Static spherically symmetric perfect fluid solutions in $f(R)$ theories of gravity

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Static spherically symmetric perfect fluid solutions are studied in metric $f(R)$ theories of gravity. We show that pressure and density do not uniquely determine $f(R)$ i.e. given a matter distribution and an equation state, one cannot determine the functional form of $f(R)$. However, we also show that matching the outside Schwarzschild–de Sitter metric to the metric inside the mass distribution leads to additional constraints that severely limit the allowed fluid configurations.

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I. INTRODUCTION

Modern day cosmological observations such as supernovae type Ia [1], cosmic microwave background [2], and large scale structure [3] provide strong evidence against a critical density matter dominated universe. Instead, the current cosmological concordance model is a critical density universe dominated by cold dark matter and dark energy in the form of some kind effective cosmological constant. The traditional cosmological constant is maybe the leading dark energy candidate (for a review see e.g. [4]), but a large number of other alternatives have been studied in the vast literature on dark energy.

Modifying general relativity (GR) to explain the present day acceleration is an often considered avenue of research. In particular, a class of models that has been extensively studied in the recent years are the $f(R)$ gravity models that replace the Einstein-Hilbert action with an arbitrary function of the curvature scalar (see e.g. [5–13] and references therein). Modifying the gravitational action is faced with many challenges, however, and obstacles such as instabilities [14–16] as well as constraints arising from known properties of gravity in our Solar System (see e.g. [17–19] and references therein) need to be overcome. Also the large scale perturbations present a challenge for $f(R)$ gravity theories [20,21]. One of the most direct and strictest constraints on any modification of gravity comes from observations of our Solar System. This is often done conformally transforming the theory to a scalar-tensor theory and then considering the parameterized post-Newtonian limit [22,23] (see also [24,25] for a discussion). The question of Solar System constraints on $f(R)$ theories has recently been extensively discussed by a number of authors. The opinions on viability of $f(R)$ theories divide from more or less sceptical [26–29] to approving [30,31] depending on the point of view of the author.

The essence of the discussion is the validity of the Schwarzschild–de Sitter (SdS) solution in the Solar System. The SdS metric is a vacuum solution to a large class of $f(R)$ theories of gravity. However, due to the higher-derivative nature of the metric $f(R)$ theories, it is not unique and other solutions can also be constructed (see e.g. [32,33]). This fact is also present in the cosmological setting, rendering any cosmological solution nonunique and hence the form of $f(R)$ cannot be uniquely determined from the expansion history of the universe alone [34].

In recent literature this question has now been addressed without resorting to scalar-tensor theory [26–28]. The result is compatible with the scalar-tensor theory calculations: the Solar System constraints are valid in a particular limit that corresponds to the limit of light effective scalar in the equivalent scalar-tensor theory. In terms of the $f(R)$ theory, this is equivalent to requiring that one can approximate the trace of the field equations by Laplace's equation [27]. As a result, the often considered $1/R$ theory is not consistent with the Solar System constraints in this limit, if the $1/R$ term is to drive late time cosmological acceleration.

In this work, we approach the question differently by asking what kind of a mass distribution is required so that the SdS metric is the solution outside a spherically symmetric body. Note that realistic theories do not need to have the exact SdS metric outside the stellar mass distribution. Instead, the metric has to approach the SdS solution quickly enough to guarantee that Solar System and cosmological observations are satisfied, potentially increasing the number of physically acceptable $f(R)$ theories. Here we concentrate on the more restricted SdS cases i.e. given the SdS metric, what is the matter distribution that has the correct boundary conditions. Our approach is general, we do not make any assumptions about the $f(R)$ theory.

We show that, like in the cosmological setting, the mass distribution alone cannot in general determine the gravitational theory, or the functional form of $f(R)$. Imposing the SdS metric as a boundary condition does limit the allowed solutions however. We also give a prescription how one can in principle solve the mass distribution that has the SdS metric as the outside solution, given a gravitational theory and the equation of state of matter.

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II. $f(R)$ GRAVITY FORMALISM

The action for $f(R)$ gravity is ($8\pi G = 1$)

$$S = \int d^4x\sqrt{-g}(f(R) + \mathcal{L}_m). \tag{1}$$

The field equations resulting in the so-called metric approach are reached by varying with respect to $g_{\mu\nu}$:

$$F(R)R_{\mu\nu} - \tfrac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = T^m_{\mu\nu}, \tag{2}$$

where $T^m_{\mu\nu}$ is the standard minimally coupled stress-energy tensor and $F(R) \equiv df/dR$. Alternatively, this can be written in a form similar to the field equations of general relativity:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = T^c_{\mu\nu} + \tilde{T}^m_{\mu\nu}, \tag{3}$$

where the stress-energy tensor of the gravitational fluid is

$$T^c_{\mu\nu} = \frac{1}{F(R)}\bigg\{\frac{1}{2}g_{\mu\nu}(f(R) - RF(R)) + F(R)^{;\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\mu\nu}g_{\alpha\beta})\bigg\}, \tag{4}$$

and we have defined

$$\tilde{T}^m_{\mu\nu} \equiv T^m_{\mu\nu}/F(R). \tag{5}$$

Contracting Eq. (4) and assuming that we can describe the stress-energy tensor with a perfect fluid, we get

$$F(R)R - 2f(R) + 3\Box F(R) = \rho - 3p. \tag{6}$$

III. SPHERICALLY SYMMETRIC PERFECT FLUID SOLUTIONS

We are interested in static, spherically symmetric perfect fluid solutions, or as commonly referred to in the literature, SSSPF solutions. Study of such solutions can be traced back to Schwarzschild [35] and the literature is extensive (see e.g. [36] for reviews). The surface of the fluid sphere is defined by the surface of zero pressure where the interior solution is matched to the outside metric.

In spherically symmetric coordinates the metric can generally be written as

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}. \tag{7}$$

The corresponding continuity equation is

$$\frac{p'(r)}{\rho(r) + p(r)} = -\frac{1}{2}\frac{A'(r)}{A(r)}, \tag{8}$$

where prime indicates a derivation with respect to r, $' \equiv d/dr$.

A. Uniqueness

The higher-derivative nature of metric $f(R)$ theories of gravity can lead to nonuniqueness of solutions of the field equations. For example, the cosmological solution $a(t)$ for a given $f(R)$ theory is not unique but other $f(R)$ theories with the same expansion history exists [34]. This also true for vacuum solutions: even though the SdS metric is a solution of the field equations in vacuum, others also exist [33].

Here we consider the question of uniqueness in the presence of matter. This is most conveniently done by using a form of the field equations where $f(R)$ is eliminated from the equations by using the contracted equation (6):

$$FR_{\mu\nu} - \tfrac{1}{4}(FR - \Box F)g_{\mu\nu} - \nabla_\mu\nabla_\nu F = T^m_{\mu\nu} - \tfrac{1}{4}(\rho - 3p)g_{\mu\nu}. \tag{9}$$

Given a matter distribution, $\rho(r)$, $p(r)$, one can solve for A from the continuity equation and substitute into Eq. (9) giving a set of differential equations for $F(r)$ and $B(r)$.

This set can be solved algebraically for $B(r)$ and $B'(r)$ so that both $B(r)$ and $B'(r)$ can be expressed in terms of $F(r)$, $\rho(r)$, $p(r)$ and their derivatives (see the appendix for details). Differentiating the expression for $B(r)$ and equating it with the other expression for $B'(r)$, we obtain a single equation relating various derivatives of F, ρ, and p. This equation is a nonlinear third order differential equation for $F(r)$ and due to its length is not explicitly shown here. Here we adopt a shorthand notation for the equation and write

$$m\mathrm{TOV}(F, \rho, \rho', \rho'', p, p', p'', p''') = S_f, \tag{10}$$

where $S_f = S_f(F', F'', F''', \rho, \rho', \rho'', p, p', p'', p''')$ represents a source term. Explicit calculations show that in the GR limit, $F = 1$, the source term vanishes, $S_f = 0$, and Eq. (10) is satisfied, whenever ρ and p satisfy the usual Tolman-Oppenheimer-Volkov (TOV) equation. Hence, we can view Eq. (10) as a modified TOV equation of metric $f(R)$ theories of modified gravity.

Given $\rho(r)$ and $p(r)$, one can solve Eq. (10) for $F(r)$ and hence one has a solution for $R(r)$ using the expressions for $B(r)$ in terms of F. From $F(r)$ and $R(r)$ one can determine, at least in principle, $F(R)$ and finally $f(R)$. The higher-derivative nature of $f(R)$ theories is apparent in that Eq. (10) can have a number of solutions. For example, even when matter follows the ordinary TOV equation, one can find nontrivial solutions for $F(r)$ i.e. for given standard SSSPF solution of general relativity, $F(r) = 1$ is not the only possible solution. We construct explicit solutions in a later section.

B. Boundary conditions

The outside solution sets the boundary conditions for the metric components at the surface of the star, $r = r_0$. The field equations are fourth order in A and third order in B so that A_0, $A_0'A_0''$, A_0''', B_0, B_0', B_0'', where $A_0 \equiv A(r_0)$ etc. are fixed. This is in contrast to general relativity, where only A_0, A_0', B_0 are fixed by the outside solution.

In this paper we are interested in solutions which have the Schwarzschild–de Sitter space time as the outside solution. The SdS metric is

$$g_{\mu\nu} = \begin{pmatrix} s(r) & 0 & 0 & 0 \\ 0 & -1/s(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}, \tag{11}$$

where $s(r) \equiv 1 - 2M/r + r^2/12R_0$ and R_0 is the corresponding scalar curvature (in the standard SdS representation $R_0 = -4\Lambda$). In the $f(R)$-theory framework R_0 is set by the equation $R_0F(R_0) = 2f(R_0)$.

This choice of metric fixes the curvature scalar and its first derivative at the stellar surface at r_0: $R(r_0) = R_0$, $R'(r_0) = 0$. The surface of the star is set by the conditions $p(r_0) = 0$ while the energy density is left as a free parameter, $\rho(r_0) = \rho_0$.

Using these boundary conditions in the modified Einstein's equations, (2), along with the requirement $R_0F(R_0) = 2f(R_0)$, gives two independent equations:

$$s(r)\left(\rho_0 + s(r)\frac{d^2f}{dR^2}\bigg|_{R_0} R''\right) = 0, \qquad s(r)\frac{d^2f}{dR^2}\bigg|_{R_0} R'' = 0. \tag{12}$$

Since we are interested in generalized gravity, $df^2/dR^2|_{R_0} \neq 0$ and hence $R''(r_0) = 0$, $\rho_0 = 0$. Note that this is already a result different from general relativity where ρ can be discontinuous at the surface.

Derivating the modified Einstein's equations once with respect to r and applying the boundary conditions along with the new constraints $R''(r_0) = 0 = \rho_0$, gives $R'''(r_0) = 0$, $\rho'(r_0) = 0$. From the continuity equation it then straightforwardly follows that also $p''(r_0) = 0$. This process can then be continued by derivating the modified Einstein's equations once more and substituting the boundary values found at previous steps, resulting in three independent equations for the highest order derivatives of R, A, and B. These can then be solved and one can proceed to the next order derivatives. Obviously, derivatives of R and A and B are not independent but no new information is obtained from their mutual relation.

Higher derivatives of ρ and p are not uniquely defined and one is free to choose ρ'' and higher derivatives on the surface in a way that reflects the desired equation of state.

In summary, requiring that the outside metric is the SdS metric sets $\rho(r_0) = \rho'(r_0) = p(r_0) = p'(r_0) = p''(r_0) = 0$ on the surface for a general $f(R)$ theory for which $d^2f/dR^2|_{R_0} \neq 0$. We can already see here how the higher-derivative nature of $f(R)$ theories lead to a more constrained system than general relativity.

C. Polytropic stars

As a simple example of how the boundary conditions already limit the range of allowed solutions, we briefly consider polytropic stars. White dwarfs and neutron starts are often approximated by a polytropic equation of state, $p = \kappa\rho^\gamma$, where κ and $\gamma = 1 + 1/n$ are constants and n is often referred to as the polytropic index (see e.g. [37]).

The continuity equation, Eq. (8), can straightforwardly be solved for such an equation of state:

$$\rho(r) = \kappa^{1/(1-\gamma)}\left(\left(\frac{A(r)}{A_0}\right)^{(1-\gamma)/(2\gamma)} - 1\right)^{1/(\gamma-1)}, \tag{13}$$

where we have used the fact that ρ vanishes at the surface. Clearly this condition sets that $\gamma > 1$. Similarly, since ρ' is also vanishing at the boundary, we must require that $\gamma < 2$. This is again quite different from GR, where γ is unbounded from above (again see e.g. [37]).

D. Examples of matter distributions with corresponding nonunique $f(R)$ theories

General arguments quite clearly indicate the existence of different $f(R)$ theories corresponding to a fixed mass distribution, ρ and p. We have calculated some simple cases explicitly and integrated some other cases numerically. In order to construct $f(R)$ theories, we need to solve F from Eq. (10) and check the physical meaningfulness of the solution by studying the behavior of $B(r)$ and the relation between $F(r)$ and $R(r)$.

By taking a simple equation of state $p = w\rho$ (constant w) the solution of Einstein's equations reads as $\rho(r) = w/(2\pi G(4w + (1 + w)^2)r^2)$. This solution can then be inserted into Eq. (10). (Note that in this case the stellar boundary is pushed to infinity.) It is easy to find some rather trivial solutions, e.g. $F(r) \propto r^k$, $k = -\frac{3+5w}{2+4w}$, corresponding to $B(r) \equiv 0$, is an unphysical solution. If $w = 1/3$, $F(r) = 1 + F_2r^2$ (F_2 is an arbitrary constant) leading to a constant $B(r) = 7/4$ and vanishing R. Again, this solution is unphysical, because it does not properly define F as a function of curvature R. These two solutions are, however, very special cases corresponding solutions with vanishing derivatives of $F(r)$ at the origin. Numerical calculations confirm, that if $F'(0) \neq 0$, a proper relation between F and R is achieved and $B(r)$ is a well-defined function everywhere (except possibly at the origin) [38].

Another straightforward example can be found by considering a pressureless matter distribution, $p = 0$, with $\rho = 3r$. We have constructed two $f(R)$ theories that have such a matter distribution as an exact solution: $f(R) = -2\sqrt{-6 - R}$ and $f(R) = 6\sqrt{3}/\sqrt{-R}$. Both solutions correspond to nontrivial $B(r)$, $F(r)$, and $R(r)$.

These cases explicitly show that nontrivial choices of F can be made for a given density and stellar profiles. Whether the solutions constructed by using Eq. (10) are physical needs to be analyzed in each case.

E. SSSPF with a SdS boundary metric

In light of the discussion presented above, we can now consider perfect fluid matter in $f(R)$ gravity theories with the SdS metric as a boundary condition. Again, it is advantageous to consider a form of the field equations where explicit $f(R)$ dependence is eliminated in favor of $F(r)$, Eq. (9). Since $R'(r_0) = R''(r_0) = R'''(r_0) = 0$, it is clear that $F'(r)$, $F''(r)$, $F'''(r)$ vanish on the boundary i.e. $F'(r_0) = F''(r_0) = F'''(r_0) = 0$. The boundary value of F can be solved from the requirement $R_0F(r_0) = 2f(R_0)$ once $f(R)$ is given, e.g. if $f(R) = R - \mu^4/R$, $R_0^2 = 3\mu^4$, and $F_0 = F(R_0) = 4/3$. Note that with a more general choice of $f(R)$, one can in principle easily mimic general relativity i.e. $F_0 = 1$ everywhere. However, requiring $R_0F_0 = 2f(R_0)$ along with Eq. (6), implies that $\rho - 3p = 0$ everywhere inside the star making such constructions physically uninteresting.

Given a fluid sphere, $\rho(r)$, $p(r)$ with appropriate boundary conditions, one can in principle solve Eq. (10) with the aforementioned boundary values for $F(r)$. Using the solution $f(R)$ can then be reconstructed. The constant solution, $F(r) = F_0 = 1$, is a solution exactly when $\rho(r)$ and $p(r)$ satisfy the TOV equation i.e. matter is distributed like in general relativity. Vice versa, if matter does not obey the TOV equation, the S_f term acts as source for the differential equation of $F(r)$ and thus the boundary conditions are not strong enough to force F to be constant, but the solution is more general. An important restriction is, however, the boundary conditions for ρ and p making the set of allowed general relativistic SSSPF solutions more restricted in $f(R)$ gravity, e.g. the standard Shwarzschild fluid sphere with constant density is not allowed since $\rho_0 \neq 0$.

IV. CONCLUSIONS

In the present paper we have discussed spherically symmetric solutions with nontrivial matter distributions applicable to stellar systems, in particular. Although we have restricted our analysis to a simplified system described by a perfect fluid and external SdS solution, similar conclusions are expected to apply in more realistic cases.

We find that, like in the cosmological case, the distribution of matter does not determine the gravitational theory uniquely but due to the higher-derivative nature of the field equations, different gravitational theories can support the same solution. Given the matter distribution, $\rho(r)$, $p(r)$, one can, at least in principle, construct a gravitational theory that has the desired solution by solving the modified TOV equation, Eq. (10). However, some mathematically admissible solutions of the modified TOV equation are not physically plausible, but may correspond singular metrics or do not define properly f as a function of R.

By considering configurations that are matched to a Schwarzschild–de Sitter metric, we find that such configurations are more tightly constrained than those of general relativity. Again, this is due to the higher-derivative nature of the metric $f(R)$ theories of gravity that requires matching of higher order derivatives at the boundary of the fluid sphere than in general relativity.

As a result, we find that stellar configurations, approximated by a perfect fluid sphere, can be accommodated to an external SdS solution, whenever ρ, p, and $f(R)$ are related by a the modified TOV equation. The conventional TOV equations correspond exactly to the choice $f(R) \equiv R$, and departures from the standard TOV equations necessarily lead to a more general gravitational action. This phenomenon may be noteworthy whenever modifications to the Einstein-Hilbert action, $f(R) = R$, are small. Small changes to the Einstein-Hilbert action are likely to lead to only small modifications to stellar models, i.e. density and pressure of the matter may deviate only slightly from the ordinary TOV relation. On this basis it hence seems possible that realistic stellar models may be constructed also in $f(R)$ gravity models without violating constraints from the Solar System. The exact nature and whether such solutions correspond to $f(R)$ theories that can act as dark energy requires more extensive analysis.

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APPENDIX

The set of modified field equations, Eq. (9), for a known matter distribution in terms of $F(r)$ reads:

$$
\begin{aligned}
0 =& \frac{F}{2r^2} - \frac{3p}{4} - \frac{3\rho}{4} + \frac{Fp^2}{2r^2B(p+\rho)^2} + \frac{Fp\rho}{r^2B(p+\rho)^2} + \frac{F\rho^2}{2r^2B(p+\rho)^2} - \frac{FB'}{2rB^2} + \frac{F'}{2rB} - \frac{B'F'}{8B^2} + \frac{Fpp'}{rB(p+\rho)^2} + \frac{F\rho p'}{rB(p+\rho)^2} \\
&- \frac{FB'p'}{4B^2(p+\rho)} + \frac{3F'p'}{4B(p+\rho)} - \frac{Fp'^2}{B(p+\rho)^2} - \frac{Fp'\rho'}{2B(p+\rho)^2} + \frac{F''}{4B} + \frac{Fpp''}{2B(p+\rho)^2} + \frac{F\rho p''}{2B(p+\rho)^2} \\
0 =& \frac{F}{2r^2} + \frac{F}{2r^2B} + \frac{p}{4} + \frac{\rho}{4} + \frac{FB'}{2rB^2} + \frac{F'}{2rB} + \frac{3B'F'}{8B^2} - \frac{Fpp'}{rB(p+\rho)^2} - \frac{F\rho p'}{rB(p+\rho)^2} - \frac{FB'p'}{4B^2(p+\rho)} - \frac{F'p'}{4B(p+\rho)} - \frac{Fp'^2}{B(p+\rho)^2} \\
&- \frac{Fp'\rho'}{2B(p+\rho)^2} - \frac{3F''}{4B} + \frac{Fp''}{2B(p+\rho)} \\
0 =& -\frac{F}{2r^2} - \frac{F}{2r^2B} + \frac{p}{4} + \frac{\rho}{4} - \frac{F'}{2rB} - \frac{B'F'}{8B^2} + \frac{FB'p'}{4B^2(p+\rho)} - \frac{F'p'}{4B(p+\rho)} + \frac{Fp'^2}{B(p+\rho)^2} + \frac{Fp'\rho'}{2B(p+\rho)^2} + \frac{F''}{4B} - \frac{Fp''}{2B(p+\rho)}.
\end{aligned}
$$

This set of equations is not linearly independent and one can solve $B(r)$ and $B'(r)$ algebraically:

$$\begin{aligned}
B'(r) = & -2(2F^2p' + r^2(p+\rho)^2(F' - rF'') + rF(p^2 + \rho^2 + p'(F' - r^2(2p' + \rho')) + \rho(-(rp') + F'' + r^2p'') \\
& + p(2\rho - rp' + F'' + r^2p'')))(r^2(p+\rho)F'^2(p + \rho + rp') + 2F^2(p^2 + \rho^2 - r^2p'(3p' + \rho') + r^2\rho p'' \\
& + p(2\rho + r^2p'')) + rF(-(r^2F'p'(3p' + \rho')) + p^2(3F' - rF'') + \rho^2(3F' - rF'') \\
& + r\rho(-(rp'F'') + F'(2p' + rp'')) + p(\rho(6F' - 2rF'') + r(-(rp'F'') + F'(2p' + rp''))))) \\
& \times ((p+\rho)^3(-2F^2 + r^3(p+\rho)F' + rF(rp + r\rho - F' - r^2p'))^2)^{-1}, \\
B(r) = & \, r\biggl(-\biggl(\frac{rF'^2(p + \rho + rp')}{p+\rho}\biggr) + F^2\biggl(\frac{-2}{r} + \frac{6rp'^2}{(p+\rho)^2} + \frac{2rp'\rho'}{(p+\rho)^2} - \frac{2rp''}{p+\rho}\biggr) \\
& + F\biggl(\frac{r(p + \rho + rp')F''}{p+\rho} + F'\biggl(-3 + \frac{3r^2p'^2}{(p+\rho)^2} + \frac{rp'(-2p - 2\rho + r\rho')}{(p+\rho)^2} - \frac{r^2p''}{p+\rho}\biggr)\biggr)\biggr) \\
& \times (2F^2 - r^3(p+\rho)F' + rF(-(rp) - r\rho + F' + r^2p'))^{-1}. \qquad \text{(A1)}
\end{aligned}$$

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