3-point off-shell vertex in scalar QED in arbitrary gauge and dimension

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We calculate the complete one-loop off-shell three-point scalar-photon vertex in arbitrary gauge and dimension for scalar quantum electrodynamics. Explicit results are presented for the particular cases of dimensions 3 and 4 both for massive and massless scalars. We then propose nonperturbative forms of this vertex that coincide with the perturbative answer to order e^2.

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I. INTRODUCTION

The nonperturbative structure of Green functions in gauge field theories has turned out to be a challenging problem. Aside from the complicated non-Abelian scenario of quantum chromodynamics (QCD), even simpler examples such as quantum electrodynamics (QED) have proved a hard nut to crack in the nonperturbative regime. Nevertheless, gauge covariance relations, such as the Ward-Fradkin-Green-Takahashi identity (WFGTI) [1], and the Landau-Khalatnikov-Fradkin transformations (LKFT) [2] contain vital clues about the Green functions. Guided by such relations, extensive work has been carried out to construct nonperturbative Green functions [3–5]. As well, perturbation theory is a reliable guide when constraining acceptable structures in the weak coupling limit [6–11].

In the context of perturbation theory, a systematic study of spinor QED was initiated by Ball and Chiu [6]. They decomposed the vertex into a "longitudinal" part which ensures that the WFGTI is satisfied, and a "transverse" part. In a basis where kinematic singularities are avoided, they gave off-shell results for the one-loop transverse vertex in 4 dimensions in the Fermi-Feynman gauge. Later on, Kızılersü, Reenders, and Pennington extended this result to an arbitrary covariant gauge [7]. Results for massless and massive QED3 were obtained afterwards [9–12]. These results were then generalized to arbitrary dimension by Davydychev, Osland, and Saks [13] in the realm of QCD (from which all QED results can be inferred). Whereas the bare fermion-boson vertex in a minimal coupling gauge theory is merely γ^μ, in general the vertex can be expanded out in terms of 12 spin amplitudes constructed from γ^μ and two independent four-momenta [14]. The WFGTI fixes four coefficients of the 12 spin amplitudes in terms of the fermion functions comprising the longitudinal component. The transverse part thus involves eight vectors with eight unconstrained scalar coefficients that depend on the gauge parameter ξ, the space-time dimension $d = 2\ell$, fermion masses, and three kinematical invariants (k^2, p^2, q^2); so this is a complicated problem even at one-loop order.

One might hope that, in the absence of spinorial matrices, scalar quantum electrodynamics (SQED) can offer a simpler platform to study nonperturbative solutions [15]. In this scenario, the 3-point vertex can be written in terms of just two independent four-momenta. The WFGTI fixes the coefficient of one of these. Therefore, there is only one unconstrained function which defines the transverse vertex—representing an 8-fold simplification of spinor QED/QCD. The trade-off is that additional four-point interactions occur in SQED. Thus the 1-loop scalar-photon vertex involves two additional Feynman diagrams. Ball and Chiu [6] carried out this calculation for massive scalars in the Fermi-Feynman gauge ($\xi = 1$) for $d = 4$. In this article, we extend their work to arbitrary dimension d and gauge ξ involving the one-loop scalar propagator along the way.

There are several reasons why this calculation is helpful: (i) it keeps track of the correct gauge covariance properties of the Green functions; (ii) one can take on-shell limits to check the gauge invariance of physical observables—this is not possible [13] if one only has results near four dimensions; (iii) SQED anyway has interest in lower dimensions, for example, nonperturbative SQED in $2 + 1$ and $0 + 1$ dimensions has been examined by [16,17], respectively; (iv) three dimensional field theories contain several features of corresponding four dimensional field theories at high temperatures [18].

We have organized the article as follows. In Sec. II we introduce the notation to calculate the three-point vertex, discuss its decomposition in the light of WFGTI, and give the expressions for one-loop scalar propagator and the longitudinal component of the three-point vertex. In Sec. III we evaluate the complete one-loop vertex in arbitrary gauge and dimensions and hence deduce an expression for its transverse component. We suggest three simple and natural constructions for its nonperturbative counterpart in Sec. IV and finish by discussing the so-called transverse Takahashi identities in Sec. V. An appendix serves to summarize many useful expressions arising from the Feynman integrals.

II. PRELIMINARIES

We shall start by setting out the notation, discussing the WFGTI and decomposing the 3-point vertex into longitudinal and transverse components. We then make use of the scalar propagator to present the longitudinal part fully at 1-loop order. Constant reference is made here and in the next section to various (mainly hypergeometric) functions which are listed in an appendix.

A. Notation

We define the bare quantities in the usual form: the scalar propagator $S^0(p) = 1/(p^2 - m^2)$, the photon propagator $\Delta^0_{\mu\nu} = -[g_{\mu\nu}p^2 - (1-\xi)p_\mu p_\nu]/p^4$, the 3-point vertex $\Gamma^0_\mu = (k+p)_\mu$, and the 4-point double photon vertex $e^2\Gamma^0_{\mu\nu} = e^2 g_{\mu\nu}$, where ξ is the general covariant gauge parameter (such that $\xi = 0$ corresponds to Landau gauge) and e is the usual QED coupling constant. The 3-point vertex up to one-loop order is diagrammatically represented in Fig. 1; it can be written in terms of two vectors alone, namely, k^μ and p^μ or, if preferred, $P^\mu \equiv (p+k)^\mu$ and $q^\mu \equiv (k-p)^\mu$. Because of the presence of the four-point vertex, there are two additional diagrams to be calculated in addition to the usual one required for the spinor QED.

The full 3-point vertex satisfies the usual WFGTI:

$$q_\mu \Gamma^\mu(k,p) = S^{-1}(k) - S^{-1}(p) \tag{1}$$

and has the nonsingular limit

$$\Gamma^\mu(p,p) = \partial S^{-1}(p)/\partial p_\mu \tag{2}$$

when $k \to p$. We can use (1) to construct the "longitudinal" part of the vertex :

$$\Gamma^\mu_L(k,p) = \frac{S^{-1}(k) - S^{-1}(p)}{k^2 - p^2}(k+p)^\mu. \tag{3}$$

The full vertex can then be written as

$$\Gamma^\mu(k,p) = \Gamma^\mu_L(k,p) + \Gamma^\mu_T(k,p), \tag{4}$$

where the "transverse" part satisfies

$$q_\mu \Gamma^\mu_T(k,p) = 0, \qquad \Gamma^\mu_T(p,p) = 0, \tag{5}$$

and can be expanded out only in terms of the basis vector

$$T^\mu(k,p) = k\cdot q p^\mu - p\cdot q k^\mu = [q^\mu(k^2 - p^2) - (k+p)^\mu q^2]/2. \tag{6}$$

Thus the full vertex is

$$\Gamma^\mu(k,p) = \frac{S^{-1}(k) - S^{-1}(p)}{k^2 - p^2}(p+k)^\mu + \tau(k^2, p^2, q^2)T^\mu(k,p). \tag{7}$$

The coefficient τ is a Lorentz scalar function of k and p, and can be expressed in terms of the 3 invariants, k^2, p^2, and q^2. Thus knowing only one unknown function τ is sufficient to fix the full 3-point vertex completely in SQED, the rest being tied to the scalar propagator S.

B. Longitudinal vertex

At one-loop the scalar propagator is given by two diagrams but because massless tadpole type diagrams are zero in dimensional regularization (which we are adopting), only the first diagram contributes, see Fig. 2.

In arbitrary dimensions $d = 2\ell$ and gauge ξ, the inverse propagator at one-loop is given by

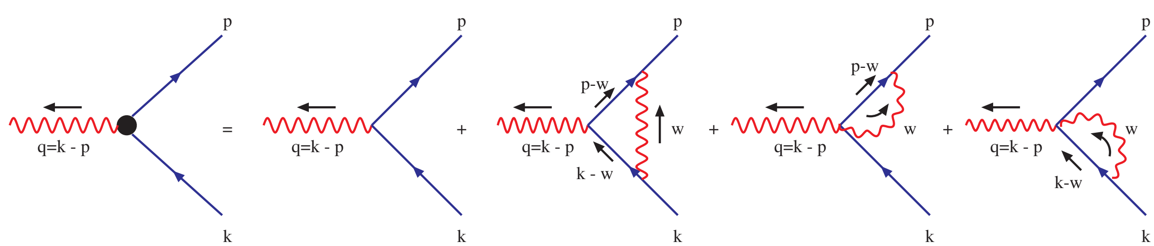

FIG. 1 (color online). One-loop 3-point vertex in SQED.

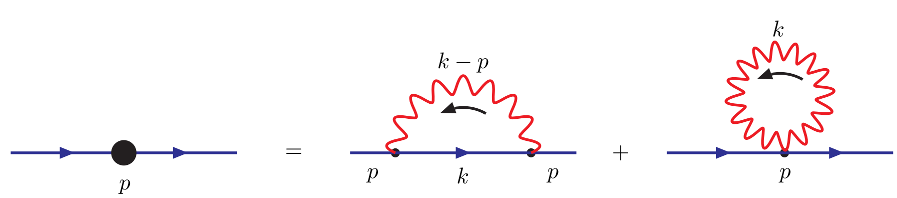

FIG. 2 (color online). One-loop scalar propagator.

$$S^{-1} = \frac{-e^2}{m^2}\left(\frac{m^2}{4\pi}\right)^{\ell}\Gamma(1-\ell) \times \left\{1 - 2\frac{(m^2+p^2)}{m^2}{}_2F_1\left(2-\ell,1;\ell;\frac{p^2}{m^2}\right) + (1-\xi)\frac{(m^2-p^2)^2}{m^4}{}_2F_1\left(3-\ell,2;\ell;\frac{p^2}{m^2}\right)\right\} \quad (8)$$

and readily yields the longitudinal part of the 3-point vertex at one loop:

$$\Gamma_L^{\mu}(k,p) = \frac{e^2\pi^2(k+p)^{\mu}}{(2\pi)^{2\ell}(k^2-p^2)}\{2Q_1(k)(m^2+k^2) - 2Q_1(p)(m^2+p^2) + (1-\xi)[(m^2-p^2)^2Q_3(p) - (m^2-k^2)^2Q_3(k)]\}, \quad (9)$$

where the functions $Q_i(p)$ are tabulated in the appendix.

III. ONE-LOOP VERTEX

In this section, we shall evaluate the complete off-shell one-loop vertex for all ξ and ℓ. Subtracting the longitudinal part from that produces the remaining transverse part.

A. The full vertex

The complete one-loop correction to the vertex is the sum of the three contributions that correspond to the last three graphs of Fig. 1:

$$\Lambda^{\mu} = \Lambda_1^{\mu}(k,p) + \Lambda_2^{\mu}(p) + \Lambda_2^{\mu}(k). \quad (10)$$

The first contribution involving only 3-point vertices is given by:

$$\Lambda_{\mu}^{1} = \frac{-ie^2}{(2\pi)^{2\ell}}\{4(k\cdot p)(k+p)_{\mu}J^{(0)} + [-8(k\cdot p)g_{\mu}{}^{\nu} - 2(k+p)_{\mu}(k+p)^{\nu}]J_{\nu}^{(1)} + 4(k+p)^{\nu}J_{\mu\nu}^{(2)} + (k+p)_{\mu}K^{(0)} - 2K_{\mu}^{(1)} + (\xi-1)[(k+p)_{\mu}K^{(0)} + 4(k+p)_{\mu}p^{\alpha}k^{\beta}I_{\alpha\beta}^{(2)} - 8p^{\alpha}k^{\beta}I_{\mu\alpha\beta}^{(3)} - 2(k+p)_{\mu} \times (k+p)^{\alpha}J_{\alpha}^{(1)} + 4(k+p)^{\alpha}J_{\mu\alpha}^{(2)} - 2K_{\mu}^{(1)}]\}, \quad (11)$$

where

$$K^{(0)} = \int d^dw\frac{1}{[(p-w)^2-m^2][(k-w)^2-m^2]}, \qquad K_{\mu}^{(1)} = \int d^dw\frac{w_{\mu}}{[(p-w)^2-m^2][(k-w)^2-m^2]},$$
$$J^{(0)} = \int d^dw\frac{1}{w^2[(p-w)^2-m^2][(k-w)^2-m^2]}, \qquad J_{\mu}^{(1)} = \int d^dw\frac{w_{\mu}}{w^2[(p-w)^2-m^2][(k-w)^2-m^2]},$$
$$J_{\mu\nu}^{(2)} = \int d^dw\frac{w_{\mu}w_{\nu}}{w^2[(p-w)^2-m^2][(k-w)^2-m^2]}, \qquad I^{(0)} = \int d^dw\frac{1}{w^4[(p-w)^2-m^2][(k-w)^2-m^2]}, \quad (12)$$
$$I_{\mu}^{(1)} = \int d^dw\frac{w_{\mu}}{w^4[(p-w)^2-m^2][(k-w)^2-m^2]}, \qquad I_{\alpha\beta}^{(2)} = \int d^dw\frac{w_{\alpha}w_{\beta}}{w^4[(p-w)^2-m^2][(k-w)^2-m^2]},$$
$$I_{\mu\alpha\beta}^{(3)} = \int d^dw\frac{w_{\mu}w_{\alpha}w_{\beta}}{w^4[(p-w)^2-m^2][(k-w)^2-m^2]}.$$

The results of computing these integrals are provided in detail in the appendix where we also compare them with other calculations in the literature for some particular cases of d. The two Λ_2^{μ} contributions contain the 4-point vertices. They are relatively simple to evaluate as they contain only propagator type loops. Thus we only quote the final result:

$$\Lambda_2^{\mu}(p) = \frac{e^2\pi^2p^{\mu}}{(2\pi)^{2\ell}}\left\{\left[3+\frac{m^2}{p^2}\right]Q_1(p) - \frac{\pi^{\ell-2}}{p^2}\Gamma(1-\ell)(m^2)^{\ell-1} + (\xi-1)\frac{(p^2-m^2)}{p^2} \times [Q_1(p) + (p^2-m^2)Q_3(p)]\right\}. \quad (13)$$

Equations (10)–(13) form the complete one-loop scalar-photon vertex for any ξ and ℓ at the one-loop level. This is a generalization to arbitrary dimension and gauge of the work of Ball and Chiu [6] who only examined the case $\xi = 1$, $\ell = 2$, using cutoff regularization. The explicit answers for the integrals (12) are stated in the appendix; as a general abbreviation we will write $X^{(0)} = i\pi^2X_0/2$ in what follows and use $\{d, m\}$ as a superscript (or subscript) to signify dimension and mass, as and when needed.

B. The transverse vertex

The transverse vertex is obtained by subtracting the longitudinal vertex, Eq. (9), from the full one, Eqs. (10)–(13), at one loop. Carrying out this exercise, we arrive at the following coefficient τ of the transverse vector T^{μ} for massive scalars:

$$\tau_{d,m}(k^2,p^2,q^2)=\frac{e^2\pi^2}{2(2\pi)^d\Delta^2}\bigg\{(k^2-2m^2+p^2-4k\cdot p)[-K_0+(m^2+k\cdot p)J_0]+\frac{2Q_1(p)}{k^2-p^2}[p^2(p^2-3k\cdot p)+k^2(k\cdot p-3p^2)-2m^2(p^2+k\cdot p)]-\frac{2Q_1(k)}{k^2-p^2}[k^2(k^2-3k\cdot p)+p^2(k\cdot p-3k^2)-2m^2(k^2+k\cdot p)]+(\xi-1)(m^2-k^2)(m^2-p^2)\bigg[J_0-(k\cdot p+m^2)I_0-\frac{2Q_3(p)}{k^2-p^2}(k\cdot p+p^2)+\frac{2Q_3(k)}{k^2-p^2}(k\cdot p+k^2)\bigg]\bigg\}, \tag{14}$$

where I_0, J_0, K_0, Q_{1-6} are explicitly stated in the appendix. $\Delta^2\equiv(k\cdot p)^2-k^2p^2=(k\cdot q)^2-k^2q^2$. Thus $4\Delta^2=\lambda(k^2,p^2,q^2)=[p^4+k^4+q^4-2p^2k^2-2p^2q^2-2k^2q^2]$, the Källén function, and is related to the $(2\times\text{area})^2$ of a triangle with sides $\sqrt{k^2}$, $\sqrt{p^2}$, $\sqrt{q^2}$. In the massless limit, the above expression reduces to

$$\tau_{d,0}=\frac{e^2\pi^2}{2(2\pi)^d\Delta^2}\bigg\{(k^2+p^2-4k\cdot p)[(k\cdot p)J_0^{d,0}-K_0^{d,0}]+\frac{2Q_1^{d,0}(p)}{k^2-p^2}[p^4-3(k^2+k\cdot p)p^2+k^2k\cdot p]-\frac{2Q_1^{d,0}(k)}{k^2-p^2}[k^4-3(p^2+k\cdot p)k^2+p^2k\cdot p]+(\xi-1)k^2p^2\bigg[J_0^{d,0}-k\cdot pI_0^{d,0}-\frac{2Q_3^{d,0}(p)}{k^2-p^2}(k\cdot p+p^2)+\frac{2Q_3^{d,0}(k)}{k^2-p^2}(k\cdot p+k^2)\bigg]\bigg\}. \tag{15}$$

In the massive case for small $\epsilon=2-\ell$ one gets

$$\tau_{4-2\epsilon,m}=\frac{\alpha}{8\pi\Delta^2}\bigg\{(k^2-2m^2+p^2-4k\cdot p)[(m^2+k\cdot p)J_0^{4-2\epsilon,m}+2\mathcal{S}]+\frac{2L(p)}{k^2-p^2}(p^2(p^2-3k\cdot p)+k^2(k\cdot p-3p^2)-2m^2(p^2+k\cdot p))-\frac{2L(k)}{k^2-p^2}(k^2(k^2-3k\cdot p)+p^2(k\cdot p-3k^2)-2m^2(k^2+k\cdot p))+(\xi-1)(m^2-k^2)\times(m^2-p^2)\bigg[J_0^{4-2\epsilon,m}-\frac{2}{\chi}(k\cdot p+m^2)\bigg(-q^2\mathcal{S}+\frac{p^2[(p^2-m^2)q^2+2m^2(k^2-p^2)]L(p)}{(p^2-m^2)^2}+\frac{k^2[(k^2-m^2)q^2-2m^2(k^2-p^2)]L(k)}{(k^2-m^2)^2}\bigg)+2\frac{(k\cdot p+p^2)(p^2+m^2)}{(k^2-p^2)(p^2-m^2)^2}L(p)-2\frac{(k\cdot p+k^2)(k^2+m^2)}{(k^2-p^2)(k^2-m^2)^2}L(k)\bigg]\bigg\}. \tag{16}$$

The quantities $\mathcal{S}$, L can be found early on in the appendix. In the massless limit for small ϵ one reads off

$$\tau_{4-2\epsilon,0}=\frac{\alpha}{8\pi\Delta^2}\bigg\{(k^2+p^2-4k\cdot p)\bigg(k\cdot pJ_0^{4-2\epsilon,0}+\ln\frac{q^4}{p^2k^2}\bigg)+\frac{(k^2+p^2)q^2-8p^2k^2}{p^2-k^2}\ln\frac{k^2}{p^2}+(\xi-1)k^2p^2\bigg[J_0^{4-2\epsilon,0}+2\frac{p^2(k^2+k\cdot p)\ln(-p^2)}{k^2-p^2}-2\frac{k^2(p^2+k\cdot p)}{k^2-p^2}\ln(-k^2)+2k\cdot p\ln(-q^2)\bigg]\bigg\}. \tag{17}$$

Note that for $\xi=1$, it agrees with Eq. (2.9) of [6]. In the massive case with $d=3$ we end up with

$$\tau_{3,m}=\frac{e^2}{16\pi\Delta^2}\bigg\{(k^2-2m^2+p^2-4k\cdot p)\bigg[(m^2+k\cdot p)J_0^{3,m}-2I\bigg(\frac{q^2}{4}\bigg)\bigg]+4(k^2(k^2-3k\cdot p)-2(k^2+k\cdot p)m^2+(k\cdot p-3k^2)p^2)\frac{I(k^2)}{(k^2-p^2)}-4(k^2(k\cdot p-3p^2)+p^2(p^2-3k\cdot p)-2m^2(k\cdot p+p^2))\frac{I(p^2)}{(k^2-p^2)}+(\xi-1)(m^2-k^2)(m^2-p^2)\bigg[J_0^{3,m}-\frac{4m(k^2+k\cdot p)}{(k^2-m^2)^2(k^2-p^2)}+\frac{4m(p^2+k\cdot p)}{(m^2-p^2)^2(k^2-p^2)}-\frac{1}{\chi}\bigg[2(m^2+k\cdot p)\bigg(\frac{J_0}{2}(m^2+k\cdot p)q^2+m\bigg(\frac{(k^2-m^2)q^2-(k^2+m^2)(k^2-p^2)}{(k^2-m^2)^2}-\frac{k^2-p^2+q^2}{m^2-p^2}\bigg)\bigg)\bigg]\bigg]\bigg\}, \tag{18}$$

where χ is a geometrical quantity listed in the appendix. The result (18) simplifies remarkably in the massless limit:

$$\tau_{3,0}=\frac{e^2}{2\mathcal{K}\mathcal{P}\mathcal{Q}}\bigg[\frac{\mathcal{K}^2+2\mathcal{Q}\mathcal{K}+\mathcal{P}^2+2\mathcal{P}\mathcal{Q}}{(\mathcal{K}+\mathcal{P})(\mathcal{K}+\mathcal{P}+\mathcal{Q})}+(\xi-1)\frac{1}{4}\bigg], \tag{19}$$

where we have adopted the Euclidian notation $\sqrt{-k^2} = \mathcal{K}$, $\sqrt{-p^2} = \mathcal{P}$, $\sqrt{-q^2} = \mathcal{Q}$.

IV. ON THE NONPERTURBATIVE VERTEX

The one-loop expression for $\tau(k^2, p^2, q^2)$ provides a guide as to its possible form in the strong coupling regime. Any nonperturbative *ansatz* for the transverse vertex should reduce to the perturbative result evaluated above. Equations (8) and (14) suggest what τ might resemble for general e^2:

$$\tau_{d,m}(k^2, p^2, q^2) = \frac{1}{4\Delta^2}\frac{[S^{-1}(k, \xi = 1) - S^{-1}(p, \xi = 1)]}{[(m^2 + k^2)Q_1(k) - (m^2 + p^2)Q_1(p)]} \times \Big\{(k^2 - 2m^2 + p^2 - 4k\cdot p)[-K_0 + (m^2 + k\cdot p)J_0] + \frac{2Q_1(p)}{k^2 - p^2}[p^2(p^2 - 3k\cdot p) + k^2(k\cdot p - 3p^2) - 2m^2(p^2 + k\cdot p)] - \frac{2Q_1(k)}{k^2 - p^2}[k^2(k^2 - 3k\cdot p) + p^2(k\cdot p - 3k^2) - 2m^2(k^2 + k\cdot p)]\Big\} + \frac{1}{2\Delta^2}\frac{[S^{-1}(k, \xi - 1) - S^{-1}(p, \xi - 1)]}{[(m^2 - k^2)^2 Q_3(k) - (m^2 - p^2)^2 Q_3(p)]}(m^2 - k^2)(m^2 - p^2) \times \Big\{J_0 - (k\cdot p + m^2)I_0 - \frac{2Q_3(p)}{k^2 - p^2}(k\cdot p + p^2) + \frac{2Q_3(k)}{k^2 - p^2}(k\cdot p + k^2)\Big\}. \tag{20}$$

The notation here means that $S(p, \xi = 1)$ is the scalar propagator in the Fermi-Feynman gauge, whereas, $S(p, \xi - 1)$ is the coefficient of the scalar propagator proportional to $(\xi - 1)$. By construction, this expression reproduces the one-loop transverse vertex in the weak coupling regime. In specific dimensions and for the massless case, it simplifies. Of course its form is not unique but it is perhaps the simplest nonperturbative extension of our earlier results for any ξ and d. We expect that an identical two-loop calculation will help us pin down the exact structure better. Because of the lack of Dirac matrices, this two-loop calculation is not as formidable a task as for spinor QED or QCD. We are currently in the process of carrying it out.

Another approach is to tie in the asymptotic behavior with the anomalous dimension of the scalar field in 4d. To see how this is done, return to the one-loop self energy as obtained previously in Eq. (8); by using contiguity relations of hypergeometric functions, this can be cast in the simpler form

$$\Sigma(p) = \frac{-e^2}{m^2}\left(\frac{m^2}{4\pi}\right)^{\ell}\Gamma(1 - \ell)\Big[1 + 2(\ell - 1)(1 - \xi) + \{(1 - 2\ell) - \xi(3 - 2\ell)\}\left(1 + \frac{p^2}{m^2}\right) \times {}_2F_1\left(2 - \ell, 1; \ell; \frac{p^2}{m^2}\right)\Big].$$

According to the procedure for self-consistent regularization by higher-order corrections [19] we then make the substitution $\ell \to 2 + \gamma$ in Eq. (21) and renormalize by ensuring that the propagator behaves as $(-p^2)^{1+\gamma}$ as $p^2 \to \infty$. This gives the self-consistent asymptotic equation

$$\left(-\frac{p^2}{m^2}\right)^{1+\gamma} \sim \frac{e^2}{16\pi^2}\frac{\Gamma^2(1 + \gamma)}{\Gamma(2\gamma + 2)}[(\xi - 3)\Gamma(-\gamma) + 2(1 - \xi)\Gamma(1 - \gamma)]\left(1 + \frac{p^2}{m^2}\right)\left(1 - \frac{p^2}{m^2}\right)^{\gamma}, \tag{21}$$

which fixes $\gamma = (3 - \xi)e^2/16\pi^2 + O(e^4)$. But anyway it produces a nonperturbative form of the 4d propagator

$$S^{-1}(p) \simeq \frac{e^2}{16(\gamma + 1)\pi^2}[(\xi - 3)\Gamma(-\gamma) + 2(1 - \xi)\Gamma(1 - \gamma)] \times \left(1 + \frac{p^2}{m^2}\right){}_2F_1(-\gamma, 1; \gamma + 2; p^2/m^2). \tag{22}$$

[This nonperturbative method succeeds in the ultraviolet but fails in the infrared limit $p^2 \to m^2$, when the propagator $S \sim 1/(p^2 - m^2)^{1+(3-\xi)e^2/8\pi^2}$. For infrared exponentiation it is much easier to resort to the gauge technique [3].] Exactly the same procedure can be applied to the transverse vertex. If τ is expressed in Feynman parametric form,

$$(p^2 - k^2)\tau(p,k,q) = \frac{4e^2\Gamma(2-\ell)}{(4\pi)^\ell}\int_0^1 d\sigma(1-\sigma)^{\ell-2}[(m^2-p^2\sigma)^{\ell-2} - (p\to k)] + \frac{e^2(2m^2+k^2+p^2-2q^2)\Gamma(3-\ell)}{(4\pi)^\ell}\int_0^1 d\sigma(1-\sigma)^{\ell-1}\int_{-1}^1 du u\mathcal{D}^{\ell-3} + \frac{e^2(\xi-1)(k^2-m^2)(p^2-m^2)\Gamma(4-\ell)}{(4\pi)^\ell}\int_0^1 d\sigma(1-\sigma)^{\ell-2}\int_{-1}^1 du u\mathcal{D}^{\ell-4}, \tag{23}$$

with $\mathcal{D} \equiv m^2 - q^2(1-\sigma)(1-u^2)/4 - p^2\sigma(1+u)/2 - k^2\sigma(1-u)$, then with the above substitution, an *ansatz* for the nonperturbative transverse vertex in 4d emerges:

$$(p^2 - k^2)\tau(p,k,q) = \frac{4e^2\Gamma(-\gamma)}{(4\pi)^2}\int_0^1 d\sigma(1-\sigma)^{\gamma}[(m^2-p^2\sigma)^{\gamma} - (p\to k)] + \frac{e^2(2m^2+k^2+p^2-2q^2)\Gamma(1-\gamma)}{(4\pi)^2}\int_0^1 d\sigma(1-\sigma)^{1+\gamma}\int_{-1}^1 du u\mathcal{D}^{\gamma-1} + \frac{e^2(\xi-1)(k^2-m^2)(p^2-m^2)\Gamma(2-\gamma)}{(4\pi)^2}\int_0^1 d\sigma(1-\sigma)^{\gamma}\int_{-1}^1 du u\mathcal{D}^{\gamma-2}, \tag{24}$$

That the anomalous dimension of the scalar field makes an appearance should come as no surprise: the WFGTI is at work. What is rather interesting about the form (24) is that for $q^2 = 0$ it takes the form $[F(p^2) - F(k^2)]/(p^2 - k^2)$ even though it is associated with the transverse piece; but for $q^2 \neq 0$ this particular structure disappears as one can see from the form of $\mathcal{D}$. Note anyway that (23) and (24) both have intrinsic dependences on all three variables p^2, k^2, and q^2 (or $p \cdot k$).

A third way of going nonperturbative relies upon dispersion relations; while these are well established for the two-point function, in the form of the Lehmann-Källén representation, they are trickier for the vertex function but can nevertheless be found as follows for graphs with triangular topology. Make the change of variable $\sigma \to m^2/W^2$ in Eq. (23)—so that W^2 runs from m^2 to ∞—in the denominator $\mathcal{D}$. This means we can generally write

$$S^{-1}(p) = \int_{m^2}^\infty dW^2 \frac{\rho(W^2)}{[p^2 - W^2 + i\epsilon]}, \quad \tau(p,k,q) = \int_{m^2}^\infty dW^2 \int_{-1}^1 du \frac{\mathcal{P}(W^2,u)}{[p^2(1+u)/2 + k^2(1-u)/2 + q^2(1-u^2)(W^2-m^2)/4 - W^2 + i\epsilon]}. \tag{25}$$

The idea is then to determine ρ and $\mathcal{P}$ self-consistently through the Schwinger-Dyson equations for the propagator and the vertex; the latter inevitably brings in the 4-point function, but we can use its own WGFTI to approximate it by connected 3-point graphs. While we have not solved this problem for $\mathcal{P}$, the idea has been taken to fruition [20] for ρ in SQED and QED, giving results that coincide with perturbation theory up to order e^4 for the charged field propagators. There is much more work involved in obtaining the spectral function $\mathcal{P}$ accurately to order e^4 and higher and this has not yet been done. These *ansätze* are unlikely to be the whole story. However, one may ask how close these are to the real vertex and how these can be compared to each other. By construction, our *ansätze* agree with perturbtion theory at the one-loop order.

(i) *Ansatz* (20) agrees with perturbation theory to $\mathcal{O}(e^2)$ in all momentum regimes, dimensions, and gauges. To see whether this relation between the vertex and the propagator survives at $\mathcal{O}(e^4)$, one needs to know these Green functions at that order.

(ii) *Ansatz* (24) would be in accord with the real vertex in 4 dimensions for large k^2 and p^2 but would fail for k^2 and $p^2 \approx m^2$. As one knows γ to $\mathcal{O}(e^2)$, it amounts to knowing the vertex to all orders in the leading logarithm approximation for asymptotically large values of momenta.

(iii) *Ansatz* (25) would agree with perturbation theory order by order depending upon the exact knowledge of the ρ and $\mathcal{P}$ functions to a given order. In principle, one could evaluate these functions nonperturbatively through SDEs. However, this exercise for $\mathcal{P}$ is a hard nut to crack.

A full two-loop calculation of the vertex should narrow down possible forms of any *ansatz*. Techniques for the two-loop vertex calculation, have been developed in [21–23]. All the master integrals for massless two-loop vertex diagram with three off-shell legs have been calculated in [24]. These advances indicate that the calculation of the two-loop transverse vertex should not be too difficult, at least for the massless case. This work is under progress.

V. ON THE TRANSVERSE TAKAHASHI IDENTITIES

Equation (20) is effectively a Ward-identity type relation linking the transverse vertex to the scalar propagator. There have been attempts to look for formal relations of this kind. Takahashi [25] discovered what are called transverse identities whose implications for the vertex have been examined for spinor QED [26–29]. In the case of SQED, as there is just one unknown which remains undetermined by the conventional WFGTI, it is tempting to look for a transverse Takahashi identity, hoping one might be able to determine the three-point vertex more realistically.

It should be noted that the general form of the vertex (7) shows that the transverse coefficient contributes to both basis vectors $P_\mu = (p+k)_\mu$ and $q_\mu = (k-p)_\mu$. The curl of the vertex $q_\mu\Gamma_\nu - q_\nu\Gamma_\mu$ will eliminate the component of Γ proportional to q, leaving us with the kinematic mixture $q_\mu P_\nu - q_\nu P_\mu$, multiplying the coefficient

$$\frac{S^{-1)}(p) - S^{-1}(k)}{p^2-k^2} + q^2\tau(p^2,k^2,q^2)/2,$$

which is unwieldy. However the same kinematic combination can also be obtained by forming the "modified curl" $P_\mu\Gamma_\nu - P_\nu\Gamma_\mu$; this has the advantage of killing off the longitudinal part of Γ and *only bringing in* τ. We suggest that new identities involving the modified curl are more appropriate and will prove more promising for SQED.

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APPENDIX

In this appendix we summarize the results for various integrals involved in the calculation of the 3-point vertex for quick reference. We write down the results for arbitrary d as well as small $\epsilon = 2-\ell$ and $d=3$. This way we aim to present all the integrals including the ones which have not been considered in the articles [6,7,9–11]. Wherever possible, we compare the results of specific cases with the ones in the above-mentioned articles.

1. Subsidiary quantities

There are a number of quantities which arise in various integrations that constantly appear later on, so we shall summarize them first and invoke them as they turn up. Unless specified we work in general dimension 2ℓ:

$$\begin{aligned}
\mathcal{S} &= \sqrt{1-\frac{4m^2}{q^2}}\ln\left(\frac{\sqrt{1-\frac{4m^2}{q^2}}+1}{\sqrt{1-\frac{4m^2}{q^2}}-1}\right)\\
&= 2\sqrt{4m^2/q^2-1}\arctan(1/\sqrt{4m^2/q^2-1}),\\
I(p^2) &\equiv (1/\sqrt{-p^2})\arctan\sqrt{-p^2/m^2},\\
L(p^2) &\equiv (1-m^2/p^2)\ln(1-p^2/m^2),\\
Q_1(k) &\equiv (\pi m^2)^{\ell-2}\Gamma(1-\ell){}_2F_1(2-\ell,1;\ell;k^2/m^2),\\
\ell Q_2(k) &\equiv (\pi m^2)^{\ell-2}\Gamma(2-\ell){}_2F_1(2-\ell,1;\ell+1,k^2/m^2),\\
m^2Q_3(k) &\equiv (\pi m^2)^{\ell-2}\Gamma(1-\ell){}_2F_1(3-\ell,2;\ell,k^2/m^2),\\
m^2Q_4(k) &\equiv (\pi m^2)^{\ell-2}(2-\ell)\Gamma(-\ell)\\
&\quad\times{}_2F_1(3-\ell,2;\ell+1;k^2/m^2),\\
Q_5(k) &\equiv -2i\pi^2(\pi m^2)^{\ell-2}\Gamma(2-\ell)\\
&\quad\times{}_2F_1(1/2,1;\ell-1;4m^2/q^2),\\
Q_6(p) &\equiv i\pi^\ell(\ell-3)(m^2)^{\ell-3}\Gamma(1-\ell)\\
&\quad\times{}_2F_1(1,4-\ell;\ell;p^2/m^2),\\
\chi &\equiv m^2(k^2-p^2)^2+(m^2-k^2)(m^2-p^2)q^2.
\end{aligned}$$

χ represents $(12\times\text{volume})^2$ of a tetrahedron constructed with base triangle lengths $\sqrt{-k^2}$, $\sqrt{-p^2}$, $\sqrt{-q^2}$ and lateral sides to the apex of lengths m, m, 0; thus it has geometrical significance. It is worth noting the zero mass limits of the Q_i as we will consider such situations later:

$$\begin{aligned}
Q_1^{d,0}(k) &= -(-\pi k^2)^{\ell-2}\Gamma^2(\ell-1)\Gamma(2-\ell)/\Gamma(2\ell-2)\\
&= -Q_2^{d,0}(k),\\
2\pi Q_3^{d,0}(k) &= (-\pi k^2)^{\ell-3}\Gamma^2(\ell-2)\Gamma(3-\ell)/\Gamma(2\ell-4)\\
&= \pi Q_4^{d,0}(k)\\
(m^2)^{2-\ell}Q_5^{d,0}(k) &\to -2i\pi^\ell\Gamma(2-\ell),\\
Q_6^{d,0}(k)/m^2 &\to i\pi^\ell(-k^2)^{\ell-4}\Gamma(\ell)\Gamma(\ell-2)/\Gamma(2\ell-4)\\
\chi^{d,0} &= k^2p^2q^2 \text{ very simply.}
\end{aligned}$$

2. The K integrals

$K^{(0)}$ in the list (12) for arbitrary $d = 2\ell$ equals

$$\begin{aligned}
i\pi^2K_0/2 &= K^{(0)}\\
&= i\pi^\ell\Gamma(2-\ell)(m^2)^{\ell-2}{}_2F_1\left(1,2-\ell;\frac{3}{2};\frac{q^2}{4m^2}\right),
\end{aligned} \tag{A1}$$

$$K_{d,0}^{(0)} = i\pi^\ell(-q^2)^{\ell-2}\frac{\Gamma^2(\ell-1)\Gamma(2-\ell)}{\Gamma(2\ell-2)}. \tag{A2}$$

Corresponding massive and massless expressions in the neighborhood of 4d ($\epsilon = 2-\ell$) are:

$$K^{(0)}_{4-2\epsilon,m} = i\pi^2[C - \mathcal{S}],$$
$$K^{(0)}_{4-2\epsilon,0} = i\pi^2[C - \ln(-q^2/m^2)], \qquad \text{(A3)}$$

where

$$C = \frac{1}{\epsilon} - \gamma - \ln(\pi m^2) + 2. \qquad \text{(A4)}$$

The first of the results (A3) agrees with Eqs. (44, 46–48) of [7]. When $d = 3$, this integral simplifies even more:

$$K^{(0)}_{3,m} = i\pi^2 I(q^2/4), \qquad K^{(0)}_{3,0} = i\pi^3/\sqrt{-q^2}. \qquad \text{(A5)}$$

Expressions (A5) coincide with (A1) of [10] and (A3) of [11], respectively. For the $K^{(1)}_\mu$ integral in (12), it is easy to show that $K^{(1)}_\mu = (p+k)_\mu K^{(0)}/2$.

3. The J integrals

The $J^{(0)}$ integral can be found in various sources [7,8,10,11,30,31] the most general case (massive scalars and any d) has been discussed in [31]; we shall simply cite the known answers. J_0 is probably the most difficult one to work out as it brings in dilogarithmic or Spence (Sp) functions when ℓ is integer. For any ℓ the massless case has been given a completely elegant representation by Davydychev [30]:

$$J^{(0)} = 2i(-i\pi)^\ell (k^2, p^2, q^2)^{\ell-2}\frac{\Gamma^2(\ell-1)\Gamma(2-\ell)}{\Gamma(2\ell-2)} \times \left[\frac{(p^2q^2)^{2-\ell}}{p^2+q^2-k^2}\,{}_2F_1\left(1, 1/2; \ell - 1/2, -\frac{\Delta}{2(p^2+q^2-k^2)^2}\right) + \text{two perms} - \pi\frac{\Gamma(2\ell-2)}{\Gamma^2(\ell-1)}(2\Delta)^{\ell-3/2}\Theta\right] \qquad \text{(A6)}$$

For massive scalars, like we have, the result $J_0^{4,m}$ in 4d is too lengthy (and uninformative) to quote. It is given in Eq. (16) of Ref. [7] and involves Spence functions of complicated arguments. In 3d the result is [11] easy to state:

$$J_0^{3,m} = \eta(k,p)I(\eta^2(k,p)\chi/4) + \eta(p,k)I(\eta^2(p,k)\chi/4);$$
$$\eta(k,p) = \frac{m^2(k^2-p^2)(2m^2-k^2-p^2)+\chi}{\chi(m^2-k^2)}. \qquad \text{(A7)}$$

In the massless limit one obtains [8,10] from all of these forms,

$$J_0^{4,0} = \frac{2}{\Delta}\left[Sp\left(\frac{p\cdot q - \Delta}{-p^2}\right) - Sp\left(\frac{p\cdot q+\Delta}{-p^2}\right) + \frac{1}{2}\left(\frac{p^2+p\cdot q-\Delta}{p^2+p\cdot q+\Delta}\right)\ln\left(\frac{q^2}{p^2}\right)\right], \qquad \text{(A8)}$$

$$J_0^{3,0} = \pi(\mathcal{K}^2+\mathcal{P}^2)/\mathcal{K}\mathcal{P}\mathcal{Q}; \qquad \mathcal{K} \equiv \sqrt{-k^2},$$
$$\mathcal{P} \equiv \sqrt{-p^2}, \qquad \mathcal{Q} \equiv \sqrt{-q^2}. \qquad \text{(A9)}$$

4. The J_μ integral

In its most general form, this can be written as

$$J^{(1)}_\mu = \frac{i\pi^2}{2}[k_\mu J_A(k,p) + p_\mu J_B(k,p)];$$
$$J_A(k,p) = J_B(p,k).$$

We find

$$J_A(k,p) = -\frac{1}{2\Delta^2}\{[p^2 - k\cdot p]K_0 + [p^2(k^2-m^2) - k\cdot p(p^2-m^2)]J_0 + 2p^2Q_1(p) - 2k\cdot pQ_1(k)\}. \qquad \text{(A10)}$$

In the massless case,

$$J_A^{d,0}(k,p) = -\frac{1}{2\Delta^2}\{[p^2-k\cdot p]K_0^{d,0} + p^2[k^2-k\cdot p]J_0^{d,0} + 2p^2Q_1^{d,0}(p) - 2k\cdot pQ_1^{d,0}(k)\}. \qquad \text{(A11)}$$

For small $\epsilon = 2-\ell$, Eq. (A10) reduces to the following expressions (see (A16, A17, 3.9(a,b,c)) of [6] and (39–44) of [7])

$$J_A^{4-2\epsilon,m} = \frac{1}{2\Delta^2}\{-(m^2p\cdot q + p^2k\cdot q)J_0^{4-2\epsilon,m} + 2k\cdot pL(k) - 2p^2L(p) - 2p\cdot q\mathcal{S}\},$$
$$J_A^{4-2\epsilon,0} = \frac{1}{2\Delta^2}\{p^2(k\cdot p - k^2)J_0^{4-2\epsilon,0} + 2(p^2-k\cdot p) \times \ln(-q^2) - 2p^2\ln(-p^2) + 2k\cdot p\ln(-k^2)\}, \qquad \text{(A12)}$$

Similar results for $d=3$ are:

$$J_A^{3,m} = \frac{1}{2\Delta^2}\{[p^2(k\cdot p - k^2) + m^2(p^2-k\cdot p)]J_0^{3,m} + 2(k\cdot p - p^2)I(q^2/4) + 4p^2I(p^2) - 4k\cdot pI(k^2)\},$$
$$J_A^{3,0} = \frac{1}{2\Delta^2}\left\{[p^2(k\cdot p - k^2)]J_0^{3,0} - \frac{2\pi}{\sqrt{-q^2}}(p^2-k\cdot p) + \frac{2\pi p^2}{\sqrt{-p^2}} - \frac{2\pi k\cdot p}{\sqrt{-k^2}}\right\}. \qquad \text{(A13)}$$

The first of these expressions coincides with Eq. (A5) of [11] and the second with Eq. (A3) of [10] after the appropriate change of notation.

5. The $J^{(2)}_{\mu\nu}$ integral

From symmetry considerations, the integral $J^{(2)}_{\mu\nu}$ of the list (12) can be expanded out as follows:

$$J^{(2)}_{\mu\nu} = \frac{i\pi^2}{2}\left[\frac{g_{\mu\nu}}{2\ell}K_0 + \left(k_\mu k_\nu - g_{\mu\nu}\frac{k^2}{2\ell}\right)J_C + \left(p_\mu k_\nu + k_\mu p_\nu - g_{\mu\nu}\frac{(k\cdot p)}{\ell}\right)J_D + \left(p_\mu p_\nu - g_{\mu\nu}\frac{p^2}{2\ell}\right)J_E\right].$$

The coefficients J_C, J_D, and J_E in the above expressions are:

$$\begin{aligned}
J_C(k,p) &= \frac{1}{4(\ell-1)\Delta^2}\{[(2\ell-2)(p^2-m^2)k\cdot p-(2\ell-1)(k^2-m^2)p^2]J_A-(p^2-m^2)p^2J_B \\
&\quad + [(2-\ell)p^2+(\ell-1)k\cdot p]K_0 - 4(\ell-1)k\cdot pQ_2(k)\}, \\
J_E(k,p) &= J_C(p,k), \\
J_D(k,p) &= \frac{1}{8(\ell-1)\Delta^2}\{[2\ell k\cdot p(k^2-m^2)+(2-2\ell)k^2(p^2-m^2)]J_A+[2\ell(p^2-m^2)k\cdot p+(2-2\ell)p^2(k^2-m^2)]J_B \\
&\quad - [(\ell-1)q^2+2k\cdot p]K_0 + 4(\ell-1)[k^2Q_2(k)+p^2Q_2(p)]\}.
\end{aligned} \tag{A14}$$

In the massless case we obtain the following expressions:

$$\begin{aligned}
J_C^{d,0}(k,p) &= \frac{1}{4(\ell-1)\Delta^2}\{[(2\ell-2)p^2k\cdot p-(2\ell-1)k^2p^2]J_A^{d,0}-p^4J_B^{d,0}+[(2-\ell)p^2+(\ell-1)k\cdot p]K_0^{d,0} \\
&\quad - 4(\ell-1)k\cdot pQ_2^{d,0}(k)\}, \\
J_D^{d,0}(k,p) &= \frac{1}{8(\ell-1)\Delta^2}\{[2\ell k\cdot pk^2+(2-2\ell)k^2p^2]J_A^{d,0}+[2\ell p^2k\cdot p+(2-2\ell)p^2k^2]J_B^{d,0} \\
&\quad - [(\ell-1)q^2+2k\cdot p]K_0^{d,0}+4(\ell-1)[k^2Q_2^{d,0}(k)+p^2Q_2^{d,0}(p)]\}.
\end{aligned} \tag{A15}$$

Then for small $\epsilon = 2-\ell$, we arrive at (compare Eq. (A16) with Eqs. (A18–A20) of [6] and Eqs. (49–51) of [7])

$$\begin{aligned}
J_C^{4-2\epsilon,m} &= \frac{1}{4\Delta^2}\{2p^2+2m^2k\cdot p/k^2-2k\cdot p\mathcal{S}+2(k\cdot p)(1-m^2/k^2)L(k)+[2k\cdot p(p^2-m^2)+3(m^2-k^2)p^2]J_A^{4-2\epsilon,m} \\
&\quad + p^2(m^2-p^2)J_B^{4-2\epsilon,m}\}, \\
J_D^{4-2\epsilon,m} &= \frac{1}{4\Delta^2}\{2k\cdot p[(k^2-m^2)J_A^{4-2\epsilon,m}+(p^2-m^2)J_B^{4-2\epsilon,m}-1]-[m^2-k^2\mathcal{S}+(k^2-m^2)L(k) \\
&\quad + k^2(p^2-m^2)J_A^{4-2\epsilon,m}]-[m^2-p^2\mathcal{S}+(p^2-m^2)L(p)+p^2(k^2-m^2)J_B^{4-2\epsilon,m}]\}.
\end{aligned} \tag{A16}$$

$$\begin{aligned}
J_C^{4-2\epsilon,0} &= \frac{1}{4\Delta^2}\{p^2[2k\cdot p-3k^2]J_A^{4-2\epsilon,0}-p^4J_B^{4-2\epsilon,0}+2p^2+2k\cdot p\ln(k^2/q^2)\}, \\
J_D^{4-2\epsilon,0} &= \frac{1}{4\Delta^2}\{[k^2(2k\cdot p-p^2]J_A^{4-2\epsilon,0}+[p^2(2k\cdot p-k^2]J_B^{4-2\epsilon,0}+k^2\ln(q^2/k^2)+p^2\ln(q^2/p^2)-2k\cdot p\}.
\end{aligned} \tag{A17}$$

Also for $d=3$,

$$\begin{aligned}
J_C^{3,m} &= \frac{1}{2\Delta^2}\left\{[(p^2-m^2)k\cdot p-2p^2(k^2-m^2)]J_A^{3,m}-p^2(p^2-m^2)J_B^{3,m}+\frac{2k\cdot p}{k^2}(m^2-k^2)I(k^2)\right. \\
&\quad \left. + (k\cdot p+p^2)I(q^2/4)-2m\frac{k\cdot p}{k^2}\right\}, \\
J_D^{3,m} &= \frac{1}{4\Delta^2}\{[k^2(3k\cdot p-p^2)-m^2(3k\cdot p-k^2)]J_A^{3,m}+[p^2(3k\cdot p-p^2)-m^2(3k\cdot p-p^2)]J_B^{3,m} \\
&\quad - 2(m^2-k^2)I(k^2)-2(m^2-p^2)I(p^2)-(k+p)^2I(q^2/4)+4m\},
\end{aligned} \tag{A18}$$

$$\begin{aligned}
J_C^{3,0} &= \frac{1}{2\Delta^2}\{p^2(k\cdot p-2k^2)J_A^{3,0}-p^4J_B^{3,0}-\pi k\cdot p/\sqrt{-k^2}+\pi p\cdot(p+k)/\sqrt{-q^2}\}, \\
J_D^{3,0} &= \frac{1}{4\Delta^2}\{k^2(3k\cdot p-p^2)J_A^{3,0}+p^2(3k\cdot p-p^2)J_B^{3,0}+\pi k^2/\sqrt{-k^2}+\pi p^2/\sqrt{-p^2}-\pi(k+p)^2/\sqrt{-q^2}\}.
\end{aligned} \tag{A19}$$

Equation (A18) is in agreement with (A6, A7) of [11] and Eq. (A19) with (A4) of [10].

6. The $I^{(0)}$ integral

The massive I_0 integral in arbitrary dimensions is given by

$$I^{(0)} = \frac{1}{2\chi}\{4(2-\ell)q^2(m^2+k\cdot p)J^{(0)} + (4m^2-q^2)Q_5(q) + [(q^2-2m^2)(p^2-m^2)+2m^2(k^2-m^2)]Q_6(p) + [(q^2-2m^2)(k^2-m^2)+2m^2(p^2-m^2)]Q_6(k)\}. \quad \text{(A20)}$$

Hence the massless case reduces to

$$I^{(0)}_{d,0} = \frac{i\pi^2}{k^2p^2}[(2-\ell)k\cdot pJ_0^{d,0} - k^2Q_3^{d,0}(k) - p^2Q_3^{d,0}(p) + q^2Q_3^{d,0}(q)]. \quad \text{(A21)}$$

When $\epsilon = 2-\ell$ is small, we have

$$I^{(0)}_{4-2\epsilon,m} = i\pi^2\left\{\frac{1}{\chi}\left[-q^2\mathcal{S} + p^2\frac{[(p^2-m^2)q^2+2m^2(k^2-p^2)]}{(p^2-m^2)^2}L(p) + k^2\frac{[(k^2-m^2)q^2-2m^2(k^2-p^2)]}{(k^2-m^2)^2}L(k)\right] - \frac{C-2}{(p^2-m^2)(k^2-m^2)}\right\},$$

$$I^{(0)}_{4-2\epsilon,0} = \frac{i\pi^2}{k^2p^2}\left[2-C+\ln\left(\frac{k^2p^2}{q^2m^2}\right)\right], \quad \text{(A22)}$$

whereas for $d=3$,

$$I^{(0)}_{3,m} = \frac{1}{\chi}\left\{q^2(m^2+k\cdot p)J^{(0)}_{3,m} + i\pi^2m\left[\frac{q^2(k^2-m^2)-(k^2-p^2)(k^2+m^2)}{(k^2-m^2)^2} + \frac{q^2(p^2-m^2)+(k^2-p^2)(p^2+m^2)}{(p^2-m^2)^2}\right]\right\},$$

$$I^{(0)}_{3,0} = \frac{k\cdot p}{k^2p^2}J^{(0)}_{3,m}. \quad \text{(A23)}$$

The first of Eq. (A23) agrees with (A8) of [11].

7. The $I^{(1)}_\mu$ integral

In analogy with the integral $J^{(1)}_\mu$, $I^{(1)}_\mu$ can be expanded out as

$$I^{(1)}_\mu = \frac{i\pi^2}{2}[k_\mu I_A(k,p) + p_\mu I_B(k,p)];$$
$$I_B(k,p) = I_A(p,k), \quad \text{(A24)}$$

where

$$I_A(k,p) = \frac{1}{2\Delta^2}\{[k\cdot p(p^2-m^2) - p^2(k^2-m^2)]I_0 + [k\cdot p - p^2]J_0 + 2k\cdot pQ_3(k) - 2p^2Q_3(p)\}.$$

In the massless case

$$I_A^{d,0}(k,p) = \frac{1}{2\Delta^2}\{p^2[k\cdot p-k^2]I_0^{d,m} + [k\cdot p - p^2]J_0^{d,m} + 2k\cdot pQ_3^{d,0}(k) - 2p^2Q_3^{d,0}(p)\}. \quad \text{(A25)}$$

Near 4 dimensions

$$I_A^{4-2\epsilon,m} = \frac{1}{2\Delta^2}\left\{-k\cdot qJ_0^{4-2\epsilon,m} - 2q^2(m^2-k^2)\times(k^2-k\cdot p)\mathcal{S}/\chi + \frac{2L(p)}{(m^2-p^2)}[p^2-k\cdot p + p^2q^2(k^2-m^2)(m^2+k\cdot p)/\chi] + 2k^2q^2(m^2+k\cdot p)L(k)/\chi\right\}, \quad \text{(A26)}$$

$$I_A^{4-2\epsilon,0} = \frac{1}{2\Delta^2}\left\{[k\cdot p - p^2]J_0^{4-2\epsilon,0} - 2[k\cdot p-k^2]\times\frac{\ln(-q^2)}{k^2} + 2\frac{k\cdot p}{k^2}\ln(-p^2) - 2\ln(-k^2)\right\}. \quad \text{(A27)}$$

Equation (A26) is in agreement with the expression (53) of [7]. Similar answers for $d=3$ are

$$I_A^{3,m} = \frac{2}{\Delta^2}\left\{[k\cdot p(p^2-m^2) - p^2(k^2-m^2)]\frac{I_0^{3,m}}{4} + [k\cdot p-p^2]\frac{J_0^{3,m}}{4} + \frac{mp^2}{(m^2-p^2)^2} - \frac{mk\cdot p}{(m^2-k^2)^2}\right\}, \quad \text{(A28)}$$

$$I_A^{3,0} = -\frac{\pi}{k^2\sqrt{-q^2k^2p^2}}. \quad \text{(A29)}$$

Equation (A28) agrees with (A11) of [11].

8. The $I^{(2)}_{\mu\nu}$ integral

The integral $I^{(2)}_{\mu\nu}$ of the list (12) may be decomposed as follows:

$$I^{(2)}_{\mu\nu} = \frac{i\pi^2}{2}\left[\frac{g_{\mu\nu}}{2\ell}J_0 + \left(k_\mu k_\nu - g_{\mu\nu}\frac{k^2}{2\ell}\right)I_C + \left(p_\mu k_\nu + k_\mu p_\nu - g_{\mu\nu}\frac{k\cdot p}{\ell}\right)I_D + \left(p_\mu p_\nu - g_{\mu\nu}\frac{p^2}{2\ell}\right)I_E\right], \quad \text{(A30)}$$

$$I_E(k,p) = I_C(p,k), \tag{A31}$$

where, in arbitrary dimensions,

$$I_C(k,p) = \frac{1}{4(\ell-1)\Delta^2}\{[(2\ell-2)(p^2-m^2)k\cdot p - (2\ell-1)(k^2-m^2)p^2]I_A - (p^2-m^2)p^2 I_B + [(2\ell-2)k\cdot p - (2\ell-1)p^2]J_A - p^2 J_B + 2p^2 J_0 + 4(\ell-1)k\cdot p Q_4(k)\}, \tag{A32}$$

$$I_D(k,p) = \frac{1}{8(\ell-1)\Delta^2}\{[2\ell(k^2-m^2)k\cdot p - (2\ell-2)\times(p^2-m^2)k^2]I_A + [2\ell(p^2-m^2)k\cdot p - (2\ell-2)(k^2-m^2)p^2]I_B + [2\ell k\cdot p - (2\ell-2)k^2]J_A - 4k\cdot p J_0 + [2\ell k\cdot p - (2\ell-2)p^2]J_B - 4(\ell-1)\times[p^2 Q_4(p) + k^2 Q_4(k)]\}. \tag{A33}$$

For the massless case, we arrive at the following simplified results

$$I_C^{d,0} = \frac{1}{4(\ell-1)\Delta^2}\{p^2[(2\ell-2)k\cdot p - (2\ell-1)k^2]I_A^{d,0} - p^4 I_B^{d,0} + [(2\ell-2)k\cdot p - (2\ell-1)p^2]J_A^{d,0} - p^2 J_B^{d,0} + 2p^2 J_0^{d,0} + 4(\ell-1)k\cdot p Q_4^{d,0}(k)\},$$
$$I_D^{d,0} = \frac{1}{4(\ell-1)\Delta^2}\{[\ell k\cdot p - (\ell-1)p^2](J_B^{d,0} + k^2 I_A^{d,0}) + [\ell k\cdot p - (\ell-1)k^2](J_A^{d,0} + p^2 I_B^{d,0}) - 2k\cdot p J_0^{d,0} - 2(\ell-1)[p^2 Q_4^{d,0}(p) + k^2 Q_4^{d,0}(k)]\}. \tag{A34}$$

Near 4 dimensions, these expressions yield

$$I_C^{4-2\epsilon,m} = \frac{1}{4\Delta^2}\left\{2p^2 J_0^{4-2\epsilon,m} - \frac{4k\cdot p}{k^2}\left(1+\frac{m^2 L(k)}{(k^2-m^2)}\right) + \{2k\cdot p - 3p^2\}J_A^{4-2\epsilon,m} - p^2 J_B^{4-2\epsilon,m} + [-2k\cdot p(m^2-p^2) + 3p^2(m^2-k^2)]I_A^{4-2\epsilon,m} + p^2(m^2-p^2)I_B^{4-2\epsilon,m}\right\},$$
$$I_D^{4-2\epsilon,m} = \frac{1}{4\Delta^2}\left\{-2(k\cdot p)J_0^{4-2\epsilon,m} + 2\left(1+\frac{m^2 L(k)}{(k^2-m^2)}\right) + 2\left(1+\frac{m^2 L(p)}{(k^2-m^2)}\right) + (2k\cdot p - k^2)J_A^{4-2\epsilon,m} + (2k\cdot p - p^2)J_B^{4-2\epsilon,m} + [k^2(m^2-p^2) - 2k\cdot p(m^2-k^2)]I_A^{4-2\epsilon,m} + [p^2(m^2-k^2) - 2k\cdot p(m^2-p^2)]I_B^{4-2\epsilon,m}\right\}, \tag{A35}$$

$$I_C^{4-2\epsilon,0} = \frac{1}{4\Delta^2}\left\{2p^2 J_0^{4-2\epsilon,0} - 4\frac{k\cdot p}{k^2} + (2k\cdot p - 3p^2)J_A^{4-2\epsilon,0} - p^2 J_B^{4-2\epsilon,0} - p^4 I_B^{4-2\epsilon,0} + p^2(2k\cdot p - 3k^2)I_A^{4-2\epsilon,0}\right\},$$
$$I_D^{4-2\epsilon,0} = \frac{1}{4\Delta^2}\{-2k\cdot p J_0^{4-2\epsilon,0} + 4 + (2k\cdot p - k^2)J_A^{4-2\epsilon,0} + (2k\cdot p - p^2)J_B^{4-2\epsilon,0} - k^2(p^2 - 2k\cdot p)I_A^{4-2\epsilon,0} - p^2(k^2 - 2k\cdot p)I_B^{4-2\epsilon,0}\}. \tag{A36}$$

Finally for $d = 3$, we have

$$I_C^{3,m} = \frac{1}{2\Delta^2}\bigg\{2p^2 J_0^{3,m} + [p^2(k\cdot p - 2k^2) - m^2(k\cdot p - 2p^2)]I_A^{3,m} - p^2(p^2 - m^2)I_B^{3,m} + (k\cdot p - 2p^2)J_A^{3,m} - p^2 J_B^{3,m} + \frac{2mk\cdot p}{k^2(m^2-k^2)} - \frac{2k\cdot p}{k^2}I(k^2)\bigg\},$$

$$I_D^{3,m} = \frac{1}{4\Delta^2}\bigg\{-4k\cdot p J_0^{3,m} + [k^2(3k\cdot p - p^2) - m^2(3k\cdot p - k^2)]I_A^{3,m} + [p^2(3k\cdot p - k^2) - m^2(3k\cdot p - p^2)]I_B^{3,m} + (3k\cdot p - k^2)J_A^{3,m} + (3k\cdot p - p^2)J_B^{3,m} - \frac{2m}{m^2-k^2} - \frac{2m}{m^2-p^2} + 2I(k^2) + 2I(p^2)\bigg\},$$

$$I_C^{3,0} = \frac{1}{2\Delta^2}\bigg\{p^2(k\cdot p - 2k^2)I_A^{3,0} - p^4 I_B^{3,0} + (k\cdot p - 2p^2)J_A^{3,0} - p^2 J_B^{3,0} - \frac{\pi k\cdot p}{k^2\sqrt{-k^2}} - \frac{4\pi p^2}{\sqrt{-k^2p^2q^2}}\bigg\},$$

$$I_D^{3,0} = \frac{1}{4\Delta^2}\bigg\{k^2(3k\cdot p - p^2)I_A^{3,0} + p^2(3k\cdot p - k^2)I_B^{3,0} + (3k\cdot p - k^2)J_A^{3,0} + (3k\cdot p - p^2)J_B^{3,0} + \frac{\pi}{\sqrt{-k^2}} + \frac{\pi}{\sqrt{-p^2}} + \frac{8k\cdot p}{\sqrt{-k^2p^2q^2}}\bigg\}. \tag{A37}$$

9. The $I^{(3)}_{\mu\alpha\beta}$ integral

The integral $I^{(3)}_{\mu\alpha\beta}$ comes contracted with vectors p^α and k^β so it is straightforward to show that

$$-8p^\alpha k^\beta I^{(3)}_{\mu\alpha\beta} = -2p^\alpha(k^2 - m^2)I^{(2)}_{\mu\alpha} - 2p^\alpha J^{(2)}_{\mu\alpha} + i\pi^2 p_\mu[Q_2(p) - (p^2 - m^2)Q_4(p)] - 2k^\beta(p^2 - m^2)I^{(2)}_{\mu\beta} - 2k^\beta J^{(2)}_{\mu\beta} + i\pi^2 k_\mu[Q_2(k) - (k^2 - m^2)Q_4(k)]. \tag{A38}$$

Therefore, we have $p^\alpha k^\beta I^{(3)}_{\mu\alpha\beta}$ in terms of integrals we already know.

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