PHYSICAL REVIEW D **76,** 065014 (2007)

Space-time noncommutative theories at finite temperature

A. V. Strelchenko*
Dnepropetrovsk National University, 49050 Dnepropetrovsk Ukraine

D. V. Vassilevich†,‡
Instituto de Física, Universidade de São Paulo Caixa Postal 66318 CEP 05315-970, São Paulo S.P., Brazil
(Received 7 June 2007; published 24 September 2007)

We analyze renormalization and the high-temperature expansion of the one-loop effective action of the space-time noncommutative ϕ^4 theory by using the zeta-function regularization in the imaginary-time formalism (i.e., on $S^1 \times \mathbb{R}^3$). Interestingly enough, there are no mixed (nonplanar) contributions to the counterterms as well as to the power-law high-temperature asymptotics. We also study the Wick rotation and formulate assumptions under which the real and imaginary-time formalisms are equivalent.

DOI: 10.1103/PhysRevD.76.065014 PACS numbers: 11.10.Nx, 11.10.Wx

I. INTRODUCTION

Since a number of review papers have been published recently (see [1]) it is not necessary to repeat here the motivations for studying noncommutative (NC) field theories. Most of the previous works, see e.g. [2–5] and references therein, on finite-temperature NC theories analyzed the case of space-space noncommutativity (with very few exceptions [6,7]). Indeed, the case of space-time noncommutativity is most problematic because of the difficulties with unitarity and causality which were discovered some years ago [8–10]. These difficulties have not been completely resolved up to now. Space-time NC theories with compact dimensions exhibit an interesting phenomenon of discreteness of time [11].

The main purpose of this paper is to develop certain aspects of the Euclidean space formalism in space-time NC theories, including the renormalization, the transition from real to imaginary time, and the high-temperature asymptotics.

We shall start our work by analyzing the one-loop divergences in the Euclidean NC ϕ^4 on $S^1 \times \mathbb{R}^3$ to make sure that the theory which will be discussed later does exist at least at the leading order of the loop expansion. We shall use the zeta-function regularization [12,13] and the heat kernel technique [14–16]. In the context of an NC field theory the heat kernel expansion was first obtained for the operators which contain only left or only right Moyal multiplications [17,18]. Such operators were, however, insufficient to deal with some physical applications, like, for example, the ϕ^4 theory. The heat kernel expansion for generalized Laplacians containing *both* left and right Moyal multiplications was constructed in [19] on the Moyal plane and in [20] on the NC torus. Nonminimal operators were considered in [21]. We would also like to mention the calculations [22] of the heat kernel expansion in the NC ϕ^4 model modified by an oscillator-type potential.

To avoid unnecessary technical complications we shall study exclusively the case of pure space-time noncommutativity, i.e., we put to zero the NC parameter with both indices in the spatial directions, $\theta^{jk} = 0$. We shall calculate the heat kernel coefficients a_n with $n \leq 4$. It will appear that the coefficients a_2 and a_4 look very similar to the commutative theory, but a_3 is given by a complicated nonlocal expression. Fortunately, odd-numbered heat kernel coefficients do not contribute to one-loop divergences at four dimensions in the zeta-function regularization. The model will turn out to be one-loop renormalizable with temperature-independent counterterms.

Of course, we do not expect this model to be renormalizable at all loops. There are well-known problems related to the so-called UV/IR mixing [23] which should also be present in our case (though, maybe, in a relatively mild form since one of the NC directions is compact). To make the finite-temperature NC ϕ^4 renormalizable to all orders one should probably make it duality covariant [24] or use a bifermionic NC parameter [25].

An approach to finite-temperature theories on static backgrounds based on the zeta-function regularization was developed long ago by Dowker and Kennedy [26]. In particular, they established relations between spectral functions of a three-dimensional operator which defines the spectrum of fluctuations and the high-temperature asymptotics of the free energy. In our case, due to the presence of the space-time noncommutativity, such a three-dimensional operator becomes frequency dependent even on static backgrounds. Therefore, eigenfrequencies of fluctuations are defined by a sort of a nonlinear spectral problem. Fortunately for us, a technique which allows one to work with finite-temperature characteristics of the theories leading to nonlinear spectral problems has been developed relatively recently in the papers in [27]. These papers were dealing with the thermodynamics of stationary but nonstatic space-times, but, after some modifications, the

*alexstrelch@yahoo.com
†On leave from V. A. Fock Institute of Physics, St. Petersburg University, Russia
‡dmitry@dfn.if.usp.br

approach of [27] can be made suitable for space-time noncommutative theories as well. By making use of these methods we shall construct the spectral density of states in the real-time formalism and express it through the heat kernel of a frequency-dependent operator in three dimensions. Then, by using this spectral density, we shall demonstrate that the Wick rotation of the Euclidean free energy gives the canonical free energy. To come to this conclusion we shall need two assumptions. First of all, we shall have to assume that the spectral density behaves "nicely" as a function of complex frequencies. Although this assumption is very hard to justify rigorously, we shall argue that the behavior of the spectral density must not be worse than in the commutative case, and we shall also suggest a consistency check based on the high-temperature asymptotics. There is no canonical Hamiltonian in the space-time NC theories. Therefore, we have to assume that the eigenfrequncies of quantum fluctuations can replace one-particle energies in thermal distributions. This assumptions cannot be derived from the first principles of quantization basing on the present knowledge on the subject, but we can turn the problem around: the very fact that the Wick rotation of the Euclidean free energy leads to a thermal distribution over the eigenfrequences supports (a rather natural) guess that the eigenfrequences are the energies of one-particle excitations. Let us stress that the calculations we shall perform in the Euclidean space do not depend on the assumptions described above.

We shall also use the heat kernel methods to calculate the high-temperature asymptotics of the Euclidean effective action assuming that the background field is static. As in the case of the counterterms, there are nonplanar contributions. The asymptotic expansion does not depend on the NC parameter (provided it is nonzero) and looks very similar to the commutative case.

This paper is organized as follows. In the next section we consider one-loop renormalization of NC ϕ^4 on $S^1 \times \mathbb{R}^3$. Section III is devoted to the Wick rotation. High-temperature asymptotics of the effective action are calculated in Sec. IV. Some concluding remarks are contained in Sec. V.

II. NONCOMMUTATIVE QUANTUM FIELD THEORY ON $S^1 \times \mathbb{R}^3$

A. Basic definitions and notations

Let us consider a scalar ϕ^4 model on NC $S^1 \times \mathbb{R}^3$. The scalar field is periodic with respect to the compact coordinate. We use the notations $(x^\mu) = (\bar{x}, x^4) = (x^i, x^4)$, where x^4 is a coordinate on S^1, $0 \le \tau \le \beta$. Similarly for the Fourier momenta we use $k = (\bar{k}, k_4)$, $k_4 = \frac{2\pi l}{\beta}$, $l \in \mathbb{Z}$.

The action reads

$$S = \frac{1}{2}\int_0^\beta dx^4 \int_{\mathbb{R}^3} d^3\bar{x}\Big((\partial_\mu \phi)^2 + m^2\phi^2 + \frac{g}{12}\phi_\star^4\Big), \quad (1)$$

where the $\phi_\star^4 = \phi \star \phi \star \phi \star \phi$. Star denotes the Moyal product

$$f_1 \star f_2(x) = \exp\Big(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x \partial_\nu^y\Big) f_1(x) f_2(y)|_{y=x}. \quad (2)$$

To simplify the setup we assume that $\theta^{ij} = 0$, but some of $\theta^{4j} \neq 0$, i.e. we have an Euclidean space-time noncommutativity.

We wish to investigate quantum corrections to (1) by means of the background field method. To this end one has to split the field ϕ into a classical background field φ and quantum fluctuations, $\phi = \varphi + \delta\varphi$. The one-loop contribution to the effective action is defined by the part of (1) which is quadratic in quantum fluctuations:

$$S[\varphi, \delta\varphi] = \frac{1}{2}\int_0^\beta dx^4 \int_{\mathbb{R}^3} d^3\bar{x}\, \delta\varphi(D + m^2)\delta\varphi, \quad (3)$$

where the operator D is of the form (cf. [19,28]):

$$D = -(\partial_\mu \partial^\mu + E), \quad (4)$$

with

$$E = -\frac{g}{6}(L(\varphi \star \varphi) + R(\varphi \star \varphi) + L(\varphi)R(\varphi)). \quad (5)$$

The one-loop effective action can be formally written as

$$W = \tfrac{1}{2}\ln\det(D + m^2). \quad (6)$$

This equation still has to be regularized. To make use of the zeta-function regularization we have to define the heat kernel[1]

$$K(t, D) = \mathrm{Tr}(e^{-tD} - e^{tD_0}) \quad (7)$$

and the zeta-function

$$\zeta(s, D + m^2) = \mathrm{Tr}((D + m^2)^{-s} - (D_0 + m^2)^{-s}). \quad (8)$$

Here Tr is the L_2 trace. In both cases we subtracted the parts corresponding to free fields with $D_0 = -\partial_\mu \partial^\mu$ to avoid volume divergences.

The regularized one-loop effective action is defined as

$$W_s = -\frac{1}{2}\mu^{2s}\int_0^\infty \frac{dt}{t^{1-s}} e^{-tm^2} K(t, D)$$
$$= -\frac{1}{2}\mu^{2s}\Gamma(s)\zeta(s, D + m^2), \quad (9)$$

where s is a regularization parameter, μ is a constant of the dimension of mass introduced to keep proper dimension of the effective action. The regularization is removed in the limit $s \to 0$. At $s = 0$ the gamma-function has a pole, so that near $s = 0$

[1]A better name used in mathematics for this object is the heat trace, but here we use the terminology more common in physics.

$$W_s = -\frac{1}{2}\left(\frac{1}{s} - \gamma_E + \ln\mu^2\right)\zeta(0, D + m^2) - \frac{1}{2}\zeta'(0, D + m^2), \quad (10)$$

where γ_E is the Euler constant.

Let us assume that there is an asymptotic expansion of the heat kernel as $t \to +0$

$$K(t, D) = \sum_{n=1}^{\infty} t^{(n-4)/2} a_n(D). \quad (11)$$

Such an expansion exists usually (but not always) in the commutative case. On NC $S^1 \times \mathbb{R}^3$ the existence of (11) will be demonstrated in Sec. II B. For a Laplace type operator on a commutative manifold all odd-numbered heat kernel coefficients a_{2k-1} vanish. (They are typical boundary effects). As we shall see below, on NC $S^1 \times \mathbb{R}^3$ the coefficient $a_3 \neq 0$. The coefficient a_0 disappears due to the subtraction of the free-space contribution in (7).

The pole part of W_s can be now expressed through the heat kernel coefficients.

$$\zeta(0, D + m^2) = -m^2 a_2(D) + a_4(D). \quad (12)$$

Note that odd-numbered heat kernel coefficients $a_{2p-1}(D)$ do note contribute to the divergences of W_s.

B. Heat kernel expansion on $S^1 \times \mathbb{R}^3$

Let us consider the operator

$$D = -(\partial_\mu^2 + E), \qquad E = L(l_1) + R(r_1) + L(l_2)R(r_2), \quad (13)$$

on $S^1 \times \mathbb{R}^3$. This operator is slightly more general than the one in (4). The potential term (5) is reproduced by the choice

$$l_1 = r_1 = -\frac{g}{6}\varphi \star \varphi, \qquad l_2 = -r_2 = \sqrt{\frac{g}{6}}\varphi. \quad (14)$$

We are interested in the asymptotics of the heat trace (7) as $t \to +0$. To calculate the trace we, as usual, sandwich the operator between two normalized plane waves [2] and integrate over the momenta and over the manifold $\mathcal{M} = S^1 \times \mathbb{R}^3$

$$K(t; D) = \frac{1}{\beta(2\pi)^3} \sum\!\!\!\!\!\!\int dk \int_{\mathcal{M}} d^4x e^{-\imath kx}(e^{-tD} - e^{tD_0})e^{\imath kx}, \quad (15)$$

where we introduced the notation

$$\sum\!\!\!\!\!\!\int dk \equiv \sum_{k_4} \int d^3\bar{k}, \quad (16)$$

with $k_4 = 2\pi n/\beta$, $n \in \mathbb{Z}$. To evaluate the asymptotic expansion of (15) at $t \to +0$ one has to extract the factor e^{-tk^2}

$$K(t, D) = \frac{1}{\beta(2\pi)^3} \int d^4x \sum\!\!\!\!\!\!\int dk e^{-tk^2}\langle \exp(t((\partial - ik)^2 + 2ik^\mu(\partial_\mu - ik_\mu) + E)) - 1\rangle_k, \quad (17)$$

where we defined

$$\langle F \rangle_k \equiv e^{-\imath kx} \star F e^{\imath kx} \quad (18)$$

for any operator F. Next one has to expand the exponential in (17) in a power series in E and $(\partial - \imath k)$. As we shall see below, only a finite number of terms in this expansion contribute to any finite order of t in the $t \to +0$ asymptotic expansion of the heat kernel. We push all $(\partial - \imath k)$ to the right until they hit $e^{\imath kx}$ and disappear:

$$K(t, D) = \frac{1}{\beta(2\pi)^3} \int d^4x \sum\!\!\!\!\!\!\int dk e^{-tk^2}\Big\langle tE + \frac{t^2}{2}([\partial_\mu, [\partial_\mu, E]] + E^2 + 2\imath k^\mu[\partial_\mu, E]) - \frac{4t^3}{6}k^\mu k^\nu[\partial_\mu, [\partial_\nu, E]] + \ldots\Big\rangle_k. \quad (19)$$

We kept in this equation all the terms which may contribute to a_n with $n \le 4$. In the commutative case all total derivatives as well as all terms linear in k vanish. In the NC case this is less obvious because of the nonlocality, so that we kept also such terms. The commutator of ∂_μ with E is a multiplication operator, e.g., $[\partial_\mu, L(l)] = L(\partial_\mu l)$, $[\partial_\mu, L(l)R(r)] = L(\partial_\mu l)R(r) + L(l)R(\partial_\mu r)$. Therefore, Eq. (19) contains multiplication operators of two different sorts: the ones with only left or only right Moyal multiplications, and the ones containing products of left and right Moyal multiplications. The terms of different sorts will be treated differently.[3]

The terms with one type of the multiplications are easy. We shall call such terms planar borrowing the terminology from the approach based on Feynman diagrams. They can be evaluated in the same way as in [17,18]. Because of the identities

$$\int d^4x \langle R(r)\rangle_k = \int d^4x r(x), \qquad \int d^4x \langle L(l)\rangle_k = \int d^4x l(x), \quad (20)$$

only the E and E^2 terms in (19) contribute. It remains then

[2] Although we are working with a real field, it is more convenient to use complex plane waves instead of real functions $\sin(kx)$ and $\cos(kx)$. For a complex field we would have a coefficient of 1 instead of $1/2$ on the right-hand side of (6). Since D with (14) is real, this is the only difference.

[3] Formally $R(r) = L(1)R(r)$, but a constant function does not belong to $C^\infty(S^1 \times \mathbb{R}^3)$ since it does not satisfy the fall-off conditions. Consequently, the two sorts of the term discussed above indeed lead to quite different asymptotics at $t \to +0$.

to evaluate the asymptotics of the integral

$$\frac{1}{\beta(2\pi)^3}\sum\!\!\!\!\!\!\!\int dk e^{-tk^2} = (4\pi t)^{-2} + \text{e.s.t.}, \tag{21}$$

where e.s.t. denotes exponentially small terms, to obtain

$$a_2^{\text{planar}}(D) = (4\pi)^{-2}\int d^4x(l_1 + r_1), \tag{22}$$

$$a_4^{\text{planar}}(D) = (4\pi)^{-2}\int d^4x\frac{1}{2}(l_1^2 + r_1^2). \tag{23}$$

Nonplanar (mixed) contributions require considerably more work. The typical term reads

$$T(l,r) = \frac{1}{\beta(2\pi)^3}\int d^4x\sum\!\!\!\!\!\!\!\int dk e^{-tk^2}\langle L(l)R(r)\rangle_k \tag{24}$$

with some functions $r(x)$ and $l(x)$. For example, taking $l = l_2$ and $r = tr_2$ the expression (24) reproduces the first term (tE) in (19). Let us expand $r(x)$ and $l(x)$ in the Fourier integrals

$$\begin{aligned} r(x) &= \frac{1}{\beta^{1/2}(2\pi)^{3/2}}\sum\!\!\!\!\!\!\!\int dq\tilde{r}(q)e^{\iota qx},\\ l(x) &= \frac{1}{\beta^{1/2}(2\pi)^{3/2}}\sum\!\!\!\!\!\!\!\int dq'\tilde{l}(q')e^{\iota q'x}. \end{aligned} \tag{25}$$

Then

$$\begin{aligned} \langle L(l)R(r)\rangle_k = &\frac{1}{\beta(2\pi)^3}\sum\!\!\!\!\!\!\!\int dq\sum\!\!\!\!\!\!\!\int dq'\tilde{r}(q)\tilde{l}(q')e^{\iota(q+q')x}\\ &\times e^{-(\iota/2)k\wedge(q-q')-(\iota/2)(q'-k)\wedge(q+k)}, \end{aligned} \tag{26}$$

where

$$k\wedge q \equiv \theta^{\mu\nu}k_\mu q_\nu. \tag{27}$$

Next we substitute (26) in (24) and integrate over x and q' to obtain

$$T(l,r) = \frac{1}{\beta(2\pi)^3}\sum\!\!\!\!\!\!\!\int dk\sum\!\!\!\!\!\!\!\int dq e^{-tk^2}\tilde{l}(-q)\tilde{r}(q)e^{-\iota k\wedge q}. \tag{28}$$

In our case $k\wedge q = \theta^{4i}(k_4q_i - k_iq_4)$.

Next we study the integral over k. The sum over k_4 is treated with the help of the Poisson formula

$$\sum_{n\in\mathbb{Z}} f(2\pi n) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty} f(p)e^{-\iota np}dp. \tag{29}$$

We apply this formula to the sum

$$\sum_{k_4}\exp(-tk_4^2 - \iota\theta^{4j}k_4q_j), \tag{30}$$

which corresponds to the choice

$$f(p) = \exp\left(-\frac{tp^2}{\beta^2} - \frac{\iota\theta^{4j}q_jp}{\beta}\right) \tag{31}$$

in (29). The sum (30) is transformed to (after changing the integration variable $y = p/\beta$)

$$\begin{aligned} &\frac{\beta}{2\pi}\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty}dy\exp(-ty^2 - \iota y(\theta^{4j}q_j + \beta n))\\ &\quad = \frac{\beta}{2\pi}\sum_{n\in\mathbb{Z}}\sqrt{\frac{\pi}{t}}\exp\left(-\frac{(\theta^{4j}q_j + \beta n)^2}{4t}\right). \end{aligned} \tag{32}$$

The integral over k_j is Gaussian and can be easily performed. We arrive at

$$\begin{aligned} T(l,r) = &\frac{1}{(4\pi t)^2}\sum\!\!\!\!\!\!\!\int dq\sum_n\exp\left(-\frac{|\theta|^2q_4^2 + (\theta^{4j}q_j + \beta n)^2}{4t}\right)\\ &\times h(q), \end{aligned} \tag{33}$$

where

$$h(q) \equiv \tilde{l}(-q)\tilde{r}(q), \qquad |\theta|^2 \equiv \theta^{4j}\theta^{4j}. \tag{34}$$

In Eq. (33) one can still put $|\theta| = 0$ thus returning to the commutative case. The limit $|\theta| \to 0$ does not commute, however, with taking the asymptotic $t \to 0$. From now on we assume $|\theta| \neq 0$. Obviously, all terms in the sum over q_4 are exponentially small as $t \to +0$ except for $q_4 = 0$;

$$\begin{aligned} T(l,r) = &\frac{1}{(4\pi t)^2}\int d^3\bar{q}\sum_n\exp\left(-\frac{(\theta^{4j}q_j + \beta n)^2}{4t}\right)h(0,\bar{q})\\ &+ \text{e.s.t.} \end{aligned} \tag{35}$$

Let us define two projectors

$$\Pi_\parallel^{ij} = \frac{\theta^{4i}\theta^{4j}}{|\theta|^2}, \qquad \Pi_\perp^{ij} = \delta^{ij} - \Pi_\parallel^{ij}, \tag{36}$$

and split $\bar{q}$ into the parts which are parallel and perpendicular to θ^{4j}: $q_\parallel = \Pi_\parallel\bar{q}$, $q_\perp = \Pi_\perp\bar{q}$. Then $d^3\bar{q} = dq_\parallel d^2q_\perp$, and $(\theta^{4j}q_j + \beta n)^2 = (|\theta|q_\parallel + \beta n)^2$. The asymptotics of the integral over $q_\parallel$ can be calculated by the saddle-point method. For each n there is one critical point of the integrand corresponding to $q_\parallel = q_\parallel^{(n)} \equiv -\beta n/|\theta|$. We expand $h(0, q_\parallel, q_\perp)$ about these critical points and take the integral over $q_\parallel$ to obtain

$$\begin{aligned} T(l,r) = &\frac{1}{|\theta|(4\pi t)^{3/2}}\sum_{n\in\mathbb{Z}}\int d^2q_\perp\Big(h(0, q_\parallel^{(n)}, q_\perp)\\ &+ \frac{t}{|\theta|^2}h''(0, q_\parallel^{(n)}, q_\perp) + \ldots\Big), \end{aligned} \tag{37}$$

where prime denotes the derivative with respect to $q_\parallel$. This completes the calculation of small t asymptotics for $T(l,r)$. Since both $l(x)$ and $r(x)$ are supposed to be smooth, their Fourier components $\tilde{l}(q)$ and $\tilde{r}(q)$ fall off faster than any power at large momenta, and each term in the asymptotic expansion is given by a convergent sum and a convergent integral.

The expression (37) is already enough to calculate mixed (nonplanar) contributions to the heat kernel expansions from the terms inside the brackets in (19) which do not contain k. (We shall do this in a moment). Regarding the terms which do contain the momentum k, for our purposes it is enough to evaluate the power of t appearing in front of such terms. One can easily trace which modifications appear in the calculations (24)–(37) due to the presence of a polynomial of k_μ. The result is (i) we still have an expansion in $t^{1/2}$, (ii) the terms with k do not contribute to the heat kernel coefficients a_n with $n \leq 4$. In other words, the only relevant mixed heat kernel coefficient is generated by the first term in the brackets in (19), and it reads

$$a_3^{\rm mixed}(D) = \frac{1}{|\theta|(4\pi)^{3/2}} \sum_{n\in\mathbb{Z}} \int d^2q_\perp \tilde{l}_2(0, -q_\parallel^{(n)}, -q_\perp) \times \tilde{r}_2(0, q_\parallel^{(n)}, q_\perp), \tag{38}$$

where we substituted the fields appearing in E [see Eq. (13)]. Note that this expression is divergent in the commutative limit $|\theta| \to 0$. The coefficient $a_3^{\rm mixed}$ is highly nonlocal. The structure of (38), especially the sum over n, reminds us of the heat kernel coefficients on NC torus for a rational NC parameter [20]. In this latter case there is a simple geometric interpretation in terms of periodic projections [29]. No such interpretation is known for the present case of $S^1 \times \mathbb{R}^3$. However, some similarities can be found to the works in [11] discussing discretization of the coordinates which do not commute with a compact coordinate.

C. Renormalization

Let us return to our particular model (1). First we summarize the results of the previous subsection and reexpress the heat kernel coefficients $a_n = a_n^{\rm planar} + a_n^{\rm mixed}$ in terms of the background field φ by means of (14):

$$a_2(D) = -\frac{g}{48\pi^2} \int d^4x \varphi^2, \tag{39}$$

$$a_3(D) = -\frac{g}{6|\theta|(4\pi)^{3/2}} \sum_{n\in\mathbb{Z}} \int d^2q_\perp \tilde{\varphi}(0, -q_\parallel^{(n)}, -q_\perp) \times \tilde{\varphi}(0, q_\parallel^{(n)}, q_\perp), \tag{40}$$

$$a_4(D) = \frac{1}{16\pi^2} \frac{g^2}{36} \int d^4x \varphi_\star^4, \tag{41}$$

where tilde is used again to denote the Fourier components.

Next we substitute (39)–(41) in (10) and (12) to the pole part of the regularized effective action

$$W_s^{\rm pole} = -\frac{1}{2s} \int d^4x \left(\frac{g}{48\pi^2} m^2\varphi^2 + \frac{1}{16\pi^2} \frac{g^2}{36} \varphi_\star^4 \right). \tag{42}$$

This divergent part of the effective action can be cancelled by an infinite renormalization of couplings in (1)

$$\delta m^2 = \frac{gm^2}{48\pi^2} \frac{1}{s}, \qquad \delta g = \frac{g^2}{48\pi^2} \frac{1}{s}. \tag{43}$$

There can be, of course, also some finite renormalization which we do not discuss here. Our main physical observation in this subsection is the renormalization (43) does not depend on the temperature $1/\beta$.

Here some more comments are in order. It is a very attractive feature of the zeta-function regularization that the nonplanar nonlocal coefficient $a_3(D)$ does not affect the counterterms. This coefficient will, however, contribute at some other places, like the large mass expansion of the one-loop effective action (see, e.g., [16]). Moreover, $a_3(D)$ can lead to troubles in different regularization schemes. For example, if one uses the proper-time cutoff at some scale Λ defining the regularized effective action by

$$W_\Lambda = -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{dt}{t} e^{-tm^2} K(t, D), \tag{44}$$

the coefficient a_3 generates a linear divergence $\propto \Lambda a_3(D)$, which has no classical counterpart and cannot be renormalized away in the standard approach. There is a subtraction scheme (that was used in quantum field theory on curved background [30] and in Casimir energy calculations [31]) which prescribes to subtract all contributions from several leading heat kernel coefficients, including $a_3(D)$ in four dimensions. In the case of two-dimensional scalar theories this heat kernel subtraction scheme is equivalent to usual renormalization with the ''no-tadpole'' normalization condition [32]. In the present case the heat kernel subtraction is, obviously, *not* equivalent to the charge and mass renormalizations given by (43).

We restricted ourselves to the case of pure space-time noncommutativity $\theta^{ij} = 0$. However, one can try to make an educated guess on what happens for a generic nondegenerate $\theta^{\mu\nu}$. By comparing the heat kernel expansion obtained above with that on NC torus [20] and on NC plane with a nondegenerate $\theta^{\mu\nu}$ [19] we can derive (rather unrigorously) the following rule: the presence of a noncompact NC dimension increases the number of the first nontrivial field-dependent nonplanar (mixed) heat kernel coefficient by one as compared to the first nontrivial field-dependent coefficient in the commutative case. Indeed, in the commutative case the first such coefficient is a_2. On the NC torus [20] (no noncompact dimensions) the first field-dependent mixed heat kernel coefficient is also a_2. For the geometry studied in this paper (one noncompact NC dimension) this is a_3. On an n-dimensional NC plane with a nondegenerate $\theta^{\mu\nu}$ the first coefficient of interest is a_{n+2} [19]. We can expect therefore that the first mixed coefficient on $S^1 \times \mathbb{R}^3$ with a nondegenerate $\theta^{\mu\nu}$ (three compact NC dimensions) will be a_5. Such coefficient does not contribute to one-loop divergence neither in the zeta-function regularization nor in the proper-time cutoff scheme. Thus the situation in the generic case may be expected to be better than in the case of a degenerate $\theta^{\mu\nu}$

discussed above. A similar conclusion has been made for the Moyal plane in [28].

As we have already mentioned above, the counterterms do not depend on the temperature. However, if one does the calculations directly on the zero-temperature manifold $\mathbb{R}^4$, there appears problems for a degenerate NC parameter [28]. Perhaps, compactification of one of the NC directions is a proper way to regularize these problems away.

III. FROM IMAGINARY TO REAL-TIME FORMALISM

The methods which allow one to make correspondence between imaginary and real-time formalisms in the case of frequency-dependent Hamiltonians were suggested in [27] and developed further in [33]. Here we briefly outline these methods and discuss the peculiarities of their application to noncommutative theories. From now on we work with static background fields, $\partial_0\varphi = \partial_4\varphi = 0$.

A. Spectral density in the real-time formalism

Let us consider a Minkowski space counterpart of the action (1). Our rules for the continuation between Euclidean and Minkowski signatures read $\partial_4 \to \imath\partial_0$ and $\theta^{j4} \to -\imath\theta^{j0}$, where θ^{j0} is real, and $\theta^{j0}\partial_0$ corresponds to $\theta^{j4}\partial_4$. We have, therefore, a real NC parameter in the Moyal product on both Euclidean and Minkowski spaces. These rules were applied, e.g., in [9], and they follow also from the requirement of reflection positivity [34]. As we shall see below, these rules also ensure consistency between the expressions for the free energy defined in imaginary and real-time formalisms.

The wave equation for quantum fluctuations $\psi(x)$ over a static background reads

$$\Big(-\partial_0^2 + \partial_j^2 - m^2 - \frac{g}{6}(L(\varphi^2) + R(\varphi^2) + L(\varphi)R(\varphi))\Big)\psi(x) = 0. \quad (45)$$

The wave operator in (45) commutes with ∂_0. Consequently, one can look for the solutions ψ_ω whose time dependence is described by $\psi_\omega(x) \sim e^{i\omega x^0}$. They satisfy the equation

$$(P(\omega) + m^2)\psi_\omega = \omega^2\psi_\omega, \quad (46)$$

where

$$P(\omega) = -\partial_j^2 + V(\omega),$$
$$V(\omega) = \frac{g}{6}(\varphi_+^2 + \varphi_-^2 + \varphi_+\varphi_-), \quad (47)$$

and

$$\varphi_\pm(x^j) = \varphi(x^j \pm \tfrac{1}{2}\theta^{j0}\omega). \quad (48)$$

Here we used the fact that left (right) Moyal multiplication of a function of x^j by $\exp(i\omega x^0)$ is equivalent to a shift of the argument.

From now on we consider the case of positive coupling g only. Then the potential $V(\omega)$ is nonnegative, $V = (g/12)(\varphi_+^2 + \varphi_-^2 + (\varphi_+ + \varphi_-)^2) \geq 0$.

To define spectral density for the Eq. (46) we follow the works [27]. Consider an auxiliary eigenvalue problem,

$$(P(\lambda) + m^2)\psi_{\nu,\lambda} = \nu^2\psi_{\nu,\lambda}. \quad (49)$$

Obviously, the functions $\psi_{\omega,\omega}$ solve the Eq. (46).

Our next step differs from that in [27]. Let us restrict λ to $\lambda \leq \lambda_0$ for some λ_0 and put the system in a three-dimensional box with periodic boundary conditions. Let us suppose that the size of the box is $\gg \theta\lambda_0$, so that φ_+ and φ_- are localized far away from the boundaries. In this case, the spectrum of the regularized problem can be considered as an approximation to the spectrum of the initial problem for the whole range of ν. Later we shall remove the box, and the restriction $\lambda \leq \lambda_0$ will become irrelevant. In the box, the spectrum of ν in (49) becomes discrete, but, for a sufficiently large box, the spacing is small. The eigenvalues $\nu_N^2(\lambda)$ depend smoothly on λ not greater than λ_0, and we can define the density of states as

$$\frac{dn(\nu,\lambda)}{d(\nu^2)} = \frac{1}{2\nu}\frac{dn(\nu,\lambda)}{d\nu} = \sum_N \delta(\nu^2 - \nu_N^2(\lambda)), \quad (50)$$

which can be used to calculate spectral functions of $\tilde{P}(\lambda)$, where tilde reminds us that we are working with a finite-volume problem. For example,

$$\widetilde{\mathrm{Tr}}(e^{-t(\tilde{P}(\lambda)+m^2)}) = \int_m^\infty \frac{dn(\nu,\lambda)}{d\nu}e^{-t\nu^2}d\nu. \quad (51)$$

Here $\widetilde{\mathrm{Tr}}$ denotes the L_2 trace in the box. The potential V is nonnegative. Consequently, there are no eigenvalues below m.

The eigenvalues ω_N^2 of the initial problem (46) in this discretized setting appear when the line $\nu^2 = \lambda^2$ intersects $\nu_N^2(\lambda)$. We can define the density of the eigenfrequencies ω_N^2 by the formula

$$\frac{dn(\omega)}{d(\omega^2)} = \sum_N \delta(\omega^2 - \omega_N^2). \quad (52)$$

Next, we would like to relate this density to (50). This can be done by calculating the derivative of the arguments of the delta function taken for $\omega = \lambda = \nu$. We obtain

$$\frac{dn(\omega)}{d(\omega^2)} = \frac{d\hat{n}(\omega,\omega)}{d(\omega^2)}, \quad (53)$$

where

$$\frac{d\hat{n}(\nu,\lambda)}{d(\nu^2)} = \sum_N\Big(1 - \frac{d(\nu_N^2)}{d(\lambda^2)}\Big)\delta(\nu^2 - \nu_N^2). \quad (54)$$

This density admits an interpretation in terms of the heat kernel

$$\begin{aligned}\tilde{\mathrm{Tr}}\left[\left(1-\frac{1}{2\lambda}\frac{d\tilde{P}(\lambda)}{d\lambda}\right)e^{-t(\tilde{P}(\lambda)+m^2)}\right] &= \left(1+\frac{1}{2\lambda t}\frac{d}{d\lambda}\right)\tilde{\mathrm{Tr}}(e^{-t(\tilde{P}(\lambda)+m^2)}) \\ &= \int_m^\infty \frac{d\hat{n}(\nu,\lambda)}{d\nu}e^{-t\nu^2}d\nu. \end{aligned} \tag{55}$$

Next we remove the box. Most of the quantities discussed above are divergent in the infinite volume limit. In order to remove these divergences we subtract the spectral densities corresponding with the free operator $\tilde{P}_0 + m^2$ with $\tilde{P}_0 = -\partial_j^2$ [not to be confused with $\tilde{P}(0)$]. Then we perform the infinite volume limit. The limits of subtracted densities $dn(\omega)/d\omega$, $dn(\nu,\lambda)/d\lambda$, and $d\hat{n}(\nu,\lambda)/d\lambda$ will be denoted by $\rho(\omega)$, $\rho(\nu,\lambda)$, and $\varrho(\nu,\lambda)$, respectively. The following relation holds in this limit:

$$\mathrm{Tr}_3(e^{-t(P(\lambda)+m^2)})_{\mathrm{sub}} = \int_m^\infty d\omega\rho(\omega;\lambda)e^{-t\omega^2}, \tag{56}$$

where Tr_3 is the L_2 trace on $\mathbb{R}_3$ and

$$\mathrm{Tr}_3(e^{-t(P(\lambda)+m^2)})_{\mathrm{sub}} \equiv \mathrm{Tr}_3(e^{-t(P(\lambda)+m^2)} - e^{-t(-\partial_j^2+m^2)}). \tag{57}$$

We also have the relation

$$\left(1+\frac{1}{2\lambda t}\frac{d}{d\lambda}\right)\mathrm{Tr}_3(e^{-t(P(\lambda)+m^2)})_{\mathrm{sub}} = \int_m^\infty \varrho(\nu;\lambda)e^{-t\nu^2}d\nu, \tag{58}$$

which, together with (56), yields

$$\varrho(\omega;\lambda) = \rho(\omega;\lambda) + \frac{\omega}{\lambda}\int_m^\omega \partial_\lambda\rho(\sigma;\lambda)d\sigma. \tag{59}$$

To derive this formula one has to integrate by parts. Vanishing of the boundary terms is established by using the same arguments as in [27]. An infinite volume counterpart of (53) reads

$$\rho(\omega) = \varrho(\omega;\omega). \tag{60}$$

An independent calculation of the spectral densities is a very hard problem. We shall view the Eq. (56) as a definition of the subtracted spectral density $\rho(\nu;\lambda)$ through the heat kernel (an explicit formula involves the inverse Laplace transform). The other spectral densities $\varrho(\nu;\lambda)$ and $\rho(\omega)$ are then defined through (59) and (60).

Relations similar to (56) and (58)–(60) were originally obtained in [27] for a different class of frequency-dependent operators and by a somewhat different method.

B. Wick rotation

In this section we show that the Wick rotation of the free energy F defined through the Euclidean effective action coincides with the canonical free energy F_C. The methods we use are borrowed from [27], but there are some subtle points related to specific features of NC theories. By definition,

$$W(\beta) = \beta(F(\beta)+\mathcal{E}), \tag{61}$$

where $\mathcal{E}$ is the energy of vacuum fluctuations.

Our renormalization prescription (43) is an equivalent to the (minimal) subtraction of the pole term (42) in (10). Therefore, the renormalized one-loop effective action reads

$$W = -\frac{1}{2}\frac{d}{ds}\bigg|_{s=0}(\tilde{\mu}^{2s}\zeta(s,D+m^2)), \tag{62}$$

where $\tilde{\mu}^2 := \mu^2 e^{-\gamma_E}$. On a static background one can separate the frequency sum from the $L_2(\mathbb{R}^3)$ trace and rewrite the zeta function as

$$\begin{aligned}\zeta(s,D+m^2) &= \sum_l \mathrm{Tr}_3((\omega_l^2+m^2+P(\omega_l))^{-s} \\ &\quad - (\omega_l^2+m^2-\partial_j^2)^{-s}) \\ &= \sum_l \int_m^\infty d\nu\rho_E(\nu;\omega_l)(\omega_l^2+\nu^2)^{-s}. \end{aligned} \tag{63}$$

$\omega_l = 2\pi l/\beta$. The spectral density ρ_E is defined for the Euclidean space NC parameter θ^{j4}. It is related to the real-time spectral density by the formula

$$\rho_E(\nu;\omega|\theta^{j4}) = \rho(\nu;\iota\omega| - i\theta^{j4}) \tag{64}$$

according to the rules which we have discussed at the beginning of Sec. III A. We have already mentioned that the Wick rotation leaves the combination $\omega\theta$ and the potential V invariant. Therefore, both densities *coincide* as functions of their arguments ν and ω. However, we shall keep the subscript E to avoid confusion but shall drop θ from the notations for the sake of brevity. Next we use the formula

$$\sum_l f(\omega_l) = \frac{\beta}{4\pi\iota}\oint_C \cot\left(\frac{\beta z}{2}\right)f(z)dz \tag{65}$$

with the contour C consisting of two parts, C_+ running from $\iota\epsilon+\infty$ to $\iota\epsilon-\infty$ and C_- running from $-\iota\epsilon-\infty$ to $-\iota\epsilon+\infty$, to rewrite the frequency sum as an integral. Then, by using the symmetry of the integrand with respect to reflections of z we replace the integral over C by twice the integral over C_+ alone. Finally, we apply the identity

$$\cot\left(\frac{\beta z}{2}\right) = \frac{2}{\beta}\frac{d}{dz}\ln(1-e^{\iota\beta z}) - \iota \tag{66}$$

to arrive at the result

$$\zeta(s,D+m^2) = \beta\zeta_0(s,D+m^2) + \zeta_T(s,D+m^2), \tag{67}$$

where

$$\zeta_0(s, D+m^2) = \frac{1}{\pi}\int_m^\infty d\nu \int_0^\infty \rho_E(\nu; z)(\nu^2+z^2)^{-s}dz, \tag{68}$$

$$\zeta_T(s, D+m^2) = \frac{1}{\pi\imath}\int_m^\infty d\nu \oint_{C_+} dz\Big[\frac{d}{dz}\ln(1-e^{\imath\beta z})\Big] \times \rho_E(\nu; z)(\nu^2+z^2)^{-s}. \tag{69}$$

In commutative theories [27], the function ζ_T, which vanishes at zero temperature, represents the purely thermal part, while ζ_0 is responsible for the vacuum energy. In space-time NC theories there is no good definition of the canonical Hamiltonian and of the energy. Therefore, we have no other choice than to accept the same identities as in the commutative case, namely,

$$F(\beta) = -\frac{1}{2\beta}\frac{d}{ds}\Big|_{s=0}\tilde{\mu}^{2s}\zeta_T(s, D+m^2), \tag{70}$$

$$\mathcal{E} = -\frac{1}{2}\frac{d}{ds}\Big|_{s=0}\tilde{\mu}^{2s}\zeta_0(s, D+m^2). \tag{71}$$

Actually, the definition of $\mathcal{E}$ is a rather natural one since it coincides with the renormalized Euclidean one-loop effective action on $\mathbb{R}^4$. However, as we have already mentioned in Sec. II C the renormalization in NC theories depends crucially on the number of compact dimensions. Therefore, if one does the renormalization directly in $\mathbb{R}^4$, one may need the counterterms which differ from (43) obtained on $S^1\times\mathbb{R}^3$.

From now on we concentrate exclusively on $F_T(\beta)$ and ζ_T. We integrate by parts over z to obtain

$$\zeta_T(s) = -\frac{1}{\pi\imath}\int_m^\infty d\nu \oint_{C_+} dz \ln(1-e^{\imath\beta z}) \times\Big[\frac{\partial_z\rho_E(\nu; z)}{(z^2+\nu^2)^s} - \frac{2zs\rho_E(\nu; z)}{(z^2+\nu^2)^{s+1}}\Big]. \tag{72}$$

To ensure the absence of the boundary terms, we have to deform the contour C_+ by moving its ends up in the complex plane, so that $e^{\imath\beta z}$ provides the necessary damping of the integrand. We discuss the conditions on ρ_E which make such deformations of the contour legitimate below. The integration by parts over ν in the first term in the square brackets in (72) yields

$$\zeta_T(s) = \frac{s}{\pi\imath}\int_m^\infty d\nu \oint_{C_+} dz \ln(1-e^{\imath\beta z}) \times\frac{2z}{(z^2+\nu^2)^{s+1}}\varrho_E(\nu; z), \tag{73}$$

where

$$\varrho_E(\nu; z) = \rho_E(\nu; z) - \frac{\nu}{z}\int_m^\nu \partial_z\rho_E(\sigma; z)d\sigma. \tag{74}$$

The right-hand side of (73) is proportional to s. To estimate the derivative ∂_s at $s=0$ in (70) one can put $s=0$ in the rest of the expression and use the Cauchy theorem after closing the contour in the upper part of the complex plane. The result is then given by the residue at $z=i\nu$. Next we make the Wick rotation of the NC parameter, so that $\rho_E(\sigma; i\nu)$ becomes $\rho(\sigma; \nu)$, and $\varrho_E(\nu; i\nu)$ becomes $\rho(\nu;\nu)=\rho(\nu)$ [cf. Eqs. (59) and (60)]. Consequently, the Euclidean free energy is given by the equation

$$F(\beta) = \frac{1}{\beta}\int_m^\infty d\nu\rho(\nu)\ln(1-e^{-\beta\nu}), \tag{75}$$

which coincides with the canonical definition of the free energy F_C.

The equality $F=F_C$ is the main result of this section. To derive it we integrated by parts and deformed the contour C_+. The integration by part over ν is a safe operation, since for any fixed z the spectral density $\rho_E(\nu, z)$ corresponds to the Laplace operator with a smooth potential. The absence of the boundary terms can be then demonstrated by standard arguments [27] based on the heat kernel expansion. The deformations of the contour are more tricky. To justify this procedure and application of the Cauchy theorem one has to assume that $\rho_E(\nu, z)$ can be analytically continued to the upper half-plane as an entire function of z. A rigorous proof of this assumption is hardly possible even in more tractable cases of stationary commutative space-times [27]. We may argue, however, that this assumption is plausible. Consider pure imaginary values of $z=i\kappa$. All deformations of the contour are done before the Wick rotation of the NC parameter θ. Therefore, $\varphi_\pm$ becomes complex, and $\varphi_+=\varphi_-^*$. The potential $V(i\kappa)$ remains real and positive. The background field φ is assumed to fall off faster than any power of the coordinates in real directions to ensure the existence of the heat kernel expansion. Such fields typically grow in imaginary directions (one can consider $\varphi\sim e^{-cx^2}$ as an example). Large positive potentials tend to diminish the spectral density thus preventing it from the blow-up behavior. It seems, therefore, that the spectral density in our case should not behave worse than the spectral density in the commutative case. Another argument in favor of our assumption will be given at the end of the next section.

The free energy (75) is expressed through a thermal distribution over the eigenfrequencies. In the absence of a well-defined Hamiltonian it is not guaranteed that this is the same as a thermal distribution of one-particle energies. This is a known problem of space-time NC theories which is beyond the scope of this paper.

IV. HIGH-TEMPERATURE ASYMPTOTICS

As in the previous section we rewrite the regularized one-loop effective action (9) on a static background in the form

$$W_s = -\frac{1}{2}\mu^{2s}\Gamma(s)\sum_\omega \mathrm{Tr}_3(\omega^2+P(\omega)+m^2)^{-s}_{\mathrm{sub}}, \tag{76}$$

where the sum over the Matsubara frequencies is separated from the trace over the L_2 functions on $\mathbb{R}^3$. As usual, we subtracted the free-space contributions corresponding to $\varphi = 0$ in $P(\omega)$ [which is indicated by the subscript "sub" in (76)]. We remind the reader that $\omega = 2\pi l/\beta$, $l \in \mathbb{Z}$. To evaluate the high-temperature (small β) asymptotics of W_s we split the sum in (76) in two parts,

$$W_s = W_s^{l=0} + W_s^{l\neq 0}, \tag{77}$$

which will be treated separately.

We start with $W_s^{l=0}$ which reads

$$\begin{aligned} W_s^{l=0} &= -\tfrac{1}{2}\mu^{2s}\Gamma(s)\,\mathrm{Tr}_3(P(0)+m^2)^{-s}_{\mathrm{sub}} \\ &= -\tfrac{1}{2}\mu^{2s}\Gamma(s)\zeta(s, P(0)+m^2). \end{aligned} \tag{78}$$

[The subtraction of free-space contributions is included in our definition of the zeta function, cf. (8).] For each given ω the operator $P(\omega)$ is a three-dimensional Laplace operator with a scalar potential. All effects of the noncommutativity are encoded in the form of this potential. Therefore, as for all Laplace type operators on $\mathbb{R}^3$, the zeta function in (78) vanishes at $s = 0$ making $W_s^{l=0}$ finite. We can immediately take the limit $s \to 0$ to obtain the renormalized expression

$$W^{l=0} = -\tfrac{1}{2}\zeta'(P(0) + m^2). \tag{79}$$

In the rest of the frequency sum we first use an integral representation for the zeta-function

$$W_s^{l\neq 0} = -\frac{1}{2}\mu^{2s}\sum_{\omega\neq 0}\int_0^\infty dt t^{s-1}\,\mathrm{Tr}_3(e^{-t(\omega^2+m^2+P(\omega))})_{\mathrm{sub}} \tag{80}$$

and then use a trick similar to the one employed in the previous section. Namely, we replace the operator in the exponential on the right-hand side of (80) by $\omega^2 + m^2 + P(\lambda)$, expand each of the terms under the frequency sum in asymptotic series at $\omega \to \infty$ keeping λ fixed, and then put $\lambda = \omega$. The result of this procedure reads

$$\begin{aligned} W_s^{l\neq 0} &= -\frac{1}{2}\mu^{2s}\sum_{\omega\neq 0}\sum_{n=2}^{\infty}\int_0^\infty dt t^{s-1} t^{(n-3)/2} e^{-t\omega^2} \\ &\quad \times a_n(P(\omega)+m^2)_{\mathrm{sub}} \\ &= -\frac{1}{2}\mu^{2s}\sum_{\omega\neq 0}\sum_{n=2}^{\infty}|\omega|^{3-n-2s}\Gamma\left(\frac{n-3}{2}+s\right) \\ &\quad \times a_n(P(\omega)+m^2)_{\mathrm{sub}}. \end{aligned} \tag{81}$$

Some comments are in order. Here we used again the fact that $\omega^2 + m^2 + P(\lambda)$ for a fixed ω is just a usual Laplace type operator in three dimensions. The large ω expansion of the heat trace in (80) is therefore standard and, as well as the usual large mass expansion is defined by the heat kernel coefficients (see, e.g., [16]). On a manifold without boundary an asymptotic expansion (11) with the replacement $D \to P + m^2$ exists, and only even numbers n appear. The coefficient a_0 vanishes due to the subtraction, so that the sum in (81) starts with $n = 2$.

Now, we have to study the behavior of $a_n(P(\omega) + m^2)_{\mathrm{sub}}$ at large ω. These heat kernel coefficients are integrals over $\mathbb{R}^3$ of polynomials constructed from the potential $V(\omega)$ and its derivatives. We can present them as

$$\begin{aligned} a_n(P(\omega)+m^2)_{\mathrm{sub}} &= a_n(P+m^2)^{\mathrm{planar}}_{\mathrm{sub}} \\ &\quad + a_n(P(\omega)+m^2)^{\mathrm{mixed}}, \end{aligned} \tag{82}$$

where the first (planar) contribution contains all terms which are the products of either φ_+ and its derivatives only, or of φ_- and its derivatives only (but not the products of φ_+ and φ_-). The rest is collected in the second (mixed) contribution. Obviously, no subtraction for the mixed heat kernel coefficient is needed. Because of the translation invariance of the integral over $\mathbb{R}^3$, the planar coefficient does not depend on ω. For example, $\int d^3x\varphi_+^2 = \int d^3x\varphi^2$. Therefore, we drop ω from the notation.

First, let us consider the mixed contributions to (81). We assumed that the background field φ belongs to $C^\infty(S^1 \times \mathbb{R}^3)$. Therefore, it should vanish exponentially fast at large distances. Since each term in $a_n(P(\omega) + m^2)^{\mathrm{mixed}}$ contains a product of at least one φ_+ with at least one φ_-, it should be of order $C_2 e^{-C_1|\omega\theta|}$ for large ω, where C_2 and C_1 are some constants. C_1 is positive and characterizes the falloff of φ at large distances. C_2 depends on n, on the amplitude of φ, and on the functional form of a_n. Up to an inessential overall constant the contribution of a mixed coefficient to (81) can be estimated as

$$\sim \sum_{\omega\neq 0}|\omega|^{3-n}e^{-C_1|\omega\theta|} \sim \sum_{l=1}^{\infty}\beta^{n-3}l^{3-n}\exp\left(-\frac{2\pi C_1 l|\theta|}{\beta}\right) \tag{83}$$

(this sum is obviously convergent, so that one can remove the regularization parameter). If β is small enough, namely, $\beta \ll C_1|\theta|$, all terms in the sum (83) are strongly suppressed, and the value of the sum can be well approximated by the first term

$$\sim \beta^{n-3}\exp\left(-\frac{2\pi C_1|\theta|}{\beta}\right). \tag{84}$$

We conclude that the contributions of the mixed terms are exponentially small and can be neglected[4] in the high-temperature expansion of the effective action.

Since the planar heat kernel coefficients do not depend on ω, we are ready to evaluate their contribution to (81) by using precisely the same procedure as in Dowker and Kennedy [26]:

[4]If one imposes a stronger restriction on the background requiring that φ is of compact support, then the mixed terms vanish identically above certain temperatures.

$$W_s^{l\neq 0} = -\mu^{2s}\sum_{n=2}^{\infty}\sum_{l=1}^{\infty}\Gamma\left(\frac{n-3}{2}+s\right)l^{3-n-2s} \times a_n(P+m^2)_{\rm sub}^{\rm planar}\left(\frac{\beta}{2\pi}\right)^{n-3+2s} = -\mu^{2s}\sum_{n=2}^{\infty}\Gamma\left(\frac{n-3}{2}+s\right)\zeta_R(2s+n-3) \times a_n(P+m^2)_{\rm sub}^{\rm planar}\left(\frac{\beta}{2\pi}\right)^{n-3+2s} \quad (85)$$

with ζ_R being the Riemann zeta function. We remind that the index n in (85) is even. The only divergence in (85) is a pole in ζ_R for $n = 4$. The corresponding term near $s = 0$ reads

$$\frac{1}{2}a_4(P+m^2)_{\rm sub}^{\rm planar}\frac{\beta}{(4\pi)^{1/2}}\left[-\frac{1}{s}-\gamma_E-2\ln\left(\frac{\mu\beta}{4\pi}\right)\right]. \quad (86)$$

On static backgrounds there is a useful formula which relates planar heat kernel coefficients of D and P,

$$a_n(D+m^2)_{\rm sub}^{\rm planar} = \frac{\beta}{(4\pi)^{1/2}}a_n(P+m^2)_{\rm sub}^{\rm planar}. \quad (87)$$

This formula follows from the analysis of the planar heat kernel coefficients presented in Sec. II B and general formulas for the heat kernel expansion of Laplace type operators [14–16]. The coefficient β appears due to the integration of a constant function over the Euclidean time, and $(4\pi)^{1/2}$ comes from different prefactors in the heat kernel coefficients in three and four dimensions. In particular, $a_4(D+m^2)_{\rm sub}^{\rm planar} = -m^2 a_2(D) + a_4(D) = \beta/(4\pi)^{1/2}a_4(P+m^2)_{\rm sub}^{\rm planar}$ [let us remind the reader that mixed $a_2(D)$ and $a_4(D)$ vanish]. From (10) and (12) we see that the divergence in the Euclidean effective is reproduced. This divergence is then removed by the renormalization of couplings (43). After the renormalization, we collect all contributions to the effective action to obtain our final result for the high-temperature expansion of the renormalized effective action

$$W = -\frac{\pi^{3/2}}{3\beta}a_2(P+m^2)_{\rm sub}^{\rm planar} - \frac{1}{2}\zeta'(P(0)+m^2) - \frac{1}{2}a_4(P+m^2)_{\rm sub}^{\rm planar}\frac{\beta}{(4\pi)^{1/2}}\left[\gamma_E+2\ln\left(\frac{\mu\beta}{4\pi}\right)\right] - \sum_{n=6}^{\infty}\Gamma\left(\frac{n-3}{2}\right)\zeta_R(n-3)a_n(P+m^2)_{\rm sub}^{\rm planar}\left(\frac{\beta}{2\pi}\right)^{n-3}. \quad (88)$$

It is instructive to compare the expansion (88) to the one in the commutative case obtained by Dowker and Kennedy [26] (note, that the normalization of the heat kernel coefficients used in that paper differs from ours). We see that the ζ' term is the same in both cases. The terms proportional to the heat kernel coefficients for the commutative case can be obtained from the expansion above by means of the replacement $a_n(P+m^2)_{\rm sub}^{\rm planar} \to a_n(P(0)+m^2)_{\rm sub}$. (In both cases subtraction of the free-space contribution means simply deleting the highest power of m in standard analytical expressions [16]). Let us write down explicit expressions for a couple of leading heat kernel coefficients. In the NC case we have

$$a_2(P+m^2)_{\rm sub}^{\rm planar} = \frac{1}{(4\pi)^{3/2}}\int d^3x\frac{g}{3}\varphi^2, \qquad a_4(P+m^2)_{\rm sub}^{\rm planar} = \frac{1}{(4\pi)^{3/2}}\int d^3x\left[\frac{g^2}{36}\varphi^4+\frac{g}{3}m^2\varphi^2\right]. \quad (89)$$

The coefficients appearing in the commutative case are

$$a_2(P(0)+m^2)_{\rm sub} = -\frac{1}{(4\pi)^{3/2}}\int d^3x\frac{g}{2}\varphi^2, \qquad a_4(P(0)+m^2)_{\rm sub} = \frac{1}{(4\pi)^{3/2}}\int d^3x\left[\frac{g^2}{8}\varphi^4+\frac{g}{2}m^2\varphi^2\right]. \quad (90)$$

In both cases the corresponding heat kernel coefficients differ only by numerical prefactors in front of the same powers of φ.

The high-temperature expansion does not depend on θ. In the limit $\theta \to 0$ (which is a trivial operation) one does not reproduce the corresponding expansion in the commutative case. The limits $\beta \to 0$ and $\theta \to 0$ are not interchangeable because of the condition $\beta \ll C_1|\theta|$ which was imposed when studying the mixed contributions to the asymptotic expansion.

In space-space NC theories a drastic reduction of the degrees of freedom in the nonplanar sector above certain temperatures was observed in [3]. This may be related in some way to the absence of nonplanar contributions to the high-temperature power-law asymptotics in space-time NC theories found above.

One can calculate the high-temperature asymptotics also in the real-time formalism. The key observation that the nonplanar sector does not contribute remains valid also in this formalism. The heat kernel expansion in the planar sector has the standard form (though the values of the heat kernel coefficients differ from the commutative case). Therefore, one can repeat step by step the calculations of [27] and obtain an expansion for the free energy which is consistent with the expansion for the effective action derived above in the imaginary-time formalism. This is another argument in favor of the assumptions made in Sec. III B.

V. CONCLUSIONS

In this paper we considered some basic features of the finite-temperature NC ϕ^4 theory in the imaginary-time formalism. We restricted ourselves to the case of pure space-time noncommutativity, $\theta^{ij} = 0$. We used the zeta-function regularization and the heat kernel methods. Although we found highly nonlocal nonplanar heat kernel coefficients, such coefficients do not contribute either to the one-loop divergences or to the high-temperature asymptotics. The theory can be renormalized at one loop by making charge and mass renormalizations, as usual. The counterterms do not depend on the temperature (as long as it is nonzero). We expect that the renormalization of this theory at zero temperature proceeds differently. The high-temperature expansion of the one-loop effective action looks similar to the commutative case. The coefficients of this expansion do not depend on the NC parameter θ, but again, one has to assume that this parameter is nonzero.

We have also studied relations between the imaginary and real-time formulations. We found that the Wick rotation of the Euclidean free energy gives the canonical free energy modulo two assumptions. One assumption about the behavior of the spectral density on the complex plane is of a technical nature. Another one is more fundamental; it concerns the interpretation of the eigenfrequencies of perturbations as one-particle energies.

An extension of our results to more general models containing gauge fields and spinors can be done rather straightforwardly. Gauge fields are particularly important to make connections to other approaches [6,7]. Curved space-times will probably be difficult because of the problems with the heat kernel expansion. Even in the case of a two-dimensional NC space with a nontrivial metric, the heat kernel coefficients for a (rather simple) operator are known as power series in the conformal factor only [35].

Another possible development is suggested by the paper [36]. The authors of [36] calculated the heat kernel expansion for a kinetic operator which already contains a contribution from the one-loop two-point function in an NC scalar theory. This procedure effectively leads to resummation of an infinite number of diagrams, known as the ring or daisy resummation. In this way, logarithmic terms in the heat kernel expansion were found. It would be interesting to check this observation in the presence of a nontrivial background field.

ACKNOWLEDGMENTS

One of the authors (D. V. V.) is grateful to C. Dehne for helpful discussions on noncommutative theories and to D. Fursaev for answering endless questions regarding the methods of the nonlinear spectral problem. This work was supported in part by FAPESP and by Grant No. RNP 2.1.1.1112.

---

[1] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001); R. J. Szabo, Phys. Rep. **378**, 207 (2003); R. Wulkenhaar, J. Geom. Phys. **56**, 108 (2006).
[2] G. Arcioni and M. A. Vazquez-Mozo, J. High Energy Phys. 01 (2000) 028; G. Arcioni, J. L. F. Barbon, J. Gomis, and M. A. Vazquez-Mozo, J. High Energy Phys. 06 (2000) 038.
[3] W. Fischler, E. Gorbatov, A. Kashani-Poor, S. Paban, P. Pouliot, and J. Gomis, J. High Energy Phys. 05 (2000) 024; W. Fischler, E. Gorbatov, A. Kashani-Poor, R. McNees, S. Paban, and P. Pouliot, J. High Energy Phys. 06 (2000) 032; W. H. Huang, Phys. Rev. D **63**, 125004 (2001).
[4] K. Landsteiner, E. Lopez, and M. H. G. Tytgat, J. High Energy Phys. 09 (2000) 027.06 (2001) 055.
[5] F. T. Brandt, J. Frenkel, and C. Muramoto, Nucl. Phys. **B754**, 146 (2006).F. T. Brandt, A. Das, J. Frenkel, and J. C. Taylor, Phys. Rev. D **67**, 105010 (2003).F. T. Brandt, A. Das, and J. Frenkel, Phys. Rev. D **66**, 065017 (2002).
[6] F. T. Brandt, A. Das, and J. Frenkel, Phys. Rev. D **68**, 085010 (2003).A. Das, J. Frenkel, S. H. Pereira, and J. C. Taylor, Phys. Lett. B **577**, 76 (2003).
[7] F. T. Brandt, J. Frenkel, and C. Muramoto, Phys. Lett. B **649**, 483 (2007).
[8] N. Seiberg, L. Susskind, and N. Toumbas, J. High Energy Phys. 06 (2000) 044.
[9] J. Gomis and T. Mehen, Nucl. Phys. **B591**, 265 (2000).
[10] L. Alvarez-Gaume, J. L. F. Barbon, and R. Zwicky, J. High Energy Phys. 05 (2001) 057.
[11] M. Chaichian, A. Demichev, P. Presnajder, and A. Tureanu, Eur. Phys. J. C **20**, 767 (2001); A. P. Balachandran, T. R. Govindarajan, A. G. Martins, and P. Teotonio-Sobrinho, J. High Energy Phys. 11 (2004) 068.
[12] J. S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976); S. W. Hawking, Commun. Math. Phys. **55**, 133 (1977).
[13] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994).
[14] K. Kirsten, *Spectral Functions in Mathematics and Physics* (Chapman & Hall/CRC, Boca Raton, FL, 2001).
[15] P. B. Gilkey, *Asymptotic Formulae in Spectral Geometry* (Chapman & Hall/CRC, Boca Raton, FL, 2004).
[16] D. V. Vassilevich, Phys. Rep. **388**, 279 (2003).
[17] D. V. Vassilevich, Lett. Math. Phys. **67**, 185 (2004).
[18] V. Gayral and B. Iochum, J. Math. Phys. (N.Y.) **46**, 043503 (2005).
[19] D. V. Vassilevich, J. High Energy Phys. 08 (2005) 085.
[20] V. Gayral, B. Iochum, and D. V. Vassilevich, Commun. Math. Phys. **273**, 415 (2007).
[21] A. Strelchenko, Int. J. Mod. Phys. A **22**, 181 (2007).

[22] H. Grosse and M. Wohlgenannt, J. Phys. Conf. Ser. **53**, 764 (2006).

[23] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, J. High Energy Phys. 02 (2000) 020; I. Chepelev and R. Roiban, J. High Energy Phys. 03 (2001) 001; I. Y. Aref'eva, D. M. Belov, and A. S. Koshelev, arXiv:hep-th/0001215.

[24] E. Langmann and R. J. Szabo, Phys. Lett. B **533**, 168 (2002); H. Grosse and R. Wulkenhaar, Commun. Math. Phys. **256**, 305 (2005).

[25] D. M. Gitman and D. V. Vassilevich, arXiv:hep-th/0701110.

[26] J. S. Dowker and G. Kennedy, J. Phys. A **11**, 895 (1978).

[27] D. V. Fursaev, Nucl. Phys. **B596**, 365 (2001); **B664**, 403(E) (2003); D. Fursaev and A. Zelnikov, Classical Quantum Gravity **18**, 3825 (2001); D. V. Fursaev, Nucl. Phys. B, Proc. Suppl. **104**, 33 (2002).

[28] V. Gayral, J. M. Gracia-Bondia, and F. R. Ruiz, Phys. Lett. B **610**, 141 (2005).

[29] D. V. Vassilevich, Mod. Phys. Lett. A **22**, 1255 (2007).

[30] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[31] M. Bordag, K. Kirsten, and D. Vassilevich, Phys. Rev. D **59**, 085011 (1999).

[32] M. Bordag, A. S. Goldhaber, P. van Nieuwenhuizen, and D. Vassilevich, Phys. Rev. D **66**, 125014 (2002).

[33] D. V. Fursaev, Classical Quantum Gravity **19**, 3635 (2002); **20**, 565(E) (2003).

[34] C. Dehne (unpublished).

[35] D. V. Vassilevich, Nucl. Phys. **B715**, 695 (2005).

[36] A. Bytsenko, E. Elizalde, S. Zerbini, Phys. Rev. D **64**, 105024 (2001).