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Unimodular cosmology and the weight of energy

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Some models are presented in which the strength of the gravitational coupling of the potential energy relative to the same coupling for the kinetic energy is, in a precise sense, adjustable. The gauge symmetry of these models consists of those coordinate changes with unit Jacobian.

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I. INTRODUCTION

There are several aspects of the cosmological constant problem. One of them is the dynamical nature of dark energy, often parametrized by its equation of state $p = w\rho$. This shall be referred to as the *inverse cosmological constant problem*, in the sense that there are data to be explained by a theoretical construct. A pure cosmological constant corresponds to $w = -1$. This aspect is mainly cosmological, and progress on it can hopefully be made by refining the observational data, which nowadays seem to strongly favor $\Omega_\Lambda \sim 0.7$, corresponding to a mass scale

$$M_{\rm DE} \sim 10^{-12}\ \text{GeV} \tag{1}$$

and $w \sim -1$ [1].

Even if a dynamical component such as quintessence is discovered that explains fully the acceleration of the Universe, other problems remain. For example, it would still not be understood why the vacuum energy due to spontaneous symmetry breaking in quantum field theory does not produce a much larger cosmological constant. This shall be designated as the *direct cosmological constant problem*, in the sense that it refers to an unfulfilled theoretical prediction.

A drastic possibility is that the way spontaneous symmetry breaking occurs in quantum physics is not understood at all. It has been suggested, in particular, by R. L. Jaffe [2], that even the experimental confirmation of the Casimir energy [3] should not be considered as a proof of the existence of (zero point) vacuum energy in quantum field theory. This seems to be an extreme viewpoint. A clear experimental signal of vacuum energy as well as of the strength of its gravitational coupling would be most welcome.

As a matter of fact, even allegedly uncontroversial breakings contribute to the problem. For instance, chiral symmetry breaking due to the quark condensate yields a contribution of the order of $\Lambda_{\rm QCD}$, which is 11 orders of magnitude[1] above the purported range of observed values for the cosmological constant.

[1]Those figures get multiplied by a factor of four if the energy density is the quantity to be compared.

Turning again to the general aspects of the direct problem, the status is as follows. In order to not strongly disagree with observations, there are essentially two possibilities: either there is a cancellation in such a way that the low energy vacuum energy is very small (in natural units) or else there is a modification of gravity such that vacuum energy does not gravitate (or gravitates less than ordinary matter). This viewpoint has been advocated, for example, in [4], but the theories proposed there are badly nonlocal. Still (as has also been hinted at by those authors) there is an intriguing relationship between their approach and unimodular theories in general which has not been fully elucidated yet.

The purpose of the present work is to explore in some detail a setting in which the relative weight of the vacuum energy with respect to the kinetic energy can be tuned at will, and indeed in an extreme case the vacuum energy does not weigh at all. This will be done in a local theory that is a minor modification of general relativity (GR), in the sense that the gauge group is not the full set of general coordinate transformations (GC) [which will be interpreted in the active sense as diffeomorphisms spanning the group Diff(M)], but rather those that enjoy unit Jacobian. These particular transformations have been called *unimodular transformations*, and in a preceding paper the name transverse diffeomorphisms (TDiffs) spanning a subgroup TDiff(M) has been used. Please refer to [5] where other relevant references can also be found. To be specific, transverse diffeomorphisms (TDiffs) in a spacetime manifold whose points are described in a particular coordinate chart by $x^\mu(P)$, $\mu = 0, 1, \ldots, n-1$ are those diffeomorphisms

$$x^\mu \to y^\mu \tag{2}$$

such that the determinant of the Jacobian matrix equals unity:

$$D(y, x) \equiv \det\left(\frac{\partial y^\mu}{\partial x^\nu}\right) = 1. \tag{3}$$

In the linearized approximation, the Diff is expressed through a vector field

$$y^\mu = x^\mu + \xi^\mu(x), \tag{4}$$

and the above mentioned condition is equivalent to

$$\partial_\alpha \xi^\alpha = 0. \tag{5}$$

This is the reason of the qualifier *transverse* applied to them. Usually *tensor densities of weight w* are defined in such a way that they get an extra factor of the Jacobian to the power w in the tensorial transformation law. For example, a scalar density transforms as a one-dimensional representation of Diff(M), namely,

$$\phi'_w(y) = (D(y, x))^w \phi_w(x). \tag{6}$$

Whenever there is a metric (in the usual sense, a rank two tensor and not a tensor density), the determinant $g \equiv \det g_{\mu\nu}$ behaves as a scalar density of weight $w = -2$.

This means that as long as we assume that TDiff is the basic symmetry of nature, we do not distinguish tensor densities among themselves. In particular, given a certain scalar field, $\phi(x)$, all dressed fields $f(g)\phi(x)$ also behave as scalars under TDiff.

Vector fields inducing TDiffs can then be represented as

$$\xi^\alpha = \epsilon^{\alpha\mu_2\ldots\mu_n}\partial_{\mu_2}\Omega_{\mu_3\ldots\mu_n} = \epsilon^{\alpha\mu_2\ldots\mu_n}\nabla_{\mu_2}\Omega_{\mu_3\ldots\mu_n}, \tag{7}$$

where $\epsilon^{\mu_1\mu_2\ldots\mu_n}$ is the contravariant Levi-Civita tensor, and $\Omega_{\mu_3\ldots\mu_n}$ is completely antisymmetric, i.e., they are the components of a $(n-3)$-form.

II. SOME SIMPLE MODELS

We shall study a particular class of theories which enjoy TDiff (as opposed to full Diff) invariance. As explained before, we can dress the gravitational and the matter sectors with different functions of the determinant of the metric, namely,

$$S \equiv \int d^n x\left(-\frac{1}{2\kappa^2} f(g)R + f_m(g)L_m(g_{\mu\nu}, \phi_i^{(w_i)}, g)\right). \tag{8}$$

In this formula, $\kappa^2 \equiv 8\pi G_n$, G_n being the n-dimensional Newton's constant. ϕ_i $(i = 1\ldots N)$ represent matter fields, which enjoy arbitrary weights w_i on top of their own transformation properties under the group Diff(M). The determinant of the metric, g, is counted as a matter field with $w = -2$ for those purposes.

The simplest instance posits L_m as a full scalar, which will be assumed by simplicity not to depend on derivatives of the metric (so that all matter fields are minimally coupled). More general terms involving $\partial_\mu g$ are not included for the time being since they introduce additional unnecessary complications. That is, the Lagrangian to be considered in the present paper is

$$S \equiv \int d^n x\left(-\frac{1}{2\kappa^2} f(g)R + f_m(g)L_m(g_{\mu\nu}, \phi_i)\right) \tag{9}$$

(with all weights $w_i = 0$).

It should be remarked from the start than this action principle is not fully covariant; if it is assumed valid in a certain reference system (RS), in general coordinates it reads

$$S = \int d^n x \frac{1}{C(x)}\left(-\frac{1}{2\kappa^2} f(g(x)C(x)^2)R(x) + f_m(g(x)C(x)^2)L_m(g_{\mu\nu}(x), \phi_i(x))\right), \tag{10}$$

where $C(x) \equiv D(x, \bar{x})$ is the determinant of the Jacobian matrix with respect to some privileged original coordinates denoted here by $\bar{x}$. The field $C(x)$ can also be viewed as the determinant of the four vector fields $A^\mu_{(\bar{\alpha})}$, constructed from the derivatives of the coordinate functions with respect to the fiducial functions $\bar{x}^\mu(x)$. As long as the field $C(x)$ is kept in the Lagrangian, formal Diff invariance is ensured by the chain property

$$C(y) \equiv \det\frac{\partial y}{\partial \bar{x}} = \det\frac{\partial y}{\partial x}\det\frac{\partial x}{\partial \bar{x}} = D(y, x)C(x), \tag{11}$$

i.e., the composite field

$$g(x)C(x)^2 \tag{12}$$

is a gauge invariant construct under the gauge group Diff(M). This type of field is sometimes [6] called a *compensator field*. A notorious example is the Stueckelberg field which renders gauge invariant massive electrodynamics. The original theory can always be recovered in the analogue of the *unitary* gauge

$$C(x) = 1. \tag{13}$$

A. The source of gravity

There are several energy-momentum tensors of interest in this case. Actually, they are in general not true tensors under GC, only densities. The true energy-momentum tensor (that is, the source of the gravitational equations), is

$$T_{\mu\nu} \equiv \frac{\delta S_m}{\delta g^{\mu\nu}} \equiv \frac{\delta}{\delta g^{\mu\nu}}\int d^n x f_m L_m. \tag{14}$$

In order to study its conservation law, let us perform a TDiff (cf. [5]), under which

$$\delta g_{\alpha\beta} = \mathcal{L}(\xi)g_{\alpha\beta} \equiv \xi^\rho\partial_\rho g_{\alpha\beta} + g_{\alpha\rho}\partial_\beta\xi^\rho + g_{\rho\beta}\partial_\alpha\xi^\rho. \tag{15}$$

The use of covariant derivatives is best avoided for the time being. The fact that the quantities considered are not tensors under Diff has already been mentioned, and this can obscure the reasoning. Performing a TDiff on the matter action,

$$\begin{aligned}0 &= \delta_{T-\mathrm{diff}} S_m \\ &= \int d^n x (\epsilon^{\rho\mu_2\ldots\mu_n} \partial_{\mu_2} \Omega_{\mu_3\ldots\mu_n} \partial_\rho g_{\alpha\beta} \\ &\quad + g_{\alpha\rho} \partial_\beta (\epsilon^{\rho\mu_2\ldots\mu_n} \partial_{\mu_2} \Omega_{\mu_3\ldots\mu_n}) \\ &\quad + g_{\beta\rho} \partial_\alpha (\epsilon^{\rho\mu_2\ldots\mu_n} \partial_{\mu_2} \Omega_{\mu_3\ldots\mu_n})) T^{\alpha\beta}. \end{aligned} \tag{16}$$

Taking into account that $\epsilon^{\mu_1\cdots\mu_n}$ is independent of the metric, and denoting

$$\omega^{\mu\nu} \equiv \epsilon^{\mu\nu\mu_3\ldots\mu_n} \Omega_{\mu_3\ldots\mu_n}, \tag{17}$$

the aforementioned condition is equivalent to

$$0 = \int d^n x \omega^{\mu\nu} (-\partial_\mu g_{\alpha\beta} \partial_\nu T^{\alpha\beta} + 2\partial_\nu \partial_\lambda T_\mu{}^\lambda). \tag{18}$$

This means that

$$\begin{aligned}&\partial_\mu g_{\alpha\beta} \partial_\nu T^{\alpha\beta} - \partial_\nu g_{\alpha\beta} \partial_\mu T^{\alpha\beta} \\ &\quad = 2(\partial_\nu \partial_\lambda T_\mu{}^\lambda - \partial_\mu \partial_\lambda T_\nu{}^\lambda),\end{aligned} \tag{19}$$

which does imply

$$\partial_\lambda T_\mu{}^\lambda - \frac{1}{2} \partial_\mu g_{\alpha\beta} T^{\alpha\beta} = \partial_\mu \Phi, \tag{20}$$

where Φ is an arbitrary function. Using the well-known formula (valid for any symmetric tensor)[2]

$$\nabla_\nu S_\mu{}^\nu = \frac{1}{\sqrt{|g|}} \partial_\nu (\sqrt{|g|} S_\mu{}^\nu) - \frac{1}{2} \partial_\mu g_{\alpha\beta} S^{\alpha\beta}, \tag{21}$$

this can be rewritten as

$$\nabla_\nu \left(\frac{T_\mu{}^\nu}{\sqrt{|g|}} \right) = \frac{1}{\sqrt{|g|}} \partial_\mu \Phi \tag{22}$$

in the understanding that the covariant derivative is to be taken as if $T_{\alpha\beta}$ were a true tensor. Note in passing that (19) also implies

$$\partial_\lambda T_\mu{}^\lambda + \frac{1}{2} g_{\alpha\beta} \partial_\mu T^{\alpha\beta} = \partial_\mu \Phi'. \tag{23}$$

And the difference between both arbitrary functions is just the trace of the energy-momentum tensor

$$\Phi' - \Phi = g_{\alpha\beta} T^{\alpha\beta} = T. \tag{24}$$

On the other hand, it is clear from its definition that

$$T_{\mu\nu} = f_m \frac{\delta L_m}{\delta g^{\mu\nu}} - g f_m' L_m g_{\mu\nu}. \tag{25}$$

We have used the abbreviation $\frac{\delta L_m}{\delta g^{\mu\nu}}$ instead of the most accurate $\frac{\delta \int d^n x L_m}{\delta g^{\mu\nu}}$. It is interesting to study the nature of this tensor since as we have seen it is not possible to deduce its covariant conservation using only invariance under TDiffs.

[2] This formula is attributed by Eisenhart in [7] to none other than Einstein himself (1916).

B. Energy-momentum tensors

The label *energy-momentum tensor* for the above construct (25) can indeed be questioned for very good reasons. It is a metric (Rosenfeld) tensor which is not conserved, and consequently, it does not reduce in flat space to the canonical one, or to its equivalent Belinfante form (cf. [8] for a lucid discussion of the standard situation). That is, the tensor (25) does not convey the Noether current corresponding to translation invariance. In order to illustrate this, let us consider the simplest example, namely, a real scalar field without coupling to the determinant of the metric, i.e., $f_m(g) = 1$,

$$S_m \equiv \int d^n x L_m = \int d^n x \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{26}$$

The energy-momentum tensor as defined before is

$$T_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi. \tag{27}$$

Using the equation of motion (EM) of the scalar

$$\frac{\delta S_m}{\delta \phi} \equiv \partial_\mu (g^{\mu\nu} \partial_\nu \phi) = 0, \tag{28}$$

it can be shown that

$$\sqrt{|g|} \nabla_\nu \left(\frac{T_\mu{}^\nu}{\sqrt{|g|}} \right) = \frac{1}{2} \nabla_\mu L_m, \tag{29}$$

conveying that fact that this energy-momentum is not covariantly conserved, and thus it cannot act as a consistent source of Einstein's equations. What is worse, $T_{\mu\nu}$ does not reduce in flat space to the canonical one

$$T^{\mathrm{can}}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} L_m \eta_{\mu\nu}, \tag{30}$$

which is well-known to be conserved. This does not happen of course with the usual covariant Lagrangian

$$S_{\mathrm{cov}} = \int d^n x \sqrt{|g|} \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \tag{31}$$

whose energy-momentum tensor

$$T^{\mathrm{GR}}_{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} \left(\frac{1}{2} \sqrt{|g|} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} \sqrt{|g|} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right) \tag{32}$$

is both covariantly conserved thanks to the new EM and reduces to the canonical one in flat space.

C. The GR template

It is instructive to check all this setup applying it to the case of general relativity. On general grounds Diff(M) invariance forces $f = f_m = \sqrt{|g|}$. The corresponding energy-momentum tensor is then

$$T^{\mathrm{Diff}}_{\mu\nu} = \sqrt{|g|} \left(\frac{\delta L_m}{\delta g^{\mu\nu}} - \frac{1}{2} L_m g_{\mu\nu} \right). \tag{33}$$

As an exercise we will derive the well-known conservation of the GR energy-momentum tensor with our present techniques. Performing a Diff on the matter action and demanding it to be stationary,

$$0 = \delta_{\mathrm{Diff}} S = \int d^n x (\xi^\rho \partial_\rho g_{\alpha\beta} + g_{\alpha\rho}\partial_\beta \xi^\rho + g_{\rho\beta}\partial_\alpha \xi^\rho) T^{\alpha\beta}_{\mathrm{Diff}}. \quad (34)$$

Conveying the fact that

$$0 = \partial_\rho g_{\alpha\beta} T^{\alpha\beta}_{(\mathrm{Diff})} - \partial_\beta T^{(\mathrm{Diff})\beta}_{\rho} - \partial_\alpha T^{\alpha}_{(\mathrm{Diff})\rho} = -2\sqrt{|g|}\nabla_\alpha \left(\frac{T^{(\mathrm{Diff})\alpha}_{\rho}}{\sqrt{|g|}}\right). \quad (35)$$

Please note that precisely for this reason the energy-momentum tensor is usually defined without the $\sqrt{|g|}$ factor (and with a conventional factor of 2 as well),

$$T^{\mathrm{GR}}_{\alpha\beta} \equiv \frac{2}{\sqrt{|g|}} T^{\mathrm{Diff}}_{\alpha\beta}. \quad (36)$$

III. THE GRAVITATIONAL EQUATIONS OF MOTION

Once we have discussed in detail the source of gravity in these models, let us turn our attention to the complete equation. Including in the Einstein-Hilbert sector an arbitrary weight $f(g)$, the gravitational equation of motion reads

$$\frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{2\kappa^2}\left(f R_{\mu\nu} - g f' R g_{\mu\nu} - \sqrt{|g|}(\nabla_{(\mu}\nabla_{\nu)} - \nabla^2 g_{\mu\nu})\frac{f}{\sqrt{|g|}}\right) + T_{\mu\nu} = 0. \quad (37)$$

It is worth stressing that the EM (37) is not a tensor equation with respect to GC. This means that it must be solved in one RS, and the results in another RS are not the same if the transformation from one to the other is not a TDiff. This poses interesting problems of principle, many of them already discussed some time ago (cf. for example, the discussion of harmonic coordinates in [9]).[3]

[3] It is worth noticing that when

$$f = f_m = |g|^{1/n} \quad (38)$$

the EM are somewhat similar to Einstein's 1919 traceless equations (cf. [5]), namely,

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = |g|^{(n-2)/2n}(\nabla_{(\mu}\nabla_{\nu)} - \nabla^2 g_{\mu\nu})|g|^{(2-n)/2n} + 2\kappa^2 |g|^{-1/n} T_{\mu\nu}. \quad (39)$$

The second member is not automatically tracefree; consistency demands that

$$(1-n)\sqrt{|g|}\nabla^2 |g|^{-(n-2)/2} + 2\kappa^2 T = 0. \quad (40)$$

In the absence of matter (implying $T_{\mu\nu} = 0$ since we have not considered a cosmological constant term), the trace reads

$$\left(1 - n\frac{g f'}{f}\right) R = (1-n)\frac{\sqrt{|g|}}{f}\nabla^2 \frac{f}{\sqrt{|g|}}. \quad (41)$$

In the simplest case that $f = \sqrt{|g|}$ the equation is tensorial in character and coincides with the GR case. This automatically implies Ricci flatness, supposing $n \neq 2$. Concerning the general case, $f \neq \sqrt{|g|}$, in the absence of sources from the preceding trace equation we are not able to deduce that the scalar curvature vanishes. This is generically incompatible with the well-known solar system test of GR and is the main reason why we consider $f = \sqrt{|g|}$ in the following. The equation with sources is then

$$G_{\mu\nu} = \frac{2\kappa^2}{\sqrt{|g|}} T_{\mu\nu}. \quad (42)$$

A small paradox can be now disposed of. Let us assume, as seems obvious, that $G_{\mu\nu}$ is really a tensor, the Einstein tensor. Then it must obey Bianchi identities, which ensure that

$$\nabla_\alpha G^\alpha{}_\mu = 0, \quad (43)$$

so from the previous Eq. (42)[4]

$$\nabla_\alpha \left(\frac{T^\alpha{}_\mu}{\sqrt{|g|}}\right) = 0, \quad (48)$$

precisely the same integrability condition that appears in GR, where Diff invariance combined with the EM of matter imply covariant conservation of the energy-momentum tensor independently of Bianchi identities.

[4] When the true physical invariance is restricted TDiff, the same argument we used to arrive at (22) leads to

$$\nabla_\lambda \left(\frac{E_\mu{}^\lambda}{\sqrt{|g|}}\right) = \frac{1}{\sqrt{|g|}}\partial_\mu \Sigma, \quad (44)$$

where

$$E_\mu{}^\lambda \equiv f(g) R_\mu{}^\lambda - g f' R \delta^\lambda_\mu - \sqrt{|g|}(\nabla^\lambda \nabla_\mu - \nabla^2 \delta^\lambda_\mu)\frac{f}{\sqrt{|g|}} \quad (45)$$

is such that, when $f = \sqrt{|g|}$, it reduces to the Einstein tensor multiplied by the square root of the determinant of the metric:

$$E_\mu{}^\lambda = \sqrt{|g|} G_\mu{}^\lambda \quad (46)$$

and the gravitational equations of motion simply demand that

$$\partial_\mu (\Sigma - 2\kappa^2 \Phi) = 0. \quad (47)$$

It is nevertheless true that when $f = \sqrt{|g|}$ there is an enhanced Diff symmetry in the pure gravitational sector, so that it is to be expected that Bianchi identities remain valid.

From the TDiff viewpoint this is an extra condition that should be added for consistency but one that looks a bit mysterious.

The role of the compensators

In order to clarify the situation, let us consider the formally invariant theory defined with the compensator field. Introducing the compensator $C(x)$ changes the matter Lagrangian in the simplest case of a scalar to

$$S_m = \int \frac{d^n x}{C(x)} f_m(g(x)C(x)^2)L_m, \tag{49}$$

and the EM for the compensator is simply

$$\frac{\delta S_m}{\delta C(x)} = -\frac{1}{C^2} f_m L_m + \frac{1}{C}\frac{\partial f_m}{\partial C} L_m = 0, \tag{50}$$

the only solutions being either $L_m = 0$ or $f_m \sim C$. This last solution implies $f_m \sim \sqrt{|g|}$, as one should expect because it corresponds to Diff invariance and the compensator is just looking for it. If we do not want to impose Diff invariance from the very beginning, we are forced to impose the constraint

$$L_m = 0. \tag{51}$$

The meaning of this is that in this sector, the TDiff energy-momentum tensor is a consistent source of Einstein's equations: the constraint ensures the integrability condition (48) holds, although it is somewhat stricter than necessary [see (29)]. In any case, from this more elegant point of view, conservation of the energy-momentum tensor in the sense of (48) is just a consequence of compensator dynamics.

Similar results are obtained when the model is generalized by allowing matter fields $\phi^{(w)}$ to become scalar densities of weight w and considering Diff invariant Lagrangians

$$S_m = \int d^n x \frac{1}{C(x)} f_m(g(x)C(x)^2)L_m(\phi^{(w)}C(x)^{-w}). \tag{52}$$

The EM for the compensator now takes the form

$$\frac{\delta S_m}{\delta C(x)} = -\frac{1}{C^2} f_m L_m + \frac{1}{C}\frac{\partial f_m}{\partial C} L_m + \frac{1}{C} f_m \frac{\delta L_m}{\delta C} = 0. \tag{53}$$

But given the functional dependence of L_m on the combination $\phi^{(w)}C(x)^{-w}$ it is easy to convince oneself that

$$\frac{\delta L_m}{\delta C} \propto \frac{\delta L_m}{\delta \phi^{(w)}}. \tag{54}$$

Therefore, (50) is recovered under the assumption that the scalar field verifies its EM.

IV. TDIFF FRIEDMANN MODELS

A. Weight one

Let us study in detail the extreme case in which the matter Lagrangian does not couple to the determinant of the metric. This means $f_m = 1$ in our previous notation.

Let us stress that we do not expect these models to be realistic. In practice $f_m \sim f_g$. Our aim in this paragraph is a sort of *existence proof*; i.e., to show that it is possible to build theoretically consistent models in the unimodular framework in which the potential energy does not weigh at all, and consequently, models that solve the direct cosmological constant problem.

The simplest example is again a minimally coupled scalar field:

$$L_m = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi), \tag{55}$$

where $V(\phi)$ is a polynomial representing the potential energy, and which does not contribute to the gravitational EM (this is the main motivation for assuming $f_m = 1$). This point is so important that it is worth emphasizing: not all energy is a source of the gravitational field in TDiff theories, but only the kinetic part of it.

The preceding paragraph would be exactly true were it not for the compensators. Actually, all energy interacts with the gravitational field through compensator exchange. This point shall be hopefully clarified in the sequel.

In the final section of this paper we will introduce a framework which allows in principle an experimental test of this hypothesis and, in general, of the strength of the coupling of the potential energy to the gravitational field.

The energy-momentum tensor reads

$$T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi. \tag{56}$$

This means that it is not only the constant piece, but rather the full potential energy density that does not generate gravitational field. The matter EM are

$$\partial_\mu(g^{\mu\nu}\partial_\nu\phi) + V'(\phi) = 0. \tag{57}$$

Plus the EM for the compensator, which is again

$$L_m = 0. \tag{58}$$

This equation is the one that feeds back the potential energy into the gravitational equations, although in a very unusual way.

B. Friedmann

In order to get an idea of the simplest cosmological consequences of the TDiff viewpoint, it would be convenient to assume a (spatially) flat Friedmann metric:

$$ds^2 = dt^2 - R(t)^2\delta_{ij}dx^i dx^j \equiv dt^2 - R(t)^2(dx^2 + dy^2 + dz^2). \tag{59}$$

This particular form of the metric is written, however, in the so-called synchronous gauge

$$g_{00} = 1, \qquad g_{0i} = 0 \tag{60}$$

(defining what are known as Robertson-Walker[5] coordinates) which is not accessible in general using TDiffs only, since we have now less gauge freedom than in general relativity because of the transversality condition, or equivalently we have already partially fixed the gauge by choosing $C(x) = 1$.

The simplest form of a metric that can be reached with TDiff is

$$\begin{aligned} ds^2 &= a(t)dt^2 - R(t)^2\delta_{ij}dx^i dx^j \\ &\equiv a(t)dt^2 - R(t)^2(dx^2 + dy^2 + dz^2). \end{aligned} \tag{61}$$

The corresponding Einstein tensor is

$$\begin{aligned} G_{00} &= 3\frac{\dot{R}^2}{R^2} \equiv 3H(t)^2, \\ G_{ij} &= \frac{\dot{a}\dot{R}R - 2a\ddot{R}R - a\dot{R}^2}{a^2}\delta_{ij}. \end{aligned} \tag{62}$$

So the spacelike sector of Einstein's equations then imply that

$$\partial_i \phi = 0, \tag{63}$$

and the gravitational equations take the simple form (using the compensator EM)

$$\begin{aligned} &3H^2 = \frac{\kappa^2}{a^{1/2}R^3}\dot{\phi}^2 = \frac{2a^{1/2}\kappa^2}{R^3}V(\phi), \\ &3aH^2 + 2a\dot{H} - \dot{a}H = 0. \end{aligned} \tag{64}$$

Since we have now an additional constraint imposed by the EM of the compensator, or in other language by the conservation of the energy-momentum tensor, it is interesting to check the compatibility of the whole system of equations. The usual counting of degrees of freedom in GR is as follows. In four dimensions the metric has 10 independent components, 4 of which can be gauge fixed. Then one has 6 variables for 10 Einstein's equations, but again using the Bianchi identities it is certain that four of them are combinations of the others, so finally one is left with 6 equations for 6 variables.

When we consider a TDiff model such as the one above, the freedom to fix the gauge is smaller so that we have to determine 7 components of the metric. The Bianchi identities are nevertheless satisfied and then there are 6 Einstein's equations. The consistency is saved finally by the EM of the compensator that provides the 7th equation for the 7 variables. Let us prove this assertion in the particular model considered, i.e., the scalar Lagrangian (55) with the metric (61). Conservation of the energy-momentum tensor forces the Lagrangian to be a constant, so differentiating (55) with respect to time,

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\dot{\phi}^2}{a}\right) - V'(\phi)\dot{\phi} = 0. \tag{65}$$

Using the EM of the scalar to eliminate the potential, after some straightforward algebra we get the condition (supposing $\dot{\phi} \neq 0$)

$$\frac{\ddot{\phi}}{\dot{\phi}} = \frac{3}{4}\frac{\dot{a}}{a}. \tag{66}$$

On the other hand, the first of the Einstein's equations in (64) implies

$$\frac{\ddot{\phi}}{\dot{\phi}} = \frac{\dot{H}}{H} + \frac{3}{2}\frac{\dot{R}}{R} + \frac{1}{4}\frac{\dot{a}}{a}. \tag{67}$$

Finally, equating these last two expressions allows us to derive the second Einstein's equation in (64). We conclude that the system of equations must be compatible and the three functions ($\phi(t)$, $a(t)$, $R(t)$) can be determined. This same consistency is not found if we choose $a(t) \equiv 1$ from the beginning,[6] except in the particular case in which the potential is a constant, and this will be related with the impossibility to get exponential expansion, as we will see.

It is easy to find the explicit solution in the absence of potential. If the matter Lagrangian is fixed to be a constant $L_m = L$ (the freedom to fix the constant is lost when the EM for the compensator is used so that $L = 0$), then the component of the metric $a = a_0$ is also constant and the scale factor goes as

$$H(t) = \frac{H_0}{1 + \frac{3}{2}H_0 t} \Rightarrow R(t) = R_0\left(1 + \frac{3}{2}H_0 t\right)^{2/3}, \tag{68}$$

where $R_0^3 = \frac{2a_0^{1/2}\kappa^2 L}{3H_0^2}$. The time evolution of the scalar is linear and given by

$$\phi(t) = \phi_0 \pm \sqrt{2La_0}t. \tag{69}$$

Once we choose to impose the constraint $L = 0$ the only solution if the potential vanishes is a constant field with null Hubble constant, but a remains undetermined.

C. The GR template

These equations ought to be contrasted with the standard GR ones, where the synchronous gauge $a = 1$ is fully accessible. The symmetry of the situation enforces again $\partial_i \phi = 0$ so that for a general potential they read

[5]The use of a specific set of Robertson-Walker coordinates is not as innocent as it seems, and physics does depend on this choice to a certain extent in the present framework.

[6]The inconsistency is between conservation of the energy-momentum tensor, or compensator's EM, and the scalar EM.

$$3H(t)^2 = \kappa^2\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right),$$
$$2\dot{H}(t) + 3H(t)^2 = -\kappa^2\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right), \tag{70}$$

together with the GR equation of motion for the scalar field

$$\ddot{\phi} + 3H(t)\dot{\phi} + V'(\phi) = 0. \tag{71}$$

When the potential vanishes, the system is easy to solve giving a Hubble constant

$$H(t) = \frac{H_0}{1 + 3H_0 t} \Rightarrow R(t) = R_0(1 + 3H_0 t)^{1/3} \tag{72}$$

and a time dependence for the scalar field

$$\phi(t) = \phi_0 \pm \frac{\sqrt{6}}{3\kappa}\log(1 + 3H_0 t). \tag{73}$$

On the other hand, solutions that describe an exponentially expanding universe are very interesting phenomenologically. It is well-known what is the origin of exponential expansion in GR: a positive constant energy density V_0 yields

$$R(t) = R_0 e^{\kappa\sqrt{(V_0/3)}t}. \tag{74}$$

D. No TDiff exponential expansion

It is of interest to examine now the conditions under which there is exponential expansion in TDiff cosmology; that is, conditions for which TDiff behaves physically in a way similar to GR with a cosmological constant. When $\dot{H} = 0$ but $H = H_0$ itself is nonvanishing,

$$R = R_0 e^{H_0 t}, \tag{75}$$

and the second of the Eqs. (64) yields

$$a = a_0 e^{3H_0 t}. \tag{76}$$

We can immediately solve for the scalar

$$\phi(t) = \phi_0 \pm \frac{4}{9\kappa}(3a_0^{1/2}R_0^3)^{1/2}(e^{(9/4)H_0 t} - 1), \tag{77}$$

which corresponds (using the compensator's EM) to a potential

$$V(\phi) = \frac{3H_0^2 R_0^3}{2\kappa^2 a_0^{1/2}}\left(1 \pm \frac{9\kappa}{4(3a_0^{1/2}R_0^3)^{1/2}}(\phi - \phi_0)\right)^{2/3}. \tag{78}$$

This expansion is, however, not exponential with respect to comoving proper time defined by

$$dT = a^{1/2}dt. \tag{79}$$

In order for the expansion to be exponential in T, it would have to obey

$$\frac{dR}{a^{1/2}dt} = RH_0. \tag{80}$$

This yields in the second equation of the set (64)

$$3a^2H_0^2 = 0. \tag{81}$$

This means that there is no truly (proper time) exponential expansion in this class of TDiff cosmologies, so that the direct cosmological constant problem appears in a new light: not only a constant term in the potential does not gravitate, but there is no way to get the gravitational field which is produced by such a term in GR.

E. Adjustable coupling gravity/potential energy

Let us consider the general case in which the matter Lagrangian (55) is coupled to the determinant of the metric to an arbitrary power, i.e., the action takes the form

$$S_m = \int d^n x g^b\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right). \tag{82}$$

This framework allows us to search for solutions that depend on the strength of the coupling between matter and the determinant of the metric parametrized in b and at the end hopefully measurable physical consequences.

Conservation of the energy-momentum tensor in the sense of (48) forces the Lagrangian to verify the integrability condition

$$g^{b-1/2}\left(\frac{1}{2} - b\right)\partial_\mu L_m = bL_m\partial_\mu g^{b-1/2}, \tag{83}$$

where we have used the scalar equation of motion

$$\partial_\mu(g^b g^{\mu\nu}\partial_\nu\phi) + g^b V' = 0. \tag{84}$$

The previous equation is less restrictive than the one coming from the compensator EM, as we have already mentioned, but is also less motivated. Note that it is identically verified for the GR case which corresponds to $b = \frac{1}{2}$. The condition of a constant Lagrangian corresponding to $b = 0$, i.e., the model studied before, is also reproduced. Using these last two equations together with the timelike sector of Einstein's equations

$$3H^2 = 2\kappa^2 a^{b-1/2}R^{6(b-1/2)}a\left((1-b)\frac{\dot{\phi}^2}{2a} + bV\right), \tag{85}$$

one can reproduce in a similar way as before the spacelike Einstein's equation

$$\dot{a}H - 3aH^2 - 2a\dot{H} = 2b\kappa^2 a^{b-1/2}R^{6(b-1/2)}a^2\left(\frac{\dot{\phi}^2}{2a} - V\right) \tag{86}$$

under the assumption $b \neq 1, \frac{1}{2}$. The GR limit has been studied in the previous subsection, and the case $b = 1$ is somehow special. Apart of these subtleties, the consistency of the system of equations is ensured. In fact, in the absence of potential it is easy to find the solution as a function of b

$$H(t) = \frac{H_0}{1 + \frac{3(b-1/2)}{(b-1)} H_0 t} \Rightarrow R(t) = R_0 \left(1 + \frac{3(b-\frac{1}{2})}{(b-1)} H_0 t\right)^{(b-1)/3(b-1/2)}. \quad (87)$$

Moreover, the temporal component of the metric is no longer a constant but

$$a(t) = a_0 \left(1 + \frac{3(b-\frac{1}{2})}{(b-1)} H_0 t\right)^{2b/((1/2)-b)}, \quad (88)$$

and the scalar goes linearly in time independently of b

$$\phi(t) = \phi_0 \pm \sqrt{\frac{3}{\kappa^2(1-b)}} H_0 a_0^{1/2((1/2)-b)} R_0^{3((1/2)-b)} t. \quad (89)$$

V. CONCLUSIONS

A family of models has been studied with slightly smaller gauge symmetry than the full set of general coordinate transformations of general relativity. Namely, they enjoy symmetry under unimodular transformations, that is, diffeomorphisms with unit determinant of the Jacobian matrix. They generate a group that is called TDiff(M).

In some simplified TDiff cosmological models it has been found that exponential expanding solutions are inconsistent. This seems to alleviate the direct problem of the cosmological constant.

The main reason, however, why those solutions are physically interesting is that they allow us to tune, in a precise sense, the relative weight of the kinetic and potential energy. There are models, in particular, in which the potential energy (and *a fortiori* the vacuum energy) does not couple to the gravitational field, so that it appears that it does not weigh at all.[7] This provides a framework to test the gravity/potential energy coupling, which violates the *equivalence principle* inasmuch as it is different from the gravity/ kinetic energy coupling (cf. Damour's contribution in [10]). Experiments are difficult but perhaps not impossible (weighing the Casimir energy?).

The vacuum gravitational equations for the TDiff models studied in this paper are exactly the same as the GR ones, so that all solar systems tests are also fulfilled. There might be some subtle points with the derivation of the binary pulsar tests [11] worth a detailed study.

Let us finally discuss if this framework could alter the results of an Eötvös type experiment. If a WKB expansion is performed [12] in the EM for the scalar field

$$\phi = e^{\frac{i}{\epsilon}\sum \epsilon^n \phi_n}, \quad (90)$$

then, to dominant order $(1/\epsilon^2)$, and defining $k_\mu \equiv \partial_\mu \phi_0$, the mass shell condition

$$k^2 = m^2 \quad (91)$$

is recovered, as well as the geodesic equation in the form

$$k^\alpha \nabla_\alpha k_\beta = 0. \quad (92)$$

All this is quite similar to the GR template. The subtle differences in the coupling to the gravitational field in the scalar EM do not appear to this order in the WKB expansion. The physical meaning of this result is that unimodular cosmology predicts exactly the same results as GR for free falling of test bodies (to which refer current experiments on the equivalence principle).

Nevertheless, it has been claimed in [13] that the huge body of data on the Eötvös experiment for different substances puts constraints (barring accidental cancellations) on the possible violations of the equivalence principle on the potential energy (less than 10^{-10}) and the kinetic energy separately (less than 10^{-7}). This then puts corresponding constraints on the ratio $\frac{f_m}{f_g}$. Hypothetical positive results could be interpreted in the framework of our formalism.

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[7] Modulo some subtleties discussed in the main text, namely, compensator exchange.

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