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Energy absorption problem of a brane-world black hole

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We have studied the wave dynamics and the energy absorption problem for the scalar field as well as the brane-localized gravitational field in the background of a brane-world black hole. Comparing our results with the four-dimensional Schwarzschild black hole, we have observed the signature of the extra dimension in the energy absorption spectrum.

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I. INTRODUCTION

Recent developments on higher-dimensional gravity resulted in a number of interesting theoretical ideas such as the brane-world concept in which the standard model fields are confined to our four-dimensional world viewed as a hyperspace embedded in the higher-dimensional spacetime where gravity can propagate [1–3]. The simplest models in this context are proposed by Randall and Sundrum [2]. It was argued that the extra dimensions need not be compact and, in particular, it was shown that it is possible to localize gravity on a 3-brane when there is one infinite extra dimension [1,2]. A striking consequence of the theory with large extra dimensions is that it can lower the fundamental gravity scale and allow the production of mini-black-holes in the universe. Such mini-black-holes are centered on the brane and may have been created in the early universe due to density perturbations and phase transitions. Recently it was proposed that such mini-black-holes may also be produced in high energy collisions and could be probed in particle accelerator experiments in the very near future [4–7].

A great deal of effort has been expanded for the precise determination of observational signatures of such mini-black-holes. One among them is the black hole quasinormal modes (QNM), which can disclose the evolution of the perturbation around the black hole during a certain time interval and was argued carrying a unique fingerprint of the black hole existence and is expected to be detected through gravitational wave observations in the near future (see reviews on this topic and references therein [8]). Recently, the QNMs of a brane-world black hole have been studied in [9–11]. The gravitational perturbation of the mini-black-hole would be a characteristic sound and could tell us the existence of such a black hole. Another chief possibility of observing this kind of mini-black-hole is the spectrum of Hawking radiation which is expected to be detected in particle accelerator experiments [11–20]. Since such small black holes carry information of extra dimensions and have different properties compared to those of ordinary 4-dimensional black holes, these two tools of detecting mini-black-holes can help to read the extra dimensions.

Most of the available works have been done for the idealized case by studying the brane-localized modes in the QNM and Hawking radiation. One of the reasons is that in the brane-world scenario it was assumed that most standard matter fields are brane localized. In the study of Hawking radiation, it was argued that the emission on the brane is dominated compared to that off the brane [21]. This argument was supported numerically in the case of standard model field emission by the $(4 + n)$-dimensional nonrotating black holes [17,18] and also in the higher-dimensional rotating black hole background [16]. Some counterexamples to this argument have also been found [19]. However, in general it is very hard to obtain exact solutions of higher-dimensional Einstein equations so that the knowledge of the bulk is lacking; the thorough study on the field emission in the bulk is still a challenging task. Using a recently constructed exact black hole localized on a 3-brane in a world with two large extra dimensions [22], Dai *et al.* explored the Hawking decay channels with the influence of the brane tension [20]. They found for the nonrotating black holes the dominate channels are still the brane-localized modes.

In this paper we are going to study a brane-world black hole obtained in [23]. A general class of spherically symmetric and static solution to the field equation with a 5-dimensional cosmological constant can be derived by considering a general line element of the type

$$ds^2 = A(r)dt^2 - \frac{1}{B(r)}dr^2 - r^2 d\Omega^2, \qquad (1)$$

and relaxing the condition $A(r) = B^{-1}(r)$ used in obtaining the usual 4-dimensional black holes. Casadio *et al.* obtained two types of solutions by fixing either $A(r)$ or $B(r)$ and in one case with the choice $A(r) = 1 - 2M/r$; the metric reads [23]

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$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{1 - \frac{3M}{2r}}{(1 - \frac{2M}{r})(1 - \frac{\gamma M}{2r})}dr^2 - r^2 d\Omega^2. \tag{2}$$

This brane-world black hole is called the Casadio-Fabbri-Mazzacurati (CFM) black hole. It would be noted that Eq. (2) was not derived from a 5-dimensional solution, but rather was obtained as a solution to 4-dimensional equations constraining the possible form of the 5-dimensional metric. The extension of the asymptotically flat static spherically metric on the brane into the bulk has been discussed in [24]. The usual 4-dimensional Schwarzschild black hole is recovered with $\gamma = 3$. The corresponding Hawking temperature is given by [23] $T_{\rm BH} = \sqrt{1 - 3(\gamma - 3)/2}/(8\pi M)$. Thus, in comparison with the Schwarzschild black hole, the brane-world black hole will be either hotter or colder depending on the sign of $\gamma - 3$. In this work we are restricted to the case when $\gamma \leq 11/3$ as in [10] to ensure the Hawking temperature to be physical. The QNM of this kind of brane-world black hole was studied in [10]. We will concentrate on the absorption problem of this black hole. Immersed in external radiation fields, the black hole scatters the radiation and may either absorb or amplify it. The process is described by dynamical perturbation equations in the black hole background. We will study the scalar and axial gravitational perturbations on the background of this CFM black hole and investigate the absorption problem. We will extract the information on the extra dimensional influence on the absorption of the brane-world black hole by comparing to that of the usual four-dimensional black hole.

For future convenience, we adopt the notation in [25]:

$$ds^2 = e^{2\nu}dt^2 - e^{2\psi}d\varphi^2 - e^{2\mu_2}dr^2 - e^{2\mu_3}d\theta^2. \tag{3}$$

Comparing with (2), we have

$$e^{2\nu} = A(r), \qquad e^{-2\mu_2} = B(r), \qquad e^{2\mu_3} = r^2, \qquad e^{2\psi} = r^2\sin^2\theta. \tag{4}$$

In the vicinity of the event horizon r_H, the asymptotic expressions of the metric components $A(r)$ and $B(r)$ can be written in the first order,

$$A(r) \sim A_1(r/r_H - 1), \qquad B(r) \sim B_1(r/r_H - 1), \qquad (r \to r_H + 0), \tag{5}$$

where $A_1 = 1$ and $B_1 = 4 - \gamma$ are dimensionless, positive real numbers. Demanding that the spacetime is asymptotically flat, we have

$$A(r) \to 1, \qquad B(r) \to 1, \qquad (r \to \infty). \tag{6}$$

Our paper is organized as follows: in Sec. II we will go over scalar perturbation and investigate its energy absorption; in Sec. III we will derive the wave equation of the gravitational perturbation and study its energy absorption spectra. We will summarize our results in the last section.

II. SCALAR PERTURBATION AND ITS ENERGY ABSORPTION

For simplicity we first consider the brane-world black hole immersed in the massless scalar field confined on the brane. We will go over the scalar perturbation and study the energy absorption problem in the general four-dimensional spherical metric equation (1), and then apply the general results to the CFM black hole.

A. Scalar perturbation equations

The scalar perturbation is governed by the Klein-Gordon equation. For the massless scalar field, we have

$$\Box\Phi = (-g)^{-1/2}\partial_\mu[(-g)^{1/2}g^{\mu\nu}\partial_\nu\Phi] = 0. \tag{7}$$

Using the decomposition of the scalar field

$$\Phi = \sum_{l=0}^{\infty} R_l(r)P_l(\cos\theta)e^{-i\omega t}, \tag{8}$$

we have the Schrödinger-like radial wave equations,

$$\frac{d^2(rR_l)}{dr_*^2} + [\omega^2 - V_{s,l}(r)](rR_l) = 0. \tag{9}$$

r_* is the tortoise coordinate defined as $dr_* = dr/\sqrt{AB}$. The effective potential reads

$$V_{s,l} = A\frac{l(l+1)}{r^2} + \frac{1}{2r}(A'B + AB'). \tag{10}$$

B. Boundary conditions

There are freedoms in choosing boundary conditions depending on physical pictures of different problems. We will consider the plane wave scattering in our work.

1. Boundary condition at infinity

In the remote region, by virtue of the scattering problem, the scalar field at infinity is in the form

$$\Phi \sim \left[e^{i\omega r\cos\theta} + \frac{f(\theta)}{r}e^{i\omega r}\right]e^{-i\omega t}. \tag{11}$$

Referring to the scheme of partial wave method in quantum mechanics, the plane wave can be expanded into spherical waves as

$$e^{i\omega r\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(\omega r)P_l(\cos\theta). \tag{12}$$

When r tends to infinity,

$$e^{i\omega r\cos\theta} \sim \sum_{l=0}^{\infty}(-1)^{l+1}\frac{2l+1}{2\omega r}[e^{-i\omega r} - (-1)^l e^{i\omega r}]P_l(\cos\theta). \tag{13}$$

The scattered wave can also be expanded in the form

$$\frac{f(\theta)}{r}e^{i\omega r} = \sum_{l=0}^{\infty}(2l+1)f_l^s\frac{e^{i\omega r}}{2\omega r}P_l(\cos\theta), \quad (14)$$

where f_l^s are complex constants, and the superscript s indicates the case of scalar field. Inserting this equation together with Eq. (13) into Eq. (11), we have the boundary condition at infinity:

$$\Phi \sim e^{-i\omega t}\sum_{l=0}^{\infty}(-1)^{l+1}\frac{2l+1}{2\omega r}[e^{-i\omega r_*}-(-1)^l S_l^s e^{i\omega r_*}]P_l(\cos\theta), \quad (15)$$

where $S_l^s = 1 + f_l^s$, and the radius r on the exponentials is replaced by tortoise coordinate r_* for future convenience. Note that at infinity $dr_* \sim dr$ so that such replacement on the exponentials makes no difference. In accordance, we have the boundary condition of R_l at infinity:

$$R_l \sim (-1)^{l+1}\frac{2l+1}{2\omega r}[e^{-i\omega r_*}-(-1)^l S_l^s e^{i\omega r_*}]. \quad (16)$$

2. Boundary condition at horizon

Now we examine the boundary condition in the near-horizon region. We can express the solution of Eq. (9) in power series [26],

$$R_l = (r/r_H - 1)^{\rho}\sum_{n=0}^{\infty} d_{l,n}(r/r_H - 1)^n. \quad (17)$$

In the vicinity of the horizon, we have the asymptotic behavior

$$\sqrt{AB} \sim \sqrt{A_1B_1}(r/r_H - 1), \quad (18)$$

as well as

$$V_{s,l}(r) \sim O(r - r_H), \qquad (r \to r_H + 0). \quad (19)$$

Inserting the expansions, Eq. (18) and (19), into the radial equation (9), we have

$$A_1B_1[\rho(\rho-1)+\rho] + r_H^2\omega^2 = 0, \quad (20)$$

or

$$\rho = \pm\frac{i\omega r_H}{\sqrt{A_1B_1}}. \quad (21)$$

Since nothing can escape from the black hole, only the minus sign is acceptable. Thus we have the boundary condition of $R_l(r)$ near the horizon [26]:

$$R_l(r) \sim g_l(r/r_H - 1)^{-i\omega r_H/\sqrt{A_1B_1}}, \quad (22)$$

where g_l are the zeroth coefficient of the expansion of R_l: $g_l = d_{l,0}$, which is dimensionless. The boundary condition for the total perturbational scalar field Φ is given by

$$\Phi \sim e^{-i\omega t}\sum_{l=0}^{\infty} g_l(r/r_H - 1)^{-i\omega r_H/\sqrt{A_1B_1}}P_l(\cos\theta). \quad (23)$$

C. The energy flux and the absorption cross section

The energy flux is derived from the energy-momentum tensor of the scalar field. For massless scalar field, it is generally expressed as [27]

$$T_{\mu\nu} = \Phi_{;\mu}\Phi^*_{;\nu} - \tfrac{1}{2}g_{\mu\nu}\Phi^{;\alpha}\Phi^*_{;\alpha}. \quad (24)$$

The energy-momentum flow of the spacetime is defined by [28]

$$P^{\mu} = T^{\mu}_{\nu}\xi^{\nu}(t), \quad (25)$$

where vector $\xi(t) = \partial/\partial t$ is the timelike Killing vector. We now calculate the energy falling into the black hole per unit time using Gauss' theorem. In the theorem, for any appropriate four vector A^{μ} and space time sector $\mathcal{N}$, there holds

$$\int_{\mathcal{N}} A^{\mu}_{;\mu}\mathbf{\Omega}_4 = \int_{\partial\mathcal{N}} A^{\mu}n_{\mu}\mathbf{\Omega}_3, \quad (26)$$

where $\mathbf{\Omega}_4$ and $\mathbf{\Omega}_3$ are the volume elements of the spacetime sector $\mathcal{N}$, and its boundary $\partial\mathcal{N}$. n^{μ} is the normal vector of the boundary $\partial\mathcal{N}$ which satisfies $\hat{\mathbf{n}} \wedge \mathbf{\Omega}_3 = \mathbf{\Omega}_4$, where $\hat{\mathbf{n}}$ is the 1-form physically equivalent to vector n^{μ}. Here we choose the spacetime sector to be the spacial area within two spheres of radius r_1 and r_2, multiplied by the time interval between t_1 and t_2, and its boundary is denoted by $\partial\mathcal{N}$. To find out the energy falling into the hole, we focus on the hypersurface of $r = r_1$, whose normal vector is $n_{\mu} = (0, \sqrt{A/B}, 0, 0)$, and we let $r_1 \to r_H$. Thus the energy falling into the hole within the time interval between t_1 and t_2 is [25]

$$\begin{aligned} E^{(\mathrm{abs})} &= \int_{t_1}^{t_2} dt \int_{4\pi} P^{\mu}n_{\mu}r_H^2\sin\theta d\theta d\varphi \\ &= \int_{t_1}^{t_2} dt \int_{4\pi} T_{0\mu}n^{\mu}r_H^2\sin\theta d\theta d\varphi. \end{aligned} \quad (27)$$

Applying the explicit forms of $T_{0\mu}$ and n^{μ}, as well as the boundary condition, Eq. (23), we have, at the horizon,

$$T_{0\mu}n^{\mu} = -\sqrt{AB}\partial_t\Psi\partial_r\Psi^*|_{r_H} = \omega^2\left|\sum_{l=0}^{\infty} g_lP_l(\cos\theta)\right|^2. \quad (28)$$

Inserting this into Eq. (27), we have the total energy falling into the horizon per unit time:

$$\Phi^{(\mathrm{abs})} = \frac{dE^{(\mathrm{abs})}}{dt} = 4\pi\omega^2 r_H^2\sum_{l=0}^{\infty}\frac{|g_l|^2}{2l+1}. \quad (29)$$

Here, we have used the orthogonality of Legendre polynomials,

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{\delta_{ll'}}{2l+1}. \tag{30}$$

On the other hand, following the similar procedure as above, we can derive the expression of the energy flux of the incoming wave at infinity, which is simply

$$j^{(\mathrm{inc})} = \omega^2. \tag{31}$$

Therefore for the massless scalar wave the total energy absorption by the black hole is given by

$$\sigma_s^{\mathrm{abs}} = \frac{\Phi^{(\mathrm{abs})}}{j^{(\mathrm{inc})}} = 4\pi r_H^2 \sum_{l=0}^{\infty} \frac{|g_l|^2}{2l+1}. \tag{32}$$

For angular index l, the partial absorption cross section is expressed as

$$\sigma_{s,l}^{\mathrm{abs}} = 4\pi r_H^2 \frac{|g_l|^2}{2l+1}. \tag{33}$$

After we get g_l, we can obtain the final result of the absorption spectrum.

For the radial equation (9), the Wronskians for any two solutions $R_l^{(1)}$ and $R_l^{(2)}$ are

$$W[R_l^{(1)}, R_l^{(2)}] = \frac{K_{12}}{r^2\sqrt{AB}}, \tag{34}$$

with the constant K_{12} to be determined by the explicit form of the two solutions. Using the asymptotic solution of R_l at infinity, Eq. (16), we have

$$W[R_l^*, R_l] = -\frac{i(2l+1)^2}{2\omega r^2\sqrt{AB}}(1-|S_l^s|^2), \tag{35}$$

and employing the asymptotic solution at the horizon, Eq. (22), we get

$$W[R_l^*, R_l] = -\frac{2i\omega r_H^2}{r^2\sqrt{AB}}|g_l|^2. \tag{36}$$

Equating the above two equations, we find

$$|g_l|^2 = \frac{(2l+1)^2}{4\omega^2 r_H^2}(1-|S_l^s|^2). \tag{37}$$

Thus with Eq. (33) we can express the energy absorption in the form

$$\sigma_{s,l}^{\mathrm{abs}} = \frac{\pi}{\omega^2}(2l+1)\Gamma_l^s, \tag{38}$$

where Γ_l^s is defined in terms of S_l^s as

$$\Gamma_l^s = 1 - |S_l^s|^2. \tag{39}$$

D. Numerical study for the scalar perturbation

Now we proceed to calculate numerically the coefficients g_l. Instead of solving Eq. (9) directly for R_l, we choose the particular solution $\varphi_l(r)$, normalized as

$$\varphi_l(r) = g_l^{-1} R_l(r). \tag{40}$$

In the asymptotically flat region, it resolves into outgoing and ingoing waves

$$\varphi_l(r) = f_l^{(-)}\mathcal{F}_{l(+)} + f_l^{(+)}\mathcal{F}_{l(-)}, \tag{41}$$

where [18,26]

$$\mathcal{F}_{l(\pm)} = e^{\mp i\omega r_*}\sum_{n=0}^{\infty}\tau_{l,n}^{(\pm)}\left(\frac{r_H}{r}\right)^{n+1}, \tag{42}$$

satisfying $\mathcal{F}_{l(+)} = \mathcal{F}_{l(-)}^*$ and the coefficients $\tau_{l,0}^{(\pm)}$ being unity for simplicity. The combination coefficients $f_l^{(\pm)}$ are called jost functions. Comparing this to Eq. (16), we have

$$g_l = \frac{(-1)^{l+1}}{\omega}\frac{l+1/2}{f_l^{(-)}}, \tag{43}$$

$$S_l^s = (-1)^{l+1}\frac{f_l^{(+)}}{f_l^{(-)}}. \tag{44}$$

Thus if the jost functions are known, we can get the solution to our problem. Applying Eq. (34) to the two particular solutions at infinity, we have

$$W[\mathcal{F}_{l(+)}, \mathcal{F}_{l(-)}] = \frac{2i\omega r_H^2}{r^2\sqrt{AB}}, \tag{45}$$

and therefore we have the jost functions

$$f_l^{(\pm)} = \mp W[\varphi_l, \mathcal{F}_{l(\pm)}]\frac{r^2\sqrt{AB}}{2i\omega r_H^2}. \tag{46}$$

In numerical calculation, we would like to express g_l in Eq. (32) and (33) in terms of the jost functions, and the absorption cross section therefore reads

$$\sigma_{s,l}^{\mathrm{abs}} = \frac{\pi}{\omega^2}\frac{2l+1}{|f_l^{(-)}|^2}, \tag{47}$$

and

$$\sigma_s^{\mathrm{abs}} = \frac{\pi}{\omega^2}\sum_{l=0}^{\infty}\frac{2l+1}{|f_l^{(-)}|^2}. \tag{48}$$

We now report our numerical result for the massless scalar perturbation in the CFM brane-world black hole background. Inserting the metric components into Eq. (10), the effective potential has the form

$$V_{s,l}(r) = \left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + \frac{M(\gamma-3)(r^2-6Mr+6M^2)}{r^3(2r-3M)^2}\right]. \tag{49}$$

Taking $M = 1$, the behavior of the potential is shown in Fig. 1. We see that there is a potential barrier outside the horizon, and for the fixed angular index l this barrier

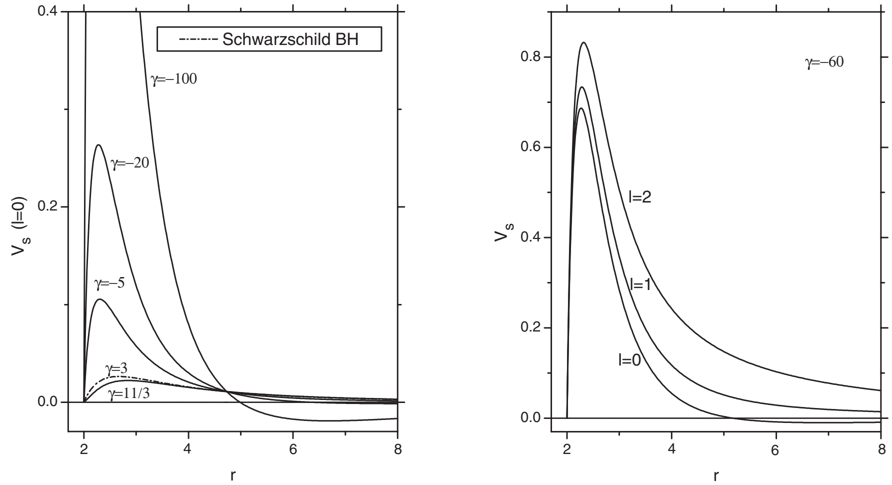

FIG. 1. The effective potential of the scalar perturbation with the change of γ and l.

increases for more negative γ. Fixing γ, we find that the potential barrier increases with the increase of the angular index. When γ becomes small, the potential has a negative well after the positive barrier. The negative well appears earlier for very negative γ. This was also observed in [10]. Comparing with the height of the positive barrier, the absolute value of the negative peak is very small for chosen γ. Thus, the potential barrier will dominate in the determination of the wave dynamics, which qualitatively leads to the general quasinormal ringing as disclosed in [10].

The spectrum of absorption cross sections for both Schwarzschild and the brane-world case with $\gamma = -5$ are shown in Fig. 2(a) for different l. Figure 2(b) displays the partial absorption cross section for chosen l and different γ. It is noticed that the absorption cross section of the brane-world black hole can be either smaller or larger than that in the Schwarzschild black hole provided that γ is smaller or bigger than 3. This result is consistent with the behavior of the potential barriers as shown in Fig. 1. The higher or lower barrier of the potential could decrease or increase the absorption of the scalar field around the brane-world black hole. For the low ω region, we observed that the absorption is higher for smaller values of γ. This could be the imprint of the negative well in the potential, since different from the barrier, the potential well enhances the absorption rate. But the overall behavior of the energy absorption shows that the potential barrier dominates the physics.

In Fig. 3, we show the spectrum of the total absorption cross section; we see that with the decrease of γ, there are fewer oscillations. For smaller γ, in the low energy band the total absorptions are suppressed. This is because of the increase in the potential barrier which plays the dominate role for smaller γ leading to the decrease of the absorption of the incoming wave. The full penetration occurs when the incoming wave with very high energy, which explains why the total absorption approaches a constant value at high frequency.

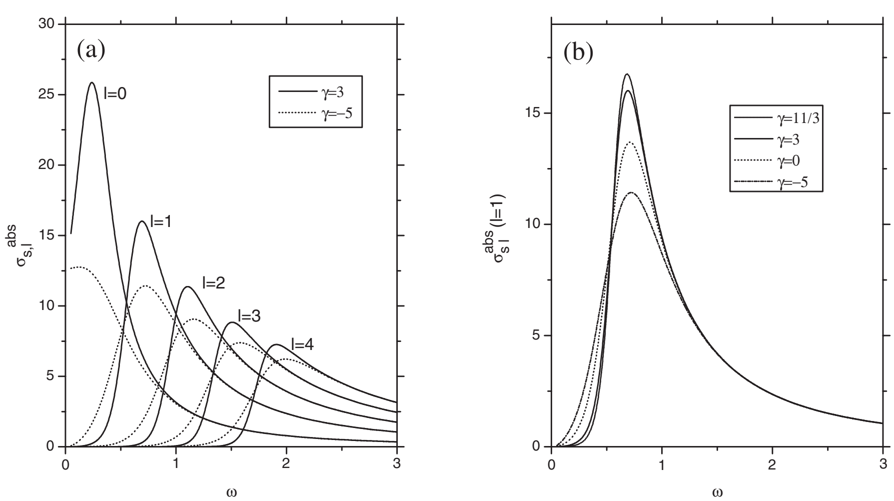

FIG. 2. The partial energy absorption of the scalar wave with the change of γ, l.

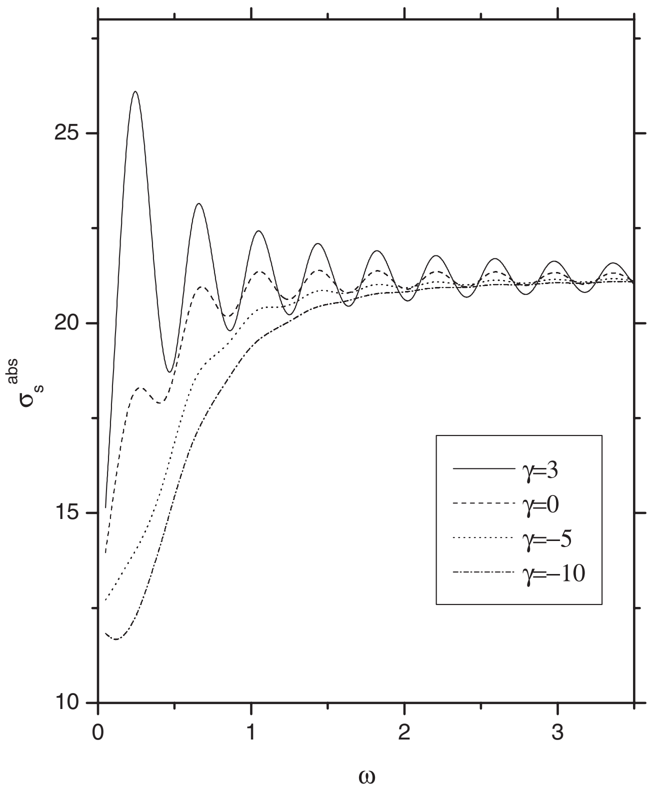

FIG. 3. The total energy absorption of the scalar wave.

III. GRAVITATIONAL PERTURBATION AND ITS ENERGY ABSORPTION CROSS SECTION

Now we start the formulation of gravitational perturbation, of which the detailed deduction requires special techniques provided in [25,29]. We will concentrate our attention here on the brane-localized axial perturbation in the background of the brane-world black hole.

A. Axial gravitational perturbation equations

Using the formalism introduced in [25], the axially perturbed metric equation (1) is

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi - \varpi dt - q_2 dx^2 - q_3 dx^3)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2. \tag{50}$$

Here

$$x^2 = r, \qquad x^3 = \theta. \tag{51}$$

ϖ and $q_i (i = 2, 3)$ are quantities that embody the perturbation. Because of the lack of knowledge of the 5D bulk metric, we restrict to the brane-localized perturbation governed by the perturbation equation

$$\delta R_{\mu\nu} = 0. \tag{52}$$

This simplification is supported by the analysis of gravitational shortcuts [30] which shows that gravitational fields do not travel deep into the bulk. It can be justified at least in a regime where the perturbation energy does not exceed the threshold of the Kaluza-Klein massive modes. By defining

$$Q_{ij} = q_{i,j} - q_{j,i}, \qquad Q_{0i} = \varpi_{,i} - q_{i,0}, \tag{53}$$

one finds equations

$$(e^{3\psi+\nu-\mu_2-\mu_3} Q_{23})_{,3} = -e^{3\psi-\nu+\mu_3-\mu_2} Q_{02,0} \tag{54}$$

from $\delta R_{12} = 0$, and

$$(e^{3\psi+\nu-\mu_2-\mu_3} Q_{23})_{,2} = e^{3\psi-\nu+\mu_2-\mu_3} Q_{03,0} \tag{55}$$

from $\delta R_{13} = 0$. We take

$$e^{3\psi+\nu-\mu_2-\mu_3} Q_{23} = r^2\sqrt{AB}\sin^3\theta Q_{23} = Q \tag{56}$$

in Eqs. (54) and (55), and then cancel ϖ, to arrive at an equation of Q:

$$-r^2 \frac{\partial^2 Q}{\partial t^2} + r^4\sqrt{AB}\frac{\partial}{\partial r}\left(\frac{\sqrt{AB}}{r^2}\frac{\partial Q}{\partial r}\right) + A\sin^3\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^3\theta}\frac{\partial Q}{\partial\theta}\right) = 0. \tag{57}$$

Analogously, this equation plays the same role as the Klein-Gordon equation in the scalar perturbation. However, the equation here cannot be separated in terms of Legendre polynomials as in the Klein-Gordon equation; the Gegenbauer polynomials are employed to carry out the separation [25]. Q is now separated as

$$Q(t, r, \theta) = \sum_{l=2}^{\infty} rZ_l(r)C_{l+2}^{-3/2}(\cos\theta)e^{-i\omega t}, \tag{58}$$

where the Gegenbauer polynomials $C_{l+2}^{-3/2}(\theta)$ satisfy the equation

$$\sin^{2\nu}\theta\frac{d}{d\theta}\left(\sin^{-2\nu}\theta\frac{dC_l^{\nu}(\cos\theta)}{d\theta}\right) + l(l + 2\nu)C_l^{\nu}(\cos\theta) = 0. \tag{59}$$

The radial Schrödinger-like equation can be derived in the form

$$\frac{d^2 Z_l}{dr_*^2} + [\omega^2 - V_{g,l}(r)]Z_l = 0. \tag{60}$$

r_* is the tortoise coordinate defined by $dr_* = dr/\sqrt{AB}$, and the effective potential is

$$V_{g,l}(r) = A\frac{(l-1)(l+2)}{r^2} + \frac{2AB}{r^2} - \frac{A'B + B'A}{2r}. \tag{61}$$

B. Boundary conditions

We set the boundary conditions to be those of plane wave scattering.

1. Boundary condition at infinity

It is expected that the quantity $Q = r^2\sqrt{AB}\sin^3\theta Q_{23}$ has the similar behavior of the scattered spherical wave since a straightforward examination shows that $Q(t, r, \theta) = e^{i(\omega r\cos\theta - \omega t)}$ is exactly a solution of Eq. (57) in the region far from the black hole. Therefore it is appropriate to write

$$Q(t, r, \theta) \sim \left[e^{i\omega r\cos\theta} + \frac{f(\theta)}{r}e^{i\omega r}\right]e^{-i\omega t}, \qquad r \to \infty. \tag{62}$$

For the infinite r, Q behaves as Eq. (62), which can also be expanded in a combination of Gegenbauer polynomials as the decomposition of Q in Eq. (58). The expansion of the plane wave part is [31]

$$e^{i\omega r\cos\theta} = -\frac{1}{3}\sum_{l=-2}^{\infty}(l + 1/2)i^l(\omega r)^2 j_l(\omega r)C_{l+2}^{-3/2}(\cos\theta). \tag{63}$$

Note that in practice we calculate only the partial absorption cross section of low angular indices, thus when ωr is a large number, the asymptotic approximation for $j_l(\omega r)$ reads

$$j_l(x) \sim \frac{1}{x}\sin(x - l\pi/2) = -\frac{i^l}{2x}[e^{-ix} - (-1)^l e^{ix}] \tag{64}$$

for $x \gg 1$ and $x \gg l$, where $x = \omega r$. There exists a certain l_0 and before this l_0 the summation terms of Eq. (63) may be replaced by the asymptotic form,

$$\begin{aligned} e^{i\omega r\cos\theta} = \frac{1}{6}\sum_{l=-2}^{l_0}&(l + 1/2)(-1)^l(\omega r) \\ &\times[e^{-i\omega r} - (-1)^l e^{i\omega r}]C_{l+2}^{-3/2}(\cos\theta) \\ &+ \left[-\frac{1}{3}\sum_{l=l_0+1}^{\infty}(l + 1/2)i^l(\omega r)^2 j_l(\omega r)\right. \\ &\left.\times C_{l+2}^{-3/2}(\cos\theta)\right]. \end{aligned} \tag{65}$$

The larger ωr is, the larger l_0 will be.

We also need the expansion of the term representing the scattered wave. Formally we may write

$$\frac{f(\theta)}{r}e^{ikr} = \sum_{l=-2}^{\infty}\chi_l(r)C_{l+2}^{-3/2}(\cos\theta). \tag{66}$$

Similar to the plane wave, we expect that, before some integer l_0', the summation coefficients in Eq. (66) are asymptotically

$$\chi_l(r) \sim \tfrac{1}{6}(l + 1/2)(\omega r)f_l^g e^{i\omega r}, \tag{67}$$

at some very large ωr, where f_l^g are complex numbers depending solely on index l. Thus we can write the expansion of Eq. (66) as

$$\begin{aligned} \frac{f(\theta)}{r}e^{i\omega r} = \frac{1}{6}&\sum_{l=-2}^{l_0'}(l + 1/2)(\omega r)f_l^g e^{i\omega r}C_{l+2}^{-3/2}(\cos\theta) \\ &+ \sum_{l=l_0'+1}^{\infty}\chi_l(r)C_{l+2}^{-3/2}(\cos\theta). \end{aligned} \tag{68}$$

Combining Eqs. (65) and (68), we have the applicable form of the asymptotic behavior of $Q(t, r, \theta)$:

$$\begin{aligned} Q \sim \frac{e^{-i\omega t}}{6}&\sum_{l=2}^{\min(l_0,l_0')}(l + 1/2)(-1)^l(\omega r) \\ &\times[e^{-i\omega r} - (-1)^l S_l^g e^{i\omega r}]C_{l+2}^{-3/2}(\cos\theta) + \cdots. \end{aligned} \tag{69}$$

Here the omitted terms are those over the upper limit for applying the asymptotic form, which are not applicable to our numerical calculation. $S_l^g = 1 - f_l^g$ is a complex number being the ratio of incoming and outgoing wave. The summation starts from index 2 for the modes from $l = -2$ to $l = 1$ are not physical in gravitational perturbation. Since $\min(l_0, l_0')$ certainly exceeds our need for numerical calculation, we may write for simplicity

$$\begin{aligned} Q \sim \frac{e^{-i\omega t}}{6}&\sum_{l=2}^{\infty}(l + 1/2)(-1)^l(\omega r) \\ &\times[e^{-i\omega r_*} - (-1)^l S_l^g e^{i\omega r_*}]C_{l+2}^{-3/2}(\cos\theta), \end{aligned} \tag{70}$$

which is valid for a limited number of angular indices. As done in the scalar case, in the exponentials we have replaced r into r_*, since in the remote region the spacetime is flat and we actually have $dr_* \sim dr$. Therefore the boundary condition for $Z_l(r)$ at infinity reads

$$Z_l(r) \sim \tfrac{1}{6}(l + 1/2)(-1)^l\omega[e^{-i\omega r_*} - (-1)^l S_l^g e^{i\omega r_*}] \qquad r \to \infty. \tag{71}$$

2. Boundary condition at the horizon

We now study the boundary condition in the vicinity of the event horizon r_H. Expanding $Z_l(r)$ near the horizon [26]

$$Z_l(r) = \left(\frac{r}{r_H} - 1\right)^{\rho}\sum_{n=0}^{\infty}c_{l,n}\left(\frac{r}{r_H} - 1\right)^n, \tag{72}$$

we have

$$\begin{aligned} \frac{dZ_l}{dr} &\sim \frac{p_l\rho}{r_H}\left(\frac{r}{r_H} - 1\right)^{\rho-1}, \\ \frac{d^2Z_l}{dr^2} &\sim \frac{p_l\rho(\rho - 1)}{r_H^2}\left(\frac{r}{r_H} - 1\right)^{\rho-2}, \end{aligned} \tag{73}$$

when $r \to r_H$. $p_l = c_{l,0}$ with the dimension of length^{-1}. Considering Eq. (18), in the vicinity of the event horizon equation (73) becomes

$$\frac{d^2Z_l}{dr_*^2} \sim A_1B_1\left(\frac{r}{r_H}-1\right)^2\frac{d^2Z_l}{dr^2}+\frac{A_1B_1}{r_H}\left(\frac{r}{r_H}-1\right)\frac{dZ_l}{dr} \sim \frac{p_l\rho^2(A_1B_1)}{r_H}\left(\frac{r}{r_H}-1\right)^\rho, \tag{74}$$

and

$$[\omega^2 - V_{\rm eff}(r)]Z_l \sim p_l\omega^2\left(\frac{r}{r_H}-1\right)^\rho. \tag{75}$$

Inserting Eqs. (74) and (75) into Eq. (60), we obtain

$$A_1B_1[\rho(\rho-1)+\rho]+r_H^2\omega^2=0, \tag{76}$$

where

$$\rho = \pm i\omega r_H/\sqrt{A_1B_1}. \tag{77}$$

Similar to the scalar case, we choose the minus sign in the ρ expression since there is only ingoing wave near the event horizon and nothing can escape from the black hole. The boundary condition at the horizon then reads

$$Z_l(r) \sim p_l(r/r_H-1)^{-i\omega r_H/\sqrt{A_1B_1}}, \qquad r\to r_H+0. \tag{78}$$

In accordance the behavior of $Q(t, r, \theta)$ at the horizon is

$$Q(t,r,\theta) \sim e^{-i\omega t}r_H(r/r_H-1)^{-i\omega r_H/\sqrt{A_1B_1}} \times \sum_{l=2}^{\infty} p_l C_{l+2}^{-3/2}(\cos\theta), \qquad r\to r_H+0. \tag{79}$$

C. Application of the Newman-Penrose formalism

The study of the gravitational perturbation and the gravitational energy absorption spectrum involves considerable algebraic complexity. There is a well developed approach to this problem provided by Teukolsky [27,28], where Newman-Penrose formalism was employed. In this section we will employ the formalism to calculate the energy flux coming from infinity and the energy falling into the black hole.

1. Formalism in the Kinnersley tetrad

This tetrad serves to calculate the incoming energy flux of the gravitational wave at infinity [32]. It is set to be

$$l^\mu = \frac{1}{A}(1, \sqrt{AB}, 0, 0), \tag{80}$$

$$n^\mu = \tfrac{1}{2}(1, -\sqrt{AB}, 0, 0), \tag{81}$$

$$m^\mu = \frac{1}{\sqrt{2}r}(0, 0, 1, i\csc\theta), \tag{82}$$

of which the components queue in the order (t, r, θ, φ). The corresponding spin coefficients are found to be

$$\kappa=\sigma=\lambda=\nu=\epsilon=\pi=\tau=0, \tag{83}$$

$$\rho=\frac{1}{r}\sqrt{\frac{B}{A}}, \qquad \alpha=-\beta=\frac{\cot\theta}{2\sqrt{2}r}, \qquad \mu=\frac{\sqrt{AB}}{2r}, \qquad \gamma=\frac{A_{,r}}{4}\sqrt{\frac{B}{A}}. \tag{84}$$

Here, taking $A = B$ recovers the Schwarzschild case.

Then we need to evaluate two contractions of the perturbation of the Riemann tensor with the Kinnersley tetrad, which play key roles in calculating the energy flux at infinity and the energy falling down the hole, and which hereafter are denoted by $\tilde{\Psi}_0^{(1)}$ and $\tilde{\Psi}_4^{(1)}$, defined as

$$\tilde{\Psi}_0^{(1)} = -\delta R_{\mu\nu\lambda\sigma}l^\mu m^\nu l^\lambda m^\sigma, \tag{85}$$

and

$$\tilde{\Psi}_4^{(1)} = -\delta R_{\mu\nu\lambda\sigma}n^\mu m^{\nu*} n^\lambda m^{\sigma*}. \tag{86}$$

To perform the contraction, a new orthonormal tetrad was introduced:

$$e^\mu_{(0)} = (e^{-\nu}, 0, 0, 0), \tag{87}$$

$$e^\mu_{(1)} = (0, e^{-\psi}, 0, 0), \tag{88}$$

$$e^\mu_{(2)} = (0, 0, e^{-\mu_2}, 0), \tag{89}$$

$$e^\mu_{(3)} = (0, 0, 0, e^{-\mu_3}), \tag{90}$$

where components are ordered by $(x^0, x^1, x^2, x^3) = (t, \varphi, r, \theta)$. The components of the Kinnersley tetrad against this tetrad basis are therefore

$$l^{(\mu)} = (e^{-\nu}, 0, e^{-\nu}, 0), \tag{91}$$

$$n^{(\mu)} = \tfrac{1}{2}(e^{\nu}, 0, -e^{\nu}, 0), \tag{92}$$

$$m^{(\mu)} = \frac{1}{\sqrt{2}}(0, i, 0, 1). \tag{93}$$

Here the indices in the bracket indicate the components against the basis of Eqs. (87)–(90), and they run through 0 to 3. Now we do the contraction of Eq. (85) and find

$$\tilde{\Psi}_0^{(1)} = -ie^{-2\nu}(\delta R_{0301}+\delta R_{2321}+\delta R_{2301}+\delta R_{0321}) - \tfrac{1}{2}e^{-2\nu}(\delta R_{0303}+2\delta R_{0323}+\delta R_{2323}-\delta R_{0101} - 2\delta R_{0121}-\delta R_{2121}), \tag{94}$$

as well as

$$\tilde{\Psi}_4^{(1)} = -\frac{i}{4}e^{2\nu}(-\delta R_{0301}-\delta R_{2321}+\delta R_{2301}+\delta R_{0321}) - \frac{1}{8}e^{2\nu}(\delta R_{0303}-2\delta R_{0323}+\delta R_{2323}-\delta R_{0101} + 2\delta R_{0121}-\delta R_{2121}). \tag{95}$$

Here, in both expressions, the second bracket would vanish in purely axial perturbation, whose demonstration was raised in Sec. 31 of [25] and applicable to all static, spherically symmetric metric. For convenience, we denote the nonvanishing part, which contains the first bracket, of Eqs. (94) and (95) to be $i\,\mathrm{Im}\tilde{\Psi}_0^{(1)}$ and $i\,\mathrm{Im}\tilde{\Psi}_4^{(1)}$, and the vanishing part, $\mathrm{Re}\tilde{\Psi}_0^{(1)}$ and $\mathrm{Re}\tilde{\Psi}_4^{(1)}$. The Riemann components therein can be simplified as (Sec. 13 of [25])

$$\delta R_{0301} = \tfrac{1}{2}e^{\psi-2\nu-\mu_3}Q_{03,0} - \tfrac{1}{2}e^{\psi-2\mu_2-\mu_3}Q_{23}\nu_{,r}, \quad (96)$$

$$\delta R_{2321} = e^{\psi-2\mu_2-\mu_3}\left[\left(\frac{1}{r}-\frac{1}{2}\mu_{2,r}\right)Q_{23}+\frac{1}{2}Q_{23,r}\right], \quad (97)$$

$$\delta R_{2301} = \tfrac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}[Q_{03,2} - Q_{02,3} + Q_{03}(\tfrac{1}{2}+\mu_{2,r}) - Q_{02}\cot\theta], \quad (98)$$

$$\delta R_{0321} = -\frac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}\left[Q_{20,3} + Q_{20}\cot\theta - Q_{23,0} + Q_{30}\left(\frac{1}{r}-\nu_{,r}\right)\right]. \quad (99)$$

Inserting these equations into Eqs. (94) and (95), we have

$$-\frac{e^{2\nu+2\mu_2}}{\sin\theta}\mathrm{Im}\tilde{\Psi}_0^{(1)} = \frac{Q_{23}}{r} + \frac{1}{2}Q_{23,r} + \frac{1}{2}e^{-\nu+\mu_2}Q_{23,0} - \frac{1}{2}(\mu_{2,r}+\nu_{,r})Q_{23} + \frac{1}{2}e^{-\nu+\mu_2}\left[Q_{03,r} + \left(\frac{2}{r}-\nu_{,r}+\mu_{2,r}\right)Q_{03}\right] + \frac{1}{2}e^{-2\nu+2\mu_2}Q_{03,0}, \quad (100)$$

and

$$-\frac{4e^{-2\nu+2\mu_2}}{\sin\theta}\mathrm{Im}\tilde{\Psi}_4^{(1)} = -\frac{Q_{23}}{r} - \frac{1}{2}Q_{23,r} + \frac{1}{2}e^{-\nu+\mu_2}Q_{23,0} + \frac{1}{2}(\mu_{2,r}+\nu_{,r})Q_{23} + \frac{1}{2}e^{-\nu+\mu_2}\left[Q_{03,r} + \left(\frac{2}{r}-\nu_{,r}+\mu_{2,r}\right)Q_{03}\right] - \frac{1}{2}e^{-2\nu+2\mu_2}Q_{03,0}. \quad (101)$$

With the aid of Eqs. (54)–(57), the two contractions can be simplified to be

$$2i\omega\,\mathrm{Im}\tilde{\Psi}_0^{(1)} = \sum_{l=2}^{\infty}\left\{\left(\frac{2\sqrt{AB}}{r} - \frac{A'B+AB'}{2\sqrt{AB}} - 2i\omega\right)\Lambda_{(-)}Z_l + \left[\frac{i\omega(AB'-A'B)}{2\sqrt{AB}} + V_{g,l}\right]Z_l\right\}\frac{C_{l+2}^{-3/2}(\cos\theta)}{rA^2\sin^2\theta}, \quad (102)$$

and

$$8i\omega\,\mathrm{Im}\tilde{\Psi}_4^{(1)} = \sum_{l=2}^{\infty}\left\{\left(\frac{2\sqrt{AB}}{r} - \frac{A'B+AB'}{2\sqrt{AB}} - 2i\omega\right)\Lambda_{(+)}Z_l + \left[\frac{i\omega(A'B-AB')}{2\sqrt{AB}} + V_{g,l}\right]Z_l\right\}\frac{C_{l+2}^{-3/2}(\cos\theta)}{r\sin^2\theta}, \quad (103)$$

where $V_{g,l}$ is the radial effective potential defined by Eq. (61) and $\Lambda_{(\pm)} = d/dr_* \pm i\omega$. It is easy to check that equating $A(r)$ and $B(r)$ would recover the result in the Schwarzschild case.

2. Formalism in the Hawking-Hartle tetrad

To calculate the gravitational wave energy flux on the horizon, the Kinnersley tetrad is no longer applicable because of their singular behavior at the horizon. We use the tetrad introduced by Hartle and Hawking instead [33], which represents a physical observer crossing the event horizon. It eliminates the singular behavior by imposing on the Kinnersley tetrad a rotation of the third class ([25], Sec. 8). In our problem the rotation parameter is $\Lambda = 2/A$. The tetrad is normalized so that on the basis of coordinate $(v = t + r_*, r, \theta, \varphi)$, $(l^{\mathrm{HH}})^v = 1$. This results in the Hawking-Hartle (HH) basis as follows:

$$(l^{\mathrm{HH}})^\mu = 1/2(1, \sqrt{AB}, 0, 0), \quad (104)$$

$$(n^{\mathrm{HH}})^\mu = (1/A, -\sqrt{B/A}, 0, 0), \quad (105)$$

and m^μ is unchanged. The components are arranged in the order of (t, r, θ, φ). Changing to the basis of $(\partial_v, \partial_r, \partial_\theta, \partial_\varphi)$, we have

$$(l^{\mathrm{HH}})^\mu = (1, \sqrt{AB}/2, 0, 0), \quad (106)$$

$$(n^{\mathrm{HH}})^\mu = (0, -\sqrt{B/A}, 0, 0), \quad (107)$$

again with m^μ unchanged. Thus, we obtain the well-behaved tetrad on the horizon, which is equivalent to a physical observer advancing along the direction l^{HH}. Hereafter we perform calculations in this Hawking-Hartle tetrad. We can derive new spin coefficients using their transformation rules under the third class of tetrad rotation ([25], Sec. 8); the one of importance is

$$\epsilon^{\rm HH} = -\frac{1}{2}\Lambda^{-2}D\Lambda = \frac{1}{4}\sqrt{\frac{B}{A}}\frac{dA}{dr}, \tag{108}$$

where D is the differential operator corresponding to the tangent vector l^μ. We define

$$\epsilon_0 = \epsilon^{\rm HH}_{r_H} = \frac{1}{4}\sqrt{\frac{B}{A}}\frac{dA}{dr}\bigg|_{r_H} = \frac{\sqrt{A_1B_1}}{4r_H}. \tag{109}$$

$2\epsilon_0$ is the surface gravity of the black hole ([25], Sec. 8). We also need to know that κ vanishes globally, ensuring the integral curves of $l^{\rm HH}$ to be null geodesics, and that $\rho^{\rm HH}$ vanishes at the horizon, since

$$\rho^{\rm HH} = \Lambda^{-1}\rho = \frac{\sqrt{AB}}{2r} \longrightarrow 0, \qquad (r \longrightarrow r_H). \tag{110}$$

Moreover, we present here two relations needed for calculating the energy falling into the hole. They are obtained with the perturbation effect, where the Hawking-Hartle tetrad is also perturbed so that $l^{\rm HH}$ is the generator of a congruence of null geodesics, and κ always stays zero. We start from two Ricci identities under any Newmann-Penrose basis, which are

$$D\sigma - \delta\kappa = \sigma(3\epsilon - \epsilon^* + \rho + \rho^*) + \kappa(\pi^* - \tau - 3\beta - \alpha^*) + \Psi_0, \tag{111}$$

$$D\rho - \delta^*\kappa = (\rho^2 + |\sigma|^2) + \rho(\epsilon + \epsilon^*) - \kappa^*\tau - \kappa(3\alpha + \beta^* - \pi) + \Phi_{00} \tag{112}$$

([25], Sec. 8). Here Ψ_0 and Φ_{00} are projections of the Weyl tensor and the Ricci tensor upon the tetrad. Linearizing these two and with the consequence from $\delta R_{\mu\nu} = 0$ that $\Phi_{00}^{(1)} = 0$, we find, under Hawking-Hartle tetrad,

$$D^{\rm HH}_{r_H}\sigma^{\rm HH(1)}_{r_H} = 2\epsilon_0\sigma^{\rm HH(1)}_{r_H} + \Psi_0^{\rm HH(1)}|_{r_H}, \tag{113}$$

$$D^{\rm HH}_{r_H}\rho^{\rm HH(1)} = 2\epsilon_0\rho^{\rm HH(1)}. \tag{114}$$

D. Gravitational wave energy absorption

1. Incoming energy flux at infinity

The general expression for the incoming energy flux is [25,27,34]

$$\frac{d^2E^{\rm (inc)}}{dtd\Omega} = \frac{1}{64\pi\omega^2}\lim_{r\to\infty} r^2|\tilde{\Psi}_0^{(1)}|^2, \tag{115}$$

where $\tilde{\Psi}_0^{(1)}$ is defined in Eq. (94). Considering the axial perturbation, we can replace $\tilde{\Psi}_0^{(1)}$ by ${\rm Im}\tilde{\Psi}_0^{(1)}$. Taking $r \to \infty$ in Eq. (102) and taking into account only the incoming wave, we have

$${\rm Im}\,\tilde{\Psi}_0^{(1)} \longrightarrow \sum_{l=2}^{\infty}(\Lambda_{(-)}Z_l^{\rm (inc)})\frac{C_{l+2}^{-3/2}(\cos\theta)}{r\sin^2\theta}, \qquad (r \longrightarrow \infty). \tag{116}$$

With the boundary condition at infinity, Eq. (71), we have

$$\Lambda_{(-)}Z_l^{\rm (inc)} \longrightarrow (-1)^{l+1}i\omega(2l+1)e^{-i\omega r_*}/6, \qquad (r \longrightarrow \infty). \tag{117}$$

Inserting Eqs. (116) and (117) into Eq. (115) and integrating the flux over all directions, we obtain the total incoming energy per unit time at infinity:

$$\Phi^{\rm (inc)} = \int_{4\pi}\frac{d^2E^{\rm (inc)}}{dtd\Omega}d\Omega = \frac{\omega^2}{2304\pi}\sum_{l=2}^{\infty}\int_{4\pi}\bigg[\frac{C_{l+2}^{-3/2}(\cos\theta)}{\sin^2\theta}\bigg]^2 d\Omega = \frac{\omega^2}{192}. \tag{118}$$

The last equality was obtained by using the orthogonality of the Gegenbauer polynomials [31],

$$\int_{-1}^{1}\frac{C_{l+2}^{-3/2}C_{l'+2}^{-3/2}}{(1-x^2)^2}dx = \frac{18\delta_{ll'}}{(2l+1)(l+2)(l+1)l(l-1)}, \tag{119}$$

and the summation

$$\sum_{l=2}^{\infty}\frac{2l+1}{(l+2)(l+1)l(l-1)} = \frac{1}{3}. \tag{120}$$

2. Energy falling into the hole

Calculating the gravitational wave energy absorption into the black hole is more complicated than that in the scalar case. We will follow Teukolsky's treatment [27] to deal with this problem.

With Eq. (114), following the procedure presented in [25], Sec. 98, we have the change in the area of the event horizon in the process of perturbation:

$$\frac{d^2\Sigma}{dvd\Omega} = \frac{r_H^2}{\epsilon_0}|\sigma^{\rm HH(1)}_{r_H}|^2, \tag{121}$$

where Σ denotes the area of event horizon, and the spin coefficient ϵ_0 is presented in Eq. (109). The first law of black hole thermodynamics relates the change of the horizon area to the change of the black hole internal energy,

$$dE = \frac{\bar{\kappa}}{8\pi}d\Sigma, \tag{122}$$

where $\bar{\kappa} = 2\epsilon_0$ is the surface gravity. Thus,

$$\frac{d^2\Sigma}{dvd\Omega} = \frac{4\pi}{\epsilon_0}\frac{d^2E^{\rm (abs)}}{dvd\Omega}. \tag{123}$$

Combining Eqs. (122) and (123), we have

$$\frac{d^2E^{(\mathrm{abs})}}{dvd\Omega} = \frac{r_H^2}{4\pi}|\sigma_{r_H}^{\mathrm{HH}(1)}|^2. \tag{124}$$

The spin coefficient $\sigma_{r_H}^{\mathrm{HH}(1)}$ is evaluated in Eq. (113), where

$$D_{r_H}^{\mathrm{HH}} = \partial_v = \partial_t \rightarrow -i\omega, \tag{125}$$

and consequently we get

$$\sigma_{r_H}^{\mathrm{HH}(1)} = -\frac{\Psi_0^{\mathrm{HH}(1)}|_{r_H}}{i\omega + 2\epsilon_0}. \tag{126}$$

Referring to the transformation of Weyl scalars under the rotation of the third class ([25], Sec. 8), we have

$$\Psi_0^{\mathrm{HH}(1)} = A^2\Psi_0^{(1)}/4. \tag{127}$$

Combining Eqs. (124), (126), and (127), we arrive at

$$\frac{d^2E^{(\mathrm{abs})}}{dvd\Omega} = \frac{r_H^2}{16\pi}\frac{|A^2\Psi_0^{(1)}|_{r_H}^2}{4\omega^2 + A_1B_1r_H^{-2}}. \tag{128}$$

Now we need to evaluate $\Psi_0^{(1)} = -\delta C_{\mu\nu\rho\lambda}l^\mu m^\nu l^\rho m^\lambda$,

$$\begin{aligned}\Psi_0^{(1)} = &-ie^{-2\nu}(\delta C_{0301} + \delta C_{2321} + \delta C_{2301} + \delta C_{0321})\\ &-\tfrac{1}{2}e^{-2\nu}(\delta C_{0303} + 2\delta C_{0323} + \delta C_{2323}\\ &-\delta C_{0101} - 2\delta C_{0121} - \delta C_{2121}).\end{aligned} \tag{129}$$

The Weyl tensors are defined as

$$\begin{aligned}C_{\mu\nu\rho\lambda} = R_{\mu\nu\rho\lambda} &- \tfrac{1}{2}(g_{\mu\rho}R_{\lambda\nu} - g_{\mu\lambda}R_{\rho\nu} + g_{\nu\rho}R_{\lambda\mu}\\ &- g_{\nu\lambda}R_{\rho\mu}) + \tfrac{1}{3}R(g_{\mu\rho}g_{\lambda\nu} - g_{\mu\lambda}g_{\rho\nu}).\end{aligned} \tag{130}$$

Three facts can help greatly simplify the evaluation, which are (i) only the diagonal components of the Ricci tensor is nonzero for a metric like Eq. (1), (ii) according to the perturbation equation, $\delta R_{\mu\nu} = 0$, and (iii) only $\delta g_{1\mu} \neq 0$ ($\mu = 0, 2, 3$). The evaluation shows that the components of the perturbed Weyl tensors in $\Psi_0^{(1)}$ all coincide with those of the Riemann's, except for δC_{0301} and δC_{2321}, which are

$$\delta C_{0301} = \delta R_{0301} + \tfrac{1}{2}R_{00}\delta g_{31} + \tfrac{1}{3}Rg_{00}\delta g_{31}, \tag{131}$$

$$\delta C_{2321} = \delta R_{2321} + \tfrac{1}{2}R_{22}\delta g_{31} + \tfrac{1}{3}Rg_{22}\delta g_{31}. \tag{132}$$

Therefore

$$\begin{aligned}\Psi_0^{(1)} = \tilde{\Psi}_0^{(1)} &+ \frac{ie^{-2\nu}}{2}(R_{00} + R_{22})\delta g_{31}\\ &+ \frac{ie^{-2\nu}}{3}R(g_{00} + g_{22})\delta g_{31}.\end{aligned} \tag{133}$$

This is further simplified by the fact that we carry out the contraction under the orthonormal basis, Eqs. (87)–(90), where for a spherical metric as Eq. (1) describes, there holds $g_{00} = -g_{22} = 1$, implying that $(g_{00} + g_{22})\delta g_{31} \equiv 0$. Also, we have the extended result from the calculation of the Schwarzschild model:

$$\begin{aligned}R_{00} &= -\frac{BA''}{2A} - \frac{A'B'}{4A} - \frac{BA'}{rA} + \frac{B}{4A^2}(A')^2;\\ R_{22} &= \frac{BA''}{2A} + \frac{A'B'}{4A} + \frac{B'}{r} - \frac{B}{4A^2}(A')^2,\end{aligned} \tag{134}$$

which gives that

$$R_{00} + R_{22} = \frac{B}{r}\left(\frac{B'}{B} - \frac{A'}{A}\right) \sim O(r/r_H - 1), \qquad (r \rightarrow r_H + 0). \tag{135}$$

That is, the term $R_{00} + R_{22}$ is of the first order small quantity as r tends to the horizon. On the other hand, it is easy to check that, near the horizon, we have $q_{3,2} \sim Q(r/r_H - 1)^{-1} \sim (r/r_H - 1)^{-i(\omega/\sqrt{A_1B_1})-1}$, so that $\delta g_{31} \sim q_3 \sim (r/r_H - 1)^{-i(\omega/\sqrt{A_1B_1})}$, and hence we have $(R_{00} + R_{22})\delta g_{31} \rightarrow 0$, as $r \rightarrow r_H + 0$. Therefore in our context

$$\Psi_0^{(1)} = i\,\mathrm{Im}\tilde{\Psi}_0^{(1)}. \tag{136}$$

Hence,

$$\frac{d^2E^{(\mathrm{abs})}}{dvd\Omega} = \frac{r_H^2}{16\pi}\frac{|A^2\,\mathrm{Im}\tilde{\Psi}_0^{(1)}|_{r_H}^2}{4\omega^2 + A_1B_1r_H^{-2}}. \tag{137}$$

In the vicinity of the event horizon, $r \rightarrow r_H + 0$, applying the boundary condition, Eq. (78), we get

$$\begin{aligned}A^2\,\mathrm{Im}\tilde{\Psi}_0^{(1)} &\rightarrow \sum_{l=2}^{\infty}[(\sqrt{A_1B_1}r_H^{-1} - 2i\omega)\Lambda_{(-)}Z_l]\frac{C_{l+2}^{-3/2}(\cos\theta)}{2i\omega r_H\sin^2\theta}\\ &\rightarrow -\sum_{l=2}^{\infty}[(\sqrt{A_1B_1}r_H^{-1} - 2i\omega)p_l(r/r_H - 1)^{-i\omega r_H/\sqrt{A_1B_1}}]\frac{C_{l+2}^{-3/2}(\cos\theta)}{r_H\sin^2\theta}.\end{aligned} \tag{138}$$

Inserting this expression into Eq. (137) and performing the integration over all directions, we have the total absorbed energy per unit time,

$$\Phi^{(\mathrm{abs})} = \int_{4\pi}\frac{d^2E^{(\mathrm{abs})}}{dvd\Omega}d\Omega = \frac{|p_l|^2}{16\pi}\int_{4\pi}\left[\frac{C_{l+2}^{-3/2}(\cos\theta)}{\sin^2\theta}\right]^2 d\Omega. \tag{139}$$

Employing the orthogonality of the Gagenbauer polynomials, Eq. (138) becomes

$$\Phi^{(\mathrm{abs})} = \frac{9}{4}\sum_{l=2}^{\infty}\frac{|p_l|^2}{2l+1}\frac{(l-2)!}{(l+2)!}. \tag{140}$$

3. The absorption cross section

We define the absorption cross section as the ratio of the total absorbed energy to the total incoming energy at infinity. Averaging the total incident energy over the event horizon, we have the incident energy flux in the analogous sense, that

$$j^{(\mathrm{inc})} = \frac{\Phi^{(\mathrm{inc})}}{4\pi r_H^2} = \frac{\omega^2}{1536\pi r_H^2}. \tag{141}$$

The total absorption cross section reads

$$\sigma_g^{\mathrm{abs}} = \frac{\Phi^{(\mathrm{abs})}}{j^{(\mathrm{inc})}} = \frac{1728\pi r_H^2}{\omega^2}\sum_{l=2}^{\infty}\frac{|p_l|^2}{2l+1}\frac{(l-2)!}{(l+2)!}. \tag{142}$$

The partial absorption cross section has the form

$$\sigma_{g,l}^{\mathrm{abs}} = \frac{1728\pi r_H^2}{\omega^2}\frac{|p_l|^2}{2l+1}\frac{(l-2)!}{(l+2)!}. \tag{143}$$

Now we proceed to evaluate p_l, which is necessary to obtain the absorption cross section.

By means of the Wronskian of the radial perturbation equation (60), for any two solutions $Z_l^{(1)}$ and $Z_l^{(2)}$, their Wronskian is

$$W[Z_l^{(1)}, Z_l^{(2)}] = \frac{C_l^{(12)}}{\sqrt{AB}}, \tag{144}$$

where $C_l^{(12)}$ is a complex number depending on the explicit form of $Z_l^{(1)}$ and $Z_l^{(2)}$. Applying this to the asymptotic solution both at infinity, as is shown in Eq. (71), and at the horizon, as is shown in Eq. (78), we have, for the region at infinity,

$$\begin{aligned} W[Z_l^*, Z_l] &= -\frac{i\omega^3(2l+1)^2}{72\sqrt{AB}}(1-|S_l^g|^2) \\ &= -\frac{i\omega^3(2l+1)^2}{72\sqrt{AB}}\Gamma_l^g, \end{aligned} \tag{145}$$

and for the region approaching the horizon,

$$W[Z_l^*, Z_l] = -\frac{2i\omega}{\sqrt{AB}}|p_l|^2. \tag{146}$$

Equating the above two equations gives

$$|p_l|^2 = \frac{\omega^2(2l+1)^2}{144}\Gamma_l^g, \tag{147}$$

where

$$\Gamma_l^g = 1 - |S_l^g|^2. \tag{148}$$

Inserting this $|p_l|^2$ into the expression of partial absorption cross sections, Eq. (143), we get

$$\sigma_{g,l}^{\mathrm{abs}} = 12\pi r_H^2(2l+1)\frac{(l-2)!}{(l+2)!}\Gamma_l^g, \tag{149}$$

which is analogous to Eq. (38) in the scalar case. In the high energy limit, the wave can easily penetrate the potential barrier and be absorbed by the black hole, $\Gamma_l^g \to 1$. Thus summing up both sides of Eq. (149) and applying the summation formula, Eq. (120), we get

$$\sigma_g^{\mathrm{abs}} \to 4\pi r_H^2, \qquad (\omega\to\infty). \tag{150}$$

E. Conservation of energy

Here we will justify the formalism for the energy absorption rate in Sec. III D 2. In the far away region, we can write Eq. (70) in the form

$$Q \sim e^{-i\omega t}\sum_{l=2}^{\infty} r(\mathcal{I}_l e^{-i\omega r_*} + \mathcal{R}_l e^{i\omega r_*})C_{l+2}^{-3/2}(\cos\theta), \qquad r\to\infty. \tag{151}$$

Near the horizon we can follow Eq. (79) and express

$$Q \sim e^{-i\omega t} r_H(r/r_H - 1)^{-i\omega r_H/\sqrt{A_1B_1}}\sum_{l=2}^{\infty}\mathcal{T}_l C_{l+2}^{-3/2}(\cos\theta), \quad r\to r_H+0, \tag{152}$$

where we have replaced p_l with $\mathcal{T}_l$ to avoid ambiguity. Our goal is to show that the difference between the incoming energy and the reflected energy by the potential barrier at infinity is all absorbed by the black hole,

$$\frac{dE^{(\mathrm{abs})}}{dt} = \frac{dE^{(\mathrm{inc})}}{dt} - \frac{dE^{(\mathrm{ref})}}{dt}. \tag{153}$$

For the incoming energy per unit time, we adopt Eq. (115), and get

$$\frac{dE^{(\mathrm{inc})}}{dt} = \sum_{l=2}^{\infty}\frac{|\mathcal{I}_l|^2}{16\pi}\Xi_l, \tag{154}$$

where we have employed the orthogonality of the Gegenbauer polynomials and we let

$$\int_{4\pi}\left[\frac{C_{l+2}^{-3/2}(\cos\theta)}{\sin^2\theta}\right]^2 d\Omega = \Xi_l \tag{155}$$

for simplicity. The reflected energy can be obtained similarly. Referring to Ref. [25], Sec. 98, the out flowing energy at infinity is

$$\frac{d^2E^{(\mathrm{ref})}}{dtd\Omega} = \frac{1}{16\pi\omega^2}\lim_{r\to\infty}|\tilde{\Psi}_4^{(1)}|^2. \tag{156}$$

Inserting Eq. (103) into it and using Eq. (151), we have

$$\frac{dE^{(\mathrm{ref})}}{dt} = \sum_{l=2}^{\infty}\frac{|\mathcal{R}_l|^2}{16\pi}\Xi_l. \tag{157}$$

The absorbed energy per unit time can be obtained by referring to the calculation in Sec. III D 2 and using Eq. (152):

$$\frac{dE^{(\mathrm{abs})}}{dt} = \sum_{l=2}^{\infty} \frac{|\mathcal{T}_l|^2}{16\pi} \Xi_l. \tag{158}$$

To complete the demonstration, we need to establish the relation among $\mathcal{I}_l$, $\mathcal{R}_l$, and $\mathcal{T}_l$. We do this by means of Wronskians of the radial equation (60). Referring to the last section where we have calculated the Wronskian, and by virtue of Eqs. (151) and (152), we have the Wronskian at infinity that

$$W[Z_l^*, Z_l] = 2i\omega(-|\mathcal{I}_l|^2 + |\mathcal{R}_l|^2)(AB)^{-1/2}, \tag{159}$$

and also the Wronskian at the horizon that

$$W[Z_l^*, Z_l] = -2i\omega|\mathcal{T}_l|^2(AB)^{-1/2}. \tag{160}$$

Equating the above two, we have the relation among $\mathcal{I}_l$, $\mathcal{R}_l$, and $\mathcal{T}_l$ that

$$|\mathcal{T}_l|^2 = |\mathcal{I}_l|^2 - |\mathcal{R}_l|^2. \tag{161}$$

Now we can relate Eqs. (154), (157), and (158) by Eq. (153), thus completing the demonstration. This demonstration shows that the application of Hawking-Hartle tetrad in our above discussions is well grounded on the law of energy conservation.

F. Numerical study for gravitational perturbation

In solving the radial equation (60), we choose a particular solution $\phi_l(r)$ to be normalized as

$$\phi_l(r) = p_l^{-1} Z_l(r). \tag{162}$$

In the remote region, it consists of the outgoing and ingoing waves, which is

$$\phi_l(r) = n_l^{(-)} \mathcal{N}_{l(+)} + n_l^{(+)} \mathcal{N}_{l(-)}, \tag{163}$$

where

$$\mathcal{N}_{l(\pm)}(r) = e^{\mp i\omega r_*} \sum_{n=0}^{\infty} \kappa_{l,n}^{(\pm)} \left(\frac{r_H}{r}\right)^n, \tag{164}$$

and the combination coefficients $n_l^{(\pm)}$ are called jost functions. To obtain the jost functions numerically, we calculate their Wronskian, by applying Eq. (144) we arrive at

$$W[\mathcal{N}_{l(+)}, \mathcal{N}_{l(-)}] = \frac{2i\omega}{\sqrt{AB}}. \tag{165}$$

The jost functions read

$$n_l^{(\pm)} = \mp \frac{\sqrt{AB}}{2i\omega} W[\phi_l, \mathcal{N}_{l(\pm)}]. \tag{166}$$

Comparing Eq. (163) with Eq. (71), we have

$$p_l = (-1)^l \frac{(2l+1)\omega}{12 n_l^{(-)}}, \tag{167}$$

$$S_l^g = (-1)^{l+1} \frac{n_l^{(+)}}{n_l^{(-)}}. \tag{168}$$

They are key quantities for determining the absorption cross section.

In practice, we insert Eq. (167) into Eqs. (142) and (143), and have

$$\sigma_{g,l}^{\mathrm{abs}} = 12\pi r_H^2 \frac{(l-2)!}{(l+2)!} \frac{2l+1}{|n_l^{(-)}|^2}, \tag{169}$$

and

$$\sigma_g^{\mathrm{abs}} = 12\pi r_H^2 \sum_{l=2}^{\infty} \frac{(l-2)!}{(l+2)!} \frac{2l+1}{|n_l^{(-)}|^2}. \tag{170}$$

Now we can do the numerical calculation. Substituting the CFM black hole metric for the gravitational perturba-

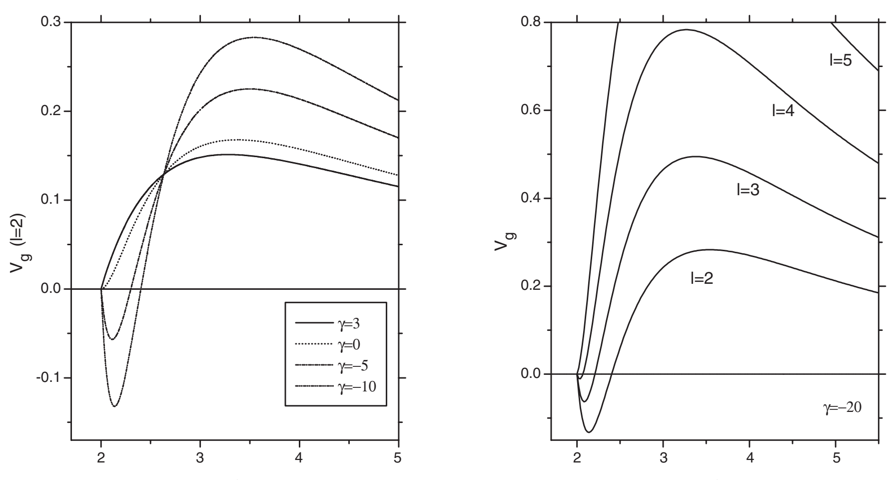

FIG. 4. The effective potentials of the gravitational perturbation for different γ and l.

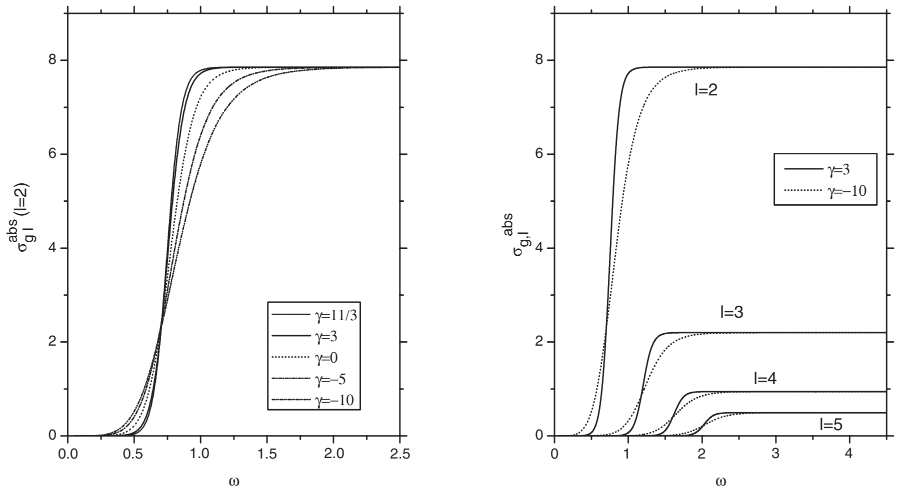

FIG. 5. Partial energy absorption of the gravitational wave for different γ, l.

tion, the effective potential has the form

$$V_{g,l}(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3} - \frac{M(\gamma - 3)(5r^2 - 20Mr + 18M^2)}{r^3(2r - 3M)^2}\right]. \quad (171)$$

Choosing $M = 1$, we display the behavior of the potential for the axial perturbation in Fig. 4. The effective potential is not positive definite. For enough negative value of γ, a negative peak will show up out of the horizon. The negative potential well becomes deeper when γ becomes more negative for fixed l. For chosen γ, the negative potential well appears for small l. Compared with the scalar case, in the gravitational perturbation, the negative wells appear before the potential barriers. It was argued that negative potential may result in the amplification of the perturbation out of the black hole and cause the spacetime to be unstable [35]. However, in the QNM study of the CFM brane-world black hole [10], it was shown that even for a very negative value of γ, the perturbative dynamics out of the black hole is always stable. The negative potential well loses the competition with the positive potential barrier and the perturbative dynamics is still dominated by the positive potential barrier.

In the study of the absorption, the potential well tends to enhance the absorption while the barrier tends to decrease it. The result on the partial absorption cross section is shown in Fig. 5. We find that, as the case observed in the QNM study [10], the potential barrier wins the competition with the well and dominates the contribution to the absorption rate. Fixing l, we observe that, for more negative γ, less gravitational wave energy is absorbed by the CFM brane-world black hole. For chosen γ, the absorption rate decreases with the increase of l due to higher potential barrier. In the low ω region, the absorption is enhanced for smaller values of γ, which is the effect of the negative potential well.

In the high energy limit, the partial absorption cross sections of the gravitational wave flatten out which is different from the case in the scalar wave. This is due to the difference in Eqs. (38) and (149), where in the scalar case the partial section is proportional to ω^{-2} in the high frequency regime, while in the gravitational wave case, the partial section is proportional to ω^0 in the high frequency

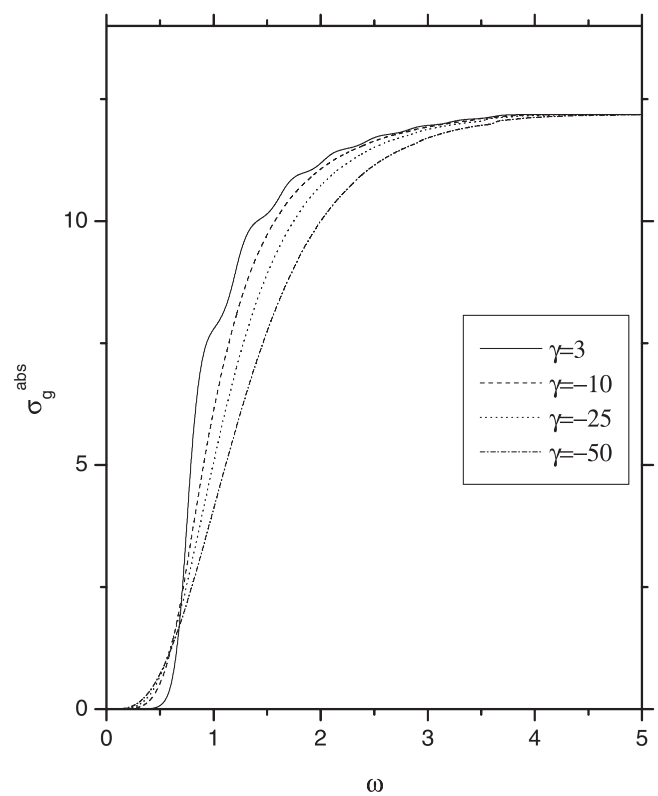

FIG. 6. Total energy absorption of the gravitational wave.

regime. The deep reason causing this difference lies in the different expansions of the plane wave. In the scalar case, the plane wave is expanded in terms of Legendre polynomials into spherical waves with a factor $(\omega r)^{-1}e^{\pm i\omega r}$ [Eq. (12)], while in the gravitational case, it is expanded in Gegenbauer polynomials into the wave with a factor $(\omega r)e^{\pm i\omega r}$ [Eq. (65)].

The result on the total absorption cross section of the gravitational wave is shown in Fig. 6. The total absorption cross section, which is the physically observable quantity, behaves in a similar manner to that in the scalar case. This shows that the difference in the gravitational partial absorption cross section from that of the scalar case is more mathematical than physical.

IV. CONCLUSION AND DISCUSSION

In this paper, we have studied the energy absorption problem of the scalar wave as well as the axial gravitational wave in the background of the CFM brane-world black hole. The CFM black hole is spherical and has only one event horizon at $r_H = 2M$. When the parameter $\gamma = 3$, the CFM brane-world black hole returns to the Schwarzschild solution. In comparison with Schwarzschild black hole, the CFM brane-world black hole will be either hotter or colder depending upon whether $\gamma < 3$ or $\gamma > 3$.

We have calculated the scalar perturbation and the axial gravitational perturbation around the brane black hole and evaluated the energy absorptions of the scalar and gravitational waves. Comparing with the 4D Schwarzschild black hole when $\gamma = 3$, we observed that the energy absorption of the brane-world black hole decreases with the decrease of γ starting from 3. While when $\gamma > 3$, we found that the energy absorption enhanced compared with that of the Schwarzschild black hole. We restricted to the case $\gamma \leq 11/3$. This result holds the same for both the scalar wave and axial gravitational wave outside the CFM hole. In both perturbations, the negative potential well appeared, however the positive potential barrier still dominated in determining the absorption rate, which has the same effect as observed in the QNM study. The result on the absorption spectrum implies that, for $\gamma < 3$, the black hole emission will be enhanced with the decrease of γ, while for $\gamma > 3$, the emission of the black hole will be suppressed, which is consistent with the behavior of the Hawking temperature of the brane-world black hole. We conclude that the energy absorption for the scalar and axial gravitational wave gives signatures of the bulk effects in the brane-world black hole, which differs from that of the 4D Schwarzschild black hole. We expect that these signatures can be observed in the future experiments, which could help us learn the properties of the extra dimensions.

In the study of the axial gravitation perturbation, the deduction of $\Psi_0^{(1)} = \tilde{\Psi}_0^{(1)}$ helps us a lot in doing the calculation, however the simplification does obviously not hold for polar perturbation. Thus, it is of interest to generalize our discussion in the future to study on the polar gravitation wave.

It needs to be emphasized that, although several spherically symmetric and static brane black hole solutions with contributions from the bulk gravity have been found, none of these are obtained as exact solutions of the full five-dimensional bulk field equations [15]. The propagation of gravity into the bulk does not permit the treatment of the brane gravitational field equations as a closed system. This fact limits the consideration of propagating modes of fields off the brane. Although the propagating modes in the bulk are hard to be obtained at the present moment, the modes on the brane are still interesting since they are the most phenomenologically interesting effects which can be detected during experiments. Furthermore, it was argued that the emission of particle modes on the brane is dominant compared to that off the brane [13]. Brane-localized modes have been investigated to disclose the information of extra dimensions in different attempts [10,11,14,17–19]. Despite that we cannot obtain the propagating modes in the bulk due to the lack of complete bulk solutions owing to the conceptually complicated gravitational field equations, our study of brane modes is still well motivated and interesting. Employing the Newman-Penrose formalism, we have provided a test of the energy absorption spectrum of a new brane-world black hole solution.

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