Dimensional duality

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We show that string theory on a compact negatively curved manifold, preserving a $U(1)^{b_1}$ winding symmetry, grows at least b_1 new effective dimensions as the space shrinks. The winding currents yield a "D-dual" description of a Riemann surface of genus h in terms of its $2h$ dimensional Jacobian torus, perturbed by a closed string tachyon arising as a potential energy term in the world sheet sigma model. D-branes on such negatively curved manifolds also reveal this structure, with a classical moduli space consisting of a b_1-torus. In particular, we present an anti-de Sitter (AdS)/conformal field theory (CFT) system which offers a nonperturbative formulation of such supercritical backgrounds. Finally, we discuss generalizations of this new string duality.

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I. INTRODUCTION

An important feature of some string dualities [1], matrix theory [2], and the anti-de Sitter (AdS)/conformal field theory (CFT) correspondence [3], is the emergence of new dimensions in a dual description. Previous work has focused on Ricci-flat and positive curvature [4] compactifications, and closely related systems. In this work, we study a new duality with a new source of emergent dimensions, arising from compact manifolds with *negative* curvature.[1] The additional dimensions arise from the rich topology of one-cycles in these manifolds.

In particular, the effective central charge $c_{\rm eff}$ is enhanced by the sum over winding sectors [5,6], due to the exponential growth of the fundamental group [7,8]. $c_{\rm eff}$, as a measure of the exponential growth of the density of single-string states, is a good operational definition of dimensionality in perturbative string theory. For $c_{\rm eff} > c_{\rm eff}^{\rm crit}$ the theory is supercritical. This raises the question of whether the system admits a useful dual description in terms of a more familiar presentation of supercritical string theory [9]. We would like to know how large $c_{\rm eff}$ becomes, and whether the system naturally reorganizes itself so that $c_{\rm eff}$ arises from additional ordinary geometrical dimensions. In this work we address this question, focusing initially on the simplest case of an expanding Riemann surface $\mathcal{M}$ of genus h, which we find admits a natural dual description in terms of its Jacobian torus.

In Sec. II we set up the system and indicate our regime of control. In Sec. III, we introduce the symmetries of the system and use them to derive a dual description. In Sec. IV, we present explicit models realizing our duality, studying them using a path integral transform along the lines of [10]. The analysis in Secs. II, III, and IV pertains in a semi-infinite range of time for which perturbative string theory applies. In Sec. V we begin by discussing generalizations and approaches to the initial singularity. We conclude by introducing an AdS/CFT system which provides a candidate nonperturbative formulation of supercritical theories arising from compact hyperbolic spaces. This makes manifest the torus corresponding to a small-radius space in the infrared regime of the CFT, and offers a concrete approach to addressing the initial singularity.

II. SETUP

We are interested in a target space containing a genus h Riemann surface $\mathcal{M}$ as a factor. There are many such backgrounds, generically time dependent, with the negative curvature driving expansion of the space.[2] It will prove particularly simple to choose the metric at some reference time X^0_r to be

$$ds^2 = \omega^a \gamma_{ab} \bar{\omega}^b dz d\bar{z}. \tag{2.1}$$

Here ω^a, $\bar{\omega}^a$ ($a = 1, \dots, h$) denote the h linearly independent holomorphic and antiholomorphic 1-forms on our Riemann surface, and γ_{ab} is a constant positive symmetric $h \times h$ matrix. The nonlinear sigma model describing the time evolution of $\mathcal{M}$ is

$$S_{\rm RS} = \int d^2\sigma(-(\partial X^0)^2 + \omega^a(z)\partial_\nu z \gamma_{ab}\bar{\omega}^b(\bar{z})\partial^\nu \bar{z} + \delta G_{z\bar{z}}(X^0, z, \bar{z})\partial_\mu z \partial^\mu \bar{z} + \mathcal{L}_\perp + \text{h.d.}), \tag{2.2}$$

where $\mathcal{L}_\perp$ describes the dynamics of the dimensions trans-

[1]These manifolds, being much more generic than flat or positively curved spaces, could be described as an underrepresented majority of compact target spaces.

[2]It is also possible to consider realistic compactifications on such spaces, metastabilizing them using extra ingredients [11], but we will stick to simpler examples here.

verse to $\mathcal{M}$. Here +h.d. refers to higher-derivative terms depending on the initial conditions, which are subleading at large radius but will play a significant role at early times. The term proportional to δG describes the time-dependent deformation away from (2.1), with initial condition $\delta G(X_r^0, z, \bar{x}) = 0$. Generically, the dilaton is sourced and the world sheet Lagrangian contains a corresponding $\int d^2\sigma \Phi(z, X^0)\mathcal{R}^{(2)}$ term.

In the bosonic string theory, one must tune the bare couplings in the model to cancel corrections to the world sheet potential (the bulk closed string tachyon). We can also study supersymmetric versions of this model, and will implement a type II GSO projection, with periodic boundary conditions around the cycles on $\mathcal{M}$. This removes bulk tachyon modes as well as tachyons from strings winding around small homology cycles.[3]

The specific metric (2.1) and the corresponding sigma model (2.2) will be convenient in formulating explicit models and in carrying out a path integral transformation in Sec. IV, but similar results hold more generally.

A. Time evolution and the string coupling

One way to control the time dependence is to study the theory in a situation where it is well approximated by renormalization group (RG) flow in the sigma model on $\mathcal{M}$—for example, a supercritical theory with large spacetime dimension D, utilizing the timelike linear dilaton solution with string coupling $g_s \sim g_0 e^{-\sqrt{D}X^0}$ becoming weak at late times [5,6,13–15].[4] The scale dependence of operators in the RG flow translates to the time dependence of the associated fields, as outlined by [5,13,14,16]. In particular, early times map to the UV regime of the flow, and late times to the IR regime. In the latter regime, RG flow is well approximated by Ricci flow. The dilaton slope adjusts, decreasing as the flow proceeds (an effect studied in detail in certain examples in [17]). We expect similar results in much more general time-dependent backgrounds with negatively curved spatial slices, such as the locally flat vacuum solution considered in [6], which will be useful in a candidate nonperturbative formulation presented in Sec. IV.

We will see that the emergence of extra dimensions in these models arises at the level of perturbative string theory. We should take some care with these world sheet arguments, since the dilaton typically varies with time; and for manifolds of nonconstant curvature, varies with space as well. However, in any such background the string coupling can be made arbitrarily small for any finite range of time (over the whole compact manifold); and if the string coupling evolves monotonically, as for linear dilaton backgrounds, the string coupling may be taken arbitrarily small over a semi-infinite range of times.[5] In such backgrounds over such time intervals, our string-tree-level discussions suffice.[6]

III. SYMMETRY AND DUALITY

The core of our duality argument is the existence of a large unbroken axial symmetry on the world sheet, and an associated broken vector group. String theory in D dimensions compactified on $\mathcal{M}$ has an unbroken $U(1)^{2h}$ gauge symmetry in the $D-2$ dimensional spacetime, under which winding strings are charged. Because the winding number is integral, the $U(1)$s are compact. The corresponding conserved world sheet currents arise from the h linearly independent holomorphic and antiholomorphic 1-forms ωa, $\bar{\omega}^a$ $(a = 1, \ldots h)$ as

$$J^a_{A\mu} \sim \epsilon_{\mu\nu}\partial^\nu z \omega^a, \qquad \bar{J}^a_{A\mu} \sim \epsilon_{\mu\nu}\partial^\nu \bar{z}\bar{\omega}^a. \tag{3.1}$$

This winding symmetry, arising from H_1 ($\mathcal{M}$), plays a key role. It is identical to the winding symmetry of a $2h$-dimensional torus. In Sec. III A we will show that a sigma model on T^{2h} perturbed by a relevant potential term constitutes the minimal UV completion of a Riemann surface preserving its winding symmetries, and in Sec. III B we will show that the winding currents translate into a description in terms of a T^{2h} much more generally. In Sec. IV we will use techniques of [10] to present explicit models exhibiting the minimal completion and deformations away from it, for a Riemann surface with metric (2.1). In Sec. V we will see that the same torus arises in an AdS/CFT formulation of our background. We will call this duality—this new set of variables appropriate to describing the small-radius surface—"D-duality."

A. A current algebra construction of D-duality

Nonlinear sigma models on negatively curved manifolds are infrared free and generically have Landau poles in the UV. In our cosmological background, this corresponds to a spacelike singularity at early times, for which the relevant degrees of freedom include a supercritical spectrum of string states [5,6]. We would like to understand the consistent UV completions of this model. This is generally a difficult question. In our case the symmetries are a power-

[3]Homologically trivial cycles still exist with antiperiodic boundary conditions. If these cycles become too small, they can produce transitions changing b_0, separating the space into multiple components [12]. In this work, we will focus on the early-time completion of histories where this does not happen.

[4]Note that this large D is independent of the additional dimensions that we will find to emerge from the Riemann surface itself.

[5]In our world sheet path integral, we consider doing the X^0 integral last; a consistency check on this procedure is the agreement with modular invariance found in [6].

[6]In the final section, we will address the very early-time physics using AdS/CFT.

ful guide: they determine the minimal completion consistent with maintaining them in the UV, as follows.[7]

A *minimal* completion of the model will reach a fixed point in the UV (otherwise $c_{\rm eff}$ will continue to grow, implying arbitrarily large numbers of degrees of freedom.) We will also assume that this minimal completion is a compact CFT (so that there are not an infinite number of winding states descending to arbitrarily low energies), and that the target space is connected. That is, the early-time background consists of a compact CFT describing the UV fixed point of $\mathcal{M}$, a linear dilaton factor, and the remaining spatial directions which we take to be flat IR^n in string frame.

If the UV completion preserves the winding symmetries, it is now straightforward to show that this minimal completion will be string theory on $\mathcal{T}^{2h}$, deformed by a tachyon. The essential reason is that the winding symmetries will be enhanced at the UV fixed point to a chiral $U(1)^{2h} \times U(1)^{2h}$ symmetry, which can be bosonized. Consider the vertex operators for the spacetime gauge bosons coupling to the winding charges in the timelike linear dilaton background. The vertex operators for gauge fields polarized along IR^n take the form $V^\mu = (\partial Y^\mu \bar{J} - J \bar{\partial} Y^\mu) \exp(ikY)$, where Y^μ are free bosons, and $J - \bar{J}$ is the current coupling to winding charge. The operators J, $\bar{J}$ must be primary operators of dimension (1, 0) and (0, 1), respectively, for the gauge field vertex operators to be dimension (1, 1). Dimension (1, 0) and (0, 1) primary operators must be chiral currents in compact CFTs[8] and so generate the affine Lie algebra $U(1)^{2h} \times U(1)^{2h}$.

These $2hU(1)$ currents can be written as ∂X^k for $2h$ free compact bosons X^k. The bosons generate a central charge $2h$, which is the order of magnitude of the contribution of the negative curvature of the Riemann surface to the dilaton beta function, when the volume of $\mathcal{M}$ is of order the string scale [5]. At later times,[9] as the volume grows and the central charge of $\mathcal{M}$ flows toward $c = 2$, the dual torus must be perturbed by a relevant operator which breaks the vector subgroup of $U(1)^{2h} \times U(1)^{2h}$. This vector subgroup is the isometry group of the torus. The most natural candidate is an X-dependent potential on the world sheet, in addition to metric perturbations which also break the isometry (and which will be generated in any case since the tachyon appears in the metric beta functions.) We will explicitly construct examples of these completions in the next section. In supersymmetric theories we require a description consistent with the type II GSO projection for (2.2)—this may add degrees of freedom, depending on the example. We will find such completions in the next section.

We conclude that a $2h$-dimensional torus perturbed by a tachyon is a natural world sheet dual to a small-radius Riemann surface which respects the winding symmetries. This torus gives the *minimal* enhancement to $c_{\rm eff}$ required to describe the UV behavior of the sigma model, given its symmetries.

Furthermore, since the winding modes in (2.2) map to the winding modes of the torus, the *Jacobian* $\mathcal{J}$ can serve as the D-dual torus in simple examples. For any genus h Riemann surface $\mathcal{M}$, the Jacobian is defined as a complex h-torus coordinatized by X^a, $a = 1 \ldots h$ with periodicity $X = X + \Omega$, where Ω is the period matrix of $\mathcal{M}$. $\mathcal{M}$ is canonically embedded into $\mathcal{J}$ via the (holomorphic) Abel-Jacobi map

$$X^a(z) = \int_{z_0}^{z} \omega^a, \tag{3.2}$$

modulo periods. Equation (3.2) identifies the 1-cycles of the two spaces. In the next section, we will find simple examples in which the relevant potential—the closed string tachyon taking the system from the minimal T^{2h} completion to the late-time Riemann surface—realizes this canonical embedding.

This choice of torus is not mathematically arbitrary. Any holomorphic map from $\mathcal{M}$ to any complex torus $\mathcal{T}$ must be the composition of the Abel-Jacobi (3.2) with an affine linear map $\mathcal{J} \to \mathcal{T}$ between the Jacobian torus $\mathcal{J}$ and the given torus $\mathcal{T}$ (see, for example, Sec. III.11.7 in [20]). Examples with (2, 2) supersymmetry in the world sheet matter sector will be constrained by holomorphy, in the RG approximation to time evolution applicable at large D. Such an example will be presented in Sec. III B.

Note that this minimal completion includes specific massive modes at the scale of the world sheet potential, which translates to specific higher-derivative operators in the late-time Riemann surface effective theory. More generally, different UV completions of the sigma model Riemann surface, with different heavy modes and higher dimension operators, may have duals with degrees of freedom in addition to the $2h$ bosons—in other words, the dual theory to $\mathcal{M}$ could have target space $\mathcal{J} \times \mathcal{T}_\perp$, with a potential that has $\mathcal{M}$ as a minimum at late times. We will find such examples in the next section; in particular, the simplest way we have found to satisfy the type II GSO

[7]Determining the early-time completion (related to the UV completion of the spatial sigma model) at the level of the perturbative string theory does not in itself resolve the spacelike singularity in situations like where the string coupling grows large at even earlier times. This requires a nonperturbative completion, such as the AdS/CFT formulation introduced in Sec. IV below.

[8]This statement has been ascribed in [18,19], but it is not mentioned explicitly. Here is an argument. $\bar{\partial} J(z)|0\rangle = \bar{L}_{-1} J(z)|0\rangle \equiv |\psi\rangle$. $\langle\psi|\psi\rangle = |0\rangle J[\bar{L}_1, \bar{L}_{-1}]J|0\rangle = \langle 0|J\bar{L}_0 J|0\rangle = 0$, since J has antiholomorphic dimension 0. Therefore $|\psi\rangle = 0$ in the physical Hilbert space. Since $|\psi\rangle$ is the image under the state-operator map of $\bar{\partial} J$, we conclude that the latter must vanish.

[9]Note that for simplicity, one can consider crossing over to the torus description at a larger scale, by tuning when the tachyon turns on. Without tuning, the tachyon condenses as soon as it can, however.

projection involves a transverse sector.[10] In a given UV completion, the higher dimension operators may contribute small corrections to the supercritical enhancement to the effective central charge (which was computed at leading order in a semiclassical expansion in [6]).

The relation between RG flow and time evolution in our example means that the choice of a UV completion is part of the choice of initial conditions intrinsic to all cosmological theories. Although there may be many consistent choices, all of them are supercritical [5,6], and we have learned here that those which maintain the symmetries of the system asymptote at minimum to a b_1-dimensional torus at early times. In Sec. IV we will find that such a torus arises naturally in an AdS/CFT setup, in the *infrared* regime of the field theory side of this correspondence.

B. Current bosonization and duality

The bosonization argument for D-duality, discussed in the previous subsection, generalizes to cases where we do not reach a minimal UV completion, and to cases where RG flow and time evolution are not closely related. The essential feature of (2.2) which leads to D-duality remains: the conserved winding currents (3.1) are *independent* operators in our world sheet field theory, in the sense that there are no relations between them in the operator algebra. This can be seen from the fact that there are $2h$ independent vertex operators for $2h$ spacetime gauge fields coupling to the $2h$ winding modes, representing independent deformations of the world sheet Hamiltonian. Since the $2h$ winding currents $J^a_{A\mu}$ are conserved, as independent operators they correspond to *scalar* degrees of freedom X^a,

$$\partial_\mu J_A^{a\mu} = 0 \Rightarrow J^a_{A\mu} = \epsilon_\mu{}^\nu \partial_\nu X^a. \tag{3.3}$$

In particular, if one wishes to work in a specific representation, their Fourier modes can be used to construct a Fock space (which will be a subspace of the Hilbert space) equivalent to that of $2h$ scalar degrees of freedom X^a, on which J can be represented as in (3.3).

The fields X^a typically are not free if either the vector or axial symmetries are not conserved; a generic bosonization of $J^a_{A,\mu}$ will contain potential energy terms $T(X^a)$, corrections to the kinetic term, and appropriate dressed higher dimension operators describing the particular history of a given model. In our models (2.2), such nontrivial interactions are required in order to reproduce the nearly critical late-time effective central charge, and to break the symmetry under translations of X^a (the dual of J_A). However, none of these corrections spoil the existence of the $2h$ independent scalar degrees of freedom.

We might also expect to construct this dual description via a generalization of Buscher's construction of T-duality [10,21]. In the next section, we consider a tractable version of this Buscher procedure, which enables us to easily exhibit the minimal UV completion discussed in Sec. II, as well as more general trajectories involving the ubiquitous T^{2h}. Before turning to that, we note how the symmetries affect D-brane probes of the system.

C. D-branes

D-brane physics singles out the Jacobian as a possible dual of (2.2). In the case of ordinary T-duality on a torus, the Wilson lines on wrapped D-branes trace out the T-dual torus. On a compact negatively curved space $\mathcal{M}_n$ of any dimension n and first homology b_1, a wrapped D-brane has as its classical moduli space of Wilson lines a torus T^{b_1} (up to curvature couplings that drop out in the low-energy Yang-Mills limit, and in the case of a flat spacetime solution as considered in [6]). Note that this torus will be the T-dual of the Jacobian—the Abel-Jacobi map sends $H_1(\mathcal{M}_n) \rightarrow H_1(\mathcal{J})$.

In Sec. V we will present an AdS/CFT system comprised of many D-branes which manifests a T^{b_1} in the regime corresponding to the IR physics of the field theory. This last construction may address the very early-time physics in the solutions to which it applies.

IV. EXPLICIT CONSTRUCTION OF DUAL PAIRS

A. Generalizations of T-duality

In constructing explicit D-duals, we follow a variant of the basic strategy used by Buscher [10,21] to derive T-duality via the path integral: that is, couple each *winding* current[11] to a gauge field, and add Lagrange multipliers X^a (which will become the same periodic scalars X^a appearing in the bosonization discussions in Sec. III) that keep the gauge field trivial. Beginning with this theory, gauge fixing and integrating out trivial degrees of freedom in different orders leads to different presentations of identical quantum theories. As in generalizations of T-duality and mirror symmetry found in e.g. [22–26], one finds that a potential $T(X^a)$ and other corrections are allowed in the dual.

In the subsections below, we will consider a tractable version of this Buscher procedure (for theories with zero, two, and four supercharges) which enables us to easily exhibit the minimal completion discussed in Sec. III, as well as more general trajectories involving a T^{2h}. These involve massive degrees of freedom that are part of the UV completion of the theory, which reduce to specific higher-derivative terms in (2.2) at late times.

B. Bosonic dual pair

We will begin with a purely bosonic world sheet theory—this will capture the qualitative features of the duality. In Sec. IV C below we will add $\mathcal{N} = (1, 1)$ world sheet

[10] We thank S. Hellerman for discussions on this point.

[11] In essence Buscher's construction is a gauging of the vector symmetry whose conserved charge is target space momentum.

supersymmetry in order to rid ourselves of additional tachyons that would destabilize the late-time behavior of the model. In Sec. IV D we will consider an $\mathcal{N} = (2,2)$ version.

We consider the bosonic theory with action (at fixed reference time X_r^0)

$$\mathcal{S} = \int d^2\sigma \gamma_{ab}\Big\{(\partial Y^a - A^a)(\partial \bar{Y}^b - \bar{A}^b)^2 - i(X^a\bar{F}^b + F^a\bar{X}^b) + i\Big[\tilde{F}^a\Big(\bar{X}^b - \int^{\bar{z}}\bar{\omega}^b\Big) + \Big(X^a - \int^z \omega^a\Big)\bar{\tilde{F}}^b\Big] + \frac{1}{e^2}\tilde{F}^a\bar{\tilde{F}}^b\Big\}, \tag{4.1}$$

where $\tilde{F}$ is the field strength for a complex gauge field $\tilde{A}$, and similarly $F = \partial_\mu A_\nu \epsilon^{\mu\nu}$. As stated above, we can consider different combinations of gauge fixing and integrating out trivial degrees of freedom; all theories which are derived in this way from (4.1) should be equivalent as quantum theories. We will look at two particular such sequences. The first sequence leads to a sigma model with variable z whose target space is the Riemann surface $\mathcal{M}_h$ (plus some higher-derivative terms corresponding to massive degrees of freedom); the other of which gives us a sigma model on a $2h$-dimensional torus, together with a nontrivial potential. Note that in (4.1), the gauge field $\tilde{A}$ couples to the axial (winding) currents of *both* z and X. In this sense (4.1) is a variant of the usual Buscher action: in [10,21], one couples the auxiliary gauge field to the target space momentum current ∂z of one variable, and the target space winding current $*\partial X$ of the dual variable.

Consider first integrating out X^a, which appears as a Lagrange multiplier. This forces $F = \tilde{F}$. If we then fix gauge by setting $Y = 0 = A = \tilde{A}$ we find a theory of massive vectors (of mass e) coupled to z:

$$\int d^2\sigma \gamma_{ab}\Big\{A^a\bar{A}^b + i\Big[F^a\int^{\bar{z}}\bar{\omega}^b + \int^z \omega^a \bar{F}^b\Big] + \frac{1}{e^2}F^a\bar{F}^b\Big\}. \tag{4.2}$$

At energies below the scale e we integrate out A to find

$$\mathcal{S} \sim \int d^2\sigma\Big\{\omega^a\partial z\gamma_{ab}\bar{\omega}^b\partial\bar{z} + \text{h.d.}\Big\}, \tag{4.3}$$

where "h.d." stands for higher-derivative terms suppressed by powers of ∂^2/e^2. Our model thus produces a particular UV completion of (2.2). At low energies it is the two-derivative nonlinear sigma model on our Riemann surface, together with a specific set of higher dimension operators which are irrelevant at large radius. At the scale e (which can be taken to be lower than the strong coupling scale of the Riemann surface nonlinear sigma model) we find additional degrees of freedom in the form of massive vector fields. The finite mass of the vector fields follows from the gauge kinetic term for $\tilde{F}$ in (4.1).[12]

Let us now integrate in a different order to obtain a second description which makes manifest the D-dual variables appropriate to early times (equivalently, scales above e). Gauge fixing $Y = 0$ and integrating over A leads to the theory

$$\tilde{\mathcal{S}} = \int d^2\sigma\gamma_{ab}\Big\{\partial X^a\partial\bar{X}^b + i\Big[\tilde{F}^a\Big(\bar{X}^b - \int^{\bar{z}}\bar{\omega}^b\Big) + \Big(X^a - \int^z\omega^a\Big)\bar{\tilde{F}}^b\Big] + \frac{1}{e^2}\tilde{F}^a\bar{\tilde{F}}^b\Big\}. \tag{4.4}$$

The gauge fields $\tilde{A}$ couple to winding currents and do not propagate. $X - \int^z \omega$ couples as an axion to $\tilde{F}$. As explained in [27,28], this theory has an effective axion potential coming from the electric field, given at one loop, for $|X^a - \int^z \omega^a| < \pi$, by

$$\mathcal{U} = e^2\Big(X^a - \int^z \omega^a\Big)\gamma_{ab}\Big(\bar{X}^b - \int^{\bar{z}}\bar{\omega}^b\Big). \tag{4.5}$$

Note that we expect the full axion potential to be periodic since (4.4) has this periodicity (due to flux quantization). This should arise from the pair production of winding states, as explained in [27].

In (4.5), z is nonpropagating and we can integrate it out. This will lead to an effective potential $V(X)$ for X. If we integrate out z classically, then $V(X)$ will vanish whenever X takes a value for which $\mathcal{U}$ vanishes; this occurs when X is at an image point of the Abel-Jacobi map. Therefore, Eq. (4.5) provides precisely the expected potential restricting the system to the Riemann surface. Note that under this map, $*\partial X$ and $*\omega\partial z$ are interchanged; the duality maps winding states in the Riemann surface into those of the Jacobian. A related point is that in contrast to T-duality, here the dilaton is the same in both descriptions; to see this directly in our sigma model note that at leading order in ∂^2/e^2 we obtain the same 1-loop determinant from integrating over A on both sides.

Upon fibering (4.1) over the time direction, $V(X)$ behaves as a spacetime tachyon and should grow at late times. At these late times, (4.3) will be a better description, while at early times the description in terms of the D-dual variables X becomes appropriate since $V(X)$ becomes smaller and smaller. That is, we have found a model realizing the minimal early-time completion determined on general grounds in Sec. III B. The path integral transform itself is fairly trivial; note that in the absence of the z fields, it simply relates massive vectors A^a on the first side to the equivalent theory of massive scalars X^a on the second side,

[12] If we gave the gauge field in [10,21] a kinetic term we would find that this term is a redundant operator. In the present case it is not—it is part of the definition of the UV completion of the theory.

and both sides manifestly reduce to a Riemann surface at low energies.

To describe trajectories different from the minimal completion of Sec. III A, we may formally add or subtract higher dimension operators from both sides. For example, if we subtract the +h.d. terms from (4.3), we obtain, at a given time slice, the basic nonlinear sigma model on a Riemann surface. By carrying through the duality transformation to the X variables, one subtracts the corresponding sum of h.d. terms, now expressed in terms of X (using the fact that these higher dimension operators are all functionals of the winding current and its derivatives, which translates into $*\partial X$ and its derivatives). We do not know *a priori* if this deformed theory in itself is UV complete, but if so we see from this transformation that it also has a dual description in terms of coordinates X on its Jacobian torus. Similar remarks apply to other trajectories obtained by deforming (4.3). This reflects the genericity of the bosonization (3.3).

C. An $\mathcal{N} = (1, 1)$ supersymmetric extension

We now wish to construct an $\mathcal{N} = (1, 1)$ supersymmetric version of (4.1), in which there is no spacetime tachyon destabilizing (2.2) at late times.

We follow the conventions of [29]. Consider the type II version on (2.2). The Riemann surface directions (at a fixed reference time X_r^0) can be described by a (1, 1) supersymmetric nonlinear sigma model involving scalar superfields Z, $\bar{Z}$:

$$\mathcal{S}_{II} = \int d^2\sigma d^2\theta \gamma_{ab}\Big\{\omega^a D_\alpha Z \bar{\omega}^b D^\alpha \bar{Z} + \text{h.d.}\Big\}, \tag{4.6}$$

where again "h.d." stands for higher-derivative operators, subleading at large radius. This theory respects a $\mathbb{Z}_2 \times \mathbb{Z}_2$ R-symmetry by which we can orbifold to implement the type II GSO projection, giving a tachyon-free spectrum.[13] In perturbative string theory, we can choose any spin structure on $\mathcal{M}$. The fully periodic spin structure avoids the decays of b_1 due to winding tachyons that can occur when $\mathcal{M}$ is small [12].

The generalization of (4.1) to (1, 1) supersymmetry is

$$\begin{aligned} S = \int d^2\sigma d^2\theta \gamma_{ab}\Big\{&(D_\alpha Y^a - \Gamma^a_\alpha)(D^\alpha \bar{Y}^b - \bar{\Gamma}^{b\alpha}) \\ &- i[X^a \bar{f}^b + \bar{X}^b f^a] + i\bigg[\bigg(X^a - \int^Z \omega^a\bigg)\bar{\tilde{f}}^b \\ &+ \bigg(X^b - \int^{\bar{Z}} \bar{\omega}^b\bigg]\tilde{f}^a\bigg] + \frac{1}{2e^2}(D^\beta D_\alpha \tilde{\Gamma}^a_\beta)(D^\delta D^\alpha \bar{\tilde{\Gamma}}^b_\delta)\Big\}. \end{aligned} \tag{4.7}$$

Here D_α is the superspace derivative, and Y^a, X^a, and Z^a are complex scalar multiplets. Γ^a, $\tilde{\Gamma}^a$ are gauge multiplets, whose scalar field strengths are $f^a \equiv D^\alpha(\sigma^3\Gamma^a)_\alpha = D_+\Gamma^a_- + D_-\Gamma^a_+$; here $\pm$ indicates the helicity in multiples of 1/2 as described in [29]. Γ and $\tilde{\Gamma}$ flip sign under each of the chiral $\mathbb{Z}_2$ R-symmetries, while the scalar multiplets are neutral.

To find a presentation of the quantum theory in which $\mathcal{M}$ is manifest, we first integrate out X, yielding $\Gamma = \tilde{\Gamma} + D\Lambda$. Fixing the gauge symmetries associated with Γ and $\tilde{\Gamma}$ can be done most simply by setting $Y = 0 = \Lambda$. Next, integrating out Γ gives

$$\Gamma - DY = D\int^Z \omega + \mathcal{O}(\partial^2/e^2), \tag{4.8}$$

which yields the Riemann surface sigma model (4.6), with the higher-derivative terms arranged in a specific power series in ∂^2/e^2.

In order to find a presentation of the quantum theory in which T^{2h} is manifest, we begin by integrating out Γ, which we can do by solving $\Gamma - DY + DX = 0$ for Γ. Plugging this into Eq. (4.7), we obtain

$$\begin{aligned} S = \int d^2\sigma d^2\theta \gamma_{ab}\Big\{&D_\alpha X^a D^\alpha \bar{X}^b + i\bigg[\bigg(X^a - \int^Z \omega^a\bigg)\bar{\tilde{f}}^b \\ &+ \text{h.c.}\bigg] + \frac{1}{2e^2}(D^\beta D_\alpha \tilde{\Gamma}^a_\beta)(D^\delta D^\alpha \bar{\tilde{\Gamma}}^b_\delta)\Big\}. \end{aligned} \tag{4.9}$$

As in the bosonic case discussed above, the electric field energy translates *á la* [27] into a potential term (4.5) for the scalar field components. There are additional terms in the supersymmetric action, involving the canonically normalized scalar $\tilde{a}_3$ in the $\tilde{\Gamma}$ gauge multiplet:

$$\begin{aligned} \mathcal{U} = e^2\bigg(X^a - \int^z \omega^a\bigg)\gamma_{ab}\bigg(\bar{X}^b - \int^{\bar{z}} \bar{\omega}^b\bigg) + e^2\gamma_{ab}\tilde{a}^a_3\bar{\tilde{a}}^b_3 \\ + e\gamma_{ab}(\bar{F}_Z\bar{\omega}^a\tilde{a}^b_3 + \bar{\tilde{a}}^a_3 F_Z\omega^b), \end{aligned} \tag{4.10}$$

where F_Z is the auxiliary field in the Z multiplet. Note that the scalar a_3 flips sign under the chiral GSO projection—this fact allows for a scalar potential consistent with a GSO projection, giving a stable late-time solution. This is reminiscent of a mechanism in [15] for obtaining dimension-changing transitions yielding the critical type II theory on an orbifold fixed locus.[14] The integral over F_Z constrains a linear combination of $\tilde{a}_3$s to vanish identically. In this theory Z is heavier than X and the remaining $\tilde{a}_3$s, and can be integrated out. Although it is difficult to carry out this path integral in practice, the form of the potential shows that at late times (when e^2 is large), it yields a theory with a tachyon which restricts the X and $\tilde{a}_3$ directions to the embedded Riemann surface. This matches the low-energy physics of the Riemann surface model (4.6).

The GSO projection also protects the model against arbitrary tachyon deformations at early times. Since $\tilde{f}^a$

[13]The time evolution produces harmless *pseudotachyonic* modes of the massless fields, as discussed recently in [6,14,15].

[14]Indeed, the existence of an example with some of these features was anticipated by S. Hellerman.

transforms under the GSO and X^a and Z do not, the only potential terms allowed for X and Z are those coming from the electric field energy arising from the axion couplings of the form $i(\bar{\tilde{f}}^a \gamma_{ab} \theta^b(X, Z) + \text{H.c.})$. In general, this yields only h complex constraints on the $h + 1$ complex variables X^a and Z, so the low-energy theory is a Riemann surface generically (without fine-tuning).

More specifically, we are considering a Riemann surface which maintains the winding symmetries of the torus. This is at least a self-consistent symmetry principle in the RG evolution of the system applicable at large D. It is worth noting also that a gas of winding strings would energetically favor the subset of tachyons preserving the winding symmetries of the torus, since tachyons which remove the corresponding cycles would render winding strings heavy.

D. An $\mathcal{N} = (2, 2)$ supersymmetric extension

At fixed reference time X_r^0, the (1, 1)-supersymmetric model just discussed has a simple extension with (2, 2) supersymmetry. Since $\mathcal{N} = (2, 2)$ SUSY implies a complex target space, the time dependence will generally break $\mathcal{N} = (2, 2)$ supersymmetry. On the other hand, if we consider the expanding Riemann surface as a factor in a highly supercritical theory in $D + 10$ dimensions, the time dependence will be slow, and we can think of the cosmology as a trajectory for an $\mathcal{N} = (2, 2)$ sigma model, with supersymmetry broken to (1, 1) by corrections of order $1/\sqrt{D}$. There are two advantages to this presentation. First, the complex target space means that the D-duality will involve a holomorphic map of $\mathcal{M}$ into T^{2h}, for which there is a natural set of mathematical objects discussed in Sec. III A. Second, gauging a symmetry in (2, 2) language removes two scalar degrees of freedom—one from the gauge fixing at low energies, and one from the D-term constraints. Thus, we need half of the gauge fields of the $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (0, 0)$ cases, to realize the duality.

The field content of the $\mathcal{N} = (2, 2)$ model is: h twisted chiral multiplets X^a, h chiral multiplets Y^a, one twisted chiral multiplet Z which lives on $\mathcal{M}$, and two sets of h vector superfields V, $\tilde{V}$ with twisted chiral field strengths $\Sigma^a = D_+ \bar{D}_- V$, $\tilde{\Sigma}^a = D_+ \bar{D}_- \tilde{V}$. The $\mathcal{N} = (2, 2)$ Lagrangian (corresponding to the (2, 2) completion of (2.2) at fixed reference time X_i^0) is

$$L = \int d^2\tilde{\theta} \frac{1}{2} \gamma_{ab} \bigg[\bar{\tilde{\Sigma}}^a \bigg(X^b - \int^Z \omega^b \bigg) - \bar{\Sigma}^a X^b \bigg] + \text{H.c.} + \int d^4\theta \gamma_{ab} \bigg[\frac{1}{2} (Y^a + \bar{Y}^a + V^a)(Y^b + \bar{Y}^b + V^b) - \frac{1}{2e^2} \tilde{\Sigma}^{a\dagger} \tilde{\Sigma}^b \bigg]. \tag{4.11}$$

Here $\int d^2\tilde{\theta} \equiv \bar{D}_+ D_-$.

To get the (2, 2) Riemann surface model, we integrate out X first, so that $\Sigma = \tilde{\Sigma}$ up to a (2, 2) supergauge transformation which we gauge fix to zero. We also choose the gauge $Y = 0$. The remaining Lagrangian, suppressing the b_1 indices which are always contracted with γ_{ab}, is

$$L(V, Z) = \int d^4\theta \bigg\{ \frac{1}{2} V^2 + \mathcal{O}(e^{-2}) \bigg\} - \frac{1}{2} \int d^2\tilde{\theta} \Sigma \int^Z \omega + \text{H.c.} \tag{4.12}$$

Consider the twisted superpotential term in (4.12). Since $\int^Z \omega$ is twisted chiral, this term is a total superspace derivative: $\frac{1}{2} \int d^2\tilde{\theta} \Sigma \int^Z \omega = \int d^4\theta V \int^Z \omega$. The resulting action is

$$L(V, Z) = \int d^4\theta \bigg\{ \frac{1}{2} V^2 - V \bigg(\int^Z \omega + \text{H.c.} \bigg) + \mathcal{O}(e^{-2}) \bigg\}. \tag{4.13}$$

Finally, integrating out V produces the Lagrangian (up to higher-derivative terms at scale e, and Kähler transformation)

$$L_{\text{RS}}(Z) = - \int d^4\theta \bigg| \int^Z \omega \bigg|^2, \tag{4.14}$$

which describes the Kähler sigma model on the Riemann surface.[15]

Performing the integrations in a different order produces a (2, 2) version of the D-dual, with the manifest T^{2h}. Let us begin by rewriting (4.11) as

$$L = \int d^4\theta \bigg[\frac{1}{2} (V + Y + \bar{Y})^2 - V(X + \bar{X}) + \tilde{V} \bigg(X - \int^Z \omega + \text{H.c.} \bigg) - \frac{1}{2e^2} \tilde{\Sigma}^\dagger \tilde{\Sigma} \bigg]. \tag{4.15}$$

Integrating out gauge multiplet V relates the X current to the Y axial current as usual in (2, 2) duality [e.g. [26]]:

$$(V + Y + \bar{Y}) - (X + \bar{X}) = 0. \tag{4.16}$$

(From this relation, we see that Y is the coordinate on the *mirror* of the Jacobian, the torus related to the Jacobian by T-duality on half the circles.)

The remaining Lagrangian is

$$L| = \int d^4\theta \bigg[-\frac{1}{2} (X + \bar{X})^2 - \tilde{V}(t + \bar{t}) - \frac{1}{2e^2} \tilde{\Sigma}^\dagger \tilde{\Sigma} \bigg], \tag{4.17}$$

where $t \equiv X - \int^Z \omega$. The dynamics of the tilded gauge multiplet generates a potential for its theta angle t, minimized at zero [28]. To see this explicitly, rewrite the Lagrangian as

$$L| = \int d^4\theta \bigg(-\frac{1}{2e^2} \tilde{\Sigma}^\dagger \tilde{\Sigma} - X^\dagger X \bigg) + \frac{1}{2} \int d^2\tilde{\theta} \, \tilde{\Sigma} \, t + \text{H.c.}, \tag{4.18}$$

[15] Note that this sign gives the correct kinetic terms for twisted chiral fields.

which is effectively a Wess-Zumino model for the twisted chiral fields $\tilde{\Sigma}$, X, Z, with twisted chiral superpotential $\tilde{W} = \tilde{\Sigma} t$ (the $\tilde{V}$ gauge symmetry does not act on anything and is trivially confined). Integrating out the auxiliary fields in the twisted chiral multiplets X, $\tilde{\Sigma}$, and rescaling $\tilde{\sigma}$ to have canonical kinetic term, the bosonic potential is

$$V(X, z, Fz) = e^2 \left| X - \int^z \omega \right|^2 + e^2 |\tilde{\sigma}|^2 + (e\tilde{\sigma} \cdot \omega F z + \text{H.c.}). \qquad (4.19)$$

This is the potential (4.5), plus a mass for the scalars $\tilde{\sigma}$. The integral over the auxiliary field F_z imposes a constraint $\sum_a \omega^a(z)\tilde{\sigma}^a = 0$.

As in the bosonic and (1, 1) models, integrating out z now generates a potential for X which vanishes when X lies in the image of the Abel-Jacobi map.

V. DISCUSSION

In this final section we briefly discuss perturbative generalizations of D-duality, and then introduce a possible nonperturbative completion using AdS/CFT.

A. Generalizations and transitions

Our construction of D-duality should work equally well for higher-dimensional target spaces $\mathcal{M}$ with negative curvature. For negatively curved manifolds with $b_1 > 0$, the current algebra arguments apply at early times to show that the minimal completion is a T^{b_1}. Mathematically, there are natural candidate D-duals. For higher-dimensional Kähler manifolds $\mathcal{M}$, the analogue of the Jacobian torus $\mathcal{J}$ of $\mathcal{M}$ is known as the "Albanese variety," and the analogue of (3.2) is known as the "Albanese map." In such cases, holomorphic maps from $\mathcal{M}$ to a complex torus $\mathcal{T}$ should be the composition of the Albanese map and map from the Albanese variety to $\mathcal{T}$. More generally, when the target manifold is not complex, we might expect a similar structure to hold for harmonic maps, relating winding modes on $\mathcal{M}$ to those on the associated T^{b_1}

Nontrivial $c_{\text{eff}} - c_{\text{crit}}$ arises essentially from the exponential growth of the first homotopy group $\pi_1(\mathcal{M})$, following from the compactness and negative curvature of our target space [5,6]. In the present work, the first homology $H_1(\mathcal{M})$ plays a key role. However, there exist negatively curved compact manifolds which are supercritical by virtue of the exponential growth of π_1, but which have $b_1 = 0$. This raises the question of what if any D-dual set of variables describes their small volume limit. Recall that in T-duality, a similar question arises for orbifolds which preserve neither winding nor momentum symmetry. The duality does extend to orbifolds, which inherit T-duality properties of the parent theory defined on the covering space. For similar reasons, we expect that our derivation will generalize to yield a dual description of any space $\mathcal{M}_n$ in terms of a $\tilde{b}_1$-dimensional torus, where $\tilde{b}_1$ is the first Betti number of any finite cover of $\mathcal{M}_n$ (known as a *virtual* first Betti number of $\mathcal{M}_n$). For example, there exist many hyperbolic 3-manifolds which are *homology spheres*, i.e., $H_1 = \{0\}$ so that $b_1 = 0$ [30]. Mathematicians have conjectured that these always have finite covers with nonzero $\tilde{b}_1$ (see, for example, [31]); this would provide a way to extend D-duality to general 3-manifolds with $c_{\text{eff}} > c_{\text{crit}}$. Conversely, it would be interesting to see if our physical methods can provide insight into the validity of the virtual first Betti number conjecture.

String theory contains dimension-changing transitions (see [14,15] for recent well-controlled examples). Having learned here and in [5,6] that string theory on negatively curved target spaces is supercritical, it is interesting to consider transitions to lower dimensions (equivalently, lower c_{eff}) in this context. The expansion of the space itself yields such a transition, since c_{eff} decreases in time. The results in [12] yield another decay mode in some cases (with spin structure chosen to be antiperiodic for spacetime fermions about some handle(s)), in which the reduction in genus reduces $c_{\text{eff}} \propto \frac{2h-2}{V_\Sigma}$..

B. Early times

The duality obtained above is essentially perturbative, and can be studied during a semi-infinite epoch of arbitrarily weak string coupling. In the timelike linear dilaton solution we considered above, the coupling ultimately grows large in the very far past, making the theory difficult to study.[16] *A priori*, there are two natural approaches to this problem: (1) change the setup to maintain weak string coupling throughout, and investigate possible resolutions of the singularity via α' effects; or (2) attempt to find a nonperturbative completion, perhaps building on existing frameworks such as [2,3].

1. Weak string coupling at early times?

Regarding approach (1), one might consider taking the early-time theory as a linear dilaton solution with $g_s \sim e^{+\sqrt{2c_{\text{eff}}/3}X^0}$ growing in time, combined with the rolling tachyon solution. The tachyon generically grows slowly toward the past and rapidly toward the future. In this case, the string coupling can be tuned arbitrarily weak during the epoch when the tachyon is small, while it is not sourced when the tachyon is large since the theory becomes a critical string compactification. However, this scenario is subject to further instabilities—the *pseudotachyonic* modes [14,15] may cause significant backreaction in the solution with the growing string coupling if they have an

[16]Note that this issue is not special to the supercritical corners of string theory; the presence of a past singularity at the level of the low-energy effective theory pertains to all nontrivial cosmological backgrounds, including ones formulated in the critical theory.

infinite time to develop.[17] Their effects are worth studying further.

2. A gauge theory dual

An avenue for approach (2) is the AdS/CFT correspondence, which has provided a useful nonperturbative formulation of a large class of backgrounds of superstring theory in the critical dimension.[18] To date, no such formulation has been derived for supercritical strings. We propose such a formulation here. The general strategy is to begin with a known AdS/CFT dual and compactify the field theory side on a time-dependent background with compact, negatively curved spatial slices, such that the spacetime dual will also contain compact, negatively curved spatial slices. We have seen here and in [5,6] that superstring theory with such factors in spacetime is supercritical.

Consider the Poincaré patch of $\mathrm{AdS}_5 \times S^5$, dual to $U(N)$ $\mathcal{N} = 4$ on Minkowski space $\mathcal{M}_{3,1}$, and consider the foliation of $\mathcal{M}_{3,1}$ by hyperbolic 3-space $\mathbb{H}_3$. Next, take an orbifold of both sides of the duality by a discrete isometry group Γ of $\mathbb{H}_3$. This has compact spatial slices $\mathcal{M}_3 = \mathbb{H}_3/\Gamma$ as the fundamental domain of Γ, with a fundamental group of exponential growth whose Abelianization is the first homology group. The CFT will then propagate on the spacetime

$$ds^2 = -dt^2 + t^2 ds^2_{\mathbb{H}/\Gamma}, \tag{5.1}$$

with the constant curvature metric descending from the metric $ds^2_{\mathbb{H}} = dy^2 + \sinh^2 y d\Omega^2$ on the covering space $\mathbb{H}$. The string theory dual is type IIB on a patch of $\mathrm{AdS}_5 \times S^5$ covered by

$$ds^2 = \frac{r^2}{\ell^2} - (dt^2 + t^2 ds_{\mathbb{H}/\Gamma}) + \ell^2 \frac{dr^2}{r^2} + ds^2_{S^5} \tag{5.2}$$

with the spacetime curvature supported by the standard N units of self-dual five-form Ramond-Ramond (RR) flux.

In the spacetime theory, the volume $\mathcal{V}_3$ of $\mathcal{M}_3$ varies both with time and with the radial direction. At fixed time t, $\mathcal{V}_3$ grows as $r \to \infty$, towards the boundary of our AdS orbifold. This is dual to the UV regime of the CFT, in which the modes are not sensitive to the compactness of the spatial slices. $\mathcal{V}_3$ shrinks as $r \to 0$; this is dual to the infrared regime of the CFT. The discussions here and in [5,6] indicate that the effective central charge of the spacetime theory grows larger as $\mathcal{V}_3$ shrinks. Thus, the *infrared* physics of the CFT characterizes the most *supercritical* regime of the string theory dual.

At this stage there are still barriers to realizing the supercritical theory more concretely. The field theory lives on a compact space with volume $\mathcal{V}_3 = L(t)^3 = t^3$. At energy scales of order the Hubble expansion scale $1/t$, we expect the field theory to develop a mass gap and spacetime to end. Furthermore, the computations in [6] and in Secs. II and III were valid in perturbative string theory in the absence of RR flux. These problems can be averted by taking the additional step of studying the theory in a state on the approximate Coulomb branch of the CFT. We consider the near-horizon limit of N D3-branes, placed on an $SO(6)$-invariant shell at some radial position R, in the directions transverse to the world volume. The branes source all N units of five-form flux, so inside the shell the flux is absent.[19] The static metric is $\mathrm{AdS}_5 \times S^5$ outside the shell, and ten dimensional flat space inside. After the orbifold, the spacetime metric is

$$ds^2 = h^{-1}(r)[-dt^2 + t^2 ds^2_{\mathbb{H}/\Gamma}] + h(r)[dr^2 + r^2 d\Omega_5^2], \tag{5.3}$$

where $i = 1, 2, 3$, and where

$$h(r) = \frac{\ell^2}{r^2}(r > R), \qquad h(r) = \frac{\ell^2}{R^2}(r < R). \tag{5.4}$$

Thus, after the orbifold, the spacetime at small r is preserved.

Inside the shell, for all $r < R$, is a regime with no RR flux and with an $\mathcal{M}_3$ factor, of proper size tR/ℓ, expanding with time. The results of Secs. II and III and of [5,6] indicate that this region is supercritical. According to the AdS/CFT dictionary, its physics is dual to the low-energy physics of the CFT on the spacetime (5.1), on the Coulomb branch.

If $\dim H_1(\mathcal{M}_3) = b_1$, the results of Sec. II show that the minimal completion of string theory on $\mathcal{M}_3$ at small r or t is a "D-dual" torus T^{b_1}. The T-dual of this torus appears naturally in the gauge theory as follows. Since all N D-branes lie on the shell, the charged W bosons are massive and low-energy dynamics of the theory is that of the $U(1)$ factors. More precisely, there is a distribution of masses of W bosons determined by the relative positions of the branes on the shell. The number n of D-branes close enough to a given brane to yield masses $m_W < 1/L$ is given by

$$n \sim N \frac{l_s^{10}}{(RL)^5}. \tag{5.5}$$

We consider $RL \gg l_s^2$ so that this number is parametrically smaller than N.

In the regime of small $g_s n$ we find that the low-energy dynamics includes Wilson lines in a range of scales between $\sqrt{g_s n}/L \ll E \ll 1/L$. These Wilson lines take values in $\mathrm{Sym}^N(T^{b_1}) \equiv (T^{b_1})^N/S_N$, with a T^{b_1} factor for each $U(1)$.

[17]We thank O. Aharony and E. Witten for discussions of this case (as well as other aspects of the duality).

[18]This includes recent proposals for resolving spacetime singularities using AdS/CFT [32,33]] (see also [34] in the context of [2]).

[19]Configurations like this have been discussed in, e.g., [35,36] (see also [37] for a different type of shell) and recently applied to a Scherk-Schwarz compactification of the CFT in [33], whose conventions we follow.

To see that the Wilson lines are relevant modes at low energies, one must first check that higher momentum modes on $\mathcal{M}_3$ do not contribute at low energies. The Wilson lines live on a compact space, and yield quantum mechanical energy levels $E_j \sim j^2 g_s/L$. For sufficiently weak string coupling, this is much smaller than the scale $1/L$ of higher momentum modes of the gauge theory.

Second, the moduli space of Wilson lines is approximate; massive W bosons contribute a potential, as studied, for example, in [38]. At low energies, we must check that this potential is small compared to the scale $1/L$. We can ensure this by making R large so that the number n of light W bosons charged under a given $U(1)$ (5.5) is small, and most W bosons are heavy compared to the scale $1/L$ of momentum modes on $\mathcal{M}_3$. The resulting potential energy scales like $g_s n \phi^a/L^2$ where ϕ_a is the canonically normalized Wilson line scalar: that is, the mass^2 of the Wilson lines is much smaller than the KK scale $1/L$. This same effect suppresses the potential on the Coulomb branch itself, and the dynamics of the scalars in the 4d gauge theory should also be taken into account.

We conclude that at low energies of the gauge theory, the gauge dynamics includes the motion of N $U(1)$ Wilson lines on an approximate moduli space $\mathrm{Sym}^N\,(T^{b_1})$. The negative curvature of the spatial slices provides Hubble friction slowing down motion on the approximate moduli space. Note that the volume of this T^{b_1} moduli space *grows* with decreasing $\mathcal{V}_3$. The torus described by the Wilson lines is the T-dual of the minimal completion derived in Sec. II from current algebra considerations. It may be tempting to identify the inevitable potential on the moduli space, arising from integrating out the W bosons, with the tachyon potential described in Secs. II and III; however, under T-duality it maps to a condensate of winding modes. The appearance of the tachyon potential of Secs. II and III from the gauge theory is an interesting question that we leave for future work.

We emphasize again that in this setup, the T^{b_1} arises in the infrared regime of the CFT, where considerations of universality and naturalness apply; whereas it appears in the *ultraviolet* regime of the world sheet matter sigma model. This suggests that the minimal completion of the system preserving the winding symmetries indeed arises naturally, in the Wilsonian sense, from the gauge theory dual. At least it implies that the other trajectories of the world sheet RG which miss the UV fixed point of the minimal completion are themselves closely related to a theory on T^{b_1}, as argued in Sec. III A. The range of possible trajectories involving the T^{b_1} may be mirrored in the dual gauge theory by the ambiguity in the choice of state on the initial light cone of singularities at $t = 0$ in (5.1).

The conformal transformation of (5.1) to

$$-\,d\tau^2 + ds^2_{\mathbb{H}/\Gamma} \tag{5.6}$$

(where $\tau = \log(t/t_0)$) may help elucidate the early-time physics, though its effect on the scalar field dynamics must also be included consistently.[20] In the conformal frame (5.6), the conformal coupling $R\phi^2$ contributes a negative effective mass squared for the scalars in the field theory [e.g. [39]]. Indeed, in the conformally rescaled background, $\phi \propto\ t \propto e^{\tau/\tau_0}$ grows like a tachyon. However, in the original conformal frame (5.1), the curvature is zero so the conformal coupling $R\phi^2$ vanishes. The late-time behavior is simply that of a quantum field theory on a slowly expanding Friedmann-Robertson-Walker (FRW) background with compact hyperbolic spatial slices. Thus it appears that the instability of the CFT on (5.6) leads to an innocuous endpoint: field theory on a late-time weakly curved cosmological spacetime. Conversely, the singularity in (5.1) as one goes back to $t = 0$ translates into the return of ϕ to the origin of the moduli space as $\tau \to -\infty$ in the frame (5.6). It will be interesting to explore the consequences of this formulation of the system via AdS/CFT.

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