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Integrability and generalized monodromy matrix

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We construct the generalized monodromy matrix $\hat{\mathcal{M}}(\omega)$ of two-dimensional string effective action by introducing the T-duality group properties. The integrability conditions with general solutions depending on spectral parameter are given. This construction is investigated for the exactly solvable Wess, Zumino, Novikov, and Witten model in pp-wave limit when $B = 0$.

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I. INTRODUCTION

Two-dimensional field theories play an important role in describing a variety of physical systems [1–3]. Some of these models possess the interesting property of integrability which corresponds to exactly solvable models [3]. The *two-dimensional nonlinear σ models* have been studied in great detail from different perspectives [2,3]. Such models are endowed with a rich symmetry structure. However, the symmetry structure of string theories is one of its most fascinating features. In particular, duality symmetries have played a very important role in the understanding of string dynamics [4–6]. The tree-level string effective action compactified on a d-dimensional torus T^d is known to be invariant under the noncompact global symmetry group $O(d, d)$. Furthermore, two-dimensional models derived from dimensional reduction of higher dimensional Einstein gravity as well as supergravity theories have been studied to bring out their integrability properties [7–9]. In this context, one may recall that the construction of the *monodromy matrix* turns out to be one of the principal objectives in the study of integrable systems [10,11]. The integrability of dimensionally reduced gravity and supergravity to two dimensions has been studied extensively by introducing the spectral parameter and constructing a set of currents which are invariant under a local $O(d) \times O(d)$ transformation and satisfy the curvaturelessness condition [6].

The purpose of this article is to construct the generalized monodromy matrix $\hat{\mathcal{M}}(\omega)$ of a two-dimensional string effective action obtained from a D-dimensional effective action, which is compactified on T^d, by considering the general *integrability conditions*. We provide the transformation property of $\hat{\mathcal{M}}(\omega)$ under $O(d, d)$ group. In this process, we can synthesize the general classical integrability properties and the *T-duality symmetry* of string theory in order to derive a generalized monodromy matrix. The special properties of the generalized monodromy matrix are given in terms of general functions depending on spectral parameter. Furthermore, the case of the WZNW model with pp-wave limit is studied where $\hat{\mathcal{M}}(\omega)$ acquires a special characteristic.

The paper is organized as follows: In Sec. II, we recall the form of the two-dimensional string effective action obtained by compactification on T^d. Then,we consider a matricial form of background fields with $O(d, d)$ transformations which leave the dimensionally reduced action invariant globally under $O(d, d)$ group and locally under $O(d) \times O(d)$. We devote Sec. III to the construction of the generalized monodromy matrix from general integrability conditions in terms of general functions depending on spectral parameter. Next, we give the transformations rules of $\hat{\mathcal{M}}(\omega)$ under T-duality symmetry, and we present its explicit form for simple background configurations.

In Sec. IV, we investigate the generalized monodromy matrix in the WZNW model with pp-wave limit. The summary and conclusion are given in Sec. V.

II. TWO-DIMENSIONAL STRING EFFECTIVE ACTION

We are interested in the two-dimensional σ model coupled to gravity, namely

$$S = \int dx^2 \sqrt{-g} e^{-\bar{\Phi}} \Big[R + (\partial\bar{\Phi})^2 + \frac{1}{8} \mathrm{Tr}(\partial_\alpha M^{-1} \partial^\alpha M) \Big], \tag{2.1}$$

which is derived by dimensional reduction of the effective action of string theory [12,13]. In this expression, g is the determinant of the metric $g_{\alpha\beta}$ where α, $\beta = 0$, 1 are the two-dimensional space-time indices and R is the associated Ricci scalar curvature. The field $\bar{\Phi}$ is the usual shifted dilaton given by

$$\bar{\Phi} = \Phi - \tfrac{1}{2} \log \det G_{ij}, \tag{2.2}$$

where G_{ij} is the metric in the internal space, corresponding to the toroidally compactified coordinates x^i; $i, j = 2, 3, \ldots (D - 1)$ with $d = D - 2$. M is a $2d \times 2d$ symmetric matrix of the form

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$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \tag{2.3}$$

where B_{ij} is the moduli coming from dimensional reduction of the Neveu-Schwartz-Neveu Schwartz sector in higher space-time dimensions. G and B parametrize the coset $O(d,d)/O(d) \times O(d)$, and the matrix M corresponds to a symmetric representation of the group $O(d,d)$.

The action (2.1) is invariant under global $O(d,d)$ transformations, namely

$$g_{\alpha\beta} \to g_{\alpha\beta} \qquad \bar{\Phi} \to \bar{\Phi} \qquad M \to \Omega^T M \Omega, \qquad \Omega \in O(d,d), \tag{2.4}$$

and the variation with respect to M leads to the conservation law

$$\partial_\alpha [e^{-\bar{\Phi}} \sqrt{-g} g^{\alpha\beta} M^{-1} \partial_\beta M] = 0. \tag{2.5}$$

In order to make apparent a local $O(d) \times O(d)$ as well as a global $O(d,d)$ transformation, it is convenient to introduce a triangular matrix V contained in $O(d,d)/O(d) \times O(d)$ of the following form:

$$V = \begin{pmatrix} E^{-1} & 0 \\ BE^{-1} & E^T \end{pmatrix} \tag{2.6}$$

such that $M = VV^T$ with $(E^T E)_{ij} = G_{ij}$ where E is the vielbein in the internal space. Consequently, the matrix V parametrizing the coset $O(d,d)/O(d) \times O(d)$ transforms nontrivially under global $O(d,d)$ and local $O(d) \times O(d)$, namely

$$V \to \Omega^T V h(x), \tag{2.7}$$

where $\Omega \in O(d,d)$ and $h(x) \in O(d) \times O(d)$.

Let us remark that the matrix M is sensitive only to a global $O(d,d)$ rotation. Furthermore, the matter sector of this model governed by the third term in (2.1) is based on a set of scalar fields which are combined into a matrix $V(x)$ taking values in a noncompact Lie group G with the maximal compact subgroup H. This subgroup can be characterized by means of a symmetric space involution η: $G \to G$ as follows

$$H = \{h \in G / \eta(h) = h\}. \tag{2.8}$$

Naturally, the involution extends to the corresponding Lie algebras $g = LieG$ and $h = LieH$ where $G = O(d,d)$ and its maximal compact subgroup $H = O(d) \times O(d)$ characterized by the involution [10]

$$\eta(g) \equiv (g^T)^{-1} \tag{2.9}$$

for $g \in G$.

However, from the matrix V, we can construct the following current: [14,15]

$$V^{-1} \partial_\alpha V = P_\alpha + Q_\alpha, \tag{2.10}$$

which belongs to the Lie algebra of $O(d,d)$. In such decomposition Q_α belongs to the Lie algebra of the maximally compact subgroup $O(d) \times O(d)$ and P_α to the complement. From the symmetric space automorphism property of the coset $O(d,d)/O(d) \times O(d)$ it follows that

$$P_\alpha^T = P_\alpha \qquad Q_\alpha^T = -Q_\alpha. \tag{2.11}$$

Therefore,

$$\begin{aligned} P_\alpha &= \tfrac{1}{2}[V^{-1}\partial_\alpha V + (V^{-1}\partial_\alpha V)^T] \\ Q_\alpha &= \tfrac{1}{2}[V^{-1}\partial_\alpha V - (V^{-1}\partial_\alpha V)^T]. \end{aligned} \tag{2.12}$$

Now it is easy to show that

$$\mathrm{Tr}[\partial_\alpha M^{-1} \partial_\beta M] = -4\,\mathrm{Tr}[P_\alpha P_\beta]. \tag{2.13}$$

The currents in (2.12) and the action (2.1) are invariant under local gauge transformation

$$V \to V h(x), \tag{2.14}$$

where $h(x) \in O(d) \times O(d)$. Consequently, the composite fields P_α and Q_α which are inert under rigid $O(d,d)$ invariance transform under $O(d) \times O(d)$ according to [15].

$$\begin{aligned} P_\alpha &\to h^{-1}(x) P_\alpha h(x) \\ Q_\alpha &\to h^{-1}(x) Q_\alpha h(x) + h^{-1}(x) \partial_\alpha h(x). \end{aligned} \tag{2.15}$$

We note that Q_α transforms like a gauge field while P_α transforms as belonging to the adjoint representation. It is clear that (2.13) is invariant under the global $O(d,d)$ as well as the local $O(d) \times O(d)$ transformations. This fact allows us to obtain the integrability conditions and the derivation of the general monodromy matrix which will be the subject of the next section.

III. CLASSICAL INTEGRABILITY AND GENERAL MONODROMY MATRIX

In this section we consider the case of the two-dimensional σ model in flat space-time defined on the coset $O(d,d)/O(d) \times O(d)$. The integrability conditions following from the currents (2.10) correspond to the zero curvature condition, namely

$$\partial_\alpha [V^{-1} \partial_\beta V] - \partial_\beta [V^{-1} \partial_\alpha V] + [(V^{-1}\partial_\alpha V), (V^{-1}\partial_\beta V)] = 0. \tag{3.1}$$

Such equations with the definitions (2.12) of P_α and Q_α are subject to the compatibility relations

$$\partial_\alpha Q_\beta - \partial_\beta Q_\alpha + [Q_\alpha, Q_\beta] = -[P_\alpha, P_\beta], \tag{3.2}$$

$$D_\alpha P_\beta - D_\beta P_\alpha = 0 \tag{3.3}$$

with

$$D_\alpha P_\beta = \partial_\alpha P_\beta + [Q_\alpha, P_\beta]. \tag{3.4}$$

For the case of flat space-time, the equation of motion (2.5) modifies to

$$\eta^{\alpha\beta}\partial_\alpha[M^{-1}\partial_\beta M] = 0, \tag{3.5}$$

which is equivalent to

$$\eta^{\alpha\beta}D_\alpha P_\beta = 0. \tag{3.6}$$

In the same way as in Refs. [14–16], we introduce a one-parameter family of matrices with a constant spectral parameter t such that

$$\hat{V}(x, t = 0) = \hat{V}(x). \tag{3.7}$$

Now, let us consider the generalized current decomposition with arbitrary functions of the spectral parameter t as follows:

$$\hat{V}^{-1}\partial_\alpha\hat{V} = Q_\alpha + f(t)P_\alpha + g(t)\varepsilon_{\alpha\beta}P^\beta, \tag{3.8}$$

where $f(t)$ and $g(t)$ are general functions satisfying the following conditions

$$f(t = 0) = 1, \qquad \lim_{t\to+\infty} f(t) = -1 \tag{3.9}$$

$$g(t = 0) = 0, \qquad \lim_{t\to+\infty} g(t) = 0 \tag{3.10}$$

in flat space-time. Let us note that these functions have to possess singularities for $t = \pm 1$ in order to recover the case of Ref [15]. Then, the integrability conditions (3.1) are rewritten as

$$\partial_\alpha[\hat{V}^{-1}\partial_\beta\hat{V}] - \partial_\beta[\hat{V}^{-1}\partial_\alpha\hat{V}] + [(\hat{V}^{-1}\partial_\alpha\hat{V}), (\hat{V}^{-1}\partial_\beta\hat{V})] = 0, \tag{3.11}$$

which can be expressed in terms of the functions f and g as follows:

$$\begin{aligned}\partial_\alpha Q_\beta - \partial_\beta Q_\alpha + 2[Q_\alpha, Q_\beta] + \partial_\alpha[f(t) + g(t)]P_\beta \\ - \partial_\beta[f(t) + g(t)]P_\alpha + [f(t) + g(t)] \\ \times (\partial_\beta P_\alpha - \partial_\alpha P_\beta + 2[P_\alpha, P_\beta]) \\ + 4f(t)g(t)[P_\alpha, P_\beta] = 0.\end{aligned} \tag{3.12}$$

By using the relation (3.2), such equations take the following form:

$$\begin{aligned}(f' + g')[\partial_\alpha t P_\beta - \partial_\beta t P_\alpha] + (f + g)(\partial_\beta P_\alpha - \partial_\alpha P_\beta \\ + 2[P_\alpha, P_\beta]) + (4fg - 1)[P_\alpha, P_\beta] + [Q_\alpha, Q_\beta] = 0.\end{aligned} \tag{3.13}$$

This leads to the integrability condition of the spectral parameter, namely

$$\begin{aligned}(\partial_\beta t)P_\alpha - (\partial_\alpha t)P_\beta = (f' + g')^{-1}\{(f + g)(\partial_\beta P_\alpha - \partial_\alpha P_\beta \\ + 2[P_\alpha, P_\beta]) + (4fg - 1)[P_\alpha, P_\beta] \\ + [Q_\alpha, Q_\beta]\},\end{aligned} \tag{3.14}$$

which can be simplified by using the light-cone indices to the following equations

$$\partial_\pm t = P_0^{-1}(P_\pm \pm H(t)) \tag{3.15}$$

with

$$\begin{aligned}H(t) = (f' + g')^{-1}\{(f + g)(\partial_1 P_0 - \partial_0 P_1 + 2[P_0, P_1]) \\ + (4fg - 1)[P_0, P_1] + [Q_0, Q_1]\}.\end{aligned} \tag{3.16}$$

Furthermore, in order to obtain the monodromy matrix in terms of generalized functions we can parametrize the $P_\pm$ quantities in terms of the vielbein as follows:

$$P_\pm = \begin{pmatrix} -E^{-1}\partial_\pm E & 0 \\ 0 & E^{-1}\partial_\pm E \end{pmatrix}, \qquad Q_\pm = 0. \tag{3.17}$$

These currents that are subject to the integrability condition can be incorporated in the matrix $\hat{V}$ as

$$\hat{V}^{-1}\partial_\pm\hat{V} = (f(t) + g(t))\begin{pmatrix} -E^{-1} & 0 \\ 0 & E^{-1} \end{pmatrix}\partial_\pm E. \tag{3.18}$$

Let us note that in this parametrization, the antisymmetric tensor field is chosen to be zero and the matrices E and G are assumed to be diagonal, namely [15]

$$\begin{aligned}E &= \mathrm{diag}(e^{(1/2)(\lambda+\psi_1)}, e^{(1/2)(\lambda+\psi_2)}, \ldots, e^{(1/2)(\lambda+\psi_d)}) \\ G &= \mathrm{diag}(e^{\lambda+\psi_1}, e^{\lambda+\psi_2}, \ldots, e^{\lambda+\psi_d})\end{aligned} \tag{3.19}$$

with $\sum\psi = 0$ so that

$$\lambda = \frac{1}{d}\log \det G. \tag{3.20}$$

We shall assume that the scalar fields live on a finite dimensional (noncompact) symmetric space G/H and that the local H invariance has been already restored. They are thus described by a matrix $V(x)$ subject to the transformation [16].

$$V(x) \to KV(x)h(x) \tag{3.21}$$

with arbitrary $K \in O(d, d)$ and $h(x) \in O(d) \times O(d)$. The dynamics is defined via the Lie algebra decomposition (2.10), and in addition to the matter fields, the model contains a dilaton field ρ which is given in the conformal gauge by

$$\rho(x) = \rho_+(x^+) + \rho_-(x^-) = e^{-\bar{\Phi}} \tag{3.22}$$

with

$$\rho = \det E,$$

where $\rho_+(x^+)$ and $\rho_-(x^-)$ are left and right moving solutions. The matter field equations of motion read

$$D^\alpha(\rho P_\alpha) = 0, \tag{3.23}$$

where D_α is the $O(d) \times O(d)$ covariant derivative. There are two first-order equations, namely

$$\partial_+\rho\partial_+\sigma = \tfrac{1}{2}\rho\,\mathrm{Tr}(P_+P_+) \qquad \partial_-\rho\partial_-\sigma = \tfrac{1}{2}\rho\,\mathrm{Tr}(P_-P_-), \tag{3.24}$$

where σ is defined by

$$\sigma \equiv \log\lambda - \tfrac{1}{2}\log(\partial_+\rho\partial_-\rho). \tag{3.25}$$

Furthermore, the matter field Eq. (3.23) for $\hat{V}(x, t)$ as well as the usual integrability conditions following from (2.10) can be recovered as the compatibility conditions of the linear system (Lax pair) [16].

$$\hat{V}^{-1}D_\alpha\hat{V} = (f(t) + g(t))P_\alpha \qquad Q_\alpha = 0, \tag{3.26}$$

where

$$D_\alpha\hat{V} = \partial_\alpha\hat{V} - \hat{V}Q_\alpha. \tag{3.27}$$

The consistency of (3.26) requires that the spectral parameter t itself be subject to a very similar system of linear differential equations as follows:

$$t^{-1}\partial_\pm t = (f(t) + g(t))\rho^{-1}\partial_\pm\rho \tag{3.28}$$

that relates ρ and t as a Bäcklund duality [16]. Indeed

$$\partial_\pm t = t(f(t) + g(t))e^{\bar{\Phi}}\partial_\pm e^{-\bar{\Phi}}. \tag{3.29}$$

Thus, from the current form, we assume that

$$f(t) + g(t) = \sqrt{\frac{\omega - \rho_-}{\omega + \rho_+}}, \tag{3.30}$$

where ω is a function of spectral parameter and represents the constant of integration of t (ω is, namely, a hidden spectral parameter) given by

$$\omega = -\rho_+ + \frac{e^{-\bar{\Phi}}}{1 - N^2(t)}, \tag{3.31}$$

where $N(t)$ must verify that $N(t) \neq \pm 1$ with $N(t) = f(t) + g(t)$. Therefore, the generalized monodromy matrix has to be of the form

$$\hat{\mathcal{M}} = \hat{V}(x, t)\hat{V}^T\left(x, \frac{1}{t}\right) = \begin{pmatrix} \mathcal{M}(\omega) & 0 \\ 0 & \mathcal{M}^{-1}(\omega) \end{pmatrix}, \tag{3.32}$$

where $\mathcal{M}(\omega)$ is diagonal with

$$\mathcal{M}_i(\omega) = \frac{\omega_i - \omega}{\omega_i + \omega} \tag{3.33}$$

and

$$\omega_i = -\rho_+ + \frac{e^{-\bar{\Phi}}}{1 - N^2(t_i)}. \tag{3.34}$$

Finally this generalized monodromy matrix can be used to examine general currents and general integrability conditions in some cosmological models. In fact, in the next section we study the WZNW model in the pp-wave limit where the generalized monodromy matrix has an interesting behavior.

IV. GENERALIZED MONODROMY MATRIX IN PP-WAVE LIMIT

The WZNW model [17,18] is one of the convenient models for constructing the generalized monodromy matrix in pp-wave limit. In fact, by investigating the previous results we can start with

$$\Omega = \begin{pmatrix} \sqrt{1+q_0} & 0 & 0 & -\sqrt{q_0} \\ 0 & \sqrt{1+q_0} & \sqrt{q_0} & 0 \\ 0 & \sqrt{q_0} & \sqrt{1+q_0} & 0 \\ -\sqrt{q_0} & 0 & 0 & \sqrt{1+q_0} \end{pmatrix} \tag{4.1}$$

which is belonging to $O(d, d)$ group and where q_0 is a constant parameter parametrizing the embedding of $U(1)$ into $E_2^c \otimes U(1)$ [18,19]. E_2^c represents the two-dimensional Euclidean group with a central extension.

As we have seen in the previous section, the one-parameter family of potentials $\hat{V}(x, t)$ for $B = 0$ satisfy

$$\hat{V}^{-1}\partial_\pm\hat{V} = (f(t) + g(t))P_\pm, \tag{4.2}$$

and its diagonal form can be written as

$$\hat{V}(x, t) = \mathrm{diag}(\hat{V}_1, \hat{V}_2, \hat{V}_3, \hat{V}_4) \tag{4.3}$$

since $\hat{V}(x, t)$ transforms globally under $O(d, d)$ transformations through Ω matrix and locally under $O(d) \times O(d)$ transformations by $h(x)$. This latter is an element of the maximal compact subgroup H of the T-duality group [19]. Hence we can write

$$\hat{V}(x, t) = \Omega^T\hat{V}^{(B=0)}h(x) = \begin{pmatrix} U_1 & 0 & 0 & U_2 \\ 0 & U_3 & U_4 & 0 \\ 0 & U_5 & U_6 & 0 \\ U_7 & 0 & 0 & U_8 \end{pmatrix} \tag{4.4}$$

where $h(x)$ is given in the Novikov-Witten (NW) model [19] by

$$h(x) = \begin{pmatrix} \cos\theta & 0 & 0 & \sin\theta \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ -\sin\theta & 0 & 0 & \cos\theta \end{pmatrix}. \tag{4.5}$$

This allows us to obtain the matrix elements of $\hat{V}(x, t)$, namely

$$
\begin{aligned}
U_1 &= \sqrt{1+q_0}\cos\theta + \sqrt{q_0}\hat{V}_4\sin\theta \\
U_2 &= \sqrt{1+q_0}\sin\theta - \sqrt{q_0}\hat{V}_4\cos\theta \\
U_3 &= \sqrt{q_0}\sin\theta + \sqrt{1+q_0}\hat{V}_2\cos\theta \\
U_4 &= \sqrt{q_0}\cos\theta - \sqrt{1+q_0}\hat{V}_2\sin\theta \\
U_5 &= \sqrt{1+q_0}\sin\theta + \sqrt{q_0}\hat{V}_2\cos\theta \\
U_6 &= \sqrt{1+q_0}\cos\theta - \sqrt{q_0}\hat{V}_2\sin\theta \\
U_7 &= -\sqrt{q_0}\cos\theta + \sqrt{1+q_0}\hat{V}_4\sin\theta \\
U_8 &= -\sqrt{q_0}\sin\theta + \sqrt{1+q_0}\hat{V}_4\cos\theta.
\end{aligned}
\tag{4.6}
$$

Consequently, this leads to the generalized monodromy matrix of the form

$$
\hat{\mathcal{M}}^{(B)} = \hat{V}^{(B)}(x,t)(\hat{V}^{(B)})^T\left(x,\frac{1}{t}\right) = \begin{pmatrix} C_1 & 0 & 0 & C_2 \\ 0 & C_3 & C_4 & 0 \\ 0 & C_4 & C_5 & 0 \\ C_2 & 0 & 0 & C_6 \end{pmatrix}
\tag{4.7}
$$

where the coefficients are obtained after some compute by

$$
\begin{aligned}
C_1 &= 1 + \frac{2q_0}{1-2\omega} \\
C_2 &= -\sqrt{q_0(1+q_0)} - \sqrt{q_0}\frac{1+2\omega}{1-2\omega} \\
C_3 &= -1 + \frac{2(1+q_0)}{1+2\omega} \\
C_4 &= \sqrt{q_0} + \sqrt{q_0(1+q_0)}\frac{1-2\omega}{1+2\omega} \\
C_5 &= 1 + \frac{2q_0}{1+2\omega} \\
C_6 &= -1 + \frac{2(1+q_0)}{1-2\omega}
\end{aligned}
\tag{4.8}
$$

with

$$
\omega = -\rho_+ + \frac{e^{-\bar{\Phi}}}{1-N^2(t)}.
$$

We remark that the generalized monodromy matrix for the particular case $B = 0$ and when $q_0 = 0$ is given by

$$
\hat{\mathcal{M}}(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-2\omega}{1+2\omega} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1+2\omega}{1-2\omega} \end{pmatrix}.
\tag{4.9}
$$

However, the generalized monodromy matrix in the pp-wave limit for this model can be constructed by following the same procedure as before but by taking $x^- \longrightarrow 0$ as the limit in consideration. In fact, the background consist of the metric with a vanishing B field, of the form [9]

$$
dS^2 = -2dx^+dx^- + \tan^{-2}\frac{x^+}{\sqrt{2}}dy^2 + \tan^2\frac{x^-}{\sqrt{2}}dz^2.
\tag{4.10}
$$

Therefore, the diagonal form of the $\hat{V}$ matrix can be given as follows :

$$
\begin{aligned}
\hat{V}^{(B=0)}(x,t) &= \mathrm{diag}(\hat{V}_1, \hat{V}_2, \hat{V}_3, \hat{V}_4) \\
&= \mathrm{diag}\left(1, \frac{(1-t)\tan x^+}{t+\tan^2 x^+}, 1, \frac{t+\tan^2 x^+}{(1-t)\tan x^+}\right).
\end{aligned}
\tag{4.11}
$$

Then, in the pp-wave limit $x^- \longrightarrow 0$, where $x = \frac{x^+}{\sqrt{2}}$, we deduce that

$$
\hat{V}_1 = -\frac{t-1}{t\tan^2\frac{x^+}{\sqrt{2}} + 1} = \hat{V}_3^{-1}
\tag{4.12}
$$

$$
\hat{V}_2 = -\frac{(t-1)\tan^2\frac{x^+}{\sqrt{2}}}{t-\tan^2\frac{x^+}{\sqrt{2}}} = \hat{V}_4^{-1}.
\tag{4.13}
$$

In particular for $t = 0$, this return to $\hat{V}$ matrix appeared before. It follows that the generalized monodromy matrix is now determined by

$$
\begin{aligned}
\hat{\mathcal{M}}^{(NW)}(\omega) &= \hat{V}(x,t)\hat{V}^T\left(x,\frac{1}{t}\right) \\
&= \begin{pmatrix} \frac{1-2\omega}{1+2\omega} & 0 & 0 & 0 \\ 0 & \frac{1-2\omega}{1+2\omega} & 0 & 0 \\ 0 & 0 & \frac{1+2\omega}{1-2\omega} & 0 \\ 0 & 0 & 0 & \frac{1+2\omega}{1-2\omega} \end{pmatrix}
\end{aligned}
\tag{4.14}
$$

where ω is given in the pp-wave limit by

$$
\omega = -\rho_+(x^+) + \frac{e^{-\bar{\Phi}}}{1-N^2(t)}
\tag{4.15}
$$

with

$$
N(t) = f(t) + g(t), \qquad N(t) \neq \pm 1.
\tag{4.16}
$$

In this case, ω depends only on one light-cone coordinate, and

$$
e^{-\bar{\Phi}} = \rho\left(\frac{x^+}{\sqrt{2}}\right) = \rho_+(x^+) + \rho_-(0).
\tag{4.17}
$$

Hence, if we choose $\rho_-(0) = 0$ we have

$$
e^{-\bar{\Phi}} = \rho\left(\frac{x^+}{\sqrt{2}}\right) = \rho_+(x^+),
\tag{4.18}
$$

which means that the dilaton is expressed with only one light-cone coordinate and admits just one part of solution. We note that the generalized monodromy matrix in the pp-wave limit which depends only on one light-cone coordinate x^+ still admits degenerate poles in the NW model.

V. CONCLUSION

In this paper, we have constructed the generalized monodromy matrix of two-dimensional string effective action by using the general integrability conditions which are expressed in terms of general functions f and g depending on the spectral parameter and satisfying conditions (3.9) in the flat space. One of our principal aims is to take into account the symmetries associated with the string effective action for constructing the generalized monodromy matrix, which contains information about these symmetries. In fact, we have shown that the generalized monodromy matrix transforms nontrivially under the noncompact T-duality group and possesses a special structure in pp-wave limit for the WZNW model. Furthermore, it is necessary to note that we have realized a connection between T-duality symmetry and general integrability properties by introducing a spectral parameter t which is space-time dependent. Indeed, we have envisaged a vielbein $E(x, t)$ in order to define the action under consideration and examine its invariance globally under G and locally under H. By including E, we have introduced an infinite potential family of matrices parametrized by general functions depending on a continuous spectral parameter necessary to obtain the generalized monodromy matrix in ordinary and pp-wave limit cases. Finally, our general results have likely exploitations and more applications in string theory and cosmological models [20–23] such as the Becchi, Rouet, Stora, and Tyutin quantization of bosonic string on $AdS_5 \times S^5$ and analysis of black holes with solutions. Furthermore the integrable models like the Calogero model for two particles and N particles and other cosmological models with noncompact Lie groups can be studied. With these results, a very large and rich class of models can be treated from this process.

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