PHYSICAL REVIEW D **76,** 066001 (2007)

Hall conductivity from dyonic black holes

Sean A. Hartnoll* and Pavel K. Kovtun†

KITP, University of California, Santa Barbara, California 93106-4030, USA

(Received 23 May 2007; published 7 September 2007)

A class of strongly interacting 2 + 1 dimensional conformal field theories in a transverse magnetic field can be studied using the AdS/CFT duality. We compute zero momentum hydrodynamic response functions of maximally supersymmetric 2 + 1 dimensional $SU(N)$ Yang-Mills theory at the conformal fixed point, in the large N limit. With background magnetic field B and electric charge density ρ, the Hall conductivity is found to be ρ/B. The result, anticipated on kinematic grounds in field theory, is obtained from perturbations of a four-dimensional AdS black hole with both electric and magnetic charges.

DOI: 10.1103/PhysRevD.76.066001 PACS numbers: 11.25.Tq, 04.70.Bw, 11.10.Wx

I. INTRODUCTION

The Hall effect is a fundamental property of a conducting medium subject to an external magnetic field. Upon application of an electric field, current flows in a direction orthogonal to both the electric and magnetic fields. Therefore the conductivity tensor, defined by $j_a = \sigma_{ab}E_b$, acquires off diagonal entries.

In a Lorentz invariant theory, the dc conductivity in the presence of an external magnetic field is completely determined by boost invariance. Suppose there is a magnetic field $\boldsymbol{B}$ in the lab frame. Consider a frame moving with small velocity $-\boldsymbol{v}$ with respect to the lab frame. In this frame there is a current $\boldsymbol{j} = \rho\boldsymbol{v}$, where ρ is the charge density of the medium in the lab frame, and an electric field

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} = -\frac{1}{\rho}\boldsymbol{j} \times \boldsymbol{B}. \tag{1}$$

If the magnetic field is $\boldsymbol{B} = (0, 0, B)$, then from (1) we have that in the xy plane

$$\sigma_{xy} = -\sigma_{yx} = \frac{\rho}{B} \quad \text{and} \quad \sigma_{xx} = \sigma_{yy} = 0. \tag{2}$$

Thus the conductivity tensor is antisymmetric: its off diagonal components are precisely the Hall conductivity.

From a microscopic point of view, σ_{ab} can be evaluated using the Kubo formula which relates electrical conductivity to the current-current retarded Green's function, evaluated in the thermal equilibrium state:

$$\sigma_{ab} = -\lim_{\omega\to 0}\frac{\mathrm{Im}G^R_{ab}(\omega)}{\omega}. \tag{3}$$

The validity of the Kubo formula only relies on linear response; in particular, it does not assume that the physical system is composed of weakly interacting quasiparticles. Thus, for strongly interacting systems without a quasiparticle description, Eq. (3) provides a first-principles way to evaluate the conductivity. We note that Kubo formulae are modified in the presence of an external magnetic field. The expression (3) remains valid however.

Motivated by understanding charge transport at quantum critical points [1], in this paper we will study conductive properties of strongly interacting 2 + 1 dimensional conformal field theories (CFTs) in the presence of an external magnetic field. A particular field theory to which our discussion applies is the infrared conformal fixed point of maximally supersymmetric $SU(N)$ Yang-Mills theory at large N. This CFT has eight supersymmetries and a global $SO(8)$ R symmetry group. The magnetic field we will turn on belongs to a $U(1)$ subgroup of $SO(8)$. The theory describes the low-energy dynamics of M2 branes in M theory. However, as emphasized in the recent work [2], it is our hope that the study of this special theory may shed light upon general aspects of strongly coupled CFTs in 2 + 1 dimensions that are of interest in quantum critical phenomena. In fact, our results will apply to any CFT with an AdS/CFT dual that may be truncated to Einstein-Maxwell theory on AdS_4.

The M2 brane theory is tractable in the large N limit because the AdS/CFT correspondence [3] provides a dual gravitational description of the system which may be treated classically in this limit. Furthermore, it has been understood how the correspondence may be used to extract hydrodynamic coefficients that describe the large scale and late time behavior of finite temperature field theories [4–6]. In 3 + 1 dimensions, that work has led to a conjectured universal lower bound on the ratio of shear viscosity to entropy density [7] and also to unanticipated connections with the fireball created at the Relativistic Heavy Ion Collider [8]. In contrast, the implications of the AdS/CFT correspondence for 2 + 1 dimensional hydrodynamics have so far been less developed, although see [2,9–11].

The behavior under a magnetic field is a basic probe of interacting 2 + 1 dimensional systems. In the following section we show how the M2 brane theory may be placed in a background magnetic field by considering a dual magnetically charged black hole in AdS_4. We go on to describe the thermodynamics and some of the hydrodynamic response functions of this theory. The AdS/CFT duality maps fluctuations of conserved currents in the 2 + 1 dimensional

*hartnoll@kitp.ucsb.edu
†kovtun@kitp.ucsb.edu

CFT to fluctuations of gauge fields in the background of the $3+1$ dimensional black hole, and provides a recipe for computing current-current correlation functions. We will use the Kubo formula (3) to compute the Hall conductivity from perturbations of the black hole spacetime and recover precisely (2).

II. DYONIC BLACK HOLE IN AdS_4

In this section we describe the four-dimensional spacetime dual to the M2 brane CFT on $\mathbb{R}^{1,2}$. We are interested in the system at finite temperature and with a background magnetic field. Finite temperature is realized in AdS/CFT by allowing the spacetime to contain a black hole [12]. We will shortly explain that a background magnetic field is obtained by allowing the black hole to carry a magnetic charge.

The fact that our CFT is relativistic implies that it has the same number of excitations with positive and negative charges. Under applied magnetic and electric fields, the charges will create opposite currents that cancel and there will be no Hall conductivity. To avoid this scenario, we need to consider the CFT in a state with a net charge density. We will recall below that this is obtained by requiring that the black hole carry an electric charge.

In summary: the background we require is a dyonic black hole in AdS_4, with both electric and magnetic charge. Such black holes have been known for some time [13]. Precisely four spacetime dimensions are necessary for a point source to be both magnetically and electrically charged.

The full supergravity context is 11-dimensional supergravity on $AdS_4 \times S^7$. This theory may be consistently truncated to Einstein-Maxwell theory on AdS_4. For details, see for instance [2]. In this reduction, the Maxwell field originates via the Kaluza-Klein mechanism as a $U(1)$ subgroup of the $SO(8)$ symmetry group of the full background. The action for Einstein-Maxwell theory with a negative cosmological constant $-1/L^2$ is

$$I = \frac{2}{\kappa_4^2}\int d^4x\sqrt{-g}\left[-\frac{1}{4}R + \frac{L^2}{4}F_{\mu\nu}F^{\mu\nu} - \frac{3}{2}\frac{1}{L^2}\right], \quad (4)$$

which implies equations of motion

$$R_{\mu\nu} = 2L^2 F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{L^2}{2}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} - \frac{3}{L^2}g_{\mu\nu}, \quad (5a)$$

$$\nabla_\mu F^{\mu\nu} = 0. \quad (5b)$$

We have included the overall normalization of the action coming from the reduction of 11-dimensional supergravity, which in terms of the field theory N is (see e.g. [2])

$$\frac{2L^2}{\kappa_4^2} = \frac{\sqrt{2}N^{3/2}}{6\pi}. \quad (6)$$

Our background metric will be a black hole in AdS_4 with planar horizon

$$\frac{1}{L^2}ds^2 = \frac{\alpha^2}{z^2}[-f(z)dt^2 + dx^2 + dy^2] + \frac{1}{z^2}\frac{dz^2}{f(z)}. \quad (7)$$

We will take the black hole to carry both electric and magnetic charge

$$F = h\alpha^2 dx \wedge dy + q\alpha dz \wedge dt, \quad (8)$$

which implies that

$$f(z) = 1 + (h^2+q^2)z^4 - (1+h^2+q^2)z^3. \quad (9)$$

This solution may be obtained, for instance, by taking the planar limit of the expressions in [13]. One can explicitly check that Eqs. (7)–(9) solve the equations of motion (5). Without loss of generality we have scaled the coordinates so that the horizon is at $z=1$. The AdS asymptopia is at $z \to 0$. The parameter α has the dimensions of mass, and determines the temperature of the black hole through

$$T = \frac{\alpha(3-h^2-q^2)}{4\pi}. \quad (10)$$

Note that for a regular horizon we have $3-h^2-q^2>0$. This inequality defines the allowed range of h and q. It will not, however, restrict the ranges of the dual variables T, B and ρ. The extremal zero temperature limit is achieved by taking $h^2+q^2 \to 3$, with α fixed. The relation of α to the mass of the black hole is given in the following section.

The dual field theory to this spacetime is the low-energy theory living on N M2 branes with worldvolume $\mathbb{R}^{1,2}$. To understand the implications of the bulk Maxwell field it is convenient to consider a potential giving $F = dA$,

$$A = h\alpha^2 x dy + q\alpha z dt. \quad (11)$$

We see that the magnetic term remains finite at the AdS boundary, $z \to 0$, whereas the electric term goes to zero. These different falloffs lead to differing dual interpretations for the charges [14]. Both falloffs of a Maxwell field in AdS_4 are normalizable, and we can choose which one to interpret as being dual to a vacuum expectation value (VEV) [14]. We will make the standard choice in which the faster falloff is dual to a VEV.

The magnetic term in the potential (11) has a slower falloff, remaining finite at the AdS boundary, and hence corresponds to an external magnetic field for a gauged $U(1)$ subgroup of the $SO(8)$ R symmetry group of the theory. Such modes are usually considered as adding the term $A^0_\mu J^\mu$ to the field theory Lagrangian, where A^0_μ is the boundary value of the field and J^μ is the dual $U(1)$ current. This is the same as gauging the $U(1)$ symmetry by adding the background field A^0_μ, which in this case is magnetic. The strength of the magnetic field in the field theory is $B = h\alpha^2$, as we can read off from taking $z \to 0$ in the expression for the bulk field strength (8).

The electric term in the potential has a faster falloff, and therefore does not correspond to a background field in the dual theory. Instead, it fixes the electric charge density of the state in the field theory to be

$$\rho \equiv \langle J^t \rangle = \frac{\delta I}{\delta A_t^0} = -\frac{\sqrt{2}N^{3/2}}{6\pi} q\alpha^2. \qquad (12)$$

Here J^t is the charge density operator for the same $U(1)$ subgroup of the R symmetry as before. We will rederive this expression for the charge density from thermodynamic considerations in the following section.

Finally, there is in fact a term missing in the potential (11). In order for the potential to be regular at the horizon of the black hole, A_t must vanish there.[1] This requires that we add to (11) the pure gauge term $-q\alpha dt$. This term remains finite as $z \to 0$ and has the dual interpretation of adding a chemical potential for the electric charge, $\mu = -q\alpha$, to the field theory. This is the chemical potential corresponding to the electric charge density ρ. Our computation of the conductivity will only depend on the bulk field strength (8) and not on this pure gauge term.

III. THERMODYNAMICS OF THE GRAND CANONICAL ENSEMBLE

Before going on to compute hydrodynamic correlators, it is useful to summarize the thermodynamic properties of the black holes we have just described. These characterize the dual field theory in thermal equilibrium.

We describe the thermodynamics of the grand canonical ensemble, where the chemical potential $\mu = -q\alpha$ is kept fixed. We also keep fixed the magnetic field $B = h\alpha^2$, this is treated as a parameter of the system rather than a thermodynamic variable. We will see that this is a consistent treatment.

We give the results in terms of the bulk spacetime variables q, h, α. There are various ways of computing thermodynamic quantities from black holes. The most elegant is holographic renormalization, in which the action is regularized using a counterterm boundary action [15,16]. No new counterterms are needed due to the Maxwell field, as the F^2 term in the action falls off sufficiently quickly near the boundary. The renormalized action is given by subtracting a boundary term from the bulk action

$$I_{\rm ren} = I - \frac{1}{\kappa_4^2}\int d^3x\sqrt{-\gamma}\theta - \frac{2}{\kappa_4^2}\frac{1}{L}\int d^3x\sqrt{-\gamma}, \qquad (13)$$

where γ is the boundary metric. We have also included the Gibbons-Hawking boundary term. Here $\theta = \gamma^{\mu\nu}\theta_{\mu\nu}$ is the trace of the extrinsic curvature $\theta_{\mu\nu} = -\frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu)$, with n an outward directed unit normal vector to the boundary.

[1] One way to see this is to look at the Euclidean black hole solution with imaginary time direction τ compactified to a circle. The radius of the circle shrinks to zero at the horizon, implying that A_τ must vanish there.

The thermodynamic potential is given by the renormalized action $I_{\rm ren}$ evaluated on the solution times the temperature (10)

$$\Omega = TI_{\rm ren} = \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{\mathcal{V}\alpha^3}{4}(-1 - q^2 + 3h^2). \qquad (14)$$

Here $\mathcal{V} = \int dxdy$ is the spatial volume.

The renormalized energy momentum tensor of the black hole is [16]

$$\frac{1}{L^3}\langle T^{\mu\nu}_{\rm b.h.}\rangle = \frac{2}{\sqrt{-\gamma}}\frac{\delta I_{\rm ren}}{\delta\gamma_{\mu\nu}} = \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{1}{2}\left[\theta^{\mu\nu} - \theta\gamma^{\mu\nu} - \frac{2}{L}\gamma^{\mu\nu}\right]. \qquad (15)$$

This is related to the field theory energy momentum tensor by $\langle T^{\mu\nu}\rangle = (\alpha/z)^5\langle T^{\mu\nu}_{\rm b.h.}\rangle$. Thus we can obtain the energy

$$E = \int d^2x\langle T^{\mu\nu}_{\rm b.h.}\rangle k_\mu \xi_\nu \sqrt{\sigma} = \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{\mathcal{V}\alpha^3}{2}(1 + q^2 + h^2). \qquad (16)$$

The integral is over the $\mathbb{R}^2$ at spatial infinity $z \to 0$, k is a unit vector normal to the spatial hypersurface $t = \text{const}$, ξ is the Killing vector ∂_t and $\sqrt{\sigma} = (\gamma_{xx}\gamma_{yy})^{1/2}$ is the volume element of the $\mathbb{R}^2$. The entropy is given by the area of the horizon times a standard normalization to be

$$S = \frac{\sqrt{2}N^{3/2}}{6}\mathcal{V}\alpha^2. \qquad (17)$$

The total electric charge may be computed by varying the free energy with respect to the chemical potential

$$Q = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,B} = \frac{\sqrt{2}N^{3/2}}{6\pi}\mathcal{V}\alpha\mu. \qquad (18)$$

This expression agrees with our previous result (12). It will be useful later to define the energy, entropy and charge densities

$$\varepsilon = \frac{E}{\mathcal{V}}, \qquad s = \frac{S}{\mathcal{V}}, \qquad \rho = \frac{Q}{\mathcal{V}}. \qquad (19)$$

Finally, the pressure in the grand canonical ensemble is simply given by

$$\Omega = -P\mathcal{V}. \qquad (20)$$

Note that in a magnetic field, P differs from $\langle T_{xx}\rangle$ by a term proportional to the magnetization. A check of the formulae we have given in this section is that they satisfy the required thermodynamic relation

$$\Omega = E - TS - \mu Q. \qquad (21)$$

The fact that this relation holds without needing to add a term for the magnetic charge shows that our treatment of h as a constant external parameter is consistent.

Analogously to the five-dimensional example in [17], we check the local thermodynamic stability of the system by considering the equation of state $\varepsilon(s, \rho)$. In the grand canonical ensemble, the condition for stability is that $\det[\partial^2_{s\rho}\varepsilon(s, \rho)] > 0$. From the formulae above it follows that

$$\varepsilon(s, \rho) = \frac{6^{1/2}}{2^{1/4}N^{3/4}} \frac{s^{3/2}}{2\pi}\left[1 + \frac{\rho^2\pi^2}{s^2} + \frac{\tilde{B}^2\pi^2}{s^2}\right], \tag{22}$$

where $\tilde{B} = B\sqrt{2}N^{3/2}/6\pi$. It is easily checked that the condition for the determinant to be positive is that $3 + 3h^2 + q^2 > 0$. This is certainly true, and therefore the system is locally thermodynamically stable at all temperatures, charges and values of the background magnetic field.

IV. FLUCTUATIONS AND ACTION

A. Equations of motion

We are aiming to compute correlators of the boundary current operators J_x and J_y, dual to the components of the bulk Maxwell potential A_x and A_y. The AdS/CFT dictionary requires that we consider fluctuations of these fields about the black hole background. In order to extract the conductivity, it will suffice to work at zero momentum in the x and y directions. [It is consistent to do so because only the background field strength (8) enters the equations, not the background potential (11). Thus, translation invariance is maintained.] That is, the perturbations are taken to be independent of x and y. In this case, it turns out that the gauge field fluctuations source fluctuations in the metric components g_{tx} and g_{ty} and no others. By linearising the Einstein-Maxwell equations about the background we obtain the following equations for the fluctuations: From the Maxwell Eq. (5b) we find

$$f(fA_x')' + w^2A_x + iwhG_y + qfG_x' = 0, \tag{23a}$$
$$f(fA_y')' + w^2A_y - iwhG_x + qfG_y' = 0. \tag{23b}$$

In these equations $G_x = g_{tx}\alpha^{-1}z^2$, and similarly for G_y. Prime denotes differentiation with respect to z. The time dependence is taken to be $e^{-i\omega t}$ for all fields. We have also introduced the dimensionless frequency $w \equiv \omega\alpha^{-1}$.

The Einstein Eq. (5a) give

$$f(G_y'/4z^2)' - h^2G_y + iwhA_x + qfA_y' = 0, \tag{24a}$$
$$f(G_x'/4z^2)' - h^2G_x - iwhA_y + qfA_x' = 0, \tag{24b}$$
$$iwG_y'/4z^2 + hfA_x' + iwqA_y + hqG_x = 0, \tag{24c}$$
$$iwG_x'/4z^2 - hfA_y' + iwqA_x - hqG_y = 0. \tag{24d}$$

It is easy to verify that these equations imply the Maxwell Eqs. (23), thus providing a consistency check. The fluctuation equations may be separated out to obtain higher order equations for the different modes. One can verify that A_x and A_y satisfy the same fifth-order equation.

We need to solve the Eqs. (24) with the condition that the solution satisfies ingoing wave boundary conditions at the horizon $z = 1$. Near the horizon, the solutions behave as $A_{x,y} \sim (1 - z)^\nu$, $G_{x,y} \sim (1 - z)^\beta$. The set of Eqs. (24) then determines the ingoing and outgoing exponents $\nu = \pm iw/(h^2 + q^2 - 3)$, $\beta = 1 + \nu$. The ingoing wave solution at the horizon corresponds to $\nu_+ = iw/(h^2 + q^2 - 3)$. Thus we will be looking for the solution of the form

$$A_x(z) = f(z)^{\nu_+}a_x(z), \tag{25a}$$
$$G_x(z) = f(z)^{1+\nu_+}g_x(z), \tag{25b}$$

and similarly for $A_y(z)$, $G_y(z)$. The functions $a_x(z)$, $g_x(z)$ are required to be regular at the horizon $z = 1$. There is also a third possible exponent at the horizon, which leads to a constant z-independent solution of Eqs. (24),

$$G_y = \frac{iw}{h}A_x, \qquad G_x = -\frac{iw}{h}A_y. \tag{26}$$

This constant solution will be important later.

B. Hydrodynamic limit

We are interested in the low-frequency, hydrodynamic behavior of the system when $\omega/T \ll \mu/T$, B/T^2. This regime of small frequencies can be achieved by letting $w \to 0$, with q and h fixed. Therefore we will solve the equations perturbatively in w:

$$a_x(z) = a_x^{(0)}(z) + wa_x^{(1)}(z) + \ldots, \tag{27}$$

$$g_x(z) = g_x^{(0)}(z) + wg_x^{(1)}(z) + \ldots, \tag{28}$$

and similarly for $a_y(z)$, $g_y(z)$. To zeroth order in w, we have[2]

$$g_x^{(0)\prime\prime}(z) + 2\frac{\psi'(z)}{\psi(z)}g_x^{(0)\prime}(z) = 0, \tag{29}$$

$$a_x^{(0)\prime}(z) + qg_x^{(0)}(z) = 0, \tag{30}$$

where $\psi(z) \equiv f(z)/z$. A general solution is of the form $g_x^{(0)\prime}(z) = \mathrm{const}/\psi(z)^2$ and the condition of regularity on the horizon implies

$$g_x^{(0)}(z) = \gamma_x, \qquad a_x^{(0)}(z) = \alpha_x - \gamma_x qz, \tag{31}$$

where γ_x, α_x are integration constants. To first order in w we find

$$g_x^{(1)\prime\prime}(z) + 2\frac{\psi'(z)}{\psi(z)}g_x^{(1)\prime}(z) = \mathcal{G}_x^{(0)}(z), \tag{32}$$

$$a_x^{(1)\prime}(z) + qg_x^{(1)}(z) = \mathcal{A}_x^{(0)}(z), \tag{33}$$

[2]Here and below we give expressions for the x components; the y components are the same but with $x \leftrightarrow y$ and $h \leftrightarrow -h$.

where the functions $\mathcal{G}_x^{(0)}(z)$, $\mathcal{A}_x^{(0)}(z)$ on the right-hand side depend on the zeroth-order solution (31). One finds that $\mathcal{G}_x^{(0)}(z)\psi(z)^2$ and $\mathcal{A}_x^{(0)}(z)f(z)$ are fourth-order polynomials in z. A general solution is of the form

$$g_x^{(1)\prime}(z) = \frac{\text{const}}{\psi(z)^2} + \frac{1}{\psi(z)^2}\int_0^z \mathcal{G}_x^{(0)}(u)\psi(u)^2 du, \tag{34}$$

and demanding regularity of $g_x^{(1)}(z)$ on the horizon now implies a relation between the integration constants:

$$\alpha_y = \frac{\gamma_x h(h^2+q^2-3) + 3\gamma_y q(1+h^2+q^2)}{4(h^2+q^2)}. \tag{35}$$

With this relation, one finds that the solution which is regular at the horizon is

$$g_x^{(1)}(z) = \tilde{\gamma}_x - i\int_0^z \frac{du}{\psi(u)^2}\gamma_x P_5(u), \tag{36a}$$

$$a_x^{(1)}(z) = \tilde{\alpha}_x - q\int_0^z g_x^{(1)}(u)du - i\int_0^z \frac{du}{f(u)}(\gamma_x Q_4(u) + \gamma_y Q_3(u)), \tag{36b}$$

where P_5, Q_3, Q_4 are polynomials in u, whose coefficients (given in the Appendix) depend on q and h only. The integration constants $\tilde{\gamma}_x$, $\tilde{\alpha}_x$ can be absorbed into γ_x and α_x respectively; as a result the solution is characterized by the two constants γ_x, γ_y. This is too few. We want a solution that is characterized by four constants: the four boundary values of the fields G_x, G_y, A_x, A_y. We must therefore add to the solution (25) the constant solution (26)[3]

$$A_x = \delta_x, \qquad A_y = \delta_y, \qquad G_x = -\frac{iw}{h}\delta_y, \qquad G_y = \frac{iw}{h}\delta_x, \tag{37}$$

where $\delta_{x,y}$ are constants. Then for the boundary values $G_x^0 \equiv G_x(z=0)$ etc., we find

$$G_x^0 = -\frac{iw}{h}\delta_y + \gamma_x, \qquad G_y^0 = \frac{iw}{h}\delta_x + \gamma_y, \tag{38}$$

$$A_x^0 = \delta_x + \alpha_x(\gamma_x,\gamma_y), \qquad A_y^0 = \delta_y + \alpha_y(\gamma_x,\gamma_y). \tag{39}$$

We can now express integration constants in terms of the boundary values of the fields[4]

$$\delta_x = A_x^0 + \frac{G_y^0 h(h^2+q^2-3) - 3G_x^0 q(1+h^2+q^2)}{4(h^2+q^2)}, \tag{40a}$$

$$\delta_y = A_y^0 - \frac{G_x^0 h(h^2+q^2-3) + 3G_y^0 q(1+h^2+q^2)}{4(h^2+q^2)}. \tag{40b}$$

To sum up: the solution to first order in w is

$$G_x(z) = -\frac{iw}{h}\delta_y + f(z)^{1+\nu_+}\Big[G_x^0 + \frac{iw}{h}\delta_y - iwG_x^0\int_0^z \frac{du}{\psi^2(u)}P_5(u)\Big], \tag{41a}$$

$$A_x(z) = \delta_x + f(z)^{\nu_+}\Big[A_x^0 - \delta_x - \Big(G_x^0 + \frac{iw}{h}\delta_y\Big)qz + iwqG_x^0\int_0^z \frac{du(z-u)}{\psi^2(u)}P_5(u) - iw\int_0^z\frac{du}{f(u)} \times (G_x^0 Q_4(u) + G_y^0 Q_3(u))\Big]. \tag{41b}$$

C. The action

The current correlators will be computed by differentiating the action evaluated on the solution with respect to the boundary values of the fields. We therefore need the renormalized action (13) to quadratic order in perturbations. The quadratic action evaluated on shell is given by the boundary term

$$I_{\text{ren}} = \lim_{z\to 0}\frac{2L^2\alpha}{\kappa_4^2}\int dtd^2x\Big[\frac{f^{1/2}-1}{2z^3 f^{1/2}}\mathbf{G}\cdot\mathbf{G} + \frac{q}{2}\mathbf{A}\cdot\mathbf{G} - \frac{1}{8z^2}\mathbf{G}\cdot\mathbf{G}' + \frac{f}{2}\mathbf{A}\cdot\mathbf{A}'\Big]. \tag{42}$$

In this expression $\mathbf{G} \equiv (G_x, G_y)$, $\mathbf{A} \equiv (A_x, A_y)$. Any possible contribution to the action from a boundary term at the horizon is neglected [5]. The action is now computed by expanding the solution (41) near the boundary $z = 0$ and substituting into Eq. (42).

The result is most cleanly expressed in terms of the Fourier transformed modes

$$A_x^0(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}A_x^0(\omega)e^{-i\omega t}, \tag{43}$$

and similarly for the other modes. The action can be written as the sum of three terms

$$I_{\text{ren}} = \frac{\sqrt{2}N^{3/2}}{6\pi}[I_{AA} + I_{AG} + I_{GG}], \tag{44}$$

where to lowest order in $\omega/T \to 0$ we have

$$I_{AA} = \frac{iq}{2h}\int\frac{d\omega}{2\pi}d^2x\,\omega\,\epsilon_{ab}A_a^0(\omega)A_b^0(-\omega), \tag{45}$$

where ϵ_{ab} is antisymmetric, with $\epsilon_{xy} = 1$. The ϵ tensor appears because of the mixing of x and y coefficients in

[3] The constant term is often not written explicitly in treatments of black hole hydrodynamics, which only give expressions for the derivatives of fields, but it is usually there when more than one field is involved. Without it, one would not have sufficient free constants at the boundary. Gauge invariant fluctuations will be purely ingoing at the horizon.

[4] The expressions (40) have $O(w)$ corrections. However, these subleading terms do not contribute to the correlators to first order in w.

(41). The term coupling the metric and gauge potential is

$$I_{AG} = \frac{-i3(1+q^2+h^2)}{4h}\int d^2x \frac{d\omega}{2\pi}\omega\epsilon_{ab}A_a^0(\omega)G_b^0(-\omega), \tag{46}$$

and finally, dropping a contact term,

$$I_{GG} = \int d^2x \frac{d\omega}{2\pi}\left[\frac{(-3+h^2+q^2)^2}{32(h^2+q^2)}\delta_{ab} + \frac{9q(1+h^2+q^2)^2}{32h(h^2+q^2)}\epsilon_{ab}\right]i\omega G_a^0(\omega)G_b^0(-\omega). \tag{47}$$

When writing these expression for the action, the boundary values for the fields $A^0_{x,y}$, $G^0_{x,y}$ in the solution (41) are taken as arbitrary functions of ω. Taking functional derivatives with respect to the boundary values allows us to compute hydrodynamic correlators and conductivity.

V. HALL CONDUCTIVITY AND HYDRODYNAMIC CORRELATORS

Recall from our discussion in Sec. II that the background magnetic field in the field theory is $B = h\alpha^2$ and the charge density of the system $\rho \propto q\alpha^2$ is given by (19). These definitions imply that ρ and B have mass dimension two, which is the correct dimensionality for charge density and field strength in three dimensions.

The AdS/CFT dictionary [5] allows us to read off the large N retarded Green's function from (45) as

$$G^R_{ab}(\omega) = -i\int d^2x dt e^{i\omega t}\theta(t)\langle[J_a(t), J_b(0)]\rangle = -i\omega\epsilon_{ab}\frac{\rho}{B}. \tag{48}$$

The conductivity is then given by the Kubo formula (3)

$$\sigma_{xy} = -\sigma_{yx} = \frac{\rho}{B}, \qquad \sigma_{xx} = \sigma_{yy} = 0. \tag{49}$$

There is no temperature dependence in this result. This expression exactly recovers the results expected on general grounds from Lorentz invariance (2).

We can also compute from (44) the retarded correlator between the momentum density and the R charge current. We have, to leading order in w and at zero spatial momentum,

$$G^R_{a\pi_b}(\omega) = -i\int d^2x dt e^{i\omega t}\theta(t)\langle[J_a(t), T_{tb}(0)]\rangle = -\frac{3\varepsilon}{2B}i\omega\epsilon_{ab}. \tag{50}$$

When extracting the correlators (50), the action I_{AG} has to be multiplied by α because $G_x = \alpha g_{tx}(z/\alpha)^2$. The final result here has been expressed in terms of the background magnetic field and the charge and energy densities of the equilibrium field theory. The expression for the momentum-momentum correlator also follows from (44) as

$$G^R_{\pi_a\pi_b}(\omega) = -i\int d^2x dt e^{i\omega t}\theta(t)\langle[T_{ta}(t), T_{tb}(0)]\rangle = \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{s^2T^2}{\rho^2+\tilde{B}^2}i\omega\delta_{ab} - \frac{9\rho\varepsilon^2}{4B(\rho^2+\tilde{B}^2)}i\omega\epsilon_{ab}. \tag{51}$$

As previously, $\tilde{B} = B\sqrt{2}N^{3/2}/6\pi$. The expressions (48), (50), and (51) are the main result of this paper. The regime of their applicability is broader than the naive hydrodynamic limit $\omega \ll T$. We have assumed that $\omega \ll \alpha$, with α implicitly defined by the relation $4\pi T = \alpha(3 - B^2/\alpha^4 - \mu^2/\alpha^2)$. This allows ω/T to take any value provided the chemical potential (or magnetic field) is sufficiently large.

VI. DISCUSSION

We have shown how a background magnetic field may be incorporated into the AdS/CFT correspondence for 2 + 1 dimensional boundary theories, by considering a dyonic black hole as the bulk spacetime. As an application, we studied low-frequency charge transport in 2 + 1 dimensional CFTs whose gravity duals contain Einstein-Maxwell theory on AdS_4. This class of conformal field theories includes maximally supersymmetric $SU(N)$ Yang-Mills theory at the conformal fixed point in the limit of large N. For the Hall conductivity, our bulk computation recovered the field theory result expected due to Lorentz covariance. In addition to the conductivity, we computed hydrodynamic response functions of charge and momentum currents at zero spatial momentum. One expects that the full study of bulk fluctuations with nonzero spatial momentum should reproduce linearized relativistic magnetohydrodynamics of the boundary field theory. These computations appear to be technically involved, and we have left them for future work.

From the fact that the diagonal dc conductivity vanishes in (49) it follows that the hydrodynamic limit $(\omega/T) \to 0$ does not commute with the limit of small magnetic fields $(B/T^2) \to 0$. It would be interesting to understand the crossover between the two regimes for the class of strongly interacting CFTs studied here.

Let us also note that turning on a background magnetic field is not the only way to introduce off diagonal conductivity in the AdS/CFT framework. A topological θ term for the four-dimensional gauge fields in the bulk gives rise to the antisymmetric contribution $\epsilon_{\mu\nu\lambda}p^\lambda$ in two point current-current correlators of the 2 + 1 dimensional CFT on the boundary [18]. Application of the Kubo formula now tells us that there is a nonzero off diagonal conductivity, proportional to θ. However, introducing a θ term in the bulk does not correspond to introducing a background magnetic field in the original CFT; rather, it means that one simply studies a different CFT [18]. Indeed, the effective Hall-like conductivity coming from the θ term is a constant, present even at zero temperature and zero charge density. In contrast, Hall conductivity due to the background magnetic field studied in this paper should be a nontrivial function of ω/T and ρ/T. As a related application, it would be interesting to see how a θ term may be obtained in AdS_4 via reduction from ten- or 11-dimensional supergravity. These terms seem to be generic in flux compactifications.

There are various other interesting phenomena that arise in $2 + 1$ dimensional theories with a background magnetic field. These range from the quantum Hall effect to the Nernst effect in superconductors. The AdS/CFT correspondence provides a unique framework in which such effects may be analytically studied in a strongly coupled field theory.

ACKNOWLEDGMENTS

It is a pleasure to thank David Berenstein, Rob Myers, Subir Sachdev, and Andrei Starinets for helpful conversations in the course of this work. We would also like to thank Chris Herzog for important comments on the preprint version of this text. This research was supported in part by the National Science Foundation under Grant No. PHY05-51164.

APPENDIX A: POLYNOMIALS

The polynomials in (36) are given by

$$\begin{aligned} P_5(u) = {} & (7h^2 + 7q^2 - 3)(1 - u)^2 \\ & - \frac{3 + 15(h^2 + q^2)^2 - 22(h^2 + q^2)}{h^2 + q^2 - 3}(1 - u)^3 \\ & + \frac{13(h^2 + q^2)^2 - 7(h^2 + q^2)}{h^2 + q^2 - 3}(1 - u)^4 \\ & - \frac{4(h^2 + q^2)^2}{h^2 + q^2 - 3}(1 - u)^5, \end{aligned} \tag{A1}$$

$$\begin{aligned} Q_3(u) = {} & \frac{h(q^2 + h^2 - 3)}{2(h^2 + q^2)}(1 - u) \\ & - \frac{3h(3h^2 + 3q^2 - 1)}{4(h^2 + q^2)}(1 - u)^2 + h(1 - u)^3, \end{aligned} \tag{A2}$$

$$\begin{aligned} Q_4(u) = {} & \frac{q(5q^2 + 5h^2 - 3)}{2(h^2 + q^2)}(1 - u) \\ & - \frac{3q(3 + 11(q^2 + h^2)^2 - 18(q^2 + h^2))}{4(q^2 + h^2)(q^2 + h^2 - 3)}(1 - u)^2 \\ & + \frac{2q(5q^2 + 5h^2 - 3)}{q^2 + h^2 - 3}(1 - u)^3 \\ & - \frac{4q(q^2 + h^2)}{q^2 + h^2 - 3}(1 - u)^4. \end{aligned} \tag{A3}$$

[1] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).

[2] C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son, Phys. Rev. D **75**, 085020 (2007).

[3] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).

[4] G. Policastro, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001).

[5] D. T. Son and A. O. Starinets, J. High Energy Phys. 09 (2002) 042.

[6] G. Policastro, D. T. Son, and A. O. Starinets, J. High Energy Phys. 09 (2002) 043.

[7] P. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).

[8] P. Steinberg, arXiv:nucl-ex/0702020.

[9] C. P. Herzog, J. High Energy Phys. 12 (2002) 026.

[10] C. P. Herzog, Phys. Rev. D **68**, 024013 (2003).

[11] O. Saremi, J. High Energy Phys. 10 (2006) 083.

[12] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).

[13] L. J. Romans, Nucl. Phys. **B383**, 395 (1992).

[14] I. R. Klebanov and E. Witten, Nucl. Phys. **B556**, 89 (1999).

[15] M. Henningson and K. Skenderis, J. High Energy Phys. 07 (1998) 023.

[16] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999).

[17] D. T. Son and A. O. Starinets, J. High Energy Phys. 03 (2006) 052.

[18] E. Witten, arXiv:hep-th/0307041.