Physically sound Hamiltonian formulation of the dynamical Casimir effect

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Recently [J. Haro and E. Elizalde, Phys. Rev. Lett. **97**, 130401 (2006)], a Hamiltonian formulation has been introduced in order to address some long-standing severe problems associated with the physical description of the dynamical Casimir effect at all times while the mirrors are moving. Here we present the complete calculation providing precise details, in particular, of the regularization procedure, which is decisive for the correct derivation of physically meaningful quantities. A basic difference when comparing with the results previously obtained by other authors is the fact that the motion force derived in our approach contains a reactive term—proportional to the mirrors' acceleration. This is of the essence in order to obtain particles with a positive energy at all times during the oscillation of the mirrors—while always satisfying the energy conservation law. A careful analysis of the interrelations among the different results previously obtained in the literature is then carried out. For simplicity, the specific case of a neutral scalar field in one dimension, with one or two partially transmitting mirrors (a fundamental proviso for the regularization issue), is considered in more detail, but our general method is shown to be generalizable, without essential problems (Sec. II of this paper), to fields of any kind in two and higher dimensions.

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I. INTRODUCTION

Moving mirrors modify the structure of the quantum vacuum, which manifests in the creation and annihilation of particles. Once the mirrors return to rest, a number of the produced particles will generically still remain, which can be interpreted as radiated particles. This flux has been calculated in the past in several situations by using different methods, such as averaging over fast oscillations [1,2], by multiple scale analysis [3], with the rotating wave approximation [4], with numerical techniques [5], and others [6]. Here we will be interested in the production of the particles and in their possible energy values, at all times while the mirrors are in movement. This is in no way a simple issue and a number of problems have recurrently appeared in the literature when trying to deal with it. To start with, it is far from clear which is the appropriate regularization to be used. Different authors tend to use different regularizations, forgetting sometimes about the need to carry out a proper (physical) renormalization procedure, in order to obtain actually meaningful quantities. Thus, it turns out that ordinarily, in the case of a single, perfectly reflecting mirror, the number of produced particles as well as their energies diverge at all times while the mirrors move. Several prescriptions have been used in order to obtain a well-defined energy; however, for some trajectories this finite energy is *not* a positive quantity and cannot be identified with the energy of the produced particles (see e.g. [7]).

Our approach relies on two very basic ingredients [8]. First, proper use of a Hamiltonian method, and second, the introduction of partially transmitting mirrors, which become transparent to very high frequencies. We will prove here, in this way, both that the number of created particles remains finite and also that their energies are always positive, for the whole trajectories corresponding to the mirrors' displacement. We will also calculate from first principles the radiation-reaction force that acts on the mirrors owing to the emission and absorption of the particles, and which is related with the field's energy through the ordinary energy conservation law. This implies, as a consequence, that the energy of the field at any time t equals, with the opposite sign, the work performed by the reaction force up to this time t [9,10]. Such force is usually split into two parts [11,12]: a dissipative force whose work equals minus the energy of the particles that remain [9], and a reactive force, which vanishes when the mirrors return to rest. We will also prove below that the radiation-reaction force calculated from the Hamiltonian approach for partially transmitting mirrors satisfies, at all times during the mirrors' oscillation, the energy conservation law and can naturally account for the creation of positive energy particles. Also, the dissipative part obtained within our procedure agrees with the one calculated by other methods, such as using the Heisenberg picture or other effective Hamiltonians. Note, however, that those methods have traditionally encountered problems with the reactive part, which, in general, yields a nonpositive energy that cannot therefore correspond to the particles created at any specific t.

The organization of the paper is as follows. In Sec. II we introduce first the canonical formulation underlying the whole procedure. In particular, we give the explicit expressions for the Hamiltonian and the corresponding energy. We do this by considering the Hamiltonian method for a

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neutral Klein-Gordon field in a cavity, generically in $(n+1)$ [although we will mainly work in $(3+1)$] dimensional space-time, with boundaries moving at a certain speed $v \ll c$. In Sec. III we deal with the case of a single partially transmitting mirror. We formulate the quantum theory based on the Hamiltonian approach and, successively, the quantum theory in the Heisenberg picture. We finish that section with a detailed comparison with early known results, e.g. those obtained with the method of Jaekel and Reynaud, and with the method of Barton and Calogeracos. In Sec. IV we study the more difficult case of two partially transmitting mirrors. In this situation, the part of the Hamiltonian that describes the interaction between the field and the mirrors depends on ϵ, which, in general, renders it quite difficult to describe this part. For that reason, the reactive part of the motion force can seldom be calculated. In any case, we prove here, in particular, that our dissipative part of the motion force exactly coincides with the dissipative force obtained by Jaekel and Reynaud [13]. Moreover, following the Hamiltonian approach we show that the problem of the negative energy that appears in the Davies-Fulling model can be resolved if partially transmitting mirrors are considered, which is, in our view, a very physical approach to the renormalization issue. The last section of the paper is devoted to a final discussion and conclusions.

II. CANONICAL FORMULATION OF THE PROBLEM

We consider a neutral massless scalar field in a cavity, Ω_t, and assume that the cavity boundary is at rest for all times $t \leq 0$ and returns to its initial position at time $t = T$, to remain there for a while. Its velocity will be given in terms of c, so that we will work with the dimensionless quantity $\epsilon = v/c \ll 1$. In a practical situation, such as the one featured in Ref. [14], this turns out to be of order 10^{-8} (more about that later).

The Lagrangian density of the field is

$$\mathcal{L}(t, \mathbf{x}) = \tfrac{1}{2}[(\partial_t \phi)^2 - |\nabla_{\mathbf{x}} \phi|^2], \quad \forall\ \mathbf{x} \in \Omega_t \subset \mathbb{R}^n, \quad \forall\ t \in \mathbb{R}. \tag{1}$$

If we use the canonical conjugated momentum

$$\xi(t, \mathbf{x}) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi(t, \mathbf{x}), \tag{2}$$

the energy density of the field is given by the expression

$$\mathcal{E}(t, \mathbf{x}) \equiv \xi \partial_t \phi - \mathcal{L}(t, \mathbf{x}) = \tfrac{1}{2}(\xi^2 + |\nabla_{\mathbf{x}} \phi|^2), \tag{3}$$

and the energy itself is

$$E(t; \epsilon) \equiv \int_{\Omega_t} d^n\mathbf{x} \mathcal{E}(t, \mathbf{x}). \tag{4}$$

A. Hamiltonian and energy

It is a well-known fact that the energy density does not generically coincide with the Hamiltonian density [15–17]. Here, to obtain the Hamiltonian density of the field, we follow the method discussed in Refs. [18,19]. First, we transform the moving boundary into a fixed one by performing a (nonconformal) change of coordinates

$$\mathcal{R} : (\bar{t}, \mathbf{y}) \rightarrow (t(\bar{t}, \mathbf{y}), \mathbf{x}(\bar{t}, \mathbf{y})) = (\bar{t}, \mathbf{R}(\bar{t}, \mathbf{y})), \tag{5}$$

that transform the domain Ω_t into a domain $\tilde{\Omega}$ which is independent of time. Making use of the coordinates $(\bar{t}, \mathbf{y})$, the action of the system behaves as

$$S = \int_{\mathbb{R}} \int_{\tilde{\Omega}} d^n\mathbf{y} d\bar{t}\, \tilde{\mathcal{L}}(\bar{t}, \mathbf{y}), \tag{6}$$

with $\tilde{\mathcal{L}}(\bar{t}, \mathbf{y}) \equiv J\mathcal{L}(\mathcal{R}(\bar{t}, \mathbf{y}))$, where we have introduced the Jacobian J of the coordinate change, defined by $d^n\mathbf{x} \equiv J d^n\mathbf{y}$.

Let us now consider the function $\tilde{\phi}$ given as $\tilde{\phi}(\bar{t}, \mathbf{y}) \equiv \sqrt{J}\phi(\mathcal{R}(\bar{t}, \mathbf{y}))$. Then, the canonical conjugated momentum is

$$\begin{aligned}\tilde{\xi}(\bar{t}, \mathbf{y}) &\equiv \frac{\partial \tilde{\mathcal{L}}}{\partial(\partial_{\bar{t}} \tilde{\phi})} \\ &= \partial_{\bar{t}} \tilde{\phi} - \frac{1}{2} \tilde{\phi} \partial_{\bar{t}}(\ln J) + \left\langle \mathbf{y}_t, \nabla_{\mathbf{y}} \tilde{\phi} - \frac{1}{2} \tilde{\phi} \nabla_{\mathbf{y}}(\ln J) \right\rangle \\ &= \sqrt{J} \partial_t \phi(\mathcal{R}(\bar{t}, \mathbf{y})),\end{aligned} \tag{7}$$

and, from here, the Hamiltonian density is obtained as

$$\begin{aligned}\tilde{\mathcal{H}}(\bar{t}, \mathbf{y}) &\equiv \tilde{\xi} \partial_{\bar{t}} \tilde{\phi} - \tilde{\mathcal{L}}(\bar{t}, \mathbf{y}) \\ &= \tfrac{1}{2}(\tilde{\xi}^2 + J|\nabla_{\mathbf{x}} \phi|^2) + \tilde{\xi}(\partial_{\bar{t}} \tilde{\phi} - \sqrt{J} \partial_t \phi).\end{aligned} \tag{8}$$

In the coordinates $(t, \mathbf{x})$, the Hamiltonian density is given by

$$\mathcal{H}(t, \mathbf{x}) \equiv \tilde{\mathcal{H}}(\mathcal{R}^{-1}(t, \mathbf{x})) \frac{d^3\mathbf{y}}{d^3\mathbf{x}} = \frac{1}{J} \tilde{\mathcal{H}}(\mathcal{R}^{-1}(t, \mathbf{x})). \tag{9}$$

Now, from expressions (2) and (7) we have that $\tilde{\xi}(\bar{t}, \mathbf{y}) = \sqrt{J}\xi(\mathcal{R}(\bar{t}, \mathbf{y}))$, and a straightforward calculation yields

$$\begin{aligned}\mathcal{H}(t, \mathbf{x}) = \mathcal{E}(t, \mathbf{x}) &+ \xi(t, \mathbf{x})\langle \partial_s \mathbf{R}(\mathcal{R}^{-1}(t, \mathbf{x})), \nabla_{\mathbf{x}} \phi(t, \mathbf{x})\rangle \\ &+ \tfrac{1}{2}\xi(t, \mathbf{x})\phi(t, \mathbf{x})\partial_s(\ln J)|_{\mathcal{R}^{-1}(t, \mathbf{x})}.\end{aligned} \tag{10}$$

B. A simple and explicit example

As a simple example, in the case of a single mirror following a prescribed trajectory $(t, \epsilon g(t))$ in a $1+1$ space-time, we can take $R(\bar{t}, y) = y + \epsilon g(\bar{t})$, and thus we explicitly get

$$\mathcal{H}(t, x) = \mathcal{E}(t, x) + \epsilon \dot{g}(t)\xi(t, x)\partial_x \phi(t, x). \tag{11}$$

III. SINGLE, PARTIALLY TRANSMITTING MIRROR

In this section we consider a single mirror in $1+1$ space-time following a prescribed trajectory $(t, \epsilon g(t))$. When the mirror is at rest, scattering of the field is described with the S matrix

$$S(\omega) = \begin{pmatrix} s(\omega) & r(\omega)e^{-2i\omega L} \\ r(\omega)e^{2i\omega L} & s(\omega) \end{pmatrix}, \tag{12}$$

where $x = L$ is the position of the mirror. The matrix S is supposed to be real in the temporal domain, as well as causal, unitary, and transparent to high frequencies [20]. More specifically, these conditions appear naturally as a consequence of the following considerations.

(1) Since the field is neutral, it should be

$$S(-\omega) = S^*(\omega). \tag{13}$$

In fact, the quantum field, in the Schrödinger picture, can be decomposed as

$$\hat{\phi}(x) = \sum_{j=R,L} \int_{\mathbb{R}} d\omega \hat{a}_{\omega,j} \tilde{g}_{\omega,j}(x),$$

where

$$\tilde{g}_{\omega,R}(x) = \frac{1}{\sqrt{4\pi\omega}}\{s(\omega)e^{-i\omega x}\theta(L-x) + (e^{-i\omega x} + r(\omega)e^{-2i\omega L}e^{i\omega x})\theta(x-L)\}, \tag{14}$$

$$\tilde{g}_{\omega,L}(x) = \frac{1}{\sqrt{4\pi\omega}}\{(e^{i\omega x} + r(\omega)e^{2i\omega L}e^{-i\omega x}) \times \theta(L-x) + s(\omega)e^{i\omega x}\theta(x-L)\} \tag{15}$$

are the right and the left incident modes, respectively [12]. As is the usual procedure in quantum field theory, when $\omega < 0$ one performs the change $\hat{a}_{\omega,j} \rightarrow \hat{a}^\dagger_{-\omega,j}$, and thus the field behaves as

$$\hat{\phi}(x) = \sum_{j=R,L} \int_0^\infty d\omega (\hat{a}_{\omega,j}\tilde{g}_{\omega,j}(x) + \hat{a}^\dagger_{\omega,j}\tilde{g}_{-\omega,j}(x)).$$

Now, since the field is neutral, it follows that $\tilde{g}_{-\omega,j}(x) = \tilde{g}^*_{\omega,j}(x)$, and finally, we conclude that

$$S(-\omega) = S^*(\omega).$$

This proves the statement.

(2) As a consequence of the commutation rule $[\hat{\phi}(t,x), \hat{\phi}(t,y)] = 0$, it follows that

$$S(\omega)S^\dagger(\omega) = Id. \tag{16}$$

This is straightforward and needs no further comment.

(3) And, as a consequence of the commutation rule $[\hat{\xi}(t,x), \hat{\phi}(t,y)] = -i\delta(x-y)$, we obtain the following causality condition:

$$\int_{\mathbb{R}} d\omega r(w)e^{-i\omega t} = \int_{\mathbb{R}} d\omega s(w)e^{-i\omega t} = 0, \quad \forall\ t < 0,$$

in a distribution sense, i.e.,

$$\lim_{\gamma\to 0}\int_{\mathbb{R}} d\omega r(w)\rho_\gamma(\omega)e^{-i\omega t} = \lim_{\gamma\to 0}\int_{\mathbb{R}} d\omega s(w)\rho_\gamma(\omega)e^{-i\omega t} = 0, \quad \forall\ t<0,$$

where ρ_γ is a frequency cutoff. This condition is satisfied when

$$S(\omega) \text{ is analytic for } \mathrm{Im}(\omega) > 0, \quad \text{and } s \text{ and } r \text{ are meromorphic functions.} \tag{17}$$

(4) A physical mirror is always transparent to high-enough incident frequencies, thus it must be

$$S(\omega) \rightarrow Id, \quad \text{when } |\omega| \rightarrow \infty. \tag{18}$$

A. Quantum theory based on the Hamiltonian approach

In order to obtain the quantum theory, we will work in the coordinates defined in the example above. In those coordinates the mirror is situated at the point $y = 0$ and the right and left incident modes are given by Eqs. (14) and (15), with $L = 0$. Then, in the coordinates (t, x), the instantaneous set of right and left incident eigenfunctions, which generalize the set used in the case of a perfectly reflecting mirror, is

$$g_{\omega,j}(t,x;\epsilon) \equiv \tilde{g}_{\omega,j}(x - \epsilon g(t)) \qquad j = R, L. \tag{19}$$

Note that, in general, we do not know explicitly the part of the Hamiltonian that describes the interaction between the field and the mirror. As a consequence, in order to obtain the quantum theory, the energy of the field $E(t) = \int_{\mathbb{R}} dx \mathcal{E}(t,x)$, which in the presence of a single mirror does not depend on ϵ, should be viewed as part of the free Hamiltonian of the system.

As is usual, working in the interaction picture, the field is expanded as follows:

$$\hat{\phi}_I(t,x;\epsilon) = \sum_{j=R,L}\int_0^\infty d\omega \hat{a}_{w,j}e^{-i\omega t}g_{\omega,j}(t,x;\epsilon) + \text{H.c.},$$

$$\hat{\xi}_I(t,x;\epsilon) = -i\sum_{j=R,L}\int_0^\infty d\omega\omega\hat{a}_{w,j}e^{-i\omega t}g_{\omega,j}(t,x;\epsilon) + \text{H.c.}, \tag{20}$$

where H.c. means, in each case, the Hermitian conjugate of the preceding expression. The quantum equation, in this picture, is given by

$$i\partial_t|\Phi\rangle = \frac{\epsilon\dot{g}(t)}{2}\left[\int_{\mathbb{R}} dx\hat{\xi}_I(t,x;\epsilon)\partial_x\hat{\phi}_I(t,x;\epsilon) + \text{H.c.}\right]|\Phi\rangle = \frac{\epsilon\dot{g}(t)}{2}\left[\int_{\mathbb{R}} dx\hat{\xi}_I(t,x;0)\partial_x\hat{\phi}_I(t,x;0) + \text{H.c.}\right]|\Phi\rangle + \mathcal{O}(\epsilon^2). \quad (21)$$

Now let $\mathcal{T}^t$ be the quantum evolution operator of the Schrödinger equation (21), and let $|0\rangle$ be the initial quantum state. Then, the average number of produced particles at time t is

$$\mathcal{N}(t) \equiv \sum_{j=R,L}\int_0^\infty d\omega\langle 0|(\mathcal{T}^t)^\dagger \hat{a}^\dagger_{\omega,j}\hat{a}_{\omega,j}\mathcal{T}^t|0\rangle, \quad (22)$$

and the dynamical energy at time t, that is, the energy of the created particles at time t, is obtained as

$$\langle\hat{E}(t)\rangle \equiv \sum_{j=R,L}\int_0^\infty d\omega\omega\langle 0|(\mathcal{T}^t)^\dagger \hat{a}^\dagger_{\omega,j}\hat{a}_{\omega,j}\mathcal{T}^t|0\rangle. \quad (23)$$

We should note that, during the movement of the mirror, the particles have sometimes been called quasiparticles, owing in part to the difficulties encountered in the past when trying to give them a physical sense (see [21]). Indeed, the concept of quasiparticle arises when the energy of the system depends on time, and this occurs when external conditions act on the quantum system. The eigenfunctions which happen to diagonalize this energy are the states corresponding to the quasiparticles (note that, when the boundary conditions cease to act, then the concept of quasiparticle coincides with that of particle). In the case when the external condition is a classical electromagnetic field, the creation of quasiparticles was studied in [21]. It was observed there that the mean number of created quasiparticles is finite, and that the semiclassical limit is a stochastic Poisson process (see [21], last reference). On the contrary, when the external conditions come from perfectly reflecting moving mirrors (by the way, nonfeasible physically), the average number of created quasiparticles diverges (see [19], second reference). This was precisely one of the main reasons to study, in this paper, the case of partially reflecting mirrors, which become transparent at high-enough frequencies. As we will see from Eq. (24) below, in this case, the average number of created quasiparticles becomes finite and experimentally checkable.

A simple but rather cumbersome calculation yields the following results:

$$\mathcal{N}(t) = \frac{\epsilon^2}{2\pi^2}\int_0^\infty\int_0^\infty \frac{d\omega d\omega'\omega\omega'}{(\omega+\omega')^2}|\widehat{\dot{g}\theta_t}(\omega+\omega')|^2 \times (|r(\omega)+r^*(\omega')|^2 + |s(\omega)-s^*(\omega')|^2) + \mathcal{O}(\epsilon^4), \quad (24)$$

$$\langle\hat{E}(t)\rangle = \frac{\epsilon^2}{4\pi^2}\int_0^\infty\int_0^\infty \frac{d\omega d\omega'\omega\omega'}{\omega+\omega'}|\widehat{\dot{g}\theta_t}(\omega+\omega')|^2 \times (|r(\omega)+r^*(\omega')|^2 + |s(\omega)-s^*(\omega')|^2) + \mathcal{O}(\epsilon^4), \quad (25)$$

where θ_t is the Heaviside step function at point t, e.g., $\theta_t(\tau) = \theta(t-\tau)$, and $\hat{f}(\omega) \equiv \int_{\mathbb{R}} d\tau f(\tau)e^{i\omega\tau}$ is the Fourier transform of the function f. These two quantities are, in general, convergent. However, for the Davies-Fulling model [7]—that is, in the case of a single perfectly reflecting mirror—such quantities are divergent when the mirror moves or when the displacement exhibits some type of discontinuity [17,22].

In this situation, in order to obtain a finite energy, several authors have used different regularization techniques [9–11]. For example, using a frequency cutoff $e^{-\omega\gamma}$ with $0 < \gamma \ll 1$, the regularized energy is (see Ref. [19])

$$\langle\hat{E}(t;\gamma)\rangle = \frac{\epsilon^2}{6\pi}\left[\frac{\dot{g}^2(t)}{\pi\gamma} - \ddot{g}(t)\dot{g}(t) + \int_0^t \ddot{g}^2(\tau)d\tau\right] + \mathcal{O}(\epsilon^4). \quad (26)$$

Thus, imposing the kinetic energy of the moving boundary to be

$$\frac{1}{2}\left(M_{\text{exp}} - \frac{1}{3\pi^2\gamma}\right)\epsilon^2\dot{g}^2(t), \quad (27)$$

where M_{exp} is the experimental mass of the mirror, those authors conclude that the renormalized dynamical energy, namely, $\hat{E}_R(t)$, is (see Refs. [7,9–11])

$$\langle\hat{E}_R(t)\rangle \equiv \frac{\epsilon^2}{6\pi}\left[-\ddot{g}(t)\dot{g}(t) + \int_0^t \ddot{g}^2(\tau)d\tau\right] + \mathcal{O}(\epsilon^4). \quad (28)$$

However, when $t \leq \delta$, with $0 < \delta \ll 1$, then such renormalized energy is negative. This shows that, when the mirror moves, the renormalized energy cannot be considered as the energy of the produced particles at time t [see also the paragraph immediately after Eq. (4.5) in Ref. [7]].

From our viewpoint, such a meaningless result is just due to the fact that a perfect reflecting mirror is used in the derivation, which is not physically feasible at any price. Physical mirrors will *always* obey a transparency condition of the kind proposed here (18), and then it comes out for free that the average number of produced particles and the dynamical energy turn out to be *well defined* and are both *positive* quantities.

We have also calculated the radiation-reaction force. For a single mirror this force is the difference between the energy density of the evolved vacuum state on both sides of the mirror, namely,

$$\langle\hat{F}_{\text{Ha}}(t)\rangle \equiv \lim_{\delta\to 0}(\langle 0|(\mathcal{T}^t)^\dagger\hat{\mathcal{E}}(t,\epsilon g(t)-|\delta|)\mathcal{T}^t|0\rangle - \langle 0|(\mathcal{T}^t)^\dagger\hat{\mathcal{E}}(t,\epsilon g(t)+|\delta|)\mathcal{T}^t|0\rangle), \quad (29)$$

where the subindex Ha means that the radiation-reaction force has been calculated in the Hamiltonian approach. We obtain

$$\langle \hat{F}_{\mathrm{Ha}}(t)\rangle = -\frac{\epsilon}{2\pi^2}\int_0^\infty\int_0^\infty \frac{d\omega d\omega'\omega\omega'}{\omega+\omega'} \times \mathrm{Re}[e^{-i(\omega+\omega')t}\widehat{\dot{g}\theta_t}(\omega+\omega')](|r(\omega)+r^*(\omega')|^2 + |s(\omega)-s^*(\omega')|^2) + \mathcal{O}(\epsilon^2). \tag{30}$$

An important remark is in order here. Note that, as a consequence of the energy conservation law, the dynamical energy at time t is equal to minus the work performed by the radiation-reaction force up to time t (see Refs. [9,10]). This law is naturally satisfied if we use the Hamiltonian approach. It is then clear that (25) and (30) are related through the formula

$$\langle \hat{E}(t)\rangle = -\epsilon\int_0^t \langle \hat{F}_{\mathrm{Ha}}(\tau)\rangle \dot{g}(\tau)d\tau. \tag{31}$$

B. Quantum theory in the Heisenberg picture

Following the method of [23], we have calculated the "in" modes when the mirror describes the prescribed trajectory $(t, \epsilon g(t))$. Using lightlike coordinates, $u \equiv t - x$ and $v \equiv t + x$, the "in" modes can be written as

$$\phi^{\mathrm{in}}_{\omega,R}(u,v;0) = \frac{1}{\sqrt{4\pi\omega}}\{[s(\omega)e^{-i\omega v} - A_\omega(v;0)]\theta(\epsilon g(t)-x) + [e^{-i\omega v} + r(\omega)e^{-i\omega u} - B_\omega(u;0)] \times\theta(x-\epsilon g(t))\} + \mathcal{O}(\epsilon^2), \tag{32}$$

$$\phi^{\mathrm{in}}_{\omega,L}(u,v;0) = \frac{1}{\sqrt{4\pi\omega}}\{[e^{-i\omega u} + r(\omega)e^{-i\omega v} + B_\omega(v;0)] \times\theta(\epsilon g(t)-x) + [s(\omega)e^{-i\omega u} + A_\omega(u;0)] \times\theta(x-\epsilon g(t))\} + \mathcal{O}(\epsilon^2), \tag{33}$$

where

$$A_\omega(y;\gamma) = \frac{i\epsilon\omega}{2\pi}\int_{\mathbb{R}} d\omega' e^{i\omega' y}\hat{g}(-\omega-\omega')[s^*(\omega') - s(\omega)]e^{-\gamma|\omega'|},$$

$$B_\omega(y;\gamma) = \frac{i\epsilon\omega}{2\pi}\int_{\mathbb{R}} d\omega' e^{i\omega' y}\hat{g}(-\omega-\omega')[r^*(\omega') + r(\omega)]e^{-\gamma|\omega'|}.$$

The average number of produced particles after the mirror returns to rest is [24]

$$\mathcal{N}(t\geq T) = \sum_{i,j=R,L}\int_0^\infty\int_0^\infty d\omega d\omega' |(\phi^{\mathrm{out}}_{\omega,i}, \phi^{\mathrm{in}*}_{\omega',j})|^2, \tag{34}$$

where $(F,G) \equiv i\int_{\mathbb{R}} dx(F^*\partial_t G - G\partial_t F^*)$. To obtain an explicit result, we calculate the Bogoliubov coefficients $\{\phi^{\mathrm{out}}_{\omega,i}, \phi^{\mathrm{in}*}_{\omega',j}\}$ in the null future infinity $\mathcal{I}^+$, because the "outgoing" modes are a very simple expression in $\mathcal{I}^+$. The final result is (see Ref. [13])

$$\mathcal{N}(t\geq T) = \frac{\epsilon^2}{2\pi^2}\int_0^\infty\int_0^\infty d\omega d\omega'\omega\omega'|\hat{g}(\omega+\omega')|^2(|r(\omega) + r^*(\omega')|^2 + |s(\omega)-s^*(\omega')|^2) + \mathcal{O}(\epsilon^4). \tag{35}$$

From this expression it is not difficult to calculate the number of particles at time t. We only need to consider the function

$$\tilde{g}_t(s) \equiv \begin{cases} g(s), & \text{when } s\leq t, \\ g(t), & \text{when } s\geq t, \end{cases}$$

because $\hat{\tilde{g}}_t(\omega+\omega') = \frac{1}{\omega+\omega'}\widehat{\dot{g}\theta_t}(\omega+\omega')$. Then, inserting this expression into (35) we obtain formula (24).

The radiation-reaction force calculated in the Heisenberg picture, namely, $\langle \hat{F}_H(t)\rangle$, is the difference between the energy density of the in vacuum state on the left and on the right sides of the mirror. A simple calculation shows that the energy density of the in vacuum [24],

$$\langle\hat{\mathcal{E}}(t,x)\rangle = \sum_{j=R,L}\int_0^\infty d\omega(\partial_u\phi^{\mathrm{in}}_{\omega,j}(u,v;0)\partial_u\phi^{\mathrm{in}*}_{\omega,j}(u,v;0) + \partial_v\phi^{\mathrm{in}}_{\omega,j}(u,v;0)\partial_v\phi^{\mathrm{in}*}_{\omega,j}(u,v;0)),$$

on both sides of the mirror, is

$$\langle\hat{\mathcal{E}}(t,x)\rangle = \int_0^\infty d\omega\omega \pm \frac{i\epsilon}{8\pi^2}\int_{\mathbb{R}^2} d\omega d\omega'\omega\omega'\hat{g}(\omega+\omega') \times\chi(\omega)(1+r(\omega)r(\omega') - s(\omega)s(\omega')) \times e^{-i(\omega+\omega')v}\theta(\pm(\epsilon g(t)-x)) + \mathcal{O}(\epsilon^2), \tag{36}$$

where $\chi(\omega) \equiv \theta(\omega) - \theta(-\omega)$ is the sign function. Note that the term of order ϵ is ill defined, because the function $\omega\omega'\hat{g}(\omega+\omega')(1+r(\omega)r(\omega')-s(\omega)s(\omega'))$ is not integrable in $\mathbb{R}^2$ and, to obtain a well-defined quantity, appropriate regularization is needed.

If we define the regularized energy by

$$\langle\hat{\mathcal{E}}(t,x;\gamma)\rangle \equiv \sum_{j=R,L}\int_0^\infty d\omega e^{-\gamma\omega}(\partial_u\phi^{\mathrm{in}}_{\omega,j}(u,v;\gamma) \times\partial_u\phi^{\mathrm{in}*}_{\omega,j}(u,v;\gamma) + \partial_v\phi^{\mathrm{in}}_{\omega,j}(u,v;\gamma)\partial_v\phi^{\mathrm{in}*}_{\omega,j}(u,v;\gamma)), \tag{37}$$

then the regularized motion force, in the Heisenberg picture, is

$$\langle\hat{F}_H(t;\gamma)\rangle = \frac{i\epsilon}{8\pi^2}\int_{\mathbb{R}^2} d\omega d\omega'\omega\omega'\hat{g}(\omega+\omega')(\chi(\omega)+\chi(\omega')) \times(1+r(\omega)r(\omega')-s(\omega)s(\omega'))e^{-\gamma(|\omega|+|\omega'|)} \times e^{-i(\omega+\omega')t} + \mathcal{O}(\epsilon^2). \tag{38}$$

This integral is convergent and already cutoff *independent*. Thus, a natural definition of the renormalized radiation-reaction force is

$$\langle \hat{F}_{H,\mathrm{ren}}(t)\rangle = \frac{i\epsilon}{8\pi^2}\int_{\mathbb{R}^2} d\omega d\omega'\omega\omega'\hat{g}(\omega+\omega')(\chi(\omega)+\chi(\omega')) \times (1+r(\omega)r(\omega')-s(\omega)s(\omega')) \times e^{-i(\omega+\omega')t} + \mathcal{O}(\epsilon^2). \tag{39}$$

Two more remarks are in order. In general, expression (39) disagrees with the radiation-reaction force (30) obtained using the Hamiltonian approach. A detailed discussion of this point will be given in the next section.

It is also important to stress that *other* definitions of the regularized energy density give rise to *different* motion forces. It is, of course, also possible to obtain the radiation-reaction force (30) within a specific, tailored regularization. To prove such statement, let us consider, for a moment, the usual case in the literature of a perfectly reflecting mirror. In the Heisenberg picture the annihilation operators on the left side of the mirror are

$$\hat{a}_{\omega,L}(t) = \hat{a}_{\omega,L}e^{-i\omega t} + e^{-i\omega t}\bigg[\epsilon\int_0^\infty d\omega'\widehat{\dot{g}\theta_t}(\omega+\omega')\hat{a}^\dagger_{\omega',L} \times \frac{\sqrt{\omega\omega'}}{\pi}\frac{1}{\omega+\omega'} - \epsilon\int_0^\infty d\omega'\widehat{\dot{g}\theta_t}(\omega-\omega')\hat{a}_{\omega',L} \times \frac{\sqrt{\omega\omega'}}{\pi}\mathcal{P}\bigg(\frac{1}{\omega-\omega'}\bigg)\bigg] + \mathcal{O}(\epsilon^2),$$

where $\mathcal{P}$ denotes Cauchy's principal value. Then, on the left side of the mirror, the field—in the Heisenberg picture—can be written as follows,

$$\hat{\phi}_H(t,x;\epsilon) = \int_0^\infty d\omega(\hat{a}_{\omega,L}(t)g_{\omega,L}(t,x;\epsilon) + \hat{a}^\dagger_{\omega,L}(t)g^*_{\omega,L}(t,x;\epsilon)),$$

and, after some algebra, we get

$$\hat{\phi}_H(t,x;\epsilon) = \int_0^\infty d\omega(\hat{a}_{\omega,L}\varphi_{\omega,L}(x,t;0) + \hat{a}^\dagger_{\omega,L}\varphi^*_{\omega,L}(x,t;0)),$$

where

$$\varphi_{\omega,L}(t,x;\gamma) = \frac{ie^{-i\omega t}}{\sqrt{\pi\omega}}[\sin(\omega x) - \epsilon g(t)\cos(\omega x)] - \frac{i\epsilon\sqrt{\omega}}{\pi\sqrt{\pi}}\int_{\mathbb{R}} d\omega' d\tau e^{i\omega' t}\widehat{\dot{g}\theta_t}(-\omega-\omega') \times \mathcal{P}\bigg(\frac{1}{\omega+\omega'}\bigg)\sin(\omega' x)e^{-\gamma|\omega'|} + \mathcal{O}(\epsilon^2).$$

Using the formula

$$\int_{\mathbb{R}} d\omega'\mathcal{P}\bigg(\frac{1}{\omega+\omega'}\bigg)e^{i\omega'(u-\tau)} = \pi i e^{-i\omega(u-\tau)}\chi(u-\tau),$$

we easily find that

$$\phi^{\mathrm{in}}_{\omega,L}(t,x;0) = \frac{i}{\sqrt{\pi\omega}}\sin(\omega x)e^{-i\omega t} - 2i\epsilon\sqrt{\frac{\omega}{4\pi}}e^{-i\omega(t-x)}g(t-x) = \varphi_{\omega,L}(t,x;0),$$

and making the changes $\epsilon\to-\epsilon$ and $x\to -x$, we obtain the expression for the right modes. That is, we have obtained an equivalent expression as for the in modes.

Then, defining the regularized energy from this new expression of the in modes,

$$\langle\hat{\mathcal{E}}(t,x;\gamma)\rangle \equiv \frac{1}{2}\sum_{j=R,L}\int_0^\infty d\omega e^{-\gamma\omega}(\partial_t\varphi_{\omega,j}(t,x;\gamma) \times \partial_t\varphi^*_{\omega,j}(t,x;\gamma) + \partial_x\varphi_{\omega,j}(t,x;\gamma)\partial_x\varphi^*_{\omega,j}(t,x;\gamma)),$$

we obtain the following regularized motion force,

$$\langle\hat{F}_H(t;\gamma)\rangle = -\frac{2\epsilon}{\pi^2}\int_0^\infty\int_0^\infty\frac{d\omega d\omega'\omega\omega'}{\omega+\omega'} \times \mathrm{Re}[e^{-i(\omega+\omega')t}\widehat{\dot{g}\theta_t}(\omega+\omega')] \times e^{-\gamma(|\omega|+|\omega'|)} + \mathcal{O}(\epsilon^2),$$

which agrees with (30), for the case of a perfectly reflecting mirror.

The fact that different (sometimes *a priori* quite reasonable) regularization procedures may lead to very different finite results is well known [25], as is also the ensuing consequence that there is generically no physics associated with an arbitrary regularization prescription, which calls for a subsequent renormalization procedure to establish contact with the physical world. This is more so when one deals with plain mathematical and physically unrealistic boundary conditions, as we have learned from a number of situations involving the ordinary Casimir effect too [26]. However, this essential point seems to have been put aside, at least to some extent, when dealing with the problem at hand, maybe due to the intrinsic mathematical difficulty of this issue here. This was the motivation for the last explicit calculation above, which we consider a sufficiently clarifying exercise that exhibits what is going on here. In the following we will proceed with a strict comparison of our results with those of other authors that have previously appeared in the literature on the subject.

C. Comparison with known results

1. The method of Jaekel and Reynaud

To study the radiation-reaction force, these authors [27] consider the following effective Hamiltonian,

$$\hat{H}_{\mathrm{JR}} \equiv -\epsilon g(t)\hat{F}(t), \tag{40}$$

where $(t, \epsilon g(t))$ is the trajectory of the mirror and $\hat{F}(t) \equiv \lim_{\delta\to 0}(\hat{\mathcal{E}}(t, -|\delta|) - \hat{\mathcal{E}}(t, +|\delta|))$ is the force operator at the point $x = 0$. A simple calculation yields

$$\langle \hat{F}_{\rm JR}(t)\rangle = -\frac{\epsilon}{2\pi^2}\int_0^\infty\int_0^\infty d\omega d\omega'\omega\omega' \times \mathrm{Im}[e^{-i(\omega+\omega')t}\widehat{g\theta_t}(\omega+\omega')] \times (|r(\omega)+r^*(\omega')|^2 + |s(\omega)-s^*(\omega')|^2) + \mathcal{O}(\epsilon^2). \tag{41}$$

Integrating by parts, we obtain

$$\langle \hat{F}_{\rm JR}(t)\rangle = \langle \hat{F}_{\rm Ha}(t)\rangle + \frac{\epsilon g(t)}{2\pi^2}\int_0^\infty\int_0^\infty \frac{d\omega d\omega'\omega\omega'}{\omega+\omega'}(|r(\omega) + r^*(\omega')|^2 + |s(\omega)-s^*(\omega')|^2) + \mathcal{O}(\epsilon^2),$$

which shows that expression (41) is divergent while the mirror moves. To obtain a regularized quantity we write (41) as follows,

$$\langle \hat{F}_{\rm JR}(t)\rangle \equiv \frac{i\epsilon}{8\pi^2}\int_{-\infty}^t d\tau g(\tau)\int_{\mathbb{R}^2} d\omega d\omega'\omega\omega'(\chi(\omega) + \chi(\omega'))[(1 + r(\omega)r(\omega') - s(\omega)s(\omega')) + (1 + r^*(\omega)r^*(\omega') - s^*(\omega)s^*(\omega'))] \times e^{-i(\omega+\omega')(t-\tau)} + \mathcal{O}(\epsilon^2),$$

and we define the regularized motion force by

$$\langle \hat{F}_{\rm JR}(t;\gamma)\rangle \equiv \frac{i\epsilon}{8\pi^2}\int_{-\infty}^t d\tau g(\tau)\int_{\mathbb{R}^2} d\omega d\omega'\omega\omega'(\chi(\omega) + \chi(\omega'))[(1 + r(\omega)r(\omega') - s(\omega)s(\omega'))\rho_\gamma(\omega,\omega') + (1 + r^*(\omega)r^*(\omega') - s^*(\omega)s^*(\omega'))\rho^*_\gamma(\omega,\omega')]e^{-i(\omega+\omega')(t-\tau)} + \mathcal{O}(\epsilon^2), \tag{42}$$

where the cutoff $\rho_\gamma(\omega,\omega')$ is a meromorphic function, analytic for $\mathrm{Im}(\omega) > 0$ and $\mathrm{Im}(\omega') > 0$.

Now, applying the causality of the S matrix [Eq. (17)], and making $\gamma \to 0$, an easy calculation leads us to the expression (39). For this reason, defining the renormalized radiation-reaction force through the formula (39), i.e.,

$$\langle \hat{F}_{\rm JR,ren}(t)\rangle \equiv \langle \hat{F}_{H,\rm ren}(t)\rangle, \tag{43}$$

one does conclude that the method of Jaekel and Reynaud is equivalent to the quantum theory in the Heisenberg picture.

Note also that

$$\epsilon\int_{\mathbb{R}} dt\langle \hat{F}_{\rm Ha}(t)\rangle\dot{g}(t) = \epsilon\int_{\mathbb{R}} dt\langle \hat{F}_{\rm JR,ren}(t)\rangle\dot{g}(t).$$

This identity proves that the dissipative parts of $\langle \hat{F}_{\rm Ha}(t)\rangle$ and $\langle \hat{F}_{\rm JR,ren}(t)\rangle$ always agree.

On the other hand, in several situations, the reactive part actually disagrees. For example, if $r(w) = -\frac{i\alpha}{\omega+i\alpha}$, and $s(w) = \frac{\omega}{\omega+i\alpha}$, with $\alpha > 0$, the following relation holds:

$$\langle \hat{F}_{\rm Ha}(t)\rangle = -\frac{\alpha\epsilon}{2\pi}\ddot{g}(t) + \langle \hat{F}_{\rm JR,ren}(t)\rangle, \tag{44}$$

where

$$\langle \hat{F}_{\rm JR,ren}(t)\rangle = \frac{\alpha\epsilon}{\pi}\int_1^\infty dz\int_{-\infty}^t d\tau\left(\frac{1}{z^2} - \frac{1}{z^3}\right)e^{-\alpha z(t-\tau)}\ddot{g}(\tau). \tag{45}$$

That is, both motion forces differ in a reactive term. Note also that, during the oscillation of the mirror, the work done by the motion force $\langle \hat{F}_{\rm JR,ren}(t)\rangle$ is *not* a negative quantity. Consequently, from the previous remark it follows that the dynamical energy is not positive, and therefore a seemingly meaningless result is obtained since, in our opinion, the dynamical energy is to be interpreted as the energy carried out by the produced particle. To avoid such difficulty the reactive term $-\frac{\alpha\epsilon}{2\pi}\ddot{g}(t)$ should not be arbitrarily suppressed but, on the contrary, has to be duly taken into account. This saves the day and endows the whole picture with physical sense, as explained in the previous section.

2. The method of Barton and Calogeracos

In Ref. [12] (see also [28,29]), these authors study the particular case $r(w) = -\frac{i\alpha}{\omega+i\alpha}$, and $s(w) = \frac{\omega}{\omega+i\alpha}$ with $\alpha > 0$. In such a situation, the interaction between the field and the mirror can be described by the Lagrangian density

$$\tfrac{1}{2}[(\partial_t\phi)^2 - (\partial_x\phi)^2] - \alpha\phi^2\delta(x - \epsilon g(t)). \tag{46}$$

Following the method discussed in Sec. II, we have obtained the quantum Hamiltonian

$$\hat{H}(t) = \int_{\mathbb{R}} dx\hat{\mathcal{E}}(t,x) + \alpha\hat{\phi}^2(t,\epsilon g(t)) + \frac{\epsilon\dot{g}(t)}{2} \times \left[\int_{\mathbb{R}} dx\hat{\xi}(t,x)\partial_x\hat{\phi}(t,x) + \text{H.c.}\right]. \tag{47}$$

Now, inserting (20) into the integral $\int_{\mathbb{R}} dx\hat{\mathcal{E}}(t,x) + \alpha\hat{\phi}^2(t,\epsilon g(t))$, we get

$$\int_{\mathbb{R}} dx\hat{\mathcal{E}}(t,x) + \alpha\hat{\phi}^2(t,\epsilon g(t)) = \sum_{j=L,R}\int_0^\infty d\omega\omega \times \left(\hat{a}^\dagger_{\omega,j}\hat{a}_{\omega,j} + \frac{1}{2}\right). \tag{48}$$

We thus conclude that the quantum equation in the interaction picture is given by expression (21). And, consequently, for these reflection and transmission coefficients, the authors would obtain the same formulas (24), (25), and (30).

It should be noted, however, that two relevant differences exist between their results and the ones we have derived here previously.

(1) In order to obtain the quantum equation, Barton and Calogeracos make a unitary transformation which does not seem to be easily generalizable to the case of two moving mirrors. In our case this can be done without the least problem, as we shall show below.

(2) In the above-mentioned paper, Ref. [12], the authors use the same technique for mass renormalization that had been employed by Barton and Eberlein in Refs. [10,11] for the case of a completely reflecting mirror, in order to eliminate the reactive part of the motion force. However, within such renormalization, the energy of the field is not a positive quantity for all time t and, consequently, the concept of particle is again ill defined during the oscillation of the mirror.

IV. TWO PARTIALLY TRANSMITTING MIRRORS

In this section we consider the situation where we have two moving mirrors which follow prescribed trajectories $(t, L_j(t;\epsilon))$, where $L_j(t;\epsilon) \equiv L_j + \epsilon g_j(t)$, with $j = 1, 2$, and we assume that $L_1(t;\epsilon) < L_2(t;\epsilon)$, $\forall\ t \in \mathbb{R}$. In this case it is impossible, in practice, to work within the Heisenberg picture, because it is indeed very complicated to obtain the in and out mode functions in the presence of the two moving mirrors. Alternatively, with the purpose to calculate the dissipative part of the motion force, the number of radiated particles, and their energy, one can use the approach due to Jaekel and Reynaud and based on the effective Hamiltonian $\hat{H}_{J,R} \equiv -\sum_{j=1,2}\epsilon g_j(t)\hat{F}_j(t)$, where $\hat{F}_j(t) \equiv \lim_{\delta\to 0}(\hat{\mathcal{E}}(t, L_j - |\delta|) - \hat{\mathcal{E}}(t, L_j + |\delta|))$ is the force operator at the point $x = L_j$ (see Refs. [13,27]). However, this method is not useful in order to calculate the reactive part of the motion force or the dynamical energy while the mirrors are in movement.

To obtain those last quantities we are led to use, once more, the Hamiltonian approach. In this case, if we consider the change

$$R(\bar{t}, y) = \frac{1}{L_2 - L_1}[L_2(\bar{t};\epsilon)(y - L_1) + L_1(\bar{t};\epsilon)(L_2 - y)],$$

the Hamiltonian density of the field is

$$\mathcal{H}(t,x) = \mathcal{E}(t,x) + \sum_{j=1,2}\frac{(-1)^j \dot{L}_j(t;\epsilon)\xi(t,x)}{L_2(t;\epsilon) - L_1(t;\epsilon)} \times\left[\partial_x\phi(t,x)(x - \bar{L}_j(t;\epsilon)) + \frac{1}{2}\phi(t,x)\right], \quad (49)$$

where $\bar{L}_{\binom{1}{2}}(t;\epsilon) \equiv L_{\binom{2}{1}}(t;\epsilon)$. In the coordinates $(\bar{t}, y)$ the set of right and left incident modes can be obtained from Eqs. (8) and (9) of Ref. [13]. We find

$$\begin{aligned}\tilde{g}_{\omega,R}(y) = \frac{1}{\sqrt{4\pi\omega}}\Bigg\{&\frac{s_1(\omega)s_2(\omega)}{d(\omega)}e^{-iwy}\theta(L_1 - y) \\ &+ \left(\frac{s_2(\omega)}{d(\omega)}e^{-iwy} + \frac{r_1(\omega)s_2(\omega)}{d(\omega)}e^{iw(y-2L_1)}\right) \\ &\times\theta(y - L_1)\theta(L_2 - y) \\ &+ \Bigg[e^{-iwy} + \Bigg(r_2(\omega)e^{-2iwL_2} \\ &+ \frac{r_1(\omega)s_2^2(\omega)}{d(\omega)}e^{-2iwL_1}\Bigg)e^{iwy}\Bigg]\theta(y - L_2)\Bigg\}, \end{aligned} \quad (50)$$

$$\begin{aligned}\tilde{g}_{\omega,L}(y) = \frac{1}{\sqrt{4\pi\omega}}\Bigg\{&\Bigg[e^{iwy} + \Bigg(r_1(\omega)e^{2iwL_1} \\ &+ \frac{r_2(\omega)s_1^2(\omega)}{d(\omega)}e^{2iwL_2}\Bigg)e^{-iwy}\Bigg]\theta(L_1 - y) \\ &+ \left(\frac{s_1(\omega)}{d(\omega)}e^{iwy} + \frac{r_2(\omega)s_1(\omega)}{d(\omega)}e^{-iw(y-2L_2)}\right) \\ &\times\theta(y - L_1)\theta(L_2 - y) \\ &+ \frac{s_1(\omega)s_2(\omega)}{d(\omega)}e^{iwy}\theta(y - L_2)\Bigg\}, \end{aligned} \quad (51)$$

where $d(\omega) \equiv 1 - r_1(\omega)r_2(\omega)e^{2i\omega(L_2 - L_1)}$. Then, the instantaneous set of right and left incident eigenfunctions in the coordinates (t, x) is

$$\begin{aligned} g_{\omega,R}(t,x;\epsilon) &= \sqrt{\frac{L_2 - L_1}{L_2(t;\epsilon) - L_1(t;\epsilon)}}\tilde{g}_{\omega,R}(y(t,x)), \\ g_{\omega,L}(t,x;\epsilon) &= \sqrt{\frac{L_2 - L_1}{L_2(t;\epsilon) - L_1(t;\epsilon)}}\tilde{g}_{\omega,L}(y(t,x)). \end{aligned} \quad (52)$$

The fields can by expanded as follows:

$$\begin{aligned}\hat{\phi}(t,x) &= \sum_{j=R,L}\int_0^\infty d\omega\hat{a}_{\omega,j}g_{\omega,j}(t,x;\epsilon) + \text{H.c.}, \\ \hat{\xi}(t,x) &= -i\sum_{j=R,L}\int_0^\infty d\omega\omega\hat{a}_{\omega,j}g_{\omega,j}(t,x;\epsilon) + \text{H.c.} \end{aligned} \quad (53)$$

In this case the energy of the fields depends on ϵ. In fact, we have

$$\begin{aligned}\hat{E}(t) &\equiv \int_{\mathbb{R}} dx\hat{\mathcal{E}}(t,x) \\ &= \frac{1}{2}\int_{\mathbb{R}} dy[(\hat{\tilde{\xi}}(y))^2 + (\partial_y\hat{\tilde{\phi}}(y))^2] \\ &\quad - \frac{\epsilon(g_2(t) - g_1(t))}{L_2 - L_1}\int_{\mathbb{R}} dy(\partial_y\hat{\tilde{\phi}}(y))^2 + \mathcal{O}(\epsilon^2), \end{aligned} \quad (54)$$

where we have introduced the "free" fields (the fields when the two mirrors are at rest)

$$\hat{\tilde{\phi}}(y) = \sum_{j=R,L} \int_0^\infty d\omega \hat{a}_{\omega,j} \tilde{g}_{\omega,j}(y) + \text{H.c.},$$
$$\hat{\tilde{\xi}}(y) = -i \sum_{j=R,L} \int_0^\infty d\omega\omega \hat{a}_{\omega,j} \tilde{g}_{\omega,j}(y) + \text{H.c.} \tag{55}$$

In the same way, the corresponding quantum Hamiltonian is obtained as

$$\begin{aligned}\hat{H}(t) &\equiv \int_{\mathbb{R}} dx \hat{\mathcal{H}}(t,x) \\ &= \frac{1}{2}\int_{\mathbb{R}} dy[(\hat{\tilde{\xi}}(y))^2 + (\partial_y \hat{\tilde{\phi}}(y))^2] \\ &\quad - \frac{\epsilon(g_2(t)-g_1(t))}{L_2-L_1}\int_{\mathbb{R}} dy(\partial_y \hat{\tilde{\phi}}(y))^2 \\ &\quad + \frac{\epsilon}{2}\bigg[\sum_{j=1,2}\int_{\mathbb{R}} dy \frac{(-1)^j \dot{g}_j(t)\hat{\tilde{\xi}}(y)}{L_2-L_1} \\ &\quad \times \bigg(\partial_y \hat{\tilde{\phi}}(y)(y-\bar{L}_j) + \frac{1}{2}\hat{\tilde{\phi}}(y)\bigg) + \text{H.c.}\bigg] + \mathcal{O}(\epsilon^2).\end{aligned} \tag{56}$$

In this case, the part of the Hamiltonian that describes the interaction between the field and the mirrors is also dependent on ϵ. However, in general, it is impossible to adequately describe this part. For that reason, the reactive part of the motion force can seldom be calculated.

For instance, if we consider the generalization to the Lagrangian density (46), i.e.

$$\frac{1}{2}((\partial_t\phi)^2 - (\partial_x\phi)^2) - \sum_{j=1,2}\alpha_j\phi^2\delta(x-L_j(t;\epsilon)), \tag{57}$$

then the part of the quantum Hamiltonian that describes the interaction is

$$\sum_{j=1,2}\alpha_j\hat{\phi}^2(t,L_j(t;\epsilon)) = \sum_{j=1,2}\alpha_j(\hat{\tilde{\phi}}(L_j))^2 - \frac{\epsilon(g_2(t)-g_1(t))}{L_2-L_1} \times \sum_{j=1,2}\alpha_j(\hat{\tilde{\phi}}(L_j))^2 + \mathcal{O}(\epsilon^2). \tag{58}$$

And, since

$$\frac{1}{2}\int_{\mathbb{R}} dy[(\hat{\tilde{\xi}}(y))^2 + (\partial_y\hat{\tilde{\phi}}(y))^2] + \sum_{j=1,2}\alpha_j(\hat{\tilde{\phi}}(L_j))^2$$

is the Hamiltonian of the system when the two mirrors are at rest, that is, the free Hamiltonian, we can conclude that, in the interaction picture, while the mirrors move, the full Hamiltonian of the system is given by

$$\begin{aligned}\hat{H}_I(t) &= -\frac{\epsilon(g_2(t)-g_1(t))}{L_2-L_1}\bigg[\int_{\mathbb{R}} dy(\partial_y\hat{\tilde{\phi}}_I(y))^2 \\ &\quad + \sum_{j=1,2}\alpha_j(\hat{\tilde{\phi}}_I(L_j))^2\bigg] \\ &\quad + \frac{\epsilon}{2}\bigg[\sum_{j=1,2}\int_{\mathbb{R}} dy\frac{(-1)^j\dot{g}_j(t)\hat{\tilde{\xi}}_I(y)}{L_2-L_1} \\ &\quad \times\bigg(\partial_y\hat{\tilde{\phi}}_I(y)(y-\bar{L}_j) + \frac{1}{2}\hat{\tilde{\phi}}_I(y)\bigg) + \text{H.c.}\bigg] \\ &\quad + \mathcal{O}(\epsilon^2).\end{aligned} \tag{59}$$

Finally, we prove that our dissipative part of the motion force coincides with the one obtained in [13]. For times τ larger than the stopping time, our quantum evolution operator, in the linear approximation, is $\mathcal{T}^\tau = Id - i\int_{\mathbb{R}} dt\hat{H}_I(t)$. We are interested in the term

$$\begin{aligned}A_1 &\equiv \frac{\epsilon}{2}\bigg[\sum_{j=1,2}\int_{\mathbb{R}} dy\frac{(-1)^j\dot{g}_j(t)\hat{\tilde{\xi}}_I(y)}{L_2-L_1}\bigg(\partial_y\hat{\tilde{\phi}}_I(y)(y-\bar{L}_j) \\ &\quad + \frac{1}{2}\hat{\tilde{\phi}}_I(y)\bigg) + \text{H.c.}\bigg].\end{aligned} \tag{60}$$

Integrating by parts, and using the fact that $\hat{\tilde{\phi}}_I$ and $\hat{\tilde{\xi}}_I$ are free fields, it follows that

$$\begin{aligned}A_1 &\equiv -\epsilon\sum_{j=1,2}\int_{\mathbb{R}} dt g_j(t)\hat{F}_j(t) \\ &\quad + \epsilon\int_{\mathbb{R}} dt\int_{\mathbb{R}} dy\frac{g_2(t)-g_1(t)}{L_2-L_1}(\partial_y\hat{\tilde{\phi}}_I(y))^2 \\ &\quad - \frac{\epsilon}{2}\lim_{\delta\to 0}\int_{\mathbb{R}} dt\frac{g_2(t)-g_1(t)}{L_2-L_1}\sum_{j=1,2}\hat{\tilde{\phi}}_I(L_j) \\ &\quad \times(\partial_y\hat{\tilde{\phi}}_I(L_j-|\delta|) - \partial_y\hat{\tilde{\phi}}_I(L_j+|\delta|)).\end{aligned} \tag{61}$$

Now, from expression (2.4) in Ref. [12],

$$\lim_{\delta\to 0}(\partial_y\hat{\tilde{\phi}}_I(L_j+|\delta|) - \partial_y\hat{\tilde{\phi}}_I(L_j-|\delta|)) = 2\alpha\hat{\tilde{\phi}}_I(L_j),$$

we obtain that

$$\begin{aligned}A_1 &\equiv -\epsilon\sum_{j=1,2}\int_{\mathbb{R}} dt g_j(t)\hat{F}_j(t) \\ &\quad + \epsilon\int_{\mathbb{R}} dt\int_{\mathbb{R}} dy\frac{g_2(t)-g_1(t)}{L_2-L_1}(\partial_y\hat{\tilde{\phi}}_I(y))^2 \\ &\quad + \epsilon\int_{\mathbb{R}} dt\frac{g_2(t)-g_1(t)}{L_2-L_1}\sum_{j=1,2}\alpha_j(\hat{\tilde{\phi}}_I(L_j))^2.\end{aligned} \tag{62}$$

And, finally, inserting this expression into $\int_{\mathbb{R}} dt\hat{H}_I(t)$, we conclude that, for times τ beyond the stopping time, we have

$$\mathcal{T}^{\tau} = Id + i\epsilon \sum_{j=1,2} \int_{\mathbb{R}} dt g_j(t) \hat{F}_j(t), \tag{63}$$

as we wanted to demonstrate.

V. DISCUSSION AND CONCLUSIONS

By considering physically plausible mirrors and with a canonical use of the Hamiltonian approach, we have showed that the problem of the negative energies that appears in the Davies-Fulling model can be avoided, the main difference with respect to their approach (and other's) being that partially transmitting mirrors, which are transparent to sufficiently high frequencies—thus providing a natural and physically sound renormalization—are being considered here.

A discussion on the change of variables used now follows. It was an old idea, clearly stated in the seminal paper by Moore [22], that in order to be able to quantize a system of perfectly reflecting moving mirrors in a cavity, only Heisenberg's image could be used, since a Hamiltonian cannot, in principle, be constructed that would give rise to a Schrödinger image. This image was only possible, according to Moore, in the case of fixed mirrors. However, it was later discovered that Moore's argument for the proof of such statement was only valid if the field was decomposed into time independent modes. However, as pointed out by Law ([16], first reference), should the field be decomposed into instantaneous modes, one then indeed gets a Hamiltonian which, in a way, does not coincide with the "energy" and can be used to create a Schrödinger picture [17]. And it can be proven that one indeed obtains, in this way, equivalent representations.

In fact, use of these modes is equivalent to working in the variables $(\bar{t}, y)$ defined in Eq. (5), whereby in this coordinate system the mirrors are at rest and, as remarked already, Schrödinger's image is available. One can see (details are given in [18]) that the Euler-Lagrange equations which are obtained starting from the Lagrangian density (1) are the same as the Hamiltonian equations that follow from the Hamiltonian density (8). Observe also that the Hamiltonian which is derived from (8) is the same one that is obtained from (10). In other words, and confirming this scheme, the Schrödinger representation which arises from the Hamiltonian (10) is in fact equivalent to the Heisenberg representation obtained from the Lagrangian density (1).

What are the advantages of using the Hamiltonian density (8) or (10), or the field decomposition in instantaneous modes, instead? It turns out that for perfectly reflecting mirrors both are exactly equivalent and there is no advantage in using one instead of the other. However, for the case of partially reflecting mirrors there is no known way of quantically treating this problem in terms of instantaneous modes (see e.g. [17], Sec. VIII). On the contrary, starting from the Hamiltonian density (10) the problem can be solved by postulating the existence of the matrix (12), with the properties (13) and (16)–(18), and, finally, decomposing the field in modes of the sort (16), for the case of a single mirror, and (50) and (51), for that of two mirrors.

We have also proved that our method provides a dissipative radiative-reaction force that fully agrees with the dissipative force derived in Refs. [12,27]. On the other hand, the motion force calculated using the Hamiltonian approach contains some reactive terms, proportional to the mirrors' acceleration, which do not appear in the results obtained in Refs. [13,27]. Those terms are fundamental in order to ensure that the energy remains positive at any time, and consequently, to guarantee the validity of the concept of particle also during the oscillation of the mirrors, which is certainly fast but, in the proposed experimental settings, still very small as compared with the speed of light [14].

We also saw explicitly that albeit a possible (and quite simple) solution to this disagreement could be to perform a mass renormalization, that completely eliminates the reactive terms proportional to the mirrors' acceleration (see Ref. [12]), it would turn out in this case that the definition of particle itself would be impossible to maintain at any time while the mirrors move, which would be indeed remarkable, in view of the relatively small velocities involved. Quite on the contrary, and in consonance with the realistic boundary conditions imposed by us on the mirrors (which led precisely to these additional terms), we take as the most reasonable and physically meaningful renormalization condition to impose, to keep those terms in full, in which case the definition of particle during the oscillation of the mirrors can be consistently preserved, as well as the fundamental laws of physics too, at any time t during the process. In plain words, as a bonus, the fundamental principle of energy conservation holds during the whole evolution towards the end state.

Our main results in the paper have been rigorously proven for $1 + 1$ dimensions; in three dimensions further work is needed, since our considerations do not apply directly. It is still true that, in the $3 + 1$ dimensional case, scattering can also be described by a matrix S with the same properties (13)–(18). But, in this case, the reflection and transmission coefficients, apart from the dependence on the frequency, also depend on the direction of propagation, which makes the computation more involved (see e.g. [20]). As in $1 + 1$ dimensions, again the case of perfectly reflecting mirrors leads here to divergent quantities (see [9,19]), and similar conclusions about the benefits of our approach, as in the low-dimensional case, can be drawn.

To finish, we must mention that the dynamical Casimir effect is being discussed right now by many groups in different contexts, and that the growing potential of this subject, both as a fundamental phenomenon and for the number and importance of its applications, is out of the question. At very large scales, it is being considered in theoretical cosmology as a most natural explanation of the observed acceleration in the Universe expansion (termed dark energy) [30]. And, in a very different context, some laboratory experiments have recently been proposed which would provide an extremely nice, alternative proof of the validity of general relativity and of some semiclassical approaches to quantum gravity. In addition, they open the way to practical applications of the Casimir effect in nano-electronics and other technologies. In those contexts some recent papers have appeared which deserve careful consideration (see e.g. [14,31]).

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