New look at the modified Coulomb potential in a strong magnetic field

N. Sadooghi[1,2,*] and A. Sodeiri Jalili[1]

[1]*Department of Physics, Sharif University of Technology, P.O. Box 11155-9161, Tehran, Iran*

[2]*Institute for Studies in Theoretical Physics and Mathematics (IPM), School of Physics, P.O. Box 19395-5531, Tehran, Iran*

(Received 30 May 2007; published 24 September 2007)

The static Coulomb potential of quantum electrodynamics (QED) is calculated in the presence of a strong magnetic field in the lowest Landau level approximation using two different methods. First, the vacuum expectation value of the corresponding Wilson loop is calculated perturbatively in two different regimes of dynamical mass $m_{\rm dyn}$, i.e., $|\mathbf{q}_{\|}^2| \ll m_{\rm dyn}^2 \ll |eB|$ and $m_{\rm dyn}^2 \ll |\mathbf{q}_{\|}^2| \ll |eB|$, where $\mathbf{q}_{\|}$ is the longitudinal component of the momentum relative to the external magnetic field B. The result is then compared with the static potential arising from Born approximation. Both results coincide. Although the arising potentials show different behavior in the aforementioned regimes, a novel dependence on the angle θ between the particle-antiparticle's axis and the direction of the magnetic field is observed. In the regime $|\mathbf{q}_{\|}^2| \ll m_{\rm dyn}^2 \ll |eB|$, for strong enough magnetic field and depending on the angle θ, a qualitative change occurs in the Coulomb-like potential; whereas for $\theta = 0, \pi$ the potential is repulsive, it exhibits a minimum for angles $\theta \in]0, \pi[$.

DOI: 10.1103/PhysRevD.76.065013 PACS numbers: 11.15.−q, 11.30.Qc, 12.20.Ds

I. INTRODUCTION

Chiral symmetry plays an important role in elementary particle physics. It has been known [1] for some time that QED, in addition to the familiar weak coupling phase, may have a nonperturbative strong coupling phase, characterized by spontaneous chiral symmetry breaking [2]. The existence of this new phase was exploited in a novel interpretation of the multiple correlated and narrow peak structures in electron and positron spectra observed at GSI several years ago [3]. According to this scenario the electron-positron peaks are due to the decay of a bound electron-positron system formed in the new QED phase induced by a strong and rapidly varying electromagnetic field which is present in the neighborhood of colliding heavy ions. It is therefore of great interest to investigate whether background fields, such as constant magnetic fields, can potentially induce chiral symmetry breaking in gauge theories and lead eventually to the formation of a chiral symmetry breaking fermion condensate $\langle\bar{\psi}\psi\rangle$ and a dynamically generated fermion mass.

Indeed, the magnetic catalysis of dynamical chiral symmetry breaking has been established as a universal phenomenon in $2 + 1$ and $3 + 1$ dimensions [2,4]. According to these results, even at the weakest attractive interaction between fermions, a constant magnetic field leads to the generation of a fermion dynamical mass. The essence of this effect is the dimensional reduction $D \to D - 2$ in the dynamics of fermion pairing in a magnetic field, which arises from the fact that the motion of charged particles is restricted in directions perpendicular to the magnetic field. It is believed that at weak coupling this dynamics is dominated by the lowest Landau level (LLL). The magnetic catalysis is not only interesting from a purely fundamental point of view, but it has potential application in condensed matter physics [5] and cosmology [6].

In this paper we are interested in the static potential between the particle and antiparticle in the presence of a strong constant magnetic field. The potential produced by a point electric charge placed into a constant magnetic field is recently calculated in [7]. It is shown that the standard Coulomb law is modified by the vacuum polarization arising in the external magnetic field. Here, since the vacuum polarization component, taken in the one-loop approximation, grows linearly with the magnetic field, a scaling regime occurs in the limit of infinite magnetic field. The scaling regime implies a short range character of interaction, expressed as a Yukawa law

$$V(\mathbf{x}) = -\frac{\alpha e^{-M_\gamma R}}{R}, \quad \text{with} \quad M_\gamma \equiv \sqrt{\frac{2\alpha|eB|N_f}{\pi}}, \tag{1.1}$$

where $R \equiv |\mathbf{x}|$ and M_γ is the photon mass.

In the present work, we will determine anew the potential $V(x)$ between the particle-antiparticle pair in the presence of a constant magnetic field in the LLL approximation. Although our method differs essentially from the analytic methods used in [7] leading to (1.1), our result is indeed consistent with this potential.

To determine the fermion-antifermion potential, we will first compute perturbatively the vacuum expectation value (VEV) of a Wilson loop of a static fermion-antifermion pair for large Euclidean time.[1] In [9], the same perturbative

[1]Here, to determine the potential in the LLL approximation, we will use the full photon propagator in the presence of a strong magnetic field [8] in the LLL approximation. In [8], it is shown that the full photon propagator depends on the dynamical mass $m_{\rm dyn}$, which is calculated nonperturbatively in [4]. Thus, although our method is a perturbative one, the result depends automatically on a parameter which is determined nonperturbatively.

*sadooghi@physics.sharif.edu

 © 2007 The American Physical Society

method is used to determine the quark-antiquark potential of ordinary QCD. At one-loop level it is shown to be

$$V(R) = -C\frac{e^2}{4\pi R}\left(1 + \frac{11e^2}{8\pi^2}\ln\frac{R}{a}\right), \quad (1.2)$$

where $C \equiv \frac{1}{4}\,\mathrm{tr}(\lambda^a\lambda^a)$ is the trace over the product of Gell-Mann matrices λ^a, and a^{-1} is the UV cutoff parameter. Using the same idea, we will determine the potential of a point charged particle in the external magnetic field. Eventually, we will compare our result with the modified Coulomb potential from a semiclassical Born approximation. In the regime of LLL dominance, we will consider two different regions of dynamical mass, $|\mathbf{q}_{\parallel}^2| \ll m_{\rm dyn}^2 \ll |eB|$ and $m_{\rm dyn}^2 \ll |\mathbf{q}_{\parallel}^2| \ll |eB|$, separately. Here, $\mathbf{q}_{\parallel}$ is the longitudinal component of the momentum with respect to the direction of the external magnetic field. We will show that the potential in the regime $|\mathbf{q}_{\parallel}^2| \ll m_{\rm dyn}^2 \ll |eB|$ has the general form of a modified Coulomb potential,

$$V_1(R,\theta) = -\frac{\alpha}{R}\left(\mathcal{A}_1(\alpha,\theta) - \frac{\gamma\mathcal{A}_2(\alpha,\theta)}{R^2} + \frac{\gamma^2\mathcal{A}_3(\alpha,\theta)}{R^4}\right),$$
$$\text{with}\quad \gamma \equiv \frac{2\alpha}{3\pi m_{\rm dyn}^2}, \quad (1.3)$$

and in the regime $m_{\rm dyn}^2 \ll |\mathbf{q}_{\parallel}^2| \ll |eB|$ has the form of a Yukawa-like potential

$$V_2(R,\theta) = -\frac{\alpha e^{-M_{\rm eff}R}}{(1-\frac{\alpha}{\pi})g(\theta)R}, \quad \text{with}$$
$$M_{\rm eff}(\theta) \equiv g(\theta)\sqrt{\frac{2\alpha|eB|}{\pi}}, \quad (1.4)$$

where $\mathcal{A}_i$, $i = 1, 2, 3$ in (1.3) and $g(\theta)$ in (1.4) will be calculated exactly in Sec. III. In (1.3) as well as (1.4), θ is the angle between the particle-antiparticle axis and the direction of the magnetic field. Up to this explicit novel dependence on the angle θ, the potential $V_2(R,\theta)$ from (1.4) is comparable with the potential (1.1) from [7]. As a consequence of this θ-dependence, the effective photon mass $M_{\rm eff}(\theta)$ in (1.4) is, in contrast to the photon mass M_γ in (1.1), a function of θ.

In the regime $|\mathbf{q}_{\parallel}^2| \ll m_{\rm dyn}^2 \ll |eB|$, it can be shown that for large enough magnetic field and depending on the angle θ, a qualitative change occurs in the Coulomb-like potential $V_1(R,\theta)$; whereas for $\theta = 0, \pi$ the potential is repulsive, it exhibits a minimum for angles $\theta \in]0, \pi[$ and distances $R \leq 0.005$ fm. The position of this minimum is proportional to $1/\sqrt{B}$ and the depth of the potential at $R_{\rm min}$ increases with the magnetic field. The exact value of the strong magnetic field will be determined in Sec. V. We interpret the appearance of such a minimum as a possibility for bound state formation in $D = 4$ dimensions. A rigorous proof of the bound state formation in the above potentials V_1 and V_2 is the subject of a separate investigation and is beyond the scope of this paper.[2]

The organization of the paper is as follows. In Sec. II, a brief review of QED in a strong magnetic field containing some important results from [4] will be presented. In Sec. III, the static potential of QED in a strong magnetic field will be calculated perturbatively by determining the VEV of the Wilson loop of a static fermion-antifermion pair in two regimes of the dynamical mass $|\mathbf{q}_{\parallel}^2| \ll m_{\rm dyn}^2 \ll |eB|$ and $m_{\rm dyn}^2 \ll |\mathbf{q}_{\parallel}^2| \ll |eB|$ in the LLL. In Sec. IV a semiclassical Born approximation will be used to determine the same particle-antiparticle potentials. In Sec. V, a qualitative analysis of these potentials will be performed and the role played by the angle θ will be discussed in detail. Sec. VI summarizes our results.

II. QED IN A STRONG MAGNETIC FIELD

In this section we will briefly review the characteristics of fermions and photons in a constant external magnetic field. To this purpose, let us start with the QED Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^\mu(i\partial_\mu + eA_\mu)\psi - m\bar{\psi}\psi, \quad (2.1)$$

where the vector field $A_\mu = a_\mu + A_\mu^{\rm ext}$, where a_μ is an Abelian quantum gauge field, $F_{\mu\nu}$ is the corresponding field strength, and $A_\mu^{\rm ext}$ describes an external electromagnetic field. In this paper we will choose the symmetric gauge for $A_\mu^{\rm ext}$, i.e.,

$$A_\mu^{\rm ext} = \frac{B}{2}(0, x_2, -x_1, 0). \quad (2.2)$$

This leads to a magnetic field in the x_3 direction. From now on, the longitudinal (0, 3) directions will be denoted by $\mathbf{x}_{\parallel}$ and the transverse directions (1, 2) by $\mathbf{x}_{\perp}$. Using the Schwinger proper time formalism [10], it is possible to derive the fermion and photon propagator in this gauge. As for the fermion propagator, it is given by

$$\mathcal{S}_F(x,y) = \exp\left(\frac{ie}{2}(x-y)^\mu A_\mu^{\rm ext}(x+y)\right)S(x-y)$$
$$= \exp\left(\frac{ieB}{2}\epsilon^{ab}x_a y_b\right)S(x-y), \qquad a, b = 1, 2. \quad (2.3)$$

Here, the first factor containing the external gauge field, $A_\mu^{\rm ext}$, is the Schwinger line integral [10]. The Fourier transform of the translational invariant part $S(x-y)$ reads

[2]A nonperturbative analysis of the corresponding Schrödinger equation describing the Nambu-Goldstone modes and arising from a Bethe-Salpeter equation for bound states shows that at least one bound state can be formed in the attractive potential in $D = 4$ dimensions [4] (for more details, see the explanation in the last paragraph of Sec. II).

$$\tilde{S}(k) = i\int_0^\infty ds e^{-ism^2}\exp\left(is\left[k_\parallel^2 - \frac{k_\perp^2}{eBs\cot(eBs)}\right]\right) \times \{(m + \gamma^\parallel\cdot\mathbf{k}_\parallel)(1 - \gamma^1\gamma^2\tan(eBS)) - \gamma^\perp\cdot\mathbf{k}_\perp(1 + \tan^2(eBs))\}, \tag{2.4}$$

where $\mathbf{k}_\parallel = (k_0, k_3)$ and $\gamma_\parallel = (\gamma_0, \gamma_3)$ and $\mathbf{k}_\perp = (k_1, k_2)$ and $\gamma_\perp = (\gamma_1, \gamma_2)$. After performing the integral over s, $\tilde{S}(k)$ can be decomposed as follows:

$$\tilde{S}(k) = ie^{-(k_\perp^2/|eB|)}\sum_{n=0}^\infty (-1)^n \frac{D_n(eB,k)}{k_\parallel^2 - m^2 - 2|eB|n}, \tag{2.5}$$

with $D_n(eB, k)$ expressed through the generalized Laguerre polynomials L_m^α,

$$D_n(eB,k) = (\gamma^\parallel\cdot\mathbf{k}_\parallel + m)\{2\mathcal{O}[L_n(2\rho) - L_{n-1}(2\rho)] + 4\gamma^\perp\cdot k_\perp L_{n-1}^1(2\rho)\}. \tag{2.6}$$

Here, we have introduced $\rho \equiv \frac{k_\perp^2}{|eB|}$ and

$$\mathcal{O} \equiv \frac{1}{2}(1 - i\gamma^1\gamma^2\mathrm{sign}(eB)). \tag{2.7}$$

Relation (2.5) suggests that in the IR region, with $|\mathbf{k}_\parallel|$, $|\mathbf{k}_\perp| \ll \sqrt{|eB|}$, all the higher Landau levels with $n \geq 1$ decouple and only the LLL with $n = 0$ is relevant. In the strong magnetic field limit, the full fermion propagator (2.3) can therefore be decomposed into two independent transverse and longitudinal parts [4]:

$$\mathcal{S}_F(x,y) = S_\parallel(\mathbf{x}_\parallel - \mathbf{y}_\parallel)P(\mathbf{x}_\perp, \mathbf{y}_\perp), \tag{2.8a}$$

with the longitudinal part

$$S_\parallel(\mathbf{x}_\parallel - \mathbf{y}_\parallel) = \int\frac{d^2k_\parallel}{(2\pi)^2}e^{i\mathbf{k}_\parallel\cdot(\mathbf{x}-\mathbf{y})^\parallel}\frac{i\mathcal{O}}{\gamma^\parallel\cdot\mathbf{k}_\parallel - m} \tag{2.8b}$$

and the transverse part

$$P(\mathbf{x}_\perp, \mathbf{y}_\perp) = \frac{|eB|}{2\pi}\exp\left(\frac{ieB}{2}\epsilon^{ab}x^a y^b - \frac{|eB|}{4}(\mathbf{x}_\perp - \mathbf{y}_\perp)^2\right), \quad a, b = 1, 2. \tag{2.8c}$$

The photon propagator $\mathcal{D}_{\mu\nu}$ of QED in an external constant magnetic field in one-loop approximation with fermions from LLL is calculated explicitly in [4,8]. It is given by

$$i\mathcal{D}_{\mu\nu}(q) = \frac{g_{\mu\nu}^\perp}{q^2} + \frac{q_\mu^\parallel q_\nu^\parallel}{q^2 q_\parallel^2} + \frac{(g_{\mu\nu}^\parallel - q_\mu^\parallel q_\nu^\parallel/q_\parallel^2)}{q^2 + q_\parallel^2\Pi(q_\perp^2, q_\parallel^2)} - \xi\frac{q_\mu q_\nu}{(q^2)^2}, \tag{2.9}$$

where ξ is an arbitrary gauge parameter. Since the LLL fermions couple only to the longitudinal (0, 3) components of the photon fields, no polarization effects are present in the transverse (1, 2) components of $\mathcal{D}_{\mu\nu}(q)$. Therefore, the full photon propagator in the LLL approximation is given by the Feynman-like covariant propagator [4]

$$i\tilde{\mathcal{D}}_{\mu\nu}(q) = \frac{g_{\mu\nu}^\parallel}{q^2 + \mathbf{q}_\parallel^2\Pi(\mathbf{q}_\parallel^2, \mathbf{q}_\perp^2)}, \tag{2.10}$$

with $\Pi(q_\perp^2, q_\parallel^2)$ having the form [11]

$$\Pi(q_\perp^2, q_\parallel^2) = \frac{2\alpha|eB|N_f}{\mathbf{q}_\parallel^2}e^{-(q_\perp^2/2|eB|)}H\left(\frac{\mathbf{q}_\parallel^2}{4m_{\mathrm{dyn}}^2}\right). \tag{2.11}$$

Here N_f is the number of fermion flavors and $\alpha \equiv \frac{e^2}{4\pi}$, where e is the running coupling. Further, $H(z)$ in (2.11) is defined by

$$H(z) \equiv \frac{1}{2\sqrt{z(z-1)}}\ln\left(\frac{\sqrt{1-z} + \sqrt{-z}}{\sqrt{1-z} - \sqrt{-z}}\right) - 1. \tag{2.12}$$

Expanding this expression for $|\mathbf{q}_\parallel^2| \ll m_{\mathrm{dyn}}^2 \ll |eB|$ and $m_{\mathrm{dyn}}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$ leads to

$$\Pi(\mathbf{q}_\perp^2, \mathbf{q}_\parallel^2) \simeq +\frac{\alpha|eB|N_f}{3\pi m_{\mathrm{dyn}}^2}e^{-(q_\perp^2/2|eB|)} \quad \text{for } |\mathbf{q}_\parallel^2| \ll m_{\mathrm{dyn}}^2 \ll |eB|, \tag{2.13}$$

$$\Pi(\mathbf{q}_\perp^2, \mathbf{q}_\parallel^2) \simeq -\frac{2\alpha|eB|N_f}{\pi\mathbf{q}_\parallel^2}e^{-(q_\perp^2/2|eB|)} \quad \text{for } m_{\mathrm{dyn}}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|. \tag{2.14}$$

In [4], it is shown that the kinematic region mostly responsible for generating the fermion mass is with the dynamical mass, m_{dyn}, satisfying $m_{\mathrm{dyn}}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$. Plugging (2.14) in the full photon propagator (2.10) and assuming that $|\mathbf{q}_\perp^2| \ll |eB|$, we get

$$\tilde{\mathcal{D}}_{\mu\nu}(q) \approx -\frac{ig_{\mu\nu}^\parallel}{q^2 - M_\gamma^2}, \quad \text{with} \quad M_\gamma^2 = \frac{2\alpha|eB|N_f}{\pi}. \tag{2.15}$$

The appearance of a finite photon mass M_γ is the result of the dimensional reduction $3 + 1 \to 1 + 1$ in the presence of a constant magnetic field. This phenomenon can be understood as reminiscent of the Higgs effect in the 1 + 1-dimensional Schwinger model, where the photon acquires also a finite mass.

As for the dynamically generated fermion mass m_{dyn}, it can be determined by solving the corresponding Schwinger-Dyson equation in the rainbow (ladder) approximation, where the effects of dynamical fermions are neglected. In this approximation, the gauge invariant dynamical mass is shown to have the form [4]

$$m_{\mathrm{dyn}} = C\sqrt{eB}\exp\left(-\frac{\pi}{2}\left(\frac{\pi}{2\alpha}\right)^{1/2}\right), \tag{2.16}$$

where the constant C is of order one.[3] In the improved rainbow approximation, however, the expression for m_{dyn} takes the following form [4]:

[3]The problem of gauge invariance of chiral symmetry breaking induced by the magnetic field found in the ladder QED is investigated in [12].

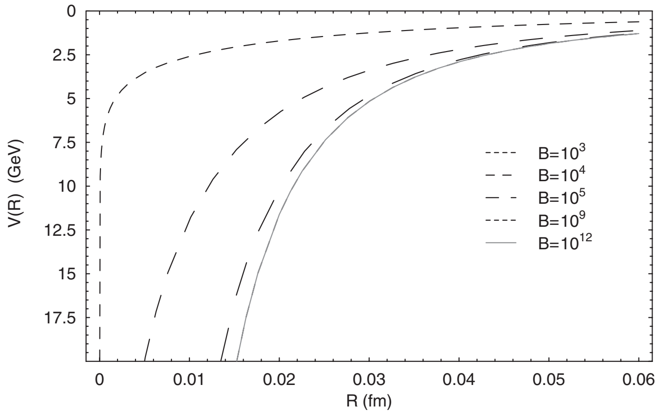

FIG. 1. Potential $V(\mathbf{x})$ from (2.19) of a particle-antiparticle pair for different magnetic fields. According to this result, whereas for $B \leq 10^9$ at each given distance the absolute value of the potential increases with increasing the magnetic field, the shape of the potential does not change for $B \geq 10^9$.

$$m_{\rm dyn} = \tilde{C}\sqrt{|eB|}\,F(\alpha)\exp\left(-\frac{\pi}{\alpha\ln(C_1/\alpha N_f)}\right), \qquad (2.17)$$

where $F(\alpha) \simeq (N_f\alpha)^{1/3}$, $C_1 \simeq 1.82 \pm 0.06$ and $\tilde{C} \sim O(1)$.

The dynamical mass, $m_{\rm dyn}$, plays the role of energy in the two-dimensional Schrödinger equation

$$(-\Delta + m_{\rm dyn}^2 + V(\mathbf{x}))\Psi(\mathbf{x}) = 0, \quad \text{with} \quad \Delta \equiv \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}. \qquad (2.18)$$

Here, the attractive potential has the form [4]

$$V(\mathbf{x}) = \frac{\alpha|eB|}{\pi}\exp\left(\frac{R^2|eB|}{2}\right)\mathrm{Ei}\left(-\frac{R^2|eB|}{2}\right), \qquad (2.19)$$

and behaves as

$$\begin{aligned} V(\mathbf{x}) &\simeq -\frac{2\alpha}{\pi}\frac{1}{R^2}, \qquad R \to \infty,\\ V(\mathbf{x}) &\simeq -\frac{\alpha|eB|}{\pi}\left(\gamma_E + \ln\frac{2}{R^2|eB|}\right), \qquad R \to 0, \end{aligned} \qquad (2.20)$$

where $\gamma_E \simeq 0.577$ is the Euler constant. To determine the long and short range behavior of the potential, the asymptotic behavior of $\mathrm{Ei}(x)$ is used [13]. In this context, the problem concerning bound state formation is reduced to finding the spectrum of bound states of the Schrödinger equation (2.18) with the attractive potential (2.19). In [4], it is shown that in $D = 4$ dimensions at least one bound state can be formed. Figure 1 shows the behavior of $V(\mathbf{x})$ for different $B = 10^3$, 10^4, 10^5, 10^9, and $B = 10^{12}$ (from left to right) as a function of R.[4] According to this result, whereas for $B < 10^9$ at each given distance the absolute value of the potential increases with increasing the magnetic field, the shape of the potential does not change for $B \geq 10^9$.

In the next section we will determine the static Coulomb potential between a fermion-antifermion pair in two aforementioned regimes in the LLL approximation by calculating the VEV of the Wilson loop perturbatively. Note that although our method is a perturbative one, our results depend explicitly on $m_{\rm dyn}$ which is determined nonperturbatively in the literature [4].

[4]Note that discussion here is purely qualitative. It is nevertheless possible to give the strength of the magnetic fields in Gauss through the relation $6.8 \times 10^{19} B \simeq H$ in Gauss [14,15]. Using this relation the potential $V(R)$ is given in GeV $= 10^9$ eV and the distance R in fm $= 10^{-15}$ m.

III. THE WILSON LOOP AND THE MODIFIED COULOMB POTENTIAL IN A STRONG MAGNETIC FIELD

In ordinary quantum field theory with no background magnetic field, the Wilson loop appears as one of the most efficient tools for probing the large distance properties of strong coupling QCD. It provides a natural criterion for confinement through the area law. In contrast to QED, where the field lines connecting a pair of opposite charges are allowed to spread, one expects that in QCD the quarks within a hadron are the sources of a chromoelectric flux which is concentrated within narrow tubes connecting the constituents. Since the energy is not allowed to spread, the potential of a quark-antiquark pair $\langle\bar{q}q\rangle$ will increase with their separation, as long as vacuum polarization effects do not screen their color charge. This picture of confinement can be checked by computing the nonperturbative potential between a static quark-antiquark pair in the path integral formalism.

It is the main purpose of this section to investigate the properties of the Wilson loop concerning the bound state problem of QED in the presence of a strong magnetic field in the LLL approximation. Here, the magnetic field plays the role of a strong catalyst and even the weakest attractive potential between fermions is enough for dynamical mass generation and bound state formation. Before calculating the modified Coulomb potential by determining the VEV of the Wilson line of a static fermion-antifermion pair in two regimes of dynamical mass, $|\mathbf{q}_\parallel^2| \ll m_{\rm dyn}^2 \ll |eB|$ and $m_{\rm dyn}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$, we briefly review the path integral formulation leading to the Coulomb potential between slowly moving particles in ordinary QED. Here, we keep closely to the notations of [16] and the classical review article [9].

Imagine creating a fermion-antifermion pair, $\bar{\psi}\psi$, at space-time point $x_\mu = 0$ and adiabatically separate them to a relative distance R. This configuration will be held for a time $T \to \infty$. Finally we bring the $\bar{\psi}\psi$ pair back together and let them annihilate. The Euclidean amplitude for this process is the matrix element of the Hamiltonian evolution operator e^{-HT} between the initial and final states, $|i\rangle$ and $|f\rangle$, respectively,

$$\langle i|e^{-HT}|f\rangle. \tag{3.1}$$

Here, $|i\rangle$ and $|f\rangle$ represents a $\bar{\psi}\psi$ pair a distance R apart and H is the Hamiltonian (for more details concerning the exact mathematical structure of the initial and final states, $|i\rangle$ and $|f\rangle$, see [16]). In the path integral representation the matrix element (3.1) can be expressed by

$$\langle i|e^{-HT}|f\rangle = \frac{\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S+ie\int A_\mu(x) j_\mu(x) d^4x}}{\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S}}, \tag{3.2}$$

where we have already skipped the subscript E for the Euclidean action S. Here, the current j_μ should describe the closed worldline C of the creation and annihilation of $\bar{\psi}\psi$ pair. For a closed worldline of the heavy fermions (3.2) simplifies to [16]

$$\langle i|e^{-HT}|f\rangle = \frac{\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S+ie\oint_C A_\mu(x) dx_\mu}}{\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S}}. \tag{3.3}$$

Since in a closed path $|i\rangle$ and $|f\rangle$ are identical and since the process is static, the Hamiltonian H is purely potential and the left-hand side (LHS) of (3.3) is

$$\langle i|e^{-HT}|f\rangle = e^{-V(R)T}. \tag{3.4}$$

Here, $V(R)$ is the fermion-antifermion potential. Taking the logarithm of (3.3) by plugging (3.4) on its LHS, it is given by

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \ln\langle W_C[A]\rangle, \tag{3.5}$$

where we have introduced the "loop-correlation" function—the *Wilson loop* [17–19],

$$W_C[A] \equiv e^{ie\oint_C A_\mu(x) dx_\mu}, \tag{3.6}$$

and used the definition

$$\langle \mathcal{O}\rangle \equiv \frac{\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{O} e^{-S}}{\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S}}. \tag{3.7}$$

For S being the QED action, the Coulomb potential $V(R) = -\frac{e^2}{4\pi R}$ can be analytically calculated in the quenched approximation, i.e., when the vacuum polarization effects arising from the presence of dynamical fermions are neglected. To calculate this Coulomb potential, one expands $\langle W_C[A]\rangle$ in powers of the background field A_μ, to get

$$\langle W_C[A]\rangle = \Big\langle 1 + ie\oint_C dx_\mu A_\mu(x) - \frac{e^2}{2}\oint_C\oint_C dx_\mu dy_\nu A_\mu(x)A_\nu(y) + \cdots\Big\rangle. \tag{3.8}$$

Here, the second term including only one gauge field does not contribute. As for the other terms, the terms with an odd number of external photon lines do not contribute to the above expansion. This is because of Furry's theorem that holds in ordinary QED in contrary to QED in the presence of external magnetic field.[5] Plugging (3.8) in (3.5), the Coulomb potential is given by

$$\begin{aligned} V(R) &= \lim_{T\to\infty} -\frac{1}{T}\ln\Big(1 - \frac{e^2}{2}\oint_C\oint_C dx_\mu dy_\nu D_{\mu\nu}(x,y) + \cdots\Big) \\ &= \lim_{T\to\infty} \frac{e^2}{2T}\oint_C\oint_C dx_\mu dy_\nu D_{\mu\nu}(x,y) + \mathcal{O}(e^3), \end{aligned} \tag{3.9}$$

where we have expanded the $\ln(1-\cdots)$ for weak coupling constant e. The integrand is the photon propagator in the coordinate space,

$$D_{\mu\nu}(x,y) \equiv \langle A_\mu(x)A_\nu(y)\rangle = \frac{\delta_{\mu\nu}}{4\pi^2(x-y)^2}. \tag{3.10}$$

To compute the double integral (3.9), one chooses a rectangular contour. Because in the Euclidean space the integrand (3.10) is proportional to $\delta_{\mu\nu}$, the double integral receives a contribution when x and y are located on the segments of integration contour which are parallel to each other. The double integral therefore reduces to

$$\begin{aligned} V(R) &= \lim_{T\to\infty} \frac{-e^2}{T}\int_0^T dT_1 \int_0^T dT_2 D_{00}(R, T_1 - T_2) \\ &= \lim_{T\to\infty} \frac{-e^2}{T}\int_0^T dT_1 \int_0^T dT_2 \frac{1}{4\pi^2[R^2 + (T_1 - T_2)^2]}. \end{aligned} \tag{3.11}$$

Here $R \equiv |\mathbf{x} - \mathbf{y}|$. Performing now the double integration, one arrives first at [16]

$$V(R) = \lim_{T\to\infty} -\frac{e^2}{2\pi^2}\Big(\frac{1}{R}\arctan\frac{T}{R} + \frac{1}{2T}\ln\Big(1 + \frac{T^2}{R^2}\Big)\Big), \tag{3.12}$$

and then after taking the limit $T \to \infty$, at the Coulomb potential of a fermion-antifermion pair

$$V(R) = -\frac{\alpha}{R}. \tag{3.13}$$

After this brief excursion, let us now turn back to our original problem of computing the modified Coulomb potential of QED in the presence of external strong magnetic field. In the regime of LLL, the full photon propagator $\tilde{\mathcal{D}}_{\mu\nu}$ in the momentum space is given by (2.10), where $\Pi(\mathbf{q}_\parallel^2, \mathbf{q}_\perp^2)$ is defined in (2.11). It is indeed a very difficult task to calculate the potential of a static fermion-antifermion pair for full photon propagator (2.10). We will therefore consider only two different regimes of dynamical mass $|\mathbf{q}_\parallel^2| \ll m_{\text{dyn}}^2 \ll |eB|$ and $m_{\text{dyn}}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$, with

[5] The phenomenon of photon splitting $\gamma \to 2\gamma$ in the presence of an external magnetic field [20] is the best counterexample for the validity of Furry's theorem for QED in external magnetic field.

$\Pi(\mathbf{q}_{\parallel}^2, \mathbf{q}_{\perp}^2)$ given in (2.13) and (2.14), respectively, and will calculate the modified Coulomb potential $V(x)$ in these two regimes.

A. Modified Coulomb potential in $|\mathbf{q}_{\parallel}^2| \ll m_{\rm dyn}^2 \ll |eB|$ regime

Substituting (2.13) with $N_f = 1$ in (2.10), the full photon propagator in this regimes reads

$$\tilde{\mathcal{D}}_{\mu\nu}(q) = -i\frac{g_{\mu\nu}^{\parallel}}{q^2 + \frac{\alpha|eB|}{3\pi m_{\rm dyn}^2}\mathbf{q}_{\parallel}^2 \exp(-\frac{\mathbf{q}_{\perp}^2}{2|eB|})}. \tag{3.14}$$

To calculate the Coulomb potential by (3.9), we need the LLL photon propagator in the coordinate space, i.e., the Fourier transform of (3.14):

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = -ig_{\mu\nu}^{\parallel}\int\frac{d^4q}{(2\pi)^4}\frac{e^{iqx}}{q^2 + \frac{\alpha|eB|}{3\pi m_{\rm dyn}^2}\mathbf{q}_{\parallel}^2 \exp(-\frac{\mathbf{q}_{\perp}^2}{2|eB|})}. \tag{3.15}$$

After a lengthy but straightforward calculation of the above integral over q (see Appendix A for more details), we arrive at the LLL photon propagator in the coordinate space

$$\begin{aligned}\tilde{\mathcal{D}}_{\mu\nu}(R,\theta,T) = \frac{\delta_{\mu\nu}^{\parallel}}{4\pi^2 a_1}\bigg[&\bigg(1 + \frac{\gamma}{a_1} - \frac{4a_2 + \gamma R^2\sin^2\theta}{2\beta a_1^2} + \frac{3a_2R^2\sin^2\theta}{2\beta^2 a_1^3}\bigg) + \frac{4\gamma^2}{a_1^2}\bigg(1 - \frac{3R^2\sin^2\theta}{2\beta a_1} + \frac{3R^4\sin^4\theta}{8\beta^2 a_1^2}\bigg)\\ &- \frac{12\gamma a_2}{\beta a_1^3}\bigg(2 - \frac{2}{\beta a_1}\bigg(2R^2\sin^2\theta + \frac{a_2}{\gamma}\bigg) + \frac{5}{4\beta^2 a_1^2}\bigg(R^4\sin^4\theta + \frac{4a_2R^2\sin^2\theta}{\gamma}\bigg) - \frac{15a_2R^4\sin^4\theta}{8\gamma\beta^3 a_1^3}\bigg)\\ &- \frac{\gamma}{|eB|\beta a_1^2}\bigg(1 - \frac{3}{\beta a_1}\bigg(\frac{R^2\sin^2\theta}{2} + \frac{a_2}{\gamma}\bigg) + \frac{6}{\beta^2 a_1^2}\bigg(\frac{R^4\sin^4\theta}{16} + \frac{a_2R^2\sin^2\theta}{\gamma}\bigg) - \frac{15a_2R^4\sin^4\theta}{8\gamma\beta^3 a_1^3}\bigg)\bigg].\end{aligned} \tag{3.16}$$

Here,

$$\beta^{-1} \equiv 4\bigg(1 + \frac{\alpha|eB|}{3\pi m_{\rm dyn}^2}\bigg) \quad \text{and} \quad \gamma(\alpha) \equiv \frac{2\alpha}{3\pi m_{\rm dyn}^2} \tag{3.17}$$

are constant c-numbers, and $a_i = a_i(R,\theta,T)$, $i = 1, 2$ are defined by

$$\begin{aligned} a_1(R,\theta,T) &\equiv T^2 + R^2 f^2(\alpha,\theta), \quad \text{with}\\ f^2(\alpha,\theta) &\equiv 1 + \frac{\gamma|eB|}{2}\sin^2\theta,\\ a_2(R,\theta,T) &\equiv \beta\gamma(T^2 + R^2\cos^2\theta).\end{aligned} \tag{3.18}$$

In all the above expressions $R \equiv |\mathbf{x}|$ is the distance between the static fermion and antifermion pair, θ is the angle between the particle-antiparticle axis and the direction of the magnetic field B, i.e., the x_3 direction, and $T \equiv ix_0$ is the Euclidean time. Note that here, to determine the LLL photon propagator, we have to use the approximation $\mathbf{q}_{\perp}^2 \ll |eB|$, which is valid in the regime of strong magnetic field.

To determine the static potential between the $\bar{\psi}\psi$ pair, we have to calculate the double integral [see (3.11) leading to the expression of the ordinary Coulomb potential]

$$V(R,\theta) = \lim_{T\to\infty} -\frac{e^2}{T}\int_0^T dT_1\int_0^T dT_2\tilde{\mathcal{D}}_{00}(R,\theta,T_1 - T_2), \tag{3.19}$$

which can be easily simplified to

$$V(R,\theta) = -2e^2\int_0^\infty dT\tilde{\mathcal{D}}_{00}(R,\theta,T). \tag{3.20}$$

Using now the definition of $\tilde{\mathcal{D}}_{\mu\nu}(R,\theta,T)$ in (3.16), and performing the integration over T, the modified Coulomb potential in the regime $|\mathbf{q}_{\parallel}^2| \ll m_{\rm dyn}^2 \ll |eB|$ can be given by

$$V(R,\theta) = -\frac{\alpha}{R}\bigg(\mathcal{A}_1(\alpha,\theta) - \frac{\gamma\mathcal{A}_2(\alpha,\theta)}{R^2} + \frac{\gamma^2\mathcal{A}_3(\alpha,\theta)}{R^4}\bigg), \tag{3.21}$$

with

$$\begin{aligned}\mathcal{A}_1(\alpha,\theta) &\equiv \frac{1}{f}, \qquad \mathcal{A}_2(\alpha,\theta) \equiv -\frac{1}{4f^3}\bigg(1 - \frac{3\cos^2\theta}{f^2} - \frac{3\sin^2\theta}{8\beta f^2} + \frac{15\sin^2\theta\cos^2\theta}{8\beta f^4}\bigg),\\ \mathcal{A}_3(\alpha,\theta) &\equiv +\frac{9}{16f^5}\bigg(1 - \frac{5\sin^2\theta}{4\beta f^2} + \frac{35\sin^4\theta}{128\beta^2 f^4}\bigg)\\ &\quad - \frac{15\cos^2\theta}{8f^7}\bigg(3 - \frac{7}{4\beta f^2}(2\beta\cos^2\theta + 3\sin^2\theta) + \frac{63\sin^2\theta}{8\beta^2 f^4}\bigg(\beta\cos^2\theta + \frac{3\sin^2\theta}{16}\bigg) - \frac{693\cos^2\theta\sin^4\theta}{256\beta^2 f^6}\bigg)\\ &\quad - \frac{3}{16|eB|f^5\gamma\beta}\bigg(1 - \frac{5}{4\beta f^2}(4\beta\cos^2\theta + \sin^2\theta) + \frac{35\sin^2\theta}{4\beta^2 f^4}\bigg(\beta\cos^2\theta + \frac{\sin^2\theta}{32}\bigg) - \frac{315\cos^2\theta\sin^4\theta}{128\beta^2 f^6}\bigg).\end{aligned} \tag{3.22}$$

In Sec. V we will present a qualitative analysis of the above potential emphasizing the role played by the angle θ in a possible bound state formation.

B. Modified Coulomb potential in $m_{\rm dyn}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$ regime

To compute the interparticle potential in this regime, we substitute (2.14) in (2.10). For $N_f = 1$, the full photon propagator in the coordinate space reads

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = -ig_{\mu\nu}^{\parallel} \int \frac{d^4q}{(2\pi)^4} \frac{e^{iqx}}{q^2 - \frac{2\alpha|eB|}{\pi}\exp(-\frac{\mathbf{q}_\perp^2}{2|eB|})}. \tag{3.23}$$

The integration over q can be performed using the approximation $\mathbf{q}_\perp^2 \ll |eB|$ which is valid in the regime of LLL dominance. After a straightforward calculation (see Appendix B for more details), the propagator is given by

$$\tilde{\mathcal{D}}_{\mu\nu}(R,\theta,T) = \frac{\delta_{\mu\nu}^{\parallel}}{4\pi^2(1-\frac{\alpha}{\pi})} \times \frac{\zeta}{\sqrt{T^2+R^2g^2(\theta)}} K_1(\zeta\sqrt{T^2+R^2g^2(\theta)}), \tag{3.24}$$

where

$$\zeta \equiv \sqrt{\frac{2\alpha|eB|}{\pi}}, \quad \text{and} \quad g^2(\theta) \equiv \cos^2\theta + \frac{\sin^2\theta}{1-\frac{\alpha}{\pi}}. \tag{3.25}$$

The fermion-antifermion potential in this regime is then calculated using the expression (3.19) or equivalently (3.20) and reads

$$V(R,\theta) = -\frac{e^2\zeta}{2\pi^2(1-\frac{\alpha}{\pi})} \times \int_0^\infty dT \frac{1}{\sqrt{T^2+R^2g^2(\theta)}} K_1(\sqrt{T^2+R^2g^2(\theta)}). \tag{3.26}$$

To evaluate the integral we make use of

$$\int_0^\infty d\tau \frac{\tau^{2\mu+1}}{\sqrt{(\tau^2+z^2)^\nu}} K_\nu(\zeta\sqrt{\tau^2+z^2}) = \frac{2^\mu\Gamma(\mu+1)}{\zeta^{\mu+1}z^{\nu-\mu-1}} K_{\nu-\mu-1}(\zeta z), \tag{3.27}$$
$$\zeta > 1, \mathrm{Re}(\mu) > -1.$$

Choosing $\tau = T$, $z = Rg(\theta)$, $\nu = 1$, and $\mu = -\frac{1}{2}$, we arrive at

$$V(R,\theta) = -\frac{\alpha}{(1-\frac{\alpha}{\pi})g(\theta)R} e^{-\zeta g(\theta)R}. \tag{3.28}$$

Here we have used $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

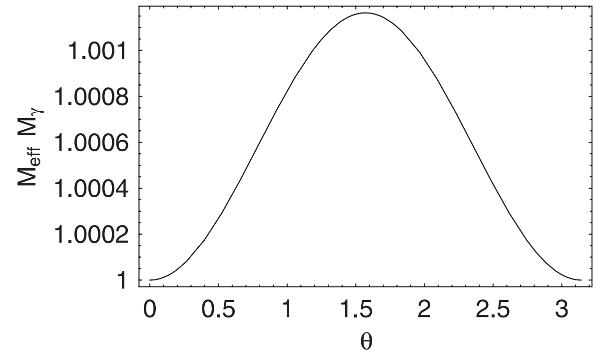

FIG. 2. The ratio $\frac{M_{\rm eff}}{M_\gamma} = g(\theta)$ for $\theta \in [0,\pi]$ in the regime $m_{\rm dyn}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$ of LLL dominance.

Apart from its θ-dependence our result is comparable with (1.1) from [7].

The potential (3.28) is indeed comparable with the ordinary attractive Yukawa potential

$$V_{\rm Yukawa}(R) = -\frac{\alpha}{R}e^{-mR}, \tag{3.29}$$

where $\alpha \to \frac{\alpha}{(1-\frac{\alpha}{\pi})g(\theta)}$ and the effective photon mass $m \to M_{\rm eff}(\theta) \equiv \zeta g(\theta)$. This result is also in agreement with the general result about the massive photons in a strong magnetic fields in the LLL approximation. As we have seen in Sec. II, in the regime $m_{\rm dyn}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$, the $3+1$ dimensional QED in the LLL approximation is reduced to a $1+1$ dimensional Schwinger model where the photon acquires a finite mass $M_\gamma = \sqrt{\frac{2\alpha|eB|N_f}{\pi}}$, with N_f the number of flavors [see (2.15) for more details]. For $N_f = 1$, we have $M_\gamma = \zeta$. Thus the effective photon mass in this regime is given by $M_{\rm eff} = M_\gamma g(\theta)$ and depends explicitly on the angle θ between the particle axis and the direction of the external magnetic field. Figure 2 shows the θ-dependence of $g(\theta)$ which is understood to be the ratio $\frac{M_{\rm eff}}{M_\gamma}$. For all values of $\theta \in [0,\pi]$ the effective mass $M_{\rm eff}$ and the aforementioned photon mass M_γ are almost equal. Here α is fixed to be $\alpha = 1/137$. A qualitative analysis of the potential (3.28) emphasizing the role played by the angle θ will be presented in Sec. V.

In Sec. IV we will use the Born approximation to determine the static Coulomb potential in two regimes of dynamical mass in the LLL approximation. Eventually we will study the behavior of the potentials (3.21) and (3.26) as a function of the angle θ for two limits of large and small distances.

IV. BORN APPROXIMATION AND THE STATIC COULOMB POTENTIAL IN A STRONG MAGNETIC FIELD

In the nonrelativistic quantum mechanics the relation between the scattering amplitude $\mathcal{M}$ and the potential is given by the Born approximation

$$\langle p'|i\mathcal{M}|p\rangle = -iV(\mathbf{q})(2\pi)\delta(E_{p'} - E_p), \tag{4.1}$$

where p (E_p) and p' ($E_{p'}$) are the momenta (energy) of the incoming and outgoing particles, respectively, and $\mathbf{q} = \mathbf{p}' - \mathbf{p}$. For ordinary QED with no background magnetic field, for instance, the amplitude of a particle-antiparticle scattering is given by [21]

$$i\mathcal{M} \sim -\frac{ie^2}{|\mathbf{p} - \mathbf{p}'|^2}. \tag{4.2}$$

Comparing with (4.1), the attractive (classical) Coulomb potential $V(\mathbf{q})$ is thus given by

$$V(\mathbf{q}) = -\frac{e^2}{|\mathbf{q}|^2}, \quad \text{with} \quad |\mathbf{q}| \equiv |\mathbf{p} - \mathbf{p}'|. \tag{4.3}$$

After a Fourier transformation into the coordinate space, the same potential reads

$$V(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3} V(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}} = -\frac{\alpha}{R}, \tag{4.4}$$

where $R \equiv |\mathbf{x}|$. Furthermore, to include the quantum correction into the result, the modified Coulomb potential can be calculated from

$$V(\mathbf{x}) = -\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \frac{e^2}{\mathbf{q}^2(1 - \Pi(\mathbf{q}^2))}, \tag{4.5}$$

where $\Pi(\mathbf{q})$ in the ordinary QED is defined by the vacuum polarization tensor

$$\Pi_{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2), \tag{4.6}$$

and is given by

$$\Pi(q^2) = -\frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \log\left(\frac{m^2}{m^2 - x(1-x)q^2}\right). \tag{4.7}$$

Choosing $q_0 = 0$ and plugging this relation into (4.5), after some straightforward calculation [21], one arrives at the so called *Uehling potential*

$$V(R) = -\frac{\alpha}{R}\left(1 + \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-2mR}}{(mR)^{3/2}} + \cdots\right). \tag{4.8}$$

In this section, we will use the above method to determine the potential of QED in a strong magnetic field in two regimes of dynamical mass $|\mathbf{q}_\parallel^2| \ll m_{\text{dyn}}^2 \ll |eB|$ and $m_{\text{dyn}}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$ separately. We will show that our results coincide with (3.21) and (3.22) in the regime $|\mathbf{q}_\parallel^2| \ll m_{\text{dyn}}^2 \ll |eB|$ and with (3.26) in the regime $m_{\text{dyn}}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$.

A. Modified Coulomb potential in $|\mathbf{q}_\parallel^2| \ll m_{\text{dyn}}^2 \ll |eB|$ regime

We will start using the relation (4.5) where $\Pi(\mathbf{q})$ in this regime is given by (2.13). The modified Coulomb potential in this regime is therefore given by

$$V(\mathbf{x}) = -e^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{-i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 + \frac{\gamma eB}{2} q_3^2 e^{-(\mathbf{q}_\perp^2/2|eB|)}}, \tag{4.9}$$

where we set $q_0 = 0$, and the factor $\gamma \equiv \frac{\alpha|eB|}{3\pi m_{\text{dyn}}^2}$ is defined in (3.17). The three-dimensional integral over q can be performed by making use of the same methods as was shown in Appendix A. The main steps of the calculations are as follows. First, using the relation (A2) and (A8) we write the integrand in the form

$$\begin{aligned} V(\mathbf{x}) = &-e^2 \int_0^\infty ds \int \frac{dq_\perp q_\perp dq_3 d\varphi'}{(2\pi)^3} \\ &\times e^{-i(q_3 R\cos\theta + q_\perp R\sin\theta\cos(\varphi-\varphi'))} \\ &\times \exp\left(-s\left(\mathbf{q}_\perp^2 + q_3^2 + \frac{\gamma eB}{2} q_3^2 e^{-(\mathbf{q}_\perp^2/2|eB|)}\right)\right), \end{aligned} \tag{4.10}$$

where θ is the angle between the particle-antiparticle axis and the external magnetic field. The integration over φ' and q_3 can be performed using (A10) and (A12), respectively. We arrive therefore at

$$\begin{aligned} V(\mathbf{x}) = &-e^2 \int_0^\infty ds \sqrt{\frac{\pi}{s'}} \int \frac{dq_\perp q_\perp}{(2\pi)^2} J_0(q_\perp R\sin\theta) \\ &\times e^{-s\mathbf{q}_\perp^2} e^{-(R^2\cos^2\theta/4s')}, \end{aligned} \tag{4.11}$$

where we have introduced

$$s' \equiv s\left(1 + \frac{\gamma eB}{2} e^{-(\mathbf{q}_\perp^2/2|eB|)}\right) \tag{4.12}$$

to simplify our notations. At this stage, to perform the integration over $q_\perp$ and eventually over s, one can use the approximation $|\mathbf{q}_\perp^2| \ll |eB|$, which is valid in the LLL approximation. The factors $\frac{1}{s'}$ and $\frac{1}{\sqrt{s'}}$ are therefore given by

$$\begin{aligned} \frac{1}{s'} &\simeq \frac{2}{s\xi}\left(1 + \frac{\gamma}{2\xi}\mathbf{q}_\perp^2 + \left(\frac{\gamma^2}{4\xi^2} - \frac{\gamma}{8|eB|\xi}\right)\mathbf{q}_\perp^4\right), \\ \frac{1}{\sqrt{s'}} &\simeq \sqrt{\frac{2}{s\xi}}\left(1 + \frac{\gamma}{4\xi}\mathbf{q}_\perp^2 + \left(\frac{3\gamma^2}{32\xi^2} - \frac{\gamma}{16|eB|\xi}\right)\mathbf{q}_\perp^4\right), \end{aligned} \tag{4.13}$$

where $\xi \equiv (2 + \gamma|eB|)$. Using these expressions, the potential is given by

$$V(\mathbf{x}) = -e^2 \int_0^\infty ds \sqrt{\frac{2\pi}{s\xi}} \int \frac{dq_\perp q_\perp}{(2\pi)^2} e^{-s\mathbf{q}_\perp^2} e^{-(b_1/2\xi s)} \times J_0(q_\perp R \sin\theta)\bigg\{1 + \frac{\gamma \mathbf{q}_\perp^2}{4\xi}\left(1 - \frac{b_1}{s\xi}\right) + \mathbf{q}_\perp^4\bigg[\frac{\gamma^2}{16\xi^2}\left(\frac{3}{2} - \frac{3b_1}{s\xi} + \frac{b_1^2}{2s^2\xi^2}\right) - \frac{\gamma}{16\xi|eB|}\left(1 - \frac{b_1}{s\xi}\right)\bigg]\bigg\}, \quad (4.14)$$

where $b_1 \equiv R^2\cos^2\theta$. The result is similar to (A16) and indeed we have used the same approximation which was done in (A15) and led to (A16). Using now the integrals I_i, $i = 1, 2, 3$ from (A21)–(A23), (4.14) can be written as

$$V(\mathbf{x}) = -e^2 \int_0^\infty \frac{ds}{(2\pi)^2} \sqrt{\frac{2\pi}{s\xi}} e^{-(b_1/2\xi s)} \bigg\{I_1 + \frac{\gamma \mathbf{q}_\perp^2}{4\xi} \times \left(1 - \frac{b_1}{s\xi}\right) I_2 + \mathbf{q}_\perp^4\bigg[\frac{\gamma^2}{16\xi^2}\left(\frac{3}{2} - \frac{3b_1}{s\xi} + \frac{b_1^2}{2s^2\xi^2}\right) - \frac{\gamma}{16\xi|eB|}\left(1 - \frac{b_1}{s\xi}\right)\bigg] I_3\bigg\}. \quad (4.15)$$

Here, the integration over s can be performed using (A25). After replacing

$$R^2\sin^2\theta + \frac{2R^2\cos^2\theta}{\xi} \rightarrow 4\beta R^2 f^2(\theta), \quad (4.16)$$

where $\beta \equiv (2\xi)^{-1}$ and $f^2(\alpha, \theta)$ are defined in (3.17) and (3.18), we arrive at the same potential (3.21) and (3.22). The potential has therefore the general form

$$V(R, \theta) = -\frac{\alpha}{R}\left(\mathcal{A}_1(\alpha, \theta) - \frac{\gamma \mathcal{A}_2(\alpha, \theta)}{R^2} + \frac{\gamma^2 \mathcal{A}_3(\alpha, \theta)}{R^4}\right),$$

with $\mathcal{A}_i$, $i = 1, 2, 3$ from (3.22).

B. Modified Coulomb potential in $m_{\rm dyn}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$ regime

Starting from (4.5) and plugging $\Pi(q)$ corresponding to the relevant regime $m_{\rm dyn}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$ from (2.14) and choosing $q_0 = 0$, we arrive first at

$$V(\mathbf{x}) = -e^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{-i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 + \zeta^2 e^{-(\mathbf{q}_\perp^2/2|eB|)}}, \quad (4.17)$$

where $\zeta \equiv \sqrt{\frac{2\alpha|eB|}{\pi}}$ is already defined in (3.25). To perform the three-dimensional integral over $\mathbf{q}$, we will use the same methods described in Appendix B to evaluate the full photon propagator in the coordinate space. First using Schwinger's parametrization technique (B2), the potential reads

$$V(\mathbf{x}) = -e^2 \int_0^\infty ds \int \frac{dq_\perp q_\perp dq_3 d\varphi'}{(2\pi)^3} \times e^{-i(q_\perp R \sin\theta \cos(\varphi-\varphi') + q_3 R\cos\theta)} \times \exp(-s(\mathbf{q}^2 + \zeta^2 e^{-(\mathbf{q}_\perp^2/2|eB|)})), \quad (4.18)$$

where we have used (A8) to bring $\mathbf{q}\cdot\mathbf{x}$ in a useful form. Now integrating over φ' and q_3 using (A10) and (A12), we arrive at

$$V(\mathbf{x}) = -e^2 \int_0^\infty ds \sqrt{\frac{\pi}{s}} \int \frac{dq_\perp q_\perp}{(2\pi)^2} e^{-(R^2\cos^2\theta/4s)} \times J_0(q_\perp R \sin\theta) e^{-s\zeta^2} e^{-s(1-[\alpha/\pi])\mathbf{q}_\perp^2}, \quad (4.19)$$

where we have used the IR approximation $|\mathbf{q}|^2 \ll |eB|$ relevant in the LLL regime

$$e^{-s(\mathbf{q}_\perp^2 + \zeta^2 \exp(-[\mathbf{q}_\perp^2/2|eB|]))} \simeq e^{-s\zeta^2} e^{-s(1-[\alpha/\pi])\mathbf{q}_\perp^2}.$$

Next, using (B10) the integration over $q_\perp$ can be performed. The potential is therefore given by

$$V(\mathbf{x}) = -\frac{e^2\sqrt{\pi}}{8\pi^2(1-\frac{\alpha}{\pi})} \int_0^\infty \frac{ds}{s^{3/2}} e^{-(R^2 g^2(\theta)/4s) - s\zeta^2}, \quad \text{with}$$

$$g^2(\theta) = \cos^2\theta + \frac{\sin^2\theta}{(1-\frac{\alpha}{\pi})}.$$

The θ-dependent function $g(\theta)$ is defined already in (3.25). Finally, defining a new variable $s' = \zeta^2 s$, and using (B13), the potential $V(\mathbf{x})$ in the $m_{\rm dyn}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$ in the LLL dominance can be calculated and reads

$$V(R, \theta) = -\frac{\alpha}{(1-\frac{\alpha}{\pi})g(\theta)R} e^{-\zeta g(\theta)R},$$

which is the same potential as (3.28) which was found using the Wilson-loop technique.

V. A QUALITATIVE ANALYSIS OF MODIFIED COULOMB POTENTIALS IN THE LLL

In this section we will consider the potential (3.22) and (3.28)

$$V_1(R, \theta) = -\frac{\alpha}{R}\left(\mathcal{A}_1(\alpha, \theta) - \frac{\gamma \mathcal{A}_2(\alpha, \theta)}{R^2} + \frac{\gamma^2 \mathcal{A}_3(\alpha, \theta)}{R^4}\right),$$

$$V_2(R, \theta) = -\frac{\alpha}{(1-\frac{\alpha}{\pi})g(\theta)R} e^{-\zeta g(\theta)R},$$

in the first $|\mathbf{q}_\parallel^2| \ll m_{\rm dyn}^2 \ll |eB|$, and the second $m_{\rm dyn}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$ LLL regimes, respectively. We will eventually compare these potentials with the modified Coulomb potential (1.1).

As for $V_1(R, \theta)$, Fig. 3 shows this potential for different choices of the magnetic field $B = 10^5, 10^6, 10^7, 10^9$ (from right to left) and different $\theta = 0, \pi/3, 2\pi/3, \pi$.

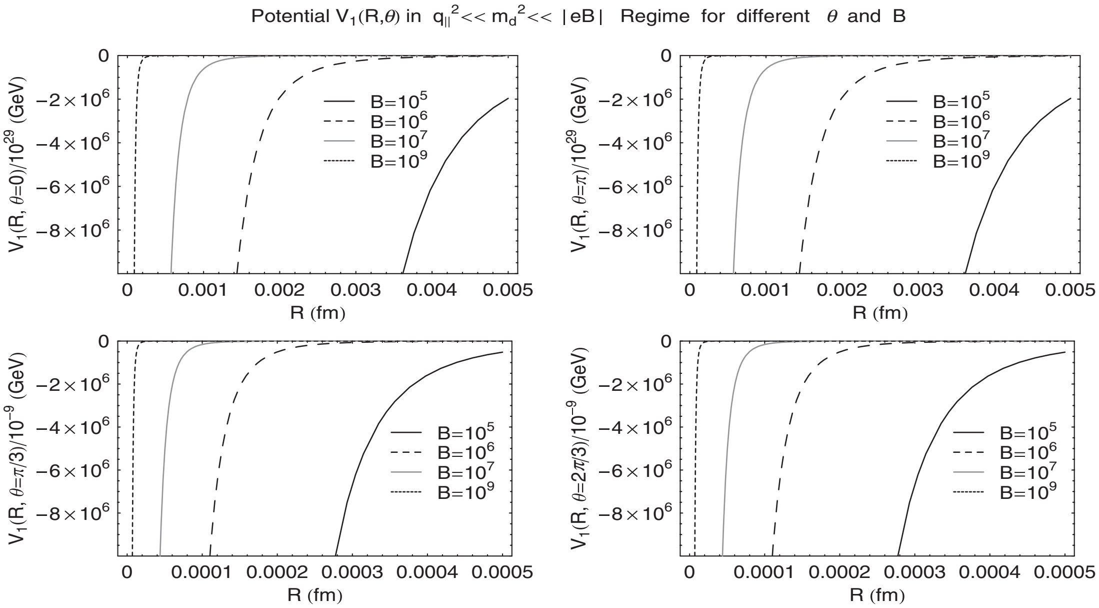

FIG. 3. Potential $V_1(R, \theta)$ for different B and θ. No qualitative changes occur by varying the angle θ.

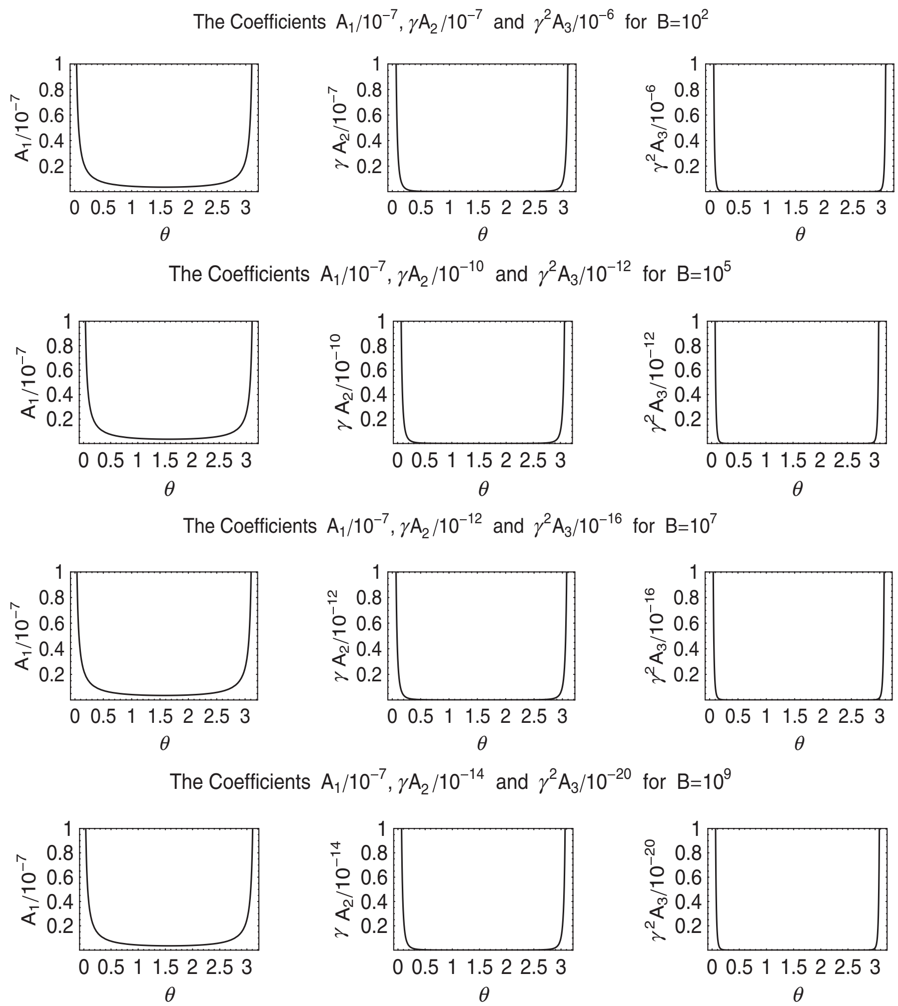

FIG. 4. Coefficients $\mathcal{A}_1$, $\gamma\mathcal{A}_2$, and $\gamma^2\mathcal{A}_3$ for different magnetic fields. As it turns out, $\mathcal{A}_3$ decreases rapidly with increasing magnetic field.

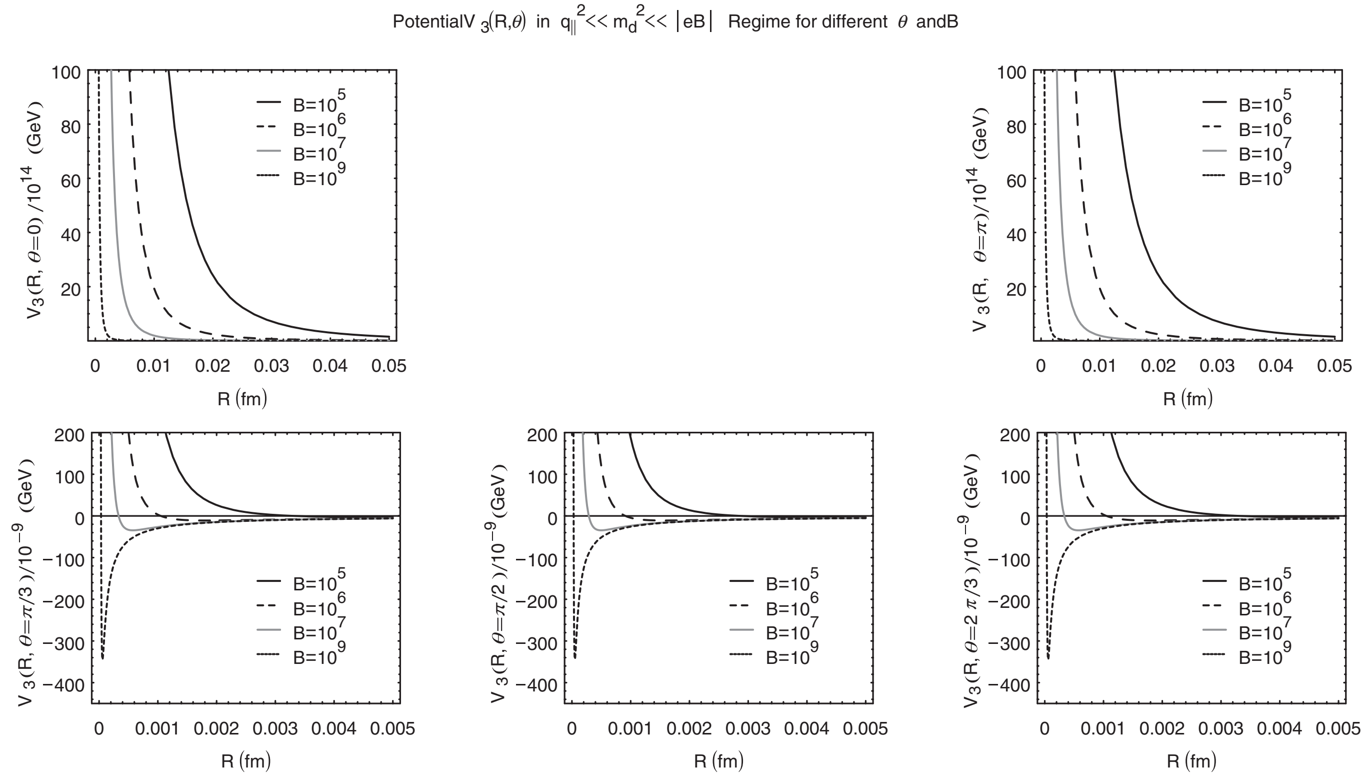

FIG. 5. Potential $V_3(R, \theta)$ for different B and θ. Bound states can be formed for $\theta \in]0, \pi[$ and for strong magnetic fields, $B \geq 10^5$, in the regime $R \leq 0.005$ fm. The depth of the potential at $R_{\min}$ increases with the magnetic field.

According to this result, for $R \longrightarrow 0$ the potential falls more rapidly to $-\infty$ the smaller the magnetic field is. Although the scales in which the potential V_1 is plotted for $\theta = 0, \pi$ are different from the scales in which it is plotted for $\theta = \pi/3, 2\pi/3$, their shapes are almost the same, i.e., no qualitative changes occur by varying the angle θ. This situation changes by neglecting the coefficient $\gamma^2 \mathcal{A}_3$ comparing with $\mathcal{A}_1$ and $\gamma \mathcal{A}_2$ in $V_1(R, \theta)$. Figure 4 shows the behavior of the coefficients $\mathcal{A}_1$, $\gamma \mathcal{A}_2$, and $\gamma^2 \mathcal{A}_3$ of the potential V_1 as functions of the angle θ for different magnetic fields $B = 10^2$, 10^5, 10^7, and $B = 10^9$.

Choosing $\alpha = 1/137$, it turns out that the coefficients $\mathcal{A}_1$, $\gamma \mathcal{A}_2$, and $\gamma^2 \mathcal{A}_3$ are positive $\forall \theta \in [0, \pi]$ and for any choice of constant magnetic field B. However, as it is shown in Fig. 4, $\mathcal{A}_3$ decreases rapidly with increasing magnetic field. Whereas for $B = 10^2$ the three coefficients $\mathcal{A}_1$, $\gamma \mathcal{A}_2$, and $\gamma^2 \mathcal{A}_3$ are comparable, for $B = 10^5$, $\gamma^2 \mathcal{A}_3$ is 10^4 times smaller than $\mathcal{A}_1$, and for $B = 10^9$ this difference is $\sim$13 order of magnitude. We conclude therefore that for strong magnetic field $B \geq 10^5$, the coefficient $\gamma^2 \mathcal{A}_3$ in $V_1(R, \theta)$ is negligible comparing with $\mathcal{A}_1$ and $\gamma \mathcal{A}_2$. Thus for $B \geq 10^5$ the potential $V_1(R, \theta)$ can be replaced by

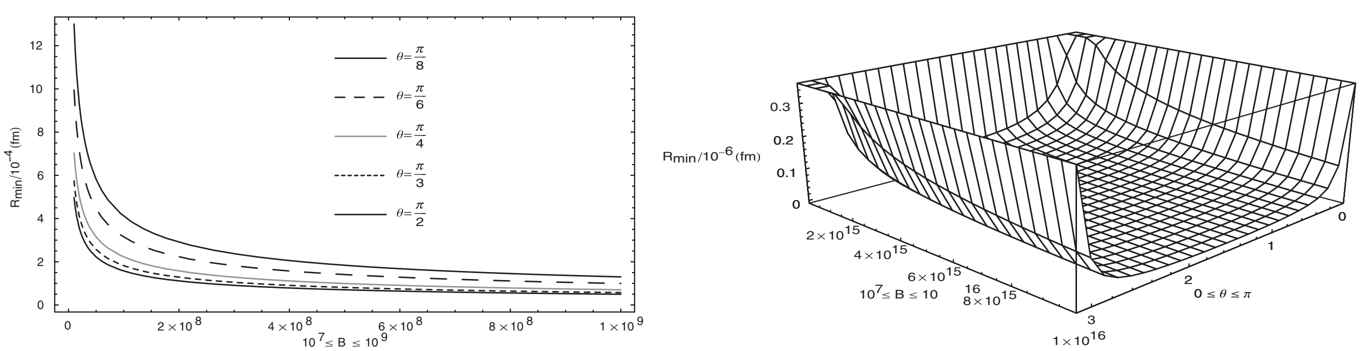

FIG. 6. Behavior of the minimum of the potential V_3 as a function of $10^7 \leq B \leq 10^{16}$ for different $0 < \theta \leq \pi/2$. For $\pi/2 \leq \theta < \pi$, $R_{\min}$ shows a symmetry in changing $\theta \rightarrow \theta + \pi/2$ (see the three-dimensional figure on the right-hand side).

$$V_3(R,\theta) = -\frac{\alpha}{R}\left(\mathcal{A}_1(\alpha,\theta) - \frac{\gamma\mathcal{A}_2(\alpha,\theta)}{R^2}\right), \qquad (5.1)$$

which has its minimum at

$$R_{\min}(B,\theta) = \sqrt{\frac{3\gamma\mathcal{A}_2}{\mathcal{A}_1}}. \qquad (5.2)$$

Figure 5 shows $V_3(R,\theta)$ for different choices of the magnetic field $B = 10^5$, 10^6, 10^7, and 10^9 (from right to left) and different $\theta = 0, \pi/3, \pi/2, 2\pi/3, \pi$. Whereas for $\theta = 0, \pi$ the potential is repulsive, it exhibits a minimum for angles $\theta \in]0,\pi[$ and distances $R \leq 0.005$ fm. The depth of the potential at $R_{\min}$ increases with the magnetic field. We interpret this effect as a possibility for bound state formation. The answer to the question concerning the existence and the number of bound states in the potential V_3 is beyond the scope of this paper.

In Fig. 6 the behavior of $R_{\min}$ from (5.2) for different θ is studied. As it turns out, for different θ, the position of the minimum of the potential is proportional to $1/\sqrt{B}$.

Let us now consider the potential $V_2(R,\theta)$ in the second regime $m_{\rm dyn}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$ of LLL. Figure 7 shows its behavior for different magnetic field B and angle θ. Again no qualitative changes occurs by varying the angle θ. As it is pointed out in the introduction, this Yukawa-like potential is comparable with the potential (1.1) from [7] in the scaling regime.

VI. SUMMARY

In this paper the static potential of QED is calculated in the presence of a strong but constant magnetic field using two different methods. First a perturbative Wilson-loop calculation is performed for two different regimes of dynamical mass $|\mathbf{q}_\parallel^2| \ll m_{\rm dyn}^2 \ll |eB|$ and $m_{\rm dyn}^2 \ll |\mathbf{q}_\parallel^2| \ll |eB|$ in the lowest Landau level (LLL). The resulting potential is then compared with the potential arising from a modified Born approximation. The results coincide. Comparing with the recently calculated potential of pointlike charges in [7], our potential shows a novel dependence on the angle θ between the particle-antiparticle axis and the direction of the magnetic field. A qualitative analysis of these modified potentials is performed in the previous Sec. V. As for the potential (3.22) from the first regime $|\mathbf{q}_\parallel^2| \ll m_{\rm dyn}^2 \ll |eB|$, it turns out that for strong magnetic field $B \geq 10^5$, the coefficient of the R^{-5} term, i.e., $\gamma^2\mathcal{A}_3$, is negligible compared to both other coefficients $\mathcal{A}_1$ and $\gamma\mathcal{A}_2$ from the expected Coulombian R^{-1} and the additional R^{-3} terms. Keeping only these two coefficients, a qualitative change occurs in the Coulomb-like potential which depends on the angle θ; whereas for $\theta = 0, \pi$ the potential is repulsive, it exhibits a minimum for angles $\theta \in]0,\pi[$ and distances $R \leq 0.005$ fm. The depth of the potential at $R_{\min}$ increases with the magnetic field and the position of the minimum is proportional to $1/\sqrt{B}$. We interpret this effect as a possibility for bound state formation. A rigorous proof of the existence

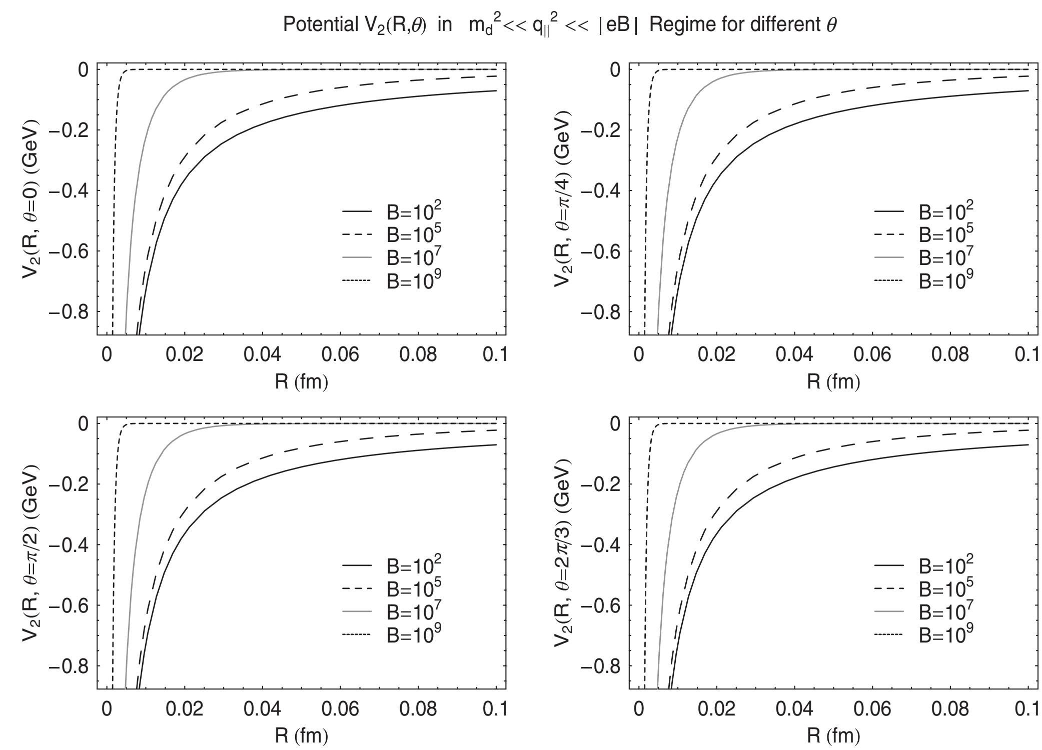

FIG. 7. Potential $V_2(R,\theta)$ for different B and θ. No qualitative changes occur by varying the angle θ. This potential is comparable with the potential (1.1) from [7] in the scaling regime.

and the number of bound states in the above potential is the subject of future investigations.

As for the potential (3.28) of the second regime $m_{\text{dyn}}^2 \ll |\mathbf{q}_{\parallel}^2| \ll |eB|$, it is comparable with a screened Yukawa potential. It is well-known that in this regime the photon acquires a finite mass proportional to $\sqrt{|eB|}$. According to our result, the photon receives also a modified effective mass depending on the angle θ between the particle axis and the direction of the magnetic field. But, this dependence is indeed negligible for fixed $\alpha = 1/137$. This result is in good agreement with the result recently found in [7].

It would be interesting to study the renormalization group (RG) improvement of these potentials, as they both depend on the coupling constant e, which is taken to be a bare parameter in our calculations. Using the RG improved potential it is possible, according to [9], to determine the Callan-Symanzik β-function of QED in the presence of strong magnetic field in the LLL approximation.

ACKNOWLEDGMENTS

The authors thank F. Ardalan and H. Arfaei for useful discussions. N. S. thanks, in particular, the referee of the paper for valuable hints concerning the potential $V(x)$ from [4]. This work is supported in part by Iranian TWAS chapter based at ISMO.

APPENDIX A: THE LLL PHOTON PROPAGATOR IN $|\mathbf{q}_{\parallel}^2| \ll m_{\text{dyn}}^2 \ll |eB|$ REGIME

In this section we will perform the integration over q in (3.15) to determine the full LLL photon propagator in the $m_{\text{dyn}}^2 \ll |\mathbf{q}_{\parallel}^2| \ll |eB|$ regime (3.16). To start, let us give the integral (3.15) in the Euclidean space

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = \delta_{\mu\nu}^{\parallel} \int \frac{d^4q}{(2\pi)^4} \times \frac{e^{-i(q_4x_4+\mathbf{q}\cdot\mathbf{x})}}{q^2 + \frac{\gamma(\alpha)|eB|}{2}(q_3^2+q_4^2)\exp(-\frac{\mathbf{q}_{\perp}^2}{2|eB|})}, \quad (A1)$$

where $\gamma(\alpha) \equiv \frac{2\alpha}{3\pi m_{\text{dyn}}^2}$. Here, the Euclidean coordinates $x_0 = -ix_4$ as well as $q_0 = -iq_4$. We have further used the notations $\mathbf{q} = (q_1, q_2, q_3)$ and $\mathbf{x} = (x_1, x_2, x_3)$. The scalar product is defined therefore by $\mathbf{q}\cdot\mathbf{x} = \sum_{i=1}^3 q_i x_i$. Using the Schwinger parametrization technique

$$\int_0^\infty ds e^{-as} = \frac{1}{a}, \quad (A2)$$

the above expression (A1) can be given by

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = \delta_{\mu\nu}^{\parallel} \int_0^\infty ds \int \frac{d^4q}{(2\pi)^4} e^{-i(q_4x_4+\mathbf{q}\cdot\mathbf{x})} \times \exp\left(-s\left(q_4^2 + \mathbf{q}^2 + \frac{\gamma(\alpha)|eB|}{2}(q_3^2+q_4^2) \times \exp\left(-\frac{\mathbf{q}_{\perp}^2}{2|eB|}\right)\right)\right). \quad (A3)$$

Let us first evaluate the integral over q_4

$$\int_{-\infty}^{+\infty} dq_4 e^{-iq_4x_4} \exp\left(-sq_4^2\left(1 + \frac{\gamma(\alpha)|eB|}{2} e^{-(\mathbf{q}_{\perp}^2/2|eB|)}\right)\right). \quad (A4)$$

For the new variable

$$s' \equiv s\left(1 + \frac{\gamma(\alpha)|eB|}{2} e^{-(\mathbf{q}_{\perp}^2/2|eB|)}\right), \quad (A5)$$

we get

$$\int_{-\infty}^{+\infty} dq_4 e^{-iq_4x_4 - s'q_4^2} = \sqrt{\frac{\pi}{s'}} e^{-(x_4^2/4s')}. \quad (A6)$$

To integrate the variable q_3 component, let us go into the polar coordinate system. Here, the product $\mathbf{q}\cdot\mathbf{x}$ is given by

$$\mathbf{q}\cdot\mathbf{x} = qR\sin\theta\sin\theta'\cos(\varphi-\varphi') + q_3R\cos\theta'\cos\theta, \quad (A7)$$

with $q \equiv |\mathbf{q}|$ and $R \equiv |\mathbf{x}|$, θ (θ') the angle between $\mathbf{x}$ ($\mathbf{q}$) and the external magnetic field, which is assumed to be in the x_3-direction. Now using $q_3 = q\cos\theta'$ and $\mathbf{q}_{\perp} = q\sin\theta'$, (A6) is given by

$$\mathbf{q}\cdot\mathbf{x} = q_{\perp}R\sin\theta\cos(\varphi-\varphi') + q_3R\cos\theta. \quad (A8)$$

In these cylindric coordinates the photon propagator (A3) is given by

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = \delta_{\mu\nu}^{\parallel} \int_0^\infty ds \int \frac{dq_{\perp}q_{\perp}dq_3d\varphi'}{(2\pi)^4} \times e^{-i(q_3R\cos\theta + q_{\perp}R\sin\theta\cos(\varphi'-\varphi))} \times \exp\left(-s\left(q_3^2 + \mathbf{q}_{\perp}^2 + \frac{\gamma(\alpha)|eB|}{2}q_3^2 \times \exp\left(-\frac{\mathbf{q}_{\perp}^2}{2|eB|}\right)\right)\right)\sqrt{\frac{\pi}{s'}}e^{-(x_4^2/4s')}, \quad (A9)$$

where we have inserted the expression from (A6). Now using the integral representation of the Bessel function J_0

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi' e^{-iz\cos(\varphi'-\varphi)}, \quad (A10)$$

and performing the integration over φ we get

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = \delta_{\mu\nu}^{\parallel} \int_0^\infty ds \int \frac{dq_{\perp}q_{\perp}dq_3}{(2\pi)^3} e^{-iq_3R\cos\theta - s'q_3^2} \times J_0(q_{\perp}R\sin\theta)e^{-s\mathbf{q}_{\perp}^2}\sqrt{\frac{\pi}{s'}}e^{-(x_4^2/4s')}. \quad (A11)$$

Performing now the integration over q_3 using

$$\int_{-\infty}^{+\infty} dq_3 e^{-iq_3 R\cos\theta - s'q_3^2} = \sqrt{\frac{\pi}{s'}} e^{-(R^2\cos^2\theta/4s')}, \quad \text{(A12)}$$

the photon propagator (A11) therefore reads

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = \delta_{\mu\nu}^{\parallel} \int_0^\infty \frac{ds}{s'} \int \frac{dq_\perp q_\perp}{8\pi^2} J_0(q_\perp R\sin\theta) \times e^{-s\mathbf{q}_\perp^2} e^{-[(x_4^2+R^2\cos^2\theta)/4s']}. \quad \text{(A13)}$$

To perform the integration over $q_\perp$ we proceed as follows. Using first the IR approximation $\mathbf{q}_\perp^2 \ll |eB|$ in the regime of LLL dominance, and expanding the factor $e^{-(\mathbf{q}_\perp^2/2|eB|)} \approx 1 - \frac{\mathbf{q}_\perp^2}{2|eB|} + \frac{1}{2!}\frac{\mathbf{q}_\perp^4}{4(eB)^2}$, the relation s'/s from (A5) can be written as

$$\frac{1}{s'} = \frac{2}{s\xi}\left(1 + \frac{\gamma\mathbf{q}_\perp^2}{2\xi} + \left(\frac{\gamma^2}{4\xi^2} - \frac{\gamma}{8\xi|eB|}\right)\mathbf{q}_\perp^4\right), \quad \text{with} \quad \xi \equiv (2 + \gamma|eB|). \quad \text{(A14)}$$

Plugging this result in (A13) the photon propagator is given by

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = \delta_{\mu\nu}^{\parallel} \int_0^\infty \frac{ds}{s\xi} \int \frac{dq_\perp q_\perp}{4\pi^2} J_0(q_\perp R\sin\theta) e^{-s\mathbf{q}_\perp^2}\left(1 + \frac{\gamma\mathbf{q}_\perp^2}{2\xi} + \left(\frac{\gamma^2}{4\xi^2} - \frac{\gamma}{8\xi|eB|}\right)\mathbf{q}_\perp^4\right) \times e^{-[(x_4^2+R^2\cos^2\theta)/2s\xi](1+[\gamma\mathbf{q}_\perp^2/2\xi]+([\gamma^2/4\xi^2]-[\gamma/8\xi|eB|])\mathbf{q}_\perp^4)}. \quad \text{(A15)}$$

Again using the IR approximation and expanding the exponent we get

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = \frac{\delta_{\mu\nu}^{\parallel}}{4\pi^2\xi} \int_0^\infty \frac{ds}{s} e^{-(b_1/2s\xi)} \int_0^\infty dq_\perp q_\perp J_0(q_\perp R\sin\theta) \times e^{-s\mathbf{q}_\perp^2}\left\{1 + \frac{\gamma\mathbf{q}_\perp^2}{2\xi}\left(1 - \frac{b_1}{2s\xi}\right) + \mathbf{q}_\perp^4\left[\frac{\gamma^2}{4\xi^2}\left(1 - \frac{b_1}{s\xi} + \frac{b_1^2}{8s^2\xi^2}\right) - \frac{\gamma}{8\xi|eB|}\left(1 - \frac{b_1}{2s\xi}\right)\right]\right\}, \quad \text{(A16)}$$

where

$$b_1 \equiv x_4^2 + R^2\cos^2\theta. \quad \text{(A17)}$$

To perform the integration over $q_\perp$ we use [13]

$$\int_0^\infty dz z^\mu e^{-\kappa z^2} J_\nu(\beta z) = \frac{\Gamma(\frac{\mu+\nu+1}{2})}{\beta\kappa^{\mu/2}\Gamma(\nu+1)} e^{-(\beta^2/8\kappa)} \times M_{(\mu/2),(\nu/2)}\left(\frac{\beta^2}{4\kappa}\right), \quad \text{(A18)}$$

where $M_{(\mu/2),(\nu/2)}(y)$ is the Whittaker function defined by

$$M_{(\mu/2),(\nu/2)}(y) = y^{(\nu+1)/2} e^{-(y/2)} \Phi\left(\frac{\nu - \mu + 1}{2}, \nu + 1; y\right), \quad \text{(A19)}$$

where

$$\Phi(\eta, \tau; y) \equiv 1 + \frac{\eta}{\tau}\frac{y}{1!} + \frac{\eta(\eta+1)}{\tau(\tau+1)}\frac{y^2}{2!} + \cdots. \quad \text{(A20)}$$

For the first term on the second line of (A16) we get therefore

$$I_1(s) \equiv \int_0^\infty dq_\perp q_\perp J_0(q_\perp R\sin\theta) e^{-s\mathbf{q}_\perp^2} = \frac{1}{2s} e^{-(R^2\sin^2\theta/4s)}, \quad \text{(A21)}$$

and for the second term we get

$$I_2(s) \equiv \int_0^\infty dq_\perp q_\perp^3 J_0(q_\perp R\sin\theta) e^{-s\mathbf{q}_\perp^2} = \frac{1}{2s^2} e^{-(R^2\sin^2\theta/4s)}\left(1 - \frac{R^2\sin^2\theta}{4s}\right), \quad \text{(A22)}$$

whereas the third term yields

$$I_3(s) \equiv \int_0^\infty dq_\perp q_\perp^5 J_0(q_\perp R\sin\theta) e^{-s\mathbf{q}_\perp^2} = \frac{1}{s^3} e^{-(R^2\sin^2\theta/4s)}\left(1 - \frac{R^2\sin^2\theta}{2s} + \frac{R^4\sin^4\theta}{32s^2}\right). \quad \text{(A23)}$$

Inserting (A21)–(A23) in (A16) we arrive first at

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = \frac{\delta_{\mu\nu}^{\parallel}}{4\pi^2\xi} \int_0^\infty \frac{ds}{s} e^{-(b_1/2s\xi)}\left\{I_1(s) + \frac{\gamma}{2\xi}\left(1 - \frac{b_1}{2s\xi}\right) \times I_2(s) + \left[\frac{\gamma^2}{4\xi^2}\left(1 - \frac{b_1}{s\xi} + \frac{b_1^2}{8s^2\xi^2}\right) - \frac{\gamma}{8\xi|eB|}\left(1 - \frac{b_1}{2s\xi}\right)\right] I_3(s)\right\}. \quad \text{(A24)}$$

Finally, the integration over s can be evaluated using [13]

$$\int_0^\infty \frac{ds}{s^n} e^{-(a/s)} = \frac{\Gamma(n-1)}{a^{n-1}}. \quad \text{(A25)}$$

Choosing the notation $b_0 = R^2\sin^2\theta + \frac{2b_1}{\xi}$, we get for the first term in (A24)

$$\frac{\delta^{\parallel}_{\mu\nu}}{8\pi^2\xi}\int\frac{ds}{s^2}e^{-(b_0/4s)}=\frac{\delta^{\parallel}_{\mu\nu}}{2\pi^2}\frac{1}{\xi b_0}, \tag{A26}$$

for the second term

$$\frac{\delta^{\parallel}_{\mu\nu}\gamma}{8\pi^2\xi^2}\int_0^\infty\frac{ds}{s}e^{-(b_0/4s)}\left(1-\frac{b_1}{2s\xi}\right)I_2(s)=\frac{\delta^{\parallel}_{\mu\nu}\gamma}{\pi^2}\left(-\frac{4b_1}{\xi^3b_0^3}+\frac{12b_1R^2\sin^2\theta}{\xi^3b_0^4}+\frac{1}{\xi^2b_0^2}-\frac{2R^2\sin^2\theta}{\xi^2b_0^3}\right), \tag{A27}$$

and the third term

$$-\frac{\delta^{\parallel}_{\mu\nu}}{4\pi^2}\int_0^\infty\frac{ds}{s^2}e^{-(b_0/4s)}\left[\frac{\gamma^2}{4\xi^2}\left(1-\frac{b_1}{s\xi}+\frac{b_1^2}{8s^2\xi^2}\right)-\frac{\gamma}{8\xi|eB|}\left(1-\frac{b_1}{2s\xi}\right)\right]I_3(s)$$
$$=\frac{8\delta^{\parallel}_{\mu\nu}\gamma^2}{\pi^2\xi^3b_0^3}\left(1-\frac{6R^2\sin^2\theta}{b_0}+\frac{6R^4\sin^4\theta}{b_0^2}\right)-\frac{96\delta^{\parallel}_{\mu\nu}\gamma^2b_1}{\pi^2\xi^4b_0^4}\left[1-\frac{8}{b_0}\left(R^2\sin^2\theta+\frac{b_1}{4\xi}\right)+\frac{20}{b_0^2}\left(\frac{R^4\sin^4\theta}{2}+\frac{R^2b_1\sin^2\theta}{\xi}\right)-\frac{30b_1R^4\sin^4\theta}{\xi b_0^3}\right]-\frac{4\delta^{\parallel}_{\mu\nu}\gamma}{\pi^2|eB|\xi^2b_0^3}\left[1-\frac{6}{b_0}\left(R^2\sin^2\theta+\frac{b_1}{\xi}\right)+\frac{6}{b_0^2}\left(R^4\sin^4\theta+\frac{8b_1R^2\sin^2\theta}{\xi}\right)-\frac{60b_1R^4\sin^4\theta}{\xi b_0^3}\right]. \tag{A28}$$

Further, to find the LLL photon propagator in the regime $|\mathbf{q}_\parallel^2|\ll m_{\rm dyn}^2\ll|eB|$ from (3.16), we will first replace

$$b_0\to 4\beta a_1(R,\theta,x_4),\qquad b_1\to\frac{a_2(R,\theta,x_4)}{\beta\gamma}\quad\text{and}\quad\xi\to(2\beta)^{-1}, \tag{A29}$$

where β and a_i, $i=1,2$ are defined in (3.18). Adding then the results from (A26)–(A28), together we arrive at the propagator (3.16).

APPENDIX B: THE LLL PHOTON PROPAGATOR IN $m_{\rm dyn}^2\ll|\mathbf{q}_\parallel^2|\ll|eB|$ REGIME

In this section we will perform the integration over q in (3.23) to determine the full LLL photon propagator in the $m_{\rm dyn}^2\ll|\mathbf{q}_\parallel^2|\ll|eB|$ regime (3.24). To start, let us give the integral (3.23) in the Euclidean space

$$\tilde{\mathcal{D}}_{\mu\nu}(x)=\delta^{\parallel}_{\mu\nu}\int\frac{d^4q}{(2\pi)^4}\frac{e^{-i(q_4x_4+\mathbf{q}\cdot\mathbf{x})}}{q_4^2+\mathbf{q}^2+\frac{2\alpha|eB|}{\pi}e^{-(\mathbf{q}_\perp^2/2|eB|)}}, \tag{B1}$$

where we have introduced the Euclidean coordinates $x_0=-ix_4$ as well as $q_0=-iq_4$. Here, $\mathbf{q}=(q_1,q_2,q_3)$ and $\mathbf{x}=(x_1,x_2,x_3)$ and $\mathbf{q}\cdot\mathbf{x}=\sum_{i=1}^3q_ix_i$. Using the Schwinger parametrization

$$\int_0^\infty ds e^{-as}=\frac{1}{a}, \tag{B2}$$

the above expression (B1) can be given by

$$\tilde{\mathcal{D}}_{\mu\nu}(x)=\delta^{\parallel}_{\mu\nu}\int_0^\infty ds\int\frac{d^4q}{(2\pi)^4}e^{-i(q_4x_4+\mathbf{q}\cdot\mathbf{x})}\times\exp\left(-s\left(q_4^2+\mathbf{q}^2+\frac{2\alpha|eB|}{\pi}e^{-(\mathbf{q}_\perp^2/2|eB|)}\right)\right). \tag{B3}$$

Performing the integration over q_4

$$\int_{-\infty}^{+\infty}dq_4e^{-sq_4^2-iq_4x_4}=\sqrt{\frac{\pi}{s}}e^{-(x_4^2/4s)}, \tag{B4}$$

we arrive at

$$\tilde{\mathcal{D}}_{\mu\nu}(x)=\delta^{\parallel}_{\mu\nu}\int_0^\infty ds\sqrt{\frac{\pi}{s}}e^{-(x_4^2/4s)}\int\frac{d^3q}{(2\pi)^4}e^{-i\mathbf{q}\cdot\mathbf{x}}\times\exp\left(-s\left(\mathbf{q}^2+\frac{2\alpha|eB|}{\pi}e^{-(\mathbf{q}_\perp^2/2|eB|)}\right)\right). \tag{B5}$$

To integrate the $\mathbf{q}$ component, we follow the same steps as in the previous section [see (A7) and (A8)] leading from (A3) to (A9). In the cylindric coordinates the photon propagator (B5) is given by

$$\tilde{\mathcal{D}}_{\mu\nu}(x)=\delta^{\parallel}_{\mu\nu}\int_0^\infty ds\sqrt{\frac{\pi}{s}}e^{-(x_4^2/4s)}\int\frac{dq_\perp q_\perp dq_3d\varphi'}{(2\pi)^4}\times e^{-i[q_\perp R\sin\theta\cos(\varphi-\varphi')+q_3R\cos\theta]}\times\exp\left(-s\left(\mathbf{q}^2+\frac{2\alpha|eB|}{\pi}e^{-(\mathbf{q}_\perp^2/2|eB|)}\right)\right). \tag{B6}$$

Here $\varphi'\in[0,2\pi]$ and $q_\perp\equiv|\mathbf{q}_\perp|\in[0,\infty)$. Now using the integral representation of the Bessel function J_0 from (A10) and performing the integration over φ we get

$$\tilde{\mathcal{D}}_{\mu\nu}(x)=\delta^{\parallel}_{\mu\nu}\int_0^\infty ds\sqrt{\frac{\pi}{s}}e^{-(x_4^2/4s)}\int\frac{dq_\perp q_\perp dq_3}{(2\pi)^3}\times e^{-iq_3R\cos\theta}J_0(q_\perp R\sin\theta)\times\exp\left(-s\left(\mathbf{q}_\perp^2+q_3^2+\frac{2\alpha|eB|}{\pi}e^{-(\mathbf{q}_\perp^2/2|eB|)}\right)\right), \tag{B7}$$

where we have written $\mathbf{q}^2=\mathbf{q}_\perp^2+q_3^2$. Performing now the integration over q_3 in the same way as the integration over q_4 [see (B4)] we get first

$$\int_{-\infty}^{+\infty} dq_3 e^{-sq_3^2 - iq_3 R\cos\theta} = \sqrt{\frac{\pi}{s}} e^{-(R^2\cos^2\theta/4s)}, \tag{B8}$$

and then

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = \delta^{\|}_{\mu\nu} \int_0^\infty \frac{ds}{s} e^{-[(x_4^2 + R^2\cos^2\theta)/4s]} \int \frac{dq_\perp q_\perp}{8\pi^2} \times J_0(q_\perp R\sin\theta) e^{-s(\mathbf{q}_\perp^2 + \zeta^2\exp(-[\mathbf{q}_\perp^2/2|eB|]))}, \tag{B9}$$

with $\zeta^2 \equiv \frac{2\alpha|eB|}{\pi}$. To perform the integration over $\mathbf{q}_\perp$, we use the approximation $\mathbf{q}_\perp^2 \ll |eB|$, which is valid in the regime of LLL dominance. After expanding the exponent

$$e^{-s(\mathbf{q}_\perp^2 + \zeta^2\exp(-[\mathbf{q}_\perp^2/2|eB|]))} \simeq e^{-s\zeta^2} e^{-s(1-[\alpha/\pi])\mathbf{q}_\perp^2},$$

the integration over $\mathbf{q}_\perp$ can be written as

$$e^{-s\zeta^2}\int_0^\infty dq_\perp q_\perp J_0(q_\perp R\sin\theta) e^{-s(1-[\alpha/\pi])\mathbf{q}_\perp^2} = \frac{e^{-s\zeta^2}}{2s(1-\frac{\alpha}{\pi})} \exp\left(-\frac{R^2\sin^2\theta}{4s(1-\frac{\alpha}{\pi})}\right). \tag{B10}$$

To evaluate the $q_\perp$ integration, we have used [13]

$$\int_0^\infty dz z^{\nu+1} J_\nu(\beta z) e^{-\gamma z^2} = \frac{\beta^\nu}{(2\gamma)^{\nu+1}} e^{-(\beta^2/4\gamma)}, \quad \mathrm{Re}(\nu) > -1, \mathrm{Re}(\alpha) > 0, \tag{B11}$$

by choosing $\gamma = s(1-\frac{\alpha}{\pi})$, $\beta = R\sin\theta$, and $\nu = 0$. Plugging this result in (B9) we arrive at

$$\tilde{\mathcal{D}}_{\mu\nu}(x) = -\frac{1}{16\pi^2(1-\frac{\alpha}{\pi})}\int_0^\infty \frac{ds}{s^2} e^{-s\zeta^2 - (1/4s)(x_4^2 + R^2 g^2(\theta))}, \tag{B12}$$

where $g(\theta)$ is defined in (3.25). Defining a new variable $s' = \zeta^2 s$, the s' integration can now be performed using

$$\int_0^\infty \frac{ds'}{s'^{\nu+1}} e^{-s' - (z^2/4s')} = \left(\frac{2}{z}\right)^\nu 2K_\nu(z). \tag{B13}$$

We arrive finally at the full LLL photon propagator in the regime $m_{\rm dyn}^2 \ll |\mathbf{q}_\|^2| \ll |eB|$, which is then given by

$$\tilde{\mathcal{D}}_{\mu\nu}(R, \theta, x_4) = \frac{\delta^{\|}_{\mu\nu}}{4\pi^2(1-\frac{\alpha}{\pi})} \times \frac{\zeta}{\sqrt{x_4^2 + R^2 g^2(\theta)}} K_1(\zeta\sqrt{x_4^2 + R^2 g^2(\theta)}). \tag{B14}$$

For the notation $x_4 \to T$ we arrive therefore at our results from (3.24).

[1] T. Maskawa and H. Nakajima, Prog. Theor. Phys. **52**, 1326 (1974); **54**, 860 (1975); R. Fukuda and T. Kugo, Nucl. Phys. **B117**, 250 (1976); V. A. Miransky, Nuovo Cimento Soc. Ital. Fis. **90A**, 149 (1985); C. N. Leung, S. T. Love, and W. A. Bardeen, Nucl. Phys. **B273**, 649 (1986); **B323**, 493 (1989); J. B. Kogut, E. Dagotto, and A. Kocic, Phys. Rev. Lett. **62**, 1001 (1989).

[2] C. N. Leung, Y. J. Ng, and A. W. Ackley, Phys. Rev. D **54**, 4181 (1996); arXiv:hep-th/9512114.

[3] Y. J. Ng and Y. Kikuchi, Phys. Rev. D **36**, 2880 (1987); D. G. Caldi and A. Chodos, Phys. Rev. D **36**, 2876 (1987); L. S. Celenza, V. K. Mishra, C. M. Shakin, and K. F. Liu, Phys. Rev. Lett. **57**, 55 (1986); D. G. Caldi, A. Chodos, K. Everding, D. A. Owen, and S. Vafaeisefat, Phys. Rev. D **39**, 1432 (1989); D. G. Caldi and S. Vafaeisefat, Phys. Lett. B **356**, 386 (1995).

[4] C. N. Leung and S. Y. Wang, Nucl. Phys. **B747**, 266 (2006); Ann. Phys. (N.Y.) **322**, 701 (2007); V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. D **67**, 107703 (2003); Nucl. Phys. **B563**, 361 (1999); Phys. Rev. Lett. **83**, 1291 (1999); Nucl. Phys. **B462**, 249 (1996); Phys. Rev. D **52**, 4747 (1995); Phys. Lett. B **349**, 477 (1995); K. G. Klimenko, Z. Phys. C **54**, 323 (1992); Teor. Mat. Fiz. **90**, 3 (1992) [Theor. Math. Phys. (Engl. Transl.) **90**, 1 (1992)]; S. Schramm, B. Muller, and A. J. Schramm, Mod. Phys. Lett. A **7**, 973 (1992); S. P. Klevansky and R. H. Lemmer, Phys. Rev. D **39**, 3478 (1989).

[5] K. Farakos, G. Koutsoumbas, and N. E. Mavromatos, Phys. Lett. B **431**, 147 (1998); K. Farakos and N. E. Mavromatos, Phys. Rev. B **57**, 3017 (1998); G. W. Semenoff, I. A. Shovkovy, and L. C. R. Wijewardhana, Mod. Phys. Lett. A **13**, 1143 (1998); E. J. Ferrer, V. P. Gusynin, and V. de la Incera, Mod. Phys. Lett. B **16**, 107 (2002); Eur. Phys. J. B **33**, 397 (2003).

[6] E. Elizalde, E. J. Ferrer, and V. de la Incera, Phys. Rev. D **70**, 043012 (2004); **68**, 096004 (2003); E. J. Ferrer and V. de la Incera, Int. J. Mod. Phys. A **19**, 5385 (2004).

[7] A. E. Shabad and V. V. Usov, Phys. Rev. Lett. **98**, 180403 (2007); arXiv:astro-ph/0607499.

[8] G. Calucci and R. Ragazzon, J. Phys. A **27**, 2161 (1994).

[9] J. B. Kogut, Rev. Mod. Phys. **55**, 775 (1983).

[10] J. S. Schwinger, Phys. Rev. **82**, 664 (1951).

[11] A. V. Kuznetsov and N. V. Mikheev, Phys. Rev. Lett. **89**, 011601 (2002).

[12] E. J. Ferrer and V. de la Incera, Phys. Rev. D **58**, 065008 (1998).

[13] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Seires and Products* (Academic Press, Orlando, 1980).

[14] D. S. Lee, C. N. Leung, and Y. J. Ng, Phys. Rev. D **55**, 6504 (1997).

[15] E. J. Ferrer and V. de la Incera, Int. J. Mod. Phys. A **14**, 3963 (1999).

[16] H. J. Rothe, *Lattice Gauge Theories*, World Scientific Lecture Notes in Physics Vol. 74 (World Scientific, Singapore, 2005).
[17] F. J. Wegner, J. Math. Phys. (N.Y.) **12**, 2259 (1971).
[18] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[19] A. M. Polyakov, Phys. Lett. **59B**, 82 (1975).
[20] S. L. Adler, Ann. Phys. (N.Y.) **67**, 599 (1971).
[21] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).