pm 0.013 M_\pi^{-1},$$
$$b_0^{(0)} = 0.285 \pm 0.012 M_\pi^{-3}. \tag{1.4}$$

These values are very stable; they compare well with the results from unitarized chiral perturbation theory (cf. the last article in Ref. [2])

$$a_0^{(0)} = 0.231^{+0.003}_{-0.006} M_\pi^{-1}, \qquad b_0^{(0)} = 0.300 \pm 0.010 M_\pi^{-3}, \tag{1.5}$$

(with only statistical errors), or with what was found from chiral perturbation theory in Ref. [7],

$$a_0^{(0)} = 0.220 \pm 0.005 M_\pi^{-1},$$
$$b_0^{(0)} = 0.276 \pm 0.006 M_\pi^{-3}, \tag{1.6}$$

although the central value of $a_0^{(0)}$ is a bit displaced.

Finally, we can find a parametrization that represents very accurately the S0 wave up to energies $s^{1/2} \simeq$ 950 MeV; see Eq. (6.1) in Sec. VI below.

The reason why we are able to present such an accurate determination of the $a_0^{(0)}$, $b_0^{(0)}$, as well as the location of the sigma pole from experiment is the availability in recent years of very precise data on the S0 wave at low energies, based on K decays and the use of a powerful model-independent fitting technique.

Before embarking on the details of the calculations, however, a few words have to be said on the meaning of the word resonance in connection with the sigma. One can give three definitions of the location of an elastic, background-free resonance mass and width: the point $s^{1/2} = \mu_0$ at which the phase shift crosses $\pi/2$, and the width of the corresponding Breit-Wigner parametrization, Γ_0 (usually called "Breit-Wigner" mass and width); the mass at which the energy derivative of the phase shift is a maximum and, for the width, the inverse of such derivative; and the pole of the partial wave amplitude in the (unphysical) Riemann sheet, at $M_\sigma - i\Gamma_\sigma/2$. Of these, the more physical one is the second: it identifies a resonance as a metastable state, whose lifetime is the inverse of the width (Wigner's time-delay theory).

In the case of *narrow* resonances, all three definitions coincide to first order in $\Gamma/2M$, but the situation for the sigma is very different. The S0 wave phase shift for $\pi\pi$ scattering, $\delta_0^{(0)}(s)$, does indeed cross $\pi/2$; but it does so [6] at an energy of $s^{1/2} = \mu_0 \simeq 800$ MeV. The energy *derivative* of $\delta_0^{(0)}(s)$ is not a maximum near this point, and the partial wave amplitude does not resemble a Breit-Wigner shape. Finally, and as we have already advanced in Eq. (1.3), there exists a pole in the second Riemann sheet, but at very low energy and with very large width. These are the reasons why the classification of the sigma as a resonance is so controversial. At any rate, in the present paper we will be concerned with the complex second Riemann

sheet pole of the partial wave amplitude, without discussing the relevance of the resonance concept for it.

II. THE RESONANCE CONDITION

In order to look for the pole on the second Riemann sheet associated with the sigma resonance, we will be using the well-known fact that such a pole, located at $\sqrt{s_\sigma} = M_\sigma - i\Gamma_\sigma/2$, corresponds exactly with a *zero* of the S-matrix partial wave $S_0^{(0)}(s)$,

$$S_0^{(0)}(s) = 1 + 2i\hat{f}_0^{(0)}(s) = e^{2i\delta_0^{(0)}},$$

at $\bar{s}_\sigma^{1/2} = M_\sigma + i\Gamma_\sigma/2$ in the *physical* Riemann sheet. Thus, we can find the location of the resonance by looking for the solutions of $S_0^{(0)}(\bar{s}_\sigma) = 0$ in the upper half of the complex plane. This zero condition may be written in a simpler manner as

$$\cot\delta_0^{(0)}(\bar{s}_\sigma) = -i. \tag{2.1}$$

Solving (2.1) requires analytical continuation from the real axis. The problem with determining the location of the resonance pole is that analytic continuation suffers from instability problems, due to the fact that the function is not known exactly on the real axis. In the old days, the low-energy experimental data for the S0 wave were hopeless: several solutions existed, and the errors were very large. This is one of the reasons for the widely different determinations of the sigma pole reported in the PDT [1]. Fortunately, and as already remarked, the situation has improved enormously in recent years. We now have reliable data from just above threshold up to the kaon mass, m_K, due to K_{l4} decays and $K_{2\pi}$ decays [8]. Moreover, and although still preliminary, we have even better very recent K_{l4} decay data [9].

In addition to this, we can profit from the fact that the determinations of the S0 wave, although not very precise, are reasonably compatible among themselves in the range 810–960 MeV; see Ref. [6], Eqs. (2.13). This is sufficient to make a fairly stable analytical extrapolation.

III. THE CONFORMAL MAPPING METHOD: THEORY

The unitarity of the S matrix imposes, in the elastic region, a constraint on partial wave amplitudes; for the S0 wave,

$$\text{Im}\,\hat{f}_0^{(0)}(s) = |\hat{f}_0^{(0)}(s)|^2.$$

If we write

$$\hat{f}_0^{(0)}(s) = \sin\delta_0^{(0)}(s)e^{i\delta_0^{(0)}(s)} = \frac{1}{s^{1/2}\psi(s)/2k - i}$$

with $k = \sqrt{s/4 - M_\pi^2}$ the c.m. momentum, then what we know of the analyticity and unitarity properties of the S0

partial wave amplitude implies that the function (*effective range function*)

$$\psi(s) \equiv (2k/s^{1/2}) \cot\delta_0^{(0)}(s)$$

is analytic in the complex plane cut from $-\infty$ to 0 and from the point where inelasticity is important, s_0, to $+\infty$ (Fig. 1): note that the *elastic cut* is absent in $\psi(s)$. In the case of the S0 wave, one can take $s_0 = 4m_K^2$.

The standard mathematical method to deal with this situation is to make a conformal mapping, from the variable s to the variable

$$w(s) = \frac{\sqrt{s} - \sqrt{4m_K^2 - s}}{\sqrt{s} + \sqrt{4m_K^2 - s}}$$

under which the cut plane is mapped into the unit disk (Fig. 1) and the analyticity properties of the effective range function in the complex plane in the variable s are *strictly equivalent* to uniform, absolute convergence of a Taylor series for the function ψ in the variable w in the disk $|w| \le 1$. For this convergence, it is necessary to separate off those poles of the effective range function that lie on the real axis, and it can also be convenient to separate the zeros of the effective range function in the same region (although this last is not necessary; see below). Of these we have one of each: a pole due to the so-called Adler zero of the partial wave amplitude, lying near the left-hand cut, at $s = \frac{1}{2}z_0^2$, $z_0 \simeq M_\pi$ (with M_π the charged pion mass); and then there is a zero due to the phase shift crossing $\pi/2$ for an energy $s^{1/2} = \mu_0 \sim 800$ MeV. Thus, we can, in all generality, write the following parametrization (we will at times simplify the notation by writing δ for $\delta_0^{(0)}$):

$$\cot\delta(s) = \frac{s^{1/2}}{2k} \frac{M_\pi^2}{s - \frac{1}{2}z_0^2} \frac{\mu_0^2 - s}{\mu_0^2} \{\hat{B}_0 + \hat{B}_1 w(s) + \hat{B}_2 w(s)^2$$
$$+ \ldots\}.$$

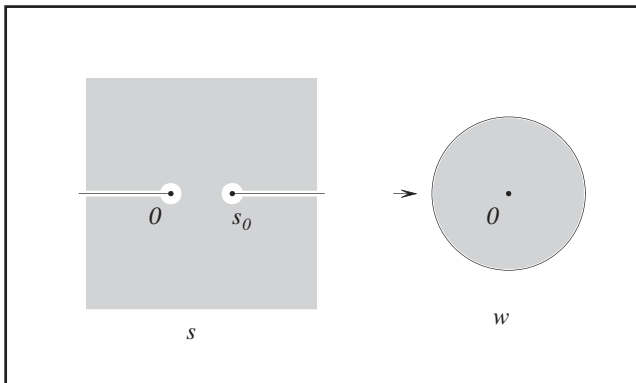However, if using this expression with a finite number of terms, a ghost is generated in the partial wave amplitude.

FIG. 1. The mapping $s \to w$.

Although this ghost is harmless, being weak and very near the left-hand cut (see the discussion in Appendix A) it is, as a matter of principle, better to use a ghost-free expansion. For this, it is sufficient to replace the formula above by

$$\cot\delta(s) = \frac{s^{1/2}}{2k} \frac{M_\pi^2}{s - \frac{1}{2}z_0^2} \frac{\mu_0^2 - s}{\mu_0^2} \left\{ \frac{z_0^2}{M_\pi\sqrt{s}} + B_0 + B_1 w(s) \right.$$
$$\left. + B_2 w(s)^2 + \ldots \right\}. \tag{3.1}$$

Another possibility is *not* to separate the zero at $s = \mu_0^2$, writing simply

$$\cot\delta(s) = \frac{s^{1/2}}{2k} \frac{M_\pi^2}{s - \frac{1}{2}z_0^2} \left\{ \frac{z_0^2}{M_\pi\sqrt{s}} + B_0 + B_1 w(s) \right.$$
$$\left. + B_2 w(s)^2 + \ldots \right\}. \tag{3.2}$$

In this last case the zero is generated by the combination of the $B_n w^n$, and we will, generally speaking, need one more term in the expansion than if we used (3.1), although the number of free parameters will be the same.

Now, the key point for us is that the expansions (3.1) and (3.2) converge in the *whole* cut plane: therefore, they can be used as they stand to solve (2.1), and hence to find the location of the resonance. In particular, the expansions converge at threshold and thus can also be used to determine the values of the scattering length and effective range parameters. As for the evaluation of the errors, we simply vary the parameters within the errors obtained in the fits to data, assuming that they are uncorrelated, and thus calculate the errors of the related observables. We have verified that, in the case of our best fit (Eq. (4.8) below) the parameters are actually almost uncorrelated, so this is a valid procedure.

It is also to be noted that, because of the nature of the singularities on the cuts, we expect the B_n to decrease like

$$|B_n| \sim (n + 1)^{-3/2} \tag{3.3}$$

for large n. This last condition is, strictly speaking, only valid when one separates off the ghost piece, i.e., for the B_n, but, because the ghost-producing piece $z_0^2/M_\pi\sqrt{s}$ is so small, it also would hold for the $\hat{B}_n$ as well. It should also be taken into account that the behavior (3.3) is expected to set in earlier if we separate out the zero, as in (3.1); if we do not separate out the zero, the first B_n are constrained to cooperate to build up the zero, so the asymptotic regime will set later. This is in fact what we observe in our fits.

We can get good fits to data with one or two B_ns in (3.1), or two to three in (3.2); no more are necessary. Generally speaking, the situation is as follows: if we include in the fit only *low* energy (K-decay) data, then a $\chi^2/$d.o.f. smaller than unity is achieved with only two parameters. If we include a third parameter in the fit, there is no substantial decrease of the $\chi^2/$d.o.f., and there appear spurious minima: hence this third parameter is rejected by the fit. On the

other hand, if we also include in the fit *high*-energy data, then one can still fit with only two parameters, but if including a third parameter, the fit remains meaningful (the $\chi^2/$d.o.f. decreases appreciably) and, moreover, it is theoretically more satisfactory. In particular, in the case where we fit with expression (3.2), the presence of terms in w, w^2 means that we uncorrelate the right-hand cut and the left-hand cut of the partial wave, while with only two B_ns we would have an *average* representation of both cuts. But even if we include high-energy data, one cannot meaningfully fit with four parameters, as spurious minima appear. All this should be clear from the values of the B_ns and the $\chi^2/$d.o.f.s of the various fits in the text.

Another technical point is that we will fix the Adler zero to $z_0 = M_\pi$ in our analysis (the fit depends very little on the precise value of z_0, provided it is near this), and we let the B_n, μ_0 vary.

Before turning to the actual calculations a few words have to be said on the matter of analytical extrapolation. In principle, if one knows an analytic function on a segment of the real line, the function is determined on the whole (cut) plane. In practice, however, the function is not known exactly. Therefore, one has to test for potential *instabilities* in the extrapolation procedure. There are two important sources of instability: first, instabilities due to small variations of the central values inside the error bars of what one may consider the best fit, and, second, we have the dependence of the results on the number of parameters used for the fits, or on the different functions used for the fits. The way to deal with this is to try extrapolations with fits to various data sets (provided they are compatible within errors), and to try fits with varying numbers of parameters and expressions. The final errors should be enlarged to encompass the results with these various fits. This is what will be done here.

IV. RESULTS

A. Fits with only low-energy (*K* decay) data

We first fit with only data from kaon decays, generically the more reliable ones. In what follows "old *K* decay data" refers to the data from K_{l4} and $K_{2\pi}$ decays of Ref. [8] (for the exact value used from the $K_{2\pi}$ analysis, see Appendix B here), and "NA48/2" refers to the K_{l4} decay data of Ref. [9]. For these last data statistical and systematic errors are taken into account; they are added quadratically. For the K_{l4} data of Pislak *et al.*, Ref. [8], we have added statistical and systematic errors in quadrature, with the exception of the higher energy point, which we will discuss in Appendix B. Let us note that all "old *K* decay data," as well as "NA48/2" data, lie well inside the convergence region of the conformal expansion, as can be seen in Fig. 2. It should also be noted that K_{l4} decay data only give the difference between the S0 and *P* wave phase shifts. However, this is not important as the *P* wave phase shift can be determined with great precision from the form
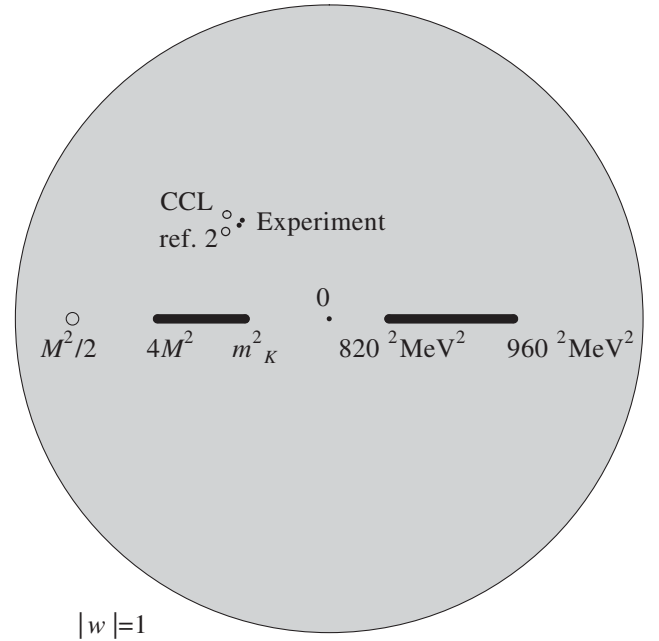

FIG. 2. The w disk, $|w| < 1$. The thick lines are the regions where one has reliable experimental data. The sigma pole is also shown; the points labeled "Experiment" are those obtained from our fits to experiment, Eqs. (4.3) and (4.10). The open point labeled CCL is the Caprini, Colangelo, Leutwyler calculation [3] with error like the thickness of the point, and the open point labeled "ref. 2" is the average quoted in Eq. (1.2).

factor of the pion [10]. Likewise, $K_{2\pi}$ decays give the difference between S0 and S2 phases, but this last is small on the kaon mass, and reasonably well known there [11].

We start with only one parameter, B_0, plus the location of the zero, μ_0, in (3.1), or two parameters B_0, B_1 in (3.2), if not separating the zero.[3] We first fit kaon decay data, including both "old *K* decay data" and "NA48/2." We find the following results ($a_0^{(0)}$ and $b_0^{(0)}$ are systematically given in units of M_π).

A-I) Separating the zero, i.e., with Eq. (3.1): $\chi^2/$d.o.f. $= 15.5/(22 - 2)$;

$$B_0 = 17.5 \pm 0.4, \qquad \mu_0 = 744 \pm 24 \text{ MeV}$$
$$a_0^{(0)} = 0.226 \pm 0.005, \qquad b_0^{(0)} = 0.287 \pm 0.008; \quad (4.1)$$
$$M_\sigma = 493 \pm 5 \text{ MeV}, \qquad \Gamma_\sigma/2 = 228 \pm 9 \text{ MeV}.$$

A-II) Not separating the zero, i.e., with Eq. (3.2): $\chi^2/$d.o.f. $= 15.8/(22 - 2)$;

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[3]This second method is less efficient in the present case as the data region, $4M_\pi^2 < s \leq m_K^2$, is very asymmetric with respect to the origin $w = 0$ around which we are expanding.

$$B_0 = 5.1 \pm 0.3, \qquad B_1 = -18.7 \pm 0.6$$

$$a_0^{(0)} = 0.221 \pm 0.006, \qquad b_0^{(0)} = 0.296 \pm 0.008; \quad (4.2)$$

$$M_\sigma = 458 \pm 5 \text{ MeV}, \qquad \Gamma_\sigma/2 = 235 \pm 3 \text{ MeV}.$$

The phase crosses $\pi/2$ at the energy $\mu_0 = 865 \pm 8$ MeV.

It is clear that the errors in (4.1) and (4.2) for the observable quantities, $a_0^{(0)}$, $b_0^{(0)}$, M_σ, and $\Gamma_\sigma/2$, are purely statistical. While the results from (4.1) and (4.2) are similar, the differences are in general larger than these nominal errors. One can get a reasonable error estimate by averaging (4.1) and (4.2), and enlarging the errors by including a systematic uncertainty which amounts to half the difference between (4.1) and (4.2). In this way we get what we consider the best result using only K decay data,

$$a_0^{(0)} = 0.224 \pm 0.006(\text{St.}) \pm 0.003(\text{Sys.}),$$

$$b_0^{(0)} = 0.292 \pm 0.008(\text{St.}) \pm 0.005(\text{Sys.});$$

$$M_\sigma = 476 \pm 5(\text{St.}) \pm 18(\text{Sys.}) \text{ MeV}, \tag{4.3}$$

$$\Gamma_\sigma/2 = 234 \pm 9(\text{St.}) \pm 4(\text{Sys.}) \text{ MeV}.$$

These fits permit a good determination of the low-energy parameters, but, as we will see later, are less reliable for extrapolation to the sigma pole; nevertheless, we still get reasonable values for M_σ and Γ_σ.

An important question is the choice of parameters. We have elected to *fix* the location of the Adler zero at $z_0 = M_\pi$, but we could, for example, in the fit separating the zero, have fixed $\mu_0 = 810$ MeV (which is the approximate value we get from fits including data at higher energies, see below) and consider, instead of it, z_0 to be the free parameter. In this case we find

$$\mu_0 \equiv 810 \text{ MeV}; \qquad B_0 = 15.7 \pm 0.3,$$

$$z_0 = 147 \pm 10 \text{ MeV}; \qquad a_0^{(0)} = 0.222 \pm 0.009,$$

$$b_0^{(0)} = 0.300 \pm 0.006; \qquad M_\sigma = 476 \pm 5 \text{ MeV}, \tag{4.4}$$

$$\Gamma_\sigma/2 = 243 \pm 2 \text{ MeV}$$

with $\chi^2/\text{d.o.f.} = 15.6/(22-2)$: Eq. (4.4) is quite compatible with (4.1), with an almost equal $\chi^2/\text{d.o.f.}$. For this reason, we will continue to fix $z_0 = M_\pi$ in the remaining fits.

B. Two-parameter fits including also higher energy data

We denote by PY05-(2.13) the set of combined data from various experiments, at energies 810 MeV $\leq s^{1/2} \leq$ 952 MeV, collected in Eqs. (2.13) in Ref. [6] (and repeated here, Appendix B, for ease of reference). Note that, as happened with the K decay data, these additional data also fall well inside the convergence region of the conformal expansion (Fig. 2).

It is possible to fit all the K decay data ("old" as well as "NA48/2"), plus PY05-(2.13), with (3.1) and only one B_0. We find the following results:

B-I) If separating the zero (with (3.1)), $\chi^2/\text{d.o.f.} = 21.5/(31-2)$ and

$$B_0 = 17.2 \pm 0.4, \qquad \mu_0 = 771 \pm 19 \text{ MeV};$$

$$a_0^{(0)} = 0.228 \pm 0.005, \qquad b_0^{(0)} = 0.286 \pm 0.007; \quad (4.5)$$

$$M_\sigma = 490 \pm 5 \text{ MeV}, \qquad \Gamma_\sigma/2 = 237 \pm 7 \text{ MeV}.$$

B-II) *Not* separating the zero at μ_0, i.e., using (3.2), one finds $\chi^2/\text{d.o.f.} = 21.3/(31-2)$ and

$$B_0 = 2.8 \pm 0.3, \qquad B_1 = -24.0 \pm 0.6;$$

$$a_0^{(0)} = 0.213 \pm 0.006, \qquad b_0^{(0)} = 0.296 \pm 0.008; \quad (4.6)$$

$$M_\sigma = 467 \pm 5 \text{ MeV}, \qquad \Gamma_\sigma/2 = 213 \pm 3 \text{ MeV}.$$

These fits with only two parameters are somewhat rigid; fortunately, it is possible to include in the fit one parameter more, which permits more flexibility.

C. Three-parameter fits including also higher energy data

When we include the PY05-(2.13) together with all (old as well as NA48/2) K decay data, it is possible to include a further parameter in the fit, thus making it more flexible, getting also a better $\chi^2/\text{d.o.f.}$; this was not possible with K decay data alone, since there would have been too few data, and including a superfluous parameter would have given spurious minima. Moreover, the fits will now be more realistic because the terms in B_1, B_2 represent (in the average) the cuts of the effective range function, and because now the data are more symmetrically distributed, lying on both sides of the point around which we are expanding, $w = 0$; see Fig. 2.

We fit as in the previous subsection and find the following results.

C-I) If we use (3.1), i.e., we separate off the zero, the $\chi^2/\text{d.o.f.}$ becomes $\chi^2/\text{d.o.f.} = 21.0/(31-3)$ and we get

$$B_0 = 19.0 \pm 4, \qquad B_1 = 4.4 \pm 0.8,$$

$$\mu_0 = 781 \pm 19 \text{ MeV}; \qquad a_0^{(0)} = 0.235 \pm 0.008,$$

$$b_0^{(0)} = 0.282 \pm 0.008; \qquad M_\sigma = 496 \pm 6 \text{ MeV}, \tag{4.7}$$

$$\Gamma_\sigma/2 = 258 \pm 8 \text{ MeV}.$$

C-II) Not separating the zero, i.e., with (3.2): the $\chi^2/\text{d.o.f.}$ improves significantly when introducing the new parameter, B_2. We find $\chi^2/\text{d.o.f.} = 18.7/(31-3)$ and
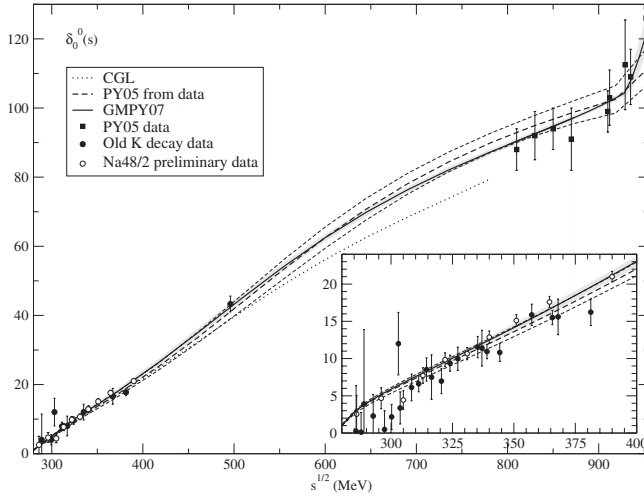
FIG. 3. The data points fitted, together with the phase shifts corresponding to our best fit, (4.8) (full line, with error given by the dark band). Dashed lines: the error band of the old PY05 fit in Ref. [6]. The dotted line is the prediction of Ref. [7], based on Roy equations and the value (1.6) for $a_0^{(0)}$. In the inserted box, a blowup of the low energy region.

$$B_0 = 4.3 \pm 0.3, \qquad B_1 = -26.7 \pm 0.6,$$
$$B_2 = -14.1 \pm 1.4; \qquad a_0^{(0)} = 0.230 \pm 0.010,$$
$$b_0^{(0)} = 0.288 \pm 0.009; \qquad M_\sigma = 474 \pm 6 \text{ MeV}$$
$$\Gamma_\sigma/2 = 254 \pm 4 \text{ MeV}.$$
(4.8)

The zero of $\cot\delta_0^{(0)}(s)$ now occurs at $s = \mu_0^2$, with $\mu_0 = 801 \pm 6$ MeV. In principle, (4.8), shown in Fig. 3, is the best fit. It takes into account the most of theoretical infor-

mation, and it has the smallest $\chi^2/$d.o.f. However, (4.7) also takes into account the same theoretical information, and its $\chi^2/$d.o.f., while larger, is quite comparable to that of (4.8); in fact, the phase shifts described by (4.7) and (4.8) overlap within their errors.

Before continuing, a few words have to be said on the errors in (4.8) (or (4.7)). The level of precision of these fits is such that the errors are comparable to the expected contributions of isospin symmetry breaking, which should therefore be considered. For the best values of the low-energy parameters and location of the sigma pole, this is not important. We will compose the results in (4.7) and (4.8), adding as systematic error half the difference, and the final uncertainty is substantially larger than the estimated effect of isospin breaking corrections; for the *parametrization*, we will discuss this in Sec. VI.

As we did for the fits with only K decay data, we average the results in (4.7) and (4.8), each weighted now with its corresponding $\chi^2/$d.o.f., and enlarge the errors including a systematic error of half the difference. In this way we find what we consider our final result, for the low-energy parameters and location of the sigma pole:

$$a_0^{(0)} = 0.233 \pm 0.010(\text{St.}) \pm 0.003(\text{Sys.})M_\pi^{-1},$$
$$b_0^{(0)}0 = 0.285 \pm 0.009(\text{St.}) \pm 0.003(\text{Sys.})M_\pi^{-3};$$
$$M_\sigma = 484 \pm 6(\text{St.}) \pm 11(\text{Sys.}) \text{ MeV},$$
$$\Gamma_\sigma/2 = 255 \pm 8(\text{St.}) \pm 2(\text{Sys.}) \text{ MeV}.$$
(4.9)

Equations (4.3) and (4.9) are *very* compatible, showing the stability of the fits, and of the extrapolations to the sigma pole, against the number of parameters used and the formulas employed.

Process	M_σ	$\Gamma_\sigma/2$	$a_0^{(0)}$	$b_0^{(0)}$
K decay data, 2 parameter (4.1)	493 ± 5	228 ± 9	0.226 ± 0.005	0.287 ± 0.008
K decay data, 2 parameter (4.2)	458 ± 5	235 ± 3	0.221 ± 0.006	0.296 ± 0.008
K decay data average	476 ± 23	234 ± 13	0.224 ± 0.009	0.292 ± 0.013
K decay data + PY05, 3 parameter (4.7)	496 ± 6	258 ± 8	0.235 ± 0.008	0.282 ± 0.008
K decay data + PY05, 3 parameter (4.8)	474 ± 6	254 ± 4	0.230 ± 0.010	0.288 ± 0.009
Best estimate	**484 ± 17**	**255 ± 10**	**0.233 ± 0.013**	**0.285 ± 0.012**

The best values for the complex zero, $\bar{s}_\sigma$, are shown, in the w plane, in Fig. 2, where we have collected the results of our fits with two parameters to K decay data only and their average, together with our three-parameter fits. The results of our best estimate and the more representative of our results for the sigma pole and low-energy parameters are collected in the accompanying Table ($a_0^{(0)}$ and $b_0^{(0)}$ in units of M_π).

One may wonder whether our choice of data at high energies may have biased our results. To test this, we will consider two alternate fits, using, instead of the PY-05 collection of data, the K decay data plus the data sets B,

C given by Grayer *et al.* [12]. We choose these solutions out of the four solutions in Ref. [12] because they are the ones that satisfy best forward dispersion relations (apart, of course, from our solution (4.8) here), as shown in Ref. [6]. For solution C we find $\chi^2/$d.o.f. $= 55.7/(48 - 3)$ and

$$B_0 = 3.57 \pm 0.17, \qquad B_1 = -24.3 \pm 0.5,$$
$$B_2 = -6.3 \pm 1.3; \qquad a_0^{(0)} = 0.226 \pm 0.008,$$
$$b_0^{(0)} = 0.299 \pm 0.007; \qquad M_\sigma = 465 \pm 5 \text{ MeV},$$
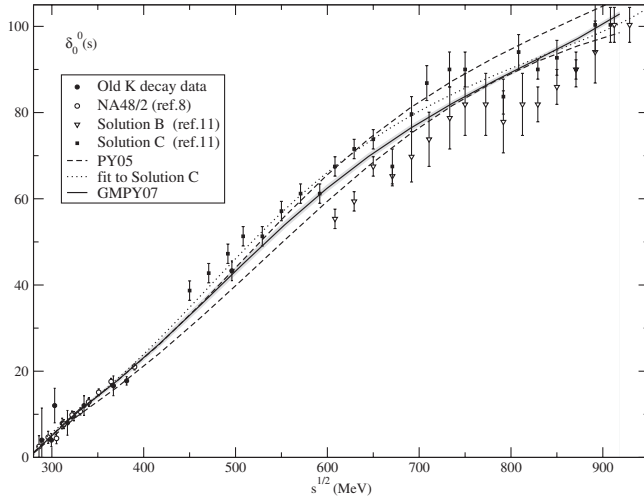$$\Gamma_\sigma = 231 \pm 3 \text{ MeV}.$$
(4.10)

FIG. 4. The fit to the data in solution C of Grayer *et al.* (Eq. (4.10), dotted line), together with the error band for the old fit in PY05 (dashed lines), and our best fit here (Eq. (4.8), continuous line and gray band). The data for solutions B and C of Grayer *et al.* are also shown.

While as for solution B, we have $\chi^2/\text{d.o.f.} = 31.4/(41 - 3)$ and

$$B_0 = 7.63 \pm 0.23, \qquad B_1 = -23.2 \pm 0.6,$$
$$B_2 = -23.0 \pm 1.4; \qquad a_0^{(0)} = 0.251 \pm 0.011,$$
$$b_0^{(0)} = 0.269 \pm 0.006; \qquad M_\sigma = 477 \pm 7 \text{ MeV},$$
$$\Gamma_\sigma = 321 \pm 6 \text{ MeV}. \tag{4.11}$$

It should be noted that the errors in Eqs. (4.10) are purely nominal. In fact, the errors in the solutions B, C of Grayer *et al.* that we have used are only the *statistical* errors, while the very existence of several solutions to the same set of

raw data in Ref. [12] shows that *systematic* errors must be substantially larger (by a factor of 3 or more; see the comments at the end of Appendix B, and Fig. 4 here). This is reflected in the incompatibility of the fits. However, and in spite of these problems with the errors, these new fits show the stability of the results for $a_0^{(0)}$, $b_0^{(0)}$, and, to a lesser extent, M_σ, while the value of Γ_σ is shown to be less reliably determined.

Finally, one may also wonder about the possibility of adding more parameters to the fits. It turns out that a fourth parameter would be redundant, as the $\chi^2/\text{d.o.f.}$ does not improve appreciably by introducing it and spurious minima appear. Specifically, we find $\chi^2/\text{d.o.f.} = 17.5/(31 - 4)$ and the central values for the parameters $B_0 = 5.8$, $B_1 = -22.4$, $B_2 = -26.3$, and $B_3 = -32.5$, which do not decrease at the expected rate.

V. THE SCATTERING LENGTH WITH THE EFFECTIVE RANGE EXPANSION

The scattering length (but of course, not the location of the sigma pole) can also be obtained with the effective range expansion. We expand the effective range function in powers of k^2,

$$\psi(s) = \frac{1}{M_\pi a_0^{(0)}} + 4R_0 k^2/M_\pi^4 + \dots. \tag{5.1}$$

Only two terms are necessary. The expansion here, however, is poorly convergent when compared to the conformal mapping one; due to the presence of the Adler zero at $s = M_\pi^2/2$, the circle of convergence only extends to $s = \frac{15}{2}M_\pi^2$, i.e., to energies $s^{1/2} \lesssim 385$ MeV. To remain well inside the region of convergence of (5.1), we only fit data at energies $s^{1/2} \leq 351$ MeV, and one then finds ($a_0^{(0)}$ and R_0 in units of M_π) [8,9]

$$\chi^2/\text{d.o.f.} = \frac{6.3}{9 - 2}; \qquad a_0^{(0)} = 0.225 \pm 0.007^{+0}_{-0.15}, \qquad R_0 = -[1.00 \pm 0.08^{+0.27}_{-0}]; \qquad [\text{Old } K \text{ decay data}].$$

$$\chi^2/\text{d.o.f.} = \frac{2.4}{7 - 2}; \qquad a_0^{(0)} = 0.229 \pm 0.005 \pm 0.024, \qquad R_0 = -[0.93 \pm 0.05^{+0.38}_{-0.22}]; \qquad [\text{NA48/2 data}] \tag{5.2}$$

$$\chi^2/\text{d.o.f.} = \frac{8.8}{17 - 2}; \qquad a_0^{(0)} = 0.229 \pm 0.004 \pm 0.021, \qquad R_0 = -[0.94 \pm 0.04^{+0.34}_{-0.21}]; \qquad [\text{Old } K \text{ decay + NA48/2}].$$

The first errors here are statistical; the second are obtained changing the point where one stops fitting to 367 MeV or to 340 MeV. Although the results are compatible with (4.8), it is clear that the effective range method is much less precise and also less stable than the conformal mapping one—not surprisingly, as the last incorporates a lot more analyticity information.

VI. SUMMARY AND DISCUSSION

In the present paper we have found an accurate representation of the S0 wave, given by the parametrization

(4.8), that we consider the best of our results. However, and as noted just after this equation, the level of precision attained requires that we consider isospin breaking effects related, for example, to whether we use the parametrization for $\pi^+\pi^-$ or $\pi^0\pi^0$ scattering.

It turns out that electromagnetic corrections, which are likely the largest part of isospin breaking corrections, are taken into account in the analysis of systematic errors in the NA48/2 experiment. This is the only place where they can be important, as the errors in all other experiments are much larger than the estimated isospin breaking effects.

For example, for the P wave, where separate fits were made in Ref. [10] to $\pi^0\pi^+$ and $\pi^+\pi^-$ processes, the corresponding corrections were found to be at the 1% level, except for the kinematical corrections. We therefore take this into account by considering that, in the parametrization in (4.8), which we repeat here for ease of reference,

$$\cot\delta_0^{(0)}(s) = \frac{s^{1/2}}{2k}\frac{M_\pi^2}{s-\frac{1}{2}z_0^2}\left\{\frac{z_0^2}{M_\pi\sqrt{s}}+B_0+B_1w(s)+B_2w(s)^2\right\}, \qquad w(s)=\frac{\sqrt{s}-\sqrt{4m_K^2-s}}{\sqrt{s}+\sqrt{4m_K^2-s}}; \tag{6.1}$$

$$B_0 = 4.3\pm0.3, \qquad B_1 = -26.7\pm0.6, \qquad B_2 = -14.1\pm1.4; \qquad z_0 = M_\pi,$$

we have to interpret

$$k=\begin{cases}\sqrt{s/4-M_\pi^2}, & M_\pi=m_{\pi^+}=139.57\text{ MeV} \quad \text{for }\pi^+\pi^- \text{ scattering,}\\ \sqrt{s/4-M_\pi^2}, & M_\pi=m_{\pi^0}=134.98\text{ MeV} \quad \text{for }\pi^0\pi^0 \text{ scattering,}\end{cases}$$

according to the process for which it is used.

The values for scattering length, effective range parameter, and location of the sigma pole from (6.1) are

$$a_0^{(0)} = 0.230\pm0.010, \qquad b_0^{(0)} = 0.288\pm0.009;$$
$$M_\sigma = 474\pm6\text{ MeV} \qquad \Gamma_\sigma/2 = 254\pm4\text{ MeV}. \tag{6.2}$$

This is perfectly compatible, both in central values and errors, with the results reported in (4.9). Because the numbers in (4.9) include a systematic error due to comparison of results with two fitting formulas (something that is of course impossible to do for the parametrization itself), they should be considered to be the more reliable results for these quantities; so we repeat them here for ease of reference:

$$a_0^{(0)} = 0.233\pm0.010(\text{St.})\pm0.003(\text{Sys.})M_\pi^{-1},$$
$$b_0^{(0)} = 0.285\pm0.009(\text{St.})\pm0.003(\text{Sys.})M_\pi^{-3};$$
$$M_\sigma = 484\pm6(\text{St.})\pm11(\text{Sys.})\text{ MeV}, \tag{6.3}$$
$$\Gamma_\sigma/2 = 255\pm8(\text{St.})\pm2(\text{Sys.})\text{ MeV}.$$

These results overlap, within errors, with what was found in Ref. [7] using chiral perturbation theory and analyticity. The value of $a_0^{(0)}$ is also compatible with experimental determinations [13] of this quantity, using a method devised by Cabibbo from $K\to\pi^+\pi^0\pi^0$ decays, or from pionium decay [14] (although our results are more precise and do not depend on knowledge of $a_0^{(2)}$).

Another question is whether one can improve our determinations. The answer is no, if we only use experimental data on the S0 wave. But it is possible to give an independent, perhaps more precise determination, calculating the function $\cot\delta_0^{(0)}(s)$ for complex s with dispersion relations (Roy equations). This necessitates input of experimental data on other waves, and input in the high-energy Regge

region, so it is not a determination based purely on S0 wave experimental data. Preliminary results indicate that one would find numbers for M_σ and $\Gamma_\sigma/2$ compatible with (1.3), and with smaller errors. This procedure could also give improved values for the scattering length and effective range parameters.

Note added.—After this work was completed, we have had notice of a calculation by J. Gasser [17][4] in which account is taken, for Ke4 decays, of the fact that in the real world isospin is broken. A full analysis is missing; but, if we take the results of the calculation at face value, we can repeat our best fit (this would correspond to Eq. (4.8))

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[4]We are grateful to Drs. Colangelo and Gasser for communicating the results prior to publication.

incorporating these corrections. We then find: $B_0 = 3.80 \pm 0.34$, $B_1 = -27.1 \pm 0.8$, $B_2 = -8.3 \pm 1.8$; $a_0^{(0)} = 0.211 \pm 0.010 M_\pi^{-1}$, $b_0^{(0)} = 0.278 \pm 0.010 M_\pi^{-3}$; $M_\sigma = 481 \pm 7$ MeV; $\Gamma_\sigma/2 = 237 \pm 5$ MeV. The phase shift itself moves very little, by less than 1° in the whole applicability range. Although, as stated, a full analysis will be necessary, it does not seem that including isospin breaking corrections will much affect our results.

APPENDIX A: ON GHOSTS

As remarked in the main text, we have separated off a term $z_0^2/M_\pi\sqrt{s}$ in Eqs. (3.1) and (3.2) to avoid the appearance of a ghost, i.e., of a spurious pole in $\hat{f}_0^{(0)}(s)$ located between the Adler zero and $s = 0$. It turns out that the effect of this ghost is negligible. To show this, we repeat our best fit, Eq. (4.8), but not eliminating the ghost; that is to say we replace Eq. (3.2) by

$$\cot\delta(s) = \frac{s^{1/2}}{2k}\frac{M_\pi^2}{s - \frac{1}{2}z_0^2}\{\hat{B}_0 + \hat{B}_1 w(s) + \hat{B}_2 w(s)^2 + \ldots\},$$
(A1)

i.e., without the term that removes the ghost. Then we find (4.8) replaced by

$$\hat{B}_0 = 4.5 \pm 0.3, \qquad \hat{B}_1 = -26.9 \pm 0.6,$$
$$\hat{B}_2 = -13.5 \pm 1.4; \qquad a_0^{(0)} = 0.231 \pm 0.010;$$
$$b_0^{(0)} = 0.287 \pm 0.008; \qquad M_\sigma = 475 \pm 6 \text{ MeV}$$
$$\Gamma_\sigma/2 = 253 \pm 5 \text{ MeV},$$
(A2)

with $\chi^2/\text{d.o.f.} = \dfrac{18.8}{31 - 3}$. This corresponds to $\mu_0 = 801 \pm 6$ MeV and is practically indistinguishable from (4.8). Removing the ghost is little more than an aesthetical requirement.

APPENDIX B: ON EXPERIMENTAL DATA

We here say a few words about the error we have taken for the datum at highest energy of Pislak et al. [8]. In this reference, the number given is

$$\delta_0^{(0)}((381.4 \text{ MeV})^2) - \delta_1((381.4 \text{ MeV})^2)$$
$$= 0.285 \pm 0.014(\text{St.}) \pm 0.003(\text{Sys.}).$$

Now, this result is suspicious. The error is the smallest of all those among the data of Pislak et al., [8], although the datum is actually an average value near the edge of phase space. Moreover, as we will see in a moment, the central value is incompatible with other determinations. For this reason, we have, in our fits, done as in Ref. [6] and multiplied the statistical error by a factor of 1.5: so, we have

taken

$$\delta_0^{(0)}((381.4 \text{ MeV})^2) - \delta_1((381.4 \text{ MeV})^2)$$
$$= 0.285 \pm (1.5 \times 0.014).$$

We could have chosen, instead of this, to add systematic and statistical error *linearly* for this datum, i.e., to fit with

$$\delta_0^{(0)}((381.4 \text{ MeV})^2) - \delta_1((381.4 \text{ MeV})^2)$$
$$= 0.285 \pm 0.017.$$

In this case, (4.8) is replaced by

$$\frac{\chi^2}{\text{dof}} = \frac{20.0}{31 - 3}; \qquad B_0 = 4.5 \pm 0.3,$$
$$B_1 = -26.9 \pm 0.6, \qquad B_2 = -15.3 \pm 1.4;$$
$$a_0^{(0)} = 0.231 \pm 0.010, \qquad b_0^{(0)} = 0.285 \pm 0.008;$$
$$\mu_0 = 804 \pm 5 \text{ MeV}; \qquad M_\sigma = 476 \pm 6 \text{ MeV}$$
$$\Gamma_\sigma/2 = 258 \pm 5 \text{ MeV},$$

that is to say, almost identical to (4.8).

To ascertain the degree of incompatibility of the datum of Pislak et al. with the others, we have repeated the fits with two other possibilities: first, adding its errors in quadrature. In this case, the χ^2 increases to 21.6, i.e., three units above what we got in (4.8). Alternatively, if we *remove* the datum from the fit, the χ^2 decreases to 15.7: the datum at 381.4 MeV of Pislak et al., with its original error, carries a penalty of increase of the χ^2 by almost six units and would certainly bias the results if included *tel quel*. This justifies treating its error as we have done.

Next, we explain the value we give for the datum at m_K^2. From the decays $K^+ \to \pi^0 \pi^+$, $K_S \to \pi^0 \pi^0$, and $K_S \to \pi^+ \pi^-$ one can obtain the difference $\delta_0^{(0)}(m_K^2) - \delta_0^{(2)}(m_K^2)$. With the latest results from KLOE and *including* radiative corrections as in Ref. [15], one finds

$$\delta_0^{(0)}(m_K^2) - \delta_0^{(2)}(m_K^2) = 57.27 \pm 0.82(\text{exp}) \pm 3(\text{radiative})$$
$$\pm 1°(\text{chiral perturbation}$$
$$\text{approximations}).$$

The meaning of the various errors is given in Ref. [15]. Subtracting the value of $\delta_0^{(2)}(m_K^2)$ obtained from fits to experiment in Ref. [6], and adding the errors linearly (as is advisable given the uncertainties in their evaluation), we arrive at the result we have been using in our fits:

$$\delta_0^{(0)}(m_K^2) = 48.7 \pm 4.9°. \qquad \text{(B1)}$$

We have verified that a very similar result is found if using

the PDT [1] decay data, and the prescription for radiative corrections of Refs. [16]

Finally, and for ease of reference, we repeat here the data PY05 (Ref. [6], Eqs. (2.13)) used in the fits:

$$\delta_0^{(0)}(0.870^2 \text{ GeV}^2) = 91 \pm 9°; \qquad \delta_0^{(0)}(0.910^2 \text{ GeV}^2) = 99 \pm 6°; \qquad \delta_0^{(0)}(0.935^2 \text{ GeV}^2) = 109 \pm 8°;$$

$$\delta_0^{(0)}(0.912^2 \text{ GeV}^2) = 103 \pm 8°; \qquad \delta_0^{(0)}(0.929^2 \text{ GeV}^2) = 112.5 \pm 13°; \qquad \delta_0^{(0)}(0.952^2 \text{ GeV}^2) = 126 \pm 16°; \quad \text{(B2)}$$

$$\delta_0^{(0)}(0.810^2 \text{ GeV}^2) = 88 \pm 6°; \qquad \delta_0^{(0)}(0.830^2 \text{ GeV}^2) = 92 \pm 7°; \qquad \delta_0^{(0)}(0.850^2 \text{ GeV}^2) = 94 \pm 6°.$$

Note that the errors here include systematic errors, estimated by comparing different determinations. They are a factor of 3 or more larger than the nominal, statistical errors of the different phase shift analyses (e.g., those in Ref. [12]).

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