

# Open string with a background $B$ field as the first order mechanics, noncommutativity, and soldering formalism

A. A. Deriglazov,<sup>1,\*</sup> C. Neves,<sup>2,†</sup> W. Oliveira,<sup>2,‡</sup> E. M. C. Abreu,<sup>3,§</sup> C. Wotzasek,<sup>4,||</sup> and C. Filgueiras<sup>5</sup>

<sup>1</sup>*Departamento de Matemática, ICE, Universidade Federal de Juiz de Fora, 36036-330 Juiz de Fora, Minas Gerais, Brazil*

<sup>2</sup>*Departamento de Física, ICE, Universidade Federal de Juiz de Fora, 36036-330 Juiz de Fora, Minas Gerais, Brazil*

<sup>3</sup>*Departamento de Física, Universidade Federal Rural do Rio de Janeiro, BR 465-07, 23851-180 Seropédica, Rio de Janeiro, Brazil*

<sup>4</sup>*Instituto de Física, Universidade Federal do Rio de Janeiro, 21945 Rio de Janeiro, Brazil*

<sup>5</sup>*Departamento de Física, CCEN, Universidade Federal da Paraíba, Cidade Universitária, 58051-970 João Pessoa, Paraíba, Brazil*

(Received 24 May 2007; published 12 September 2007)

To study noncommutativity properties of the open string with constant  $B$  field, we construct a mechanical action that reproduces classical dynamics of the string sector under consideration. It allows one to apply the Dirac quantization procedure for constrained systems in a direct and unambiguous way. The mechanical action turns out to be the first order system without taking the strong field limit  $B \rightarrow \infty$ . In particular, it is true for the zero mode of the string coordinate, which means that the noncommutativity is an intrinsic property of this mechanical system. We describe the arbitrariness in the relation existing between the mechanical and the string variables and show that noncommutativity of the string variables on the boundary can be removed. This is in correspondence with the result of Seiberg and Witten on the relation among noncommutative and ordinary Yang-Mills theories. The recently developed soldering formalism helps us to establish a connection between the original open string action and the Polyakov action.

DOI: [10.1103/PhysRevD.76.064007](https://doi.org/10.1103/PhysRevD.76.064007)

PACS numbers: 04.60.Ds, 11.15.-q, 11.25.-w, 11.30.Rd

## I. INTRODUCTION

Noncommutative Yang-Mills theory arises in a definite limit of string theory [1]. It has been extracted by Seiberg and Witten [2] starting from the open string in the presence of a  $B$  field [3–9], with the action for the corresponding sector being [2]

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma [\partial_a x^i \partial_a x^i + 2\pi\alpha' \epsilon^{ab} \partial_a x^i \partial_b x^j B_{ij}], \quad (1)$$

where we adopted Euclidean metrics both on the world sheet and in spacetime. An extremum of the action is supplied by

$$(\partial_\tau^2 - \partial_\sigma^2)x^i = 0, \quad (2)$$

$$\{\partial_\sigma x^i + 2\pi\alpha' \partial_\tau x^j B_j^i\}_{\sigma=0}^{\sigma=\pi} = 0. \quad (3)$$

An open string propagator with the boundary conditions (3) contains an antisymmetric matrix [3,6], which gives rise to noncommutativity of the string coordinate on the boundaries [2,7]. It was suggested [8–11] that the noncommutative geometry can be reproduced in the elementary framework of constrained systems as a result of the Hamiltonian quantization of the system (2) and (3), by analogy with the noncommutativity arising for coordinates

of a charged particle in the lowest Landau level [12–14]. To achieve this, rather radical modifications [8,11] of the Dirac procedure for constrained systems were proposed. There is some discrepancy among the results obtained in different approaches, which was discussed in [8–11,15–17]. Let us talk about these issues for a while in order to put our work into perspective.

In [8] the authors analyze the connection among the various appearances of noncommutativity in string theory,  $M$  theory and the  $M$ (atrix) model. The work relates the noncommutativity in  $M$ (atrix) compactification to a phenomenon of noncommutativity of space due to mixed boundary conditions in string theory (see [8] and references therein). It was shown that the noncommutativity of the canonical commutation relations of the space coordinates of the mixed membrane reflects the zero brane distribution inside the  $D$  membrane and therefore is closely related to the noncommutativity of the  $D$ -brane dynamics. The presence of a Kalb-Ramond field requires the consideration of wrapping the mixed branes over a torus in order to recover the noncommutativity of the torus of the  $M$ (atrix) model compactification. The  $T$  duality helps to obtain the mass spectrum of the open strings compactified on a noncommutative torus.

The generalization for the case of a  $D$  string on a torus in a nonzero  $B_{\mu\nu}$  background field, i.e., a moving  $D$  string with a nonzero electric field, was obtained in [9] using a Dirac-Born-Infeld action in a  $B$ -field background that gives the dynamics of  $D$  strings. The Hamiltonian formulation helps in the construction of the mass spectrum of the membrane and the full Bogomol'nyi-Prasad-Sommerfield monopoles spectrum. It was shown that the noncommuta-

\*alexei.deriglazov@ufjf.edu.br

†cneves@fisica.ufjf.br

‡wilson@fisica.ufjf.br

§evertonabreu@ufrj.br

||clovis@if.ufrj.br

tivity of brane coordinates comes about naturally in the formulation of open strings in the background  $B$  field, as well as the noncommutativity of the torus. The  $C^*$  algebra of functions on the noncommutative torus was constructed using the noncommutative open string position operators.

In both papers above it was shown that noncommutativity can be derived within the string theory by embedding branes with background  $B_{ij}$  field on the compactification torus. In [10] the authors used the basic principles of Dirac quantization in order to discuss the noncommutativity of the internal coordinates of branes. It was shown that the noncommutativity of the coordinates of the open string-brane system is intrinsic, namely, there is no gauge in which the noncommutativity can be removed. There are alterations in the noncommutativity of the system since the gauge keeps changing, but it never disappears. And the position of noncommutativity is always on the brane. In [10], the mixed boundary conditions were treated as the Dirac constraints and the quantization was obtained through the Dirac method without mode expansions.

On the other hand, in [11], the mixed boundary conditions make the canonical quantization of the theory nontrivial. An inconsistency was obtained when standard commutation relations were imposed. This inconsistency was removed through the relaxation of the commutativity of the space coordinates of the open strings along the direction of the brane described by mixed boundary conditions. The analysis was based on a time averaged symplectic form. The conclusion was that mixed open strings are noncommutative only at the end, where open strings are attached to the brane, and the center of the mass coordinates is commuting. Moreover, the noncommutative end coordinates lead to the conclusion that the  $D$ -brane worldvolume is a noncommutative space. And the  $D$ -brane worldvolume does not need to be a compact space in order to have noncommutativity. The quantization method used in [11] was based on the string mode expansion.

In [15] the authors obtained the same result as before in [11] although using the Dirac constrained quantization method. The constraints are also the boundary conditions of the open string ending on a  $D$  brane. The appearance of an ambiguity during the quantization process justifies the use of a lattice regularization by replacing the interval  $[0, \pi]$  by a lattice of  $N$  equidistant points with spacing  $\epsilon = \pi/N$ . The conclusion is that a  $D$  brane in the  $B$ -field background has a noncommutative worldvolume in the sense that its worldvolume theory is the supersymmetric Yang-Mills theory living on a noncommutative space. The authors proposed that the arbitrariness of the regularization can lead to different results in the literature. Therefore, trying to make a comparison, we can say that in [10], the nontrivial boundary condition in the presence of the  $B$  field modifies the canonical commutation relations and leads to noncommutativity on  $D$ -brane worldvolume. In [15] the authors examine the symplectic form obtained in terms of

the mode expansion of the classical solutions. The detail that characterizes the difference between both works is the fact that the canonical Hamiltonian, instead of the primary Hamiltonian, has been used in the stability condition of the primary constraint, which is different from the usual Dirac quantization formalism.

To understand the dynamics of a  $D$  brane with a  $B$  field, the author in [16] used the canonical quantization. With background  $B$  field one can obtain a mixed boundary condition, which generates an infinite number of secondary second-class constraints. These constraints were solved explicitly in [16] in order to obtain a simple Hamiltonian for the open string in the  $D$ -brane background. The Hamiltonian found was a free Hamiltonian for an open string in spacetime.

Finally, in [17] the Dirac constraint quantization of the brane on the constant antisymmetric backgrounds was reconsidered to obtain consistent noncommutative commutators of the canonical variables including momenta. The main difference between [10,15,17] is the choice of the nonvanishing multiplier in the primary Hamiltonian. The results disclose no secondary constraints and the single primary constraint itself forms the second-class constraint algebra. The final results show several kinds of interesting commutators and expected noncommutative coordinates.

The aim of our work is to quantize the system (2) and (3) following the standard methods [18,19] without any modifications. Canonical analysis of this system presents a problem since the Dirac procedure is initially formulated for the mechanical system (and then can be generalized for a field with vanishing boundary conditions). Application of the Dirac formalism to a field with nontrivial boundary conditions on a compact manifold requires more careful analysis (see [20] for the open string with the Neumann boundary conditions). Consistent treatment of such a system implies necessity to represent the initial dynamics in terms of a mechanical system. Thus, we first rewrite Eqs. (2) and (3) in the form of equations of motion for mechanical variables  $c_n(\tau)$ ,  $n \in \mathbb{Z}$ , and then restore the mechanical action that reproduces this dynamics. After that, the Dirac procedure can be applied to the mechanical system in a direct and unambiguous way. In particular,  $\delta$ -function regularization is not necessary in this case. The last step is to rewrite the results in terms of the string coordinate  $x^i(\tau, \sigma)$ , which gives the Hamiltonian formulation associated with the theory (1)–(3).

Some of the results thus obtained are as follows. The corresponding mechanical system turns out to be the first order system without taking the strong field limit  $B \rightarrow \infty$ . Thus, as a consequence of the mixed boundary conditions, dynamics of the mechanical variables is governed by equations of the first order in time derivative. In particular, it is true for the zero modes  $c_0^i(\tau)$  of the string coordinate  $x^i(\tau, \sigma)$ . As a consequence,  $c_0^i$  are canonically conjugated to each other:  $\{c_0^i, c_0^j\} \neq 0$ . It means that the noncommu-

tativity is an intrinsic property of this mechanical system. Brackets for the string coordinates turn out to be non-commutative in the bulk and on the boundary. We study the freedom in relation between the mechanical and the string variables as well as freedom in choosing brackets, which is always present in the Hamiltonian formalism. This allows one to discuss the extent to which the non-commutativity of the string coordinates can be avoided. We show that the embedding coordinates of a  $D$  brane can be made commutative; of course, one needs in this case to transform simultaneously the Hamiltonian of the system. This is in correspondence with the result of Seiberg and Witten on the equivalence of noncommutative and commutative Yang-Mills fields.

This paper is organized in the following manner: in Sec. II we present a solution for the problem (2) and (3) with our mechanical variables and consequently the equations of motion. In Sec. III we apply directly the Dirac algorithm to the action obtained in Sec. II. Second-class constraints are obtained and consequently we obtain the Dirac brackets for the mechanical variables and the respective Hamiltonian. In Sec. IV, with the results obtained in the previous sections, we construct a Hamiltonian formulation for the open string with a  $B$  field and for the coordinates on the  $D$  brane. In Sec. V we discuss the freedom in the definition of the phase space bracket in the Hamiltonian formalism. In Sec. VI we accomplish the soldering procedure between different aspects of the open string with a  $B$  field. In the final section we discuss our conclusions and perspectives.

## II. OPEN STRING WITH $B$ FIELD IN TERMS OF MECHANICAL VARIABLES

To start with, let us continue  $x^i(\sigma)$ ,  $\sigma \in [0, \pi]$  on the interval  $[0, 2\pi]$  such that  $\tilde{x}^i(\sigma) = x^i(\sigma)$  on  $[0, \pi]$ ,  $\partial_\sigma \tilde{x}|_\pi = \tilde{\partial}_\sigma \tilde{x}|_\pi, \dots$ , and  $\tilde{x}^i(2\pi) = \tilde{x}^i(0)$ ,  $\tilde{\partial}_\sigma \tilde{x}|_{2\pi} = \tilde{\partial}_\sigma \tilde{x}|_0, \dots$ . It can be further continued on  $\sigma \in (-\infty, \infty)$  as a periodic function  $\tilde{x}^i(\sigma + 2\pi n) = \tilde{x}^i(\sigma)$ . As a result, any solution of the problem (2) and (3) can be presented in the form

$$x^i(\tau, \sigma) = c_0^i(\tau) + \sum_{n=1}^{\infty} \frac{1}{n} [c_n^i(\tau) \cos n\sigma + c_{-n}^i(\tau) \sin n\sigma], \quad (4)$$

where  $c_n^i(\tau)$ ,  $n \in \mathbb{Z}$ , are our mechanical variables. Substitution in Eqs. (2) and (3) gives the equations of motion

$$\ddot{c}_0^i = 0, \quad \ddot{c}_n^i + n^2 c_n^i = 0, \quad n \neq 0, \quad (5)$$

$$\dot{c}_0^i = 0, \quad \dot{c}_n^j B_j^i + \frac{n}{2\pi\alpha'} c_{-n}^i = 0, \quad n > 0, \quad (6)$$

from which one finds the further consequence

$$\dot{c}_{-n}^i - 2\pi\alpha' n c_n^j B_j^i = 0. \quad (7)$$

Consequently, the complete dynamics can be rewritten in the equivalent form as

$$\begin{aligned} \dot{c}_0^i &= 0, & \dot{c}_n^j B_j^i + \frac{n}{2\pi\alpha'} c_{-n}^i &= 0, \\ \dot{c}_{-n}^i - 2\pi\alpha' n c_n^j B_j^i &= 0, & n > 0, \end{aligned} \quad (8)$$

which consists of equations of the first order only. Equation (8) follows from the first order action

$$S_f = \int d\tau \left[ \frac{1}{2} \dot{c}_0^i B_{ij} c_0^j + \sum_{n=1}^{\infty} f_n \left( \pi\alpha' \dot{c}_n^i B_{ij} c_n^j - \frac{1}{4\pi\alpha'} \dot{c}_{-n}^i B_{ij}^{-1} c_{-n}^j + n c_{-n}^i c_n^i \right) \right], \quad (9)$$

where  $f_n \neq 0$ ,  $n > 0$  are real numbers. While they can be removed by shift of the variables  $c_n$ , it is convenient to keep them in the action. Starting from any particular  $S_f$ , the variables  $c_n$  can be taken as the ones that generate the string coordinate (4); the latter will not depend on  $f_n$ . At the same time, brackets of  $c_n$  and the Hamiltonian  $H_f$  will depend on  $f_n$ , so there appears to be a natural arbitrariness in the induced bracket for  $x^i(\tau, \sigma)$ . Choice of some particular form of the bracket implies that the corresponding Hamiltonian  $H_f$  must be associated with the Lagrangian formulation (1)–(3).

Another possibility is to take as independent Eqs. (5) and (6) with  $n > 0$ . Then the action can be chosen in the second order form

$$S = \int d\tau \left[ \frac{1}{2} \dot{c}_0 B c_0 + \sum_{n=1}^{\infty} \frac{1}{2} \dot{c}_n \dot{c}_n - \frac{n^2}{2} c_n c_n + \left( \dot{c}_n B + \frac{n}{2\pi\alpha'} c_{-n} \right) \lambda_n \right]. \quad (10)$$

It looks less natural since it involves the Lagrangian multipliers  $\lambda_n$ .

## III. HAMILTONIAN ANALYSIS OF THE MECHANICAL ACTION

Direct application of the Dirac algorithm to the action (9), gives us the primary second-class constraints

$$\begin{aligned} G_{0i} &\equiv p_{0i} - \frac{1}{2} B_{ij} c_0^j = 0, \\ G_{ni} &\equiv p_{ni} - \pi\alpha' f_n B_{ij} c_n^j = 0, \end{aligned} \quad (11)$$

$$G_{-ni} \equiv p_{-ni} + \frac{f_n}{4\pi\alpha'} B_{ij}^{-1} c_{-n}^j = 0.$$

Their Poisson brackets are

$$\begin{aligned} \{G_{0i}, G_{0j}\} &= -B_{ij}, & \{G_{ni}, G_{mj}\} &= -2\pi\alpha' f_n B_{ij} \delta_{n-m,0}, \\ \{G_{-ni}, G_{-mj}\} &= \frac{f_n}{2\pi\alpha'} B_{ij}^{-1} \delta_{n-m,0}, \end{aligned} \quad (12)$$

and the corresponding Hamiltonian is

$$\begin{aligned}
H = & \sum_{n=1}^{\infty} -f_n n c_{-n} c_n + \lambda_0 \left( p_0 - \frac{1}{2} B c_0 \right) \\
& + \lambda_n (p_n - \pi \alpha' f_n B c_n) \\
& + \lambda_{-n} \left( p_{-n} + \frac{f_n}{4\pi \alpha'} B^{-1} c_{-n} \right). \quad (13)
\end{aligned}$$

There are no secondary constraints in the problem. From the consistency conditions that constraints do not evolve in time,  $\dot{G} = 0$ , one obtains expressions for the Lagrangian multipliers:

$$\begin{aligned}
\lambda_0 = 0, \quad \lambda_n = & -\frac{n}{2\pi \alpha'} c_{-n} B^{-1}, \\
\lambda_{-n} = & 2\pi \alpha' n c_n B. \quad (14)
\end{aligned}$$

To take into account the second-class constraints (11), one introduces the Dirac bracket

$$\begin{aligned}
\{K, P\}_D = & \{K, P\} + \{K, G_{0i}\} (B^{-1})^{ij} \{G_{0j}, P\} \\
& + \sum_{n=1}^{\infty} \{K, G_{ni}\} \frac{1}{2\pi \alpha' f_n} (B^{-1})^{ij} \{G_{nj}, P\} \\
& - \{K, G_{-ni}\} \frac{2\pi \alpha'}{f_n} B^{ij} \{G_{-nj}, P\}. \quad (15)
\end{aligned}$$

After that, the variables  $p_n^i$  can be omitted from consideration. The resulting Hamiltonian formulation for (9) consists of the physical variables  $c_n^i$  with the Dirac brackets

$$\begin{aligned}
\{c_0^i, c_0^j\}_D = & -(B^{-1})^{ij}, \\
\{c_n^i, c_m^j\}_D = & -\frac{1}{2\pi \alpha' f_n} (B^{-1})^{ij} \delta_{n-m,0}, \\
\{c_{-n}^i, c_{-m}^j\}_D = & \frac{2\pi \alpha'}{f_n} B^{ij} \delta_{n-m,0}, \quad n, m > 0, \quad (16)
\end{aligned}$$

whose dynamics is governed now by the Hamiltonian

$$H = -\sum_{n=1}^{\infty} f_n n c_{-n}^i c_n^i. \quad (17)$$

As it should be for the first order system, the Hamiltonian equations of motion that follow from (16) and (17) are the Eqs. (8). Note also that the variables  $c_n^i$  with  $n$  fixed are canonically conjugated to each other. The Eqs. (8) for the physical variables can be solved now in terms of oscillators

$$\begin{aligned}
c_n^i(\tau) = & \alpha_n^i e^{in\tau} + \alpha_{-n}^i e^{-in\tau}, \\
c_{-n}^i(\tau) = & -2i\pi \alpha' \alpha_n^i B e^{in\tau} + 2i\pi \alpha' \alpha_{-n}^i B e^{-in\tau}, \\
\alpha_n^{i*} = & \alpha_{-n}^i. \quad (18)
\end{aligned}$$

From Eq. (16) one finds their brackets

$$\begin{aligned}
\{c_0^i, c_0^j\}_D = & -(B^{-1})^{ij}, \\
\{\alpha_n^i, \alpha_m^j\}_D = & -\frac{1}{4\pi \alpha' f_{|n|}} (B^{-1})^{ij} \delta_{n+m,0}, \quad n, m \neq 0, \quad (19)
\end{aligned}$$

while the Hamiltonian (17) acquires the form

$$H_f = 4i\pi \alpha' \sum_{n=1}^{\infty} f_n n \alpha_n^i B_{ij} \alpha_{-n}^j. \quad (20)$$

It is worth noting that this procedure, being applied to the open string with the Neumann boundary conditions, leads to the standard results [21,22].

#### IV. HAMILTONIAN FORMULATION FOR THE OPEN STRING WITH A B FIELD

Using Eqs. (4) and (18), we restore the string coordinate  $x^i(\tau, \sigma)$  in terms of oscillators

$$\begin{aligned}
x^i(\tau, \sigma) = & c_0^i + \sum_{n \neq 0} \frac{e^{in\tau}}{|n|} (\alpha_n^i \cos n\sigma \\
& - 2i\pi \alpha' \alpha_n^j B_j^i \sin n\sigma). \quad (21)
\end{aligned}$$

Thus, with the system (1)–(3), one associates the Hamiltonian formulation including the variables  $c_0^i$ ,  $\alpha_n^i$ ,  $n \neq 0$  with the brackets (19) and the Hamiltonian (20). As a consequence of Eqs. (19) and (21) the induced bracket of the string coordinates turns out to be noncommutative in the bulk and on the boundary

$$\begin{aligned}
\{x^i(\tau, \sigma), x^j(\tau, \sigma')\}_D = & -(B^{-1})^{ij} - \sum_{n=1}^{\infty} \frac{1}{n^2 f_n} \\
& \times \left( \frac{1}{2\pi \alpha'} (B^1)^{ij} \times \cos n\sigma \cos n\sigma' \right. \\
& \left. - 2\pi \alpha' B^{ij} \times \sin n\sigma \sin n\sigma' \right). \quad (22)
\end{aligned}$$

In particular, the coordinates on  $D$  brane obey

$$\begin{aligned}
\{x^i(\tau, 0), x^j(\tau, 0)\}_D = & \{x^i(\tau, \pi), x^j(\tau, \pi)\}_D \\
= & -\left( 1 + \frac{1}{2\pi \alpha'} \sum_{n=1}^{\infty} \frac{1}{n^2 f_n} \right) (B^{-1})^{ij}. \quad (23)
\end{aligned}$$

It is interesting to note that the standard normalization  $f_n \sim \frac{1}{n}$  of the oscillator brackets (19) implies that one needs to use some regularization for Eq. (23). For the choice  $f_n = -\frac{\pi}{12\alpha'}$ , the  $D$ -brane plane becomes commutative, and the Hamiltonian in this case is

$$H = -\frac{i\pi^2}{3} \sum_{n=1}^{\infty} \alpha_n^i B_{ij} \alpha_{-n}^j. \quad (24)$$

#### V. D BRANE

Finally, let us discuss freedom in definition of the phase space bracket in the Hamiltonian formalism. Transition from configuration to phase space description is not a unique procedure, since one needs to define simultaneously the bracket and the Hamiltonian. One possibility is to start

from the Poisson bracket. Then the exact expression for the Hamiltonian is known for a general case [16,19,20,23]. According to the Darboux theorem, one can equally take as the starting point any nondegenerate closed two-form and then try to define a Hamiltonian from the condition that corresponding equations of motion are equivalent to the initial ones. Turning to our case (16) and (17), it will be sufficient to consider the brackets

$$\begin{aligned} \{c_0^i, c_0^j\}_D &= -E^{ij}, \\ \{c_n^i, c_m^j\}_D &= -\frac{1}{2\pi\alpha' f_n} A^{ij} \delta_{n-m,0}, \quad n, m > 0, \end{aligned} \quad (25)$$

with some antisymmetric nondegenerate constant matrices  $E$  and  $A$ . From the condition that the dynamics (8) is reproduced in this formulation, one finds

$$\begin{aligned} \{c_{-n}^i, c_{-m}^j\}_D &= \frac{2\pi\alpha'}{f_n} (AB^2)^{ij} \delta_{n-m,0}, \\ H &= -\sum_{n=1}^{\infty} f_n n c_{-n} (AB)^{-1} c_n. \end{aligned} \quad (26)$$

One can take Eqs. (25) and (26) instead of Eqs. (16) and (17) as the Hamiltonian formulation corresponding to the system (9). Repeating the previous analysis, one finds the same expression (19) for the string coordinate in terms of oscillators that obey

$$\begin{aligned} \{c_0^i, c_0^j\}_D &= -E^{ij}, \\ \{\alpha_n^i, \alpha_m^j\}_D &= -\frac{1}{8\pi\alpha' f_{|n|}} (A - \text{sgn}(nm)B^{-1}AB)^{ij} \delta_{|n|-|m|,0}, \\ n, m &\neq 0. \end{aligned} \quad (27)$$

The quantities  $f_n, E, A$  can be chosen in such a way that the noncommutativity parameter on the  $D$  brane acquires exactly the same form as was obtained from the disk propagator [2]. Namely, for the choice

$$\begin{aligned} f_n &= \frac{\pi^2}{3(2\pi\alpha')^3}, \quad E = \frac{(2\pi\alpha')^2}{2} A, \\ A &= (1 + (2\pi\alpha')^2 B^2)^{-1} B, \end{aligned} \quad (28)$$

one obtains the following Hamiltonian formulation for the system (1)–(3):

$$\begin{aligned} H &= \frac{i\pi^2}{3(2\pi\alpha')^2} \sum_{n=1}^{\infty} n \alpha_n B^{-1} (1 + (2\pi\alpha')^2 B^2) \alpha_{-n}, \\ \{c_0^i, c_0^j\}_D &= -\frac{(2\pi\alpha')^2}{2} [(1 + (2\pi\alpha')^2 B^2)^{-1} B]^{ij}, \\ \{\alpha_n^i, \alpha_m^j\}_D &= -\frac{3(2\pi\alpha')^2}{2\pi^2} [(1 + (2\pi\alpha')^2 B^2)^{-1} B]^{ij} \delta_{n+m,0}, \\ n, m &\neq 0, \end{aligned} \quad (29)$$

which gives for the coordinates on the  $D$  brane the desired expression

$$\{x^i, x^j\}_D = -(2\pi\alpha')^2 (1 - 2i\pi\alpha' B)^{-1} B (1 + 2i\pi\alpha' B)^{-1}. \quad (30)$$

## VI. SOLDERING OF OPEN STRINGS WITH THE PRESENCE OF A $B$ FIELD

The soldering formalism [24,25] is an iterative method that permits us to construct an effective action invariant under a specific gauge symmetry disclosing interesting physical features analogous to the interference phenomena. The method uses primarily the Noether gauging procedure, which helps us to fuse together the variables representing the different aspects of a theory. It works by elevating a global symmetry to a local form thanks to a mutual cancellation of the obstruction to gauge symmetry of the individual components. The gauge field introduced in the process is the agent responsible for this new symmetry and may be eliminated after the gauging is complete. In this paper we will explore just the main steps of the method. For a detailed reading, see [26].

The soldering formalism is the well suited algorithm for fusing together opposite aspects of a global symmetry, such as the right and left propagating modes of chiral bosons, to name just one application. However, the non-linear feature of the action (1) can create new difficulties. The idea is to combine two different models in order to obtain a single composite model. The final result reflects a direct connection between the old and the new theories. The soldering formalism accomplishes this mission for theories that manifest dual aspects of some symmetry, such as chirality, self-duality [27], noncommutativity [28,29], etc. The final result, i.e., the soldered action, hides the mentioned symmetry in an interesting way such that it brings new information about both theories, the old and the new ones. It can be demonstrated that the soldering and the canonical transformation procedures have a strict relation of complementarity [30].

In order to establish these interesting connections between the so-called parent and daughter actions [28] in an open string theory, let us write two different aspects of the action (1), i.e.,

$$\mathcal{L}_{k_1}^x = \partial_a x^i \partial_a x^i + 2\pi k_1 \alpha' \epsilon^{ab} \partial_a x^i \partial_b x^j B_{ij}, \quad (31)$$

$$\mathcal{L}_{k_2}^y = \partial_a y^i \partial_a y^i + 2\pi k_2 \alpha' \epsilon^{ab} \partial_a y^i \partial_b y^j B_{ij}, \quad (32)$$

where  $k_1$  and  $k_2$  are connected by the relations  $k_1 = k_2 = 1$  and  $k_1 = -k_2 = 1$ . For simplicity we will work at this stage with the Lagrangian densities.

The relations show two open strings with equal and opposed characteristics, respectively. It is clear that the first relation is the trivial one and must show nothing new. The relevant one must be the second relation. However, we will carry out both computations to prove the above observations.

So, let us gauge the following global symmetry:

$$\delta x^i = \delta y^i = \sigma^i, \quad (33)$$

where  $\sigma^i$  is the gauge parameter. The gauging of the global parameter  $\sigma^i$  will be done in the soldering process. With this symmetry, a simple calculation shows that

$$\delta \mathcal{L}_0^{k_1} = J_{x_j}^b \partial_b \sigma^j, \quad (34)$$

where  $J_x^{bj}$  is the Noether chiral current given by

$$J_{x_j}^b = 2[\partial^b x_j + 2\pi\alpha' k_1 B_{ij} \epsilon^{ab} \partial_a x^i], \quad (35)$$

which parametrizes the lack of gauge symmetry of the original action  $\mathcal{L}_0^x$ .

The next step is to introduce an auxiliary field that helps in the gauging procedure. Let us call this field by  $D^{bi}$ , known as the soldering field. We can now construct the first-iterated correction of the open string action (1) as

$$\mathcal{L}_0^{k_1} \rightarrow \mathcal{L}_1^{k_1} = \mathcal{L}_0^{k_1} - D_b^i J_{xi}^b, \quad (36)$$

Notice that we are looking for an action that is gauge-invariant under (33).

The gauge variation of  $\mathcal{L}_1^x$  is

$$\delta \mathcal{L}_1^{k_1} = -D_b^i \delta J_{xi}^b = -\delta(D_b^j)^2 - 4\pi\alpha' k_1 B_{ij} \epsilon^{ab} (\delta D_a^i) D_b^j, \quad (37)$$

where we have chosen  $\delta D_b^i = \partial_b \sigma^i$  to cancel (34) and used (33) once again.

Again we can construct the second-iterated correction of the open string action (1) as

$$\mathcal{L}_1^{k_1} \rightarrow \mathcal{L}_2^{k_1} = \mathcal{L}_0^{k_1} + (D_b^j)^2, \quad (38)$$

and therefore,

$$\delta \mathcal{L}_2^{k_1} = -4\pi\alpha' k_1 B_{ij} \epsilon^{ab} (\delta D_a^i) D_b^j, \quad (39)$$

Accomplishing the same procedure for  $\mathcal{L}_y^{k_2}$ , we have that

$$\delta \mathcal{L}_2^{k_2} = -4\pi\alpha' k_2 B_{ij} \epsilon^{ab} D_a^i (\delta D_b^j). \quad (40)$$

Because of the antisymmetric characteristic of both tensors  $B_{ij}$  and  $\epsilon_{ij}$ , it can be easily demonstrated that, for  $k_1 = k_2 = 1$  and  $k_1 = -k_2 = 1$  we have that

$$\begin{aligned} \mathcal{L}_{\text{final}}^{k_1=k_2=1} &= \mathcal{L}_2^{k_1} + \mathcal{L}_2^{k_2} + 4\pi\alpha' B_{ij} \epsilon^{ab} D_a^i D_b^j \\ \mathcal{L}_{\text{final}}^{k_1=-k_2=1} &= \mathcal{L}_2^{k_1} + \mathcal{L}_2^{k_2}. \end{aligned} \quad (41)$$

In both cases we have

$$\delta \mathcal{L}_2 = 0, \quad (42)$$

to finally obtain the desired gauge-invariant action.

In order to be brief, the next step is to eliminate the auxiliary field  $D_a^i$  through its equations of motion in each final action above separately, which demands a certain algebra. After that we have to substitute it back in both

actions (41) and finally obtain two effective actions, respectively,

$$\begin{aligned} \mathcal{S}_{\text{eff}}^{k_1=k_2=1} &= -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[ \frac{1}{2} \partial^a z^i \partial^a z^i \right. \\ &\quad \left. + \pi\alpha' B_{ij} \epsilon^{ab} \partial_a z^i \partial_b z^j \right], \end{aligned} \quad (43)$$

and

$$\mathcal{S}_{\text{eff}}^{k_1=-k_2=1} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[ \frac{1}{2} \partial^a z^i \partial^a z^i \right], \quad (44)$$

where in both cases we have a composed string coordinate given by  $z^i = x^i - y^i$ .

In the first case, the soldered action (43) shows us that the fusion (soldering) of two open strings with the aspects results in another open string. However, we have a different behavior when both strings have opposite aspects. The result (44) shows a Polyakov action that is well known to be equivalent at the classical level to the Nambu-Goto action. This last result establishes a direct connection between the parent and the daughter action, i.e., between the open string action and the Polyakov action. Moreover, we can see clearly that in the first case, the soldering coupling keeps the magnetic field  $B_{ij}$ . However, in the second case the magnetic field disappears from the effective action.

*The dual projection analysis.*— Let us analyze the Polyakov action given by

$$\mathcal{S} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ab} \partial_a x^\mu \partial_b x^\mu, \quad (45)$$

which is invariant under the manifest reparametrization invariance

$$\begin{aligned} \delta x^\mu &= \epsilon^a \partial_a x^\mu \\ \delta g^{ab} &= \epsilon^c \partial_c g^{ab} - b^{ac} \partial_c \epsilon^b - g^{bc} \partial_c \epsilon^a \\ \delta \sqrt{g} &= \partial_a (\epsilon^a \sqrt{g}). \end{aligned} \quad (46)$$

We can easily notice that the Polyakov action above resembles an action with scalar fields interacting with an external two-dimensional gravitational field.

Recently, some of us showed that the dual projection mechanism [25,31] disclosed from the scalar model embedded in a gravitational background two particles with no dynamics, well known as notons. The noton field was introduced by Hull in order to cancel off the Siegel anomaly [32].

Having in mind this idea, we can assume that the Polyakov action has a relation with the noton as

$$\begin{aligned} \mathcal{S} &= -\frac{1}{4\pi\alpha'} \int d^2\sigma [\dot{x}_{+\mu} x_+^{\prime\mu} - \eta_+ (x_+^{\prime\mu})^2 - \dot{x}_{-\mu} x_-^{\prime\mu} \\ &\quad - \eta_- (x_-^{\prime\mu})^2], \end{aligned} \quad (47)$$

where  $\eta_{\pm}$  is a function of the metric elements and  $x_{\pm}^{\mu}$  are the notons (for more details see [25,31]). So, we can see in (47) that the Polyakov action can be perceived as a combination of two notons. As the soldering formalism establishes a relation between the parent and the daughter actions, we can say from (43) that there is a relation between the noton field and the open string action.

## VII. CONCLUSIONS

In this work we quantized the open string action used by Seiberg and Witten following precisely the standard methods. We obtained the equations of motion for the mechanical variables and wrote the respective action. The mechanical variables are the ones that can be taken in order to generate the string coordinate. The equations of motion permit us to introduce the Lagrangian multipliers.

The direct application of the Dirac formalism gives us the primary second-class constraints and consequently the Poisson brackets and the corresponding Hamiltonian. Since there are no secondary constraints in the problem we can construct the Dirac brackets for the theory and the resulting Hamiltonian with the physical variables. We also constructed a Hamiltonian for the physical variables written in terms of oscillators. With this relation in terms of

oscillators, we restored the string coordinate and computed the Dirac brackets for these coordinates.

Writing the Dirac brackets in terms of antisymmetric nondegenerate constant matrices, we obtained the Hamiltonian formulation for the original system (1)–(3). These matrices can be chosen in such a way that the non-commutativity parameter on the  $D$  brane acquires precisely the same form as was obtained in [2].

Finally, the soldering formalism permits us to establish a clear relation between the original system and the Polyakov action, which is equivalent to the Nambu-Goto theory. The Polyakov action clearly resembles an action with scalar fields interacting with an external two-dimensional gravitational field. This relation brings us the idea that there must be a kind of noton particle in the Polyakov formulation, since it was demonstrated by two of us that the noton comprises the spectrum of scalar fields with gravitational background.

## ACKNOWLEDGMENTS

The authors would like to thank CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) and FAPEMIG (Brazilian financial agencies) for financial support.

- 
- [1] A. Connes, M.R. Douglas, and A. Schwarz, *J. High Energy Phys.* 02 (1998) 003.
  - [2] N. Seiberg and E. Witten, *J. High Energy Phys.* 09 (1999) 032.
  - [3] C.G. Callan, C. Lovelace, C.R. Nappi, and S.A. Yost, *Nucl. Phys.* **B288**, 525 (1987); A.A. Abouelsaood, C.G. Callan, C.R. Nappi, and S.A. Yost, *Nucl. Phys.* **B280**, 599 (1987).
  - [4] H. Arfaei and M.M. Sheikh-Jabbari, *Nucl. Phys.* **B526**, 278 (1998).
  - [5] M.M. Sheikh-Jabbari, *Phys. Lett. B* **425**, 48 (1998).
  - [6] E.S. Fradkin and A.A. Tseytlin, *Phys. Lett.* **163B**, 123 (1985).
  - [7] V. Schomerus, *J. High Energy Phys.* 06 (1999) 030.
  - [8] F. Ardalan, H. Arfaei, and M.M. Sheikh-Jabbari, arXiv:hep-th/9803067.
  - [9] F. Ardalan, H. Arfaei, and M.M. Sheikh-Jabbari, *J. High Energy Phys.* 02 (1999) 016.
  - [10] F. Ardalan, H. Arfaei, and M.M. Sheikh-Jabbari, *Nucl. Phys.* **B576**, 578 (2000).
  - [11] C.S. Chu and P.M. Ho, *Nucl. Phys.* **B550**, 151 (1999).
  - [12] S. Girvin and T. Jach, *Phys. Rev. B* **29**, 5617 (1984).
  - [13] G. Dunne and R. Jackiw, *Nucl. Phys. B, Proc. Suppl.* **33C**, 114 (1993).
  - [14] D. Bigatti and L. Susskind, *Phys. Rev. D* **62**, 066004 (2000).
  - [15] C.S. Chu and P.M. Ho, *Nucl. Phys.* **B568**, 447 (2000).
  - [16] T. Lee, *Phys. Rev. D* **62**, 024022 (2000).
  - [17] W.T. Kim and J.J. Oh, *Mod. Phys. Lett. A* **15**, 1597 (2000).
  - [18] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).
  - [19] D.M. Gitman and I.V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, Berlin, 1990).
  - [20] L. Brink and M. Henneaux, *Principles of String Theory* (Plenum Press, New York, 1988).
  - [21] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, 1987), Vols. I, II.
  - [22] J. Polchinski, *String Theory* (Cambridge University Press, Cambridge, 1998), Vols. I, II.
  - [23] A.A. Deriglazov and K.E. Evdokimov, *Int. J. Mod. Phys. A* **15**, 4045 (2000).
  - [24] M. Stone, *Phys. Rev. Lett.* **63**, 731 (1989); *Nucl. Phys.* **B327**, 399 (1989); Illinois University Report No. ILL-23/89, 1989 (unpublished); D. Depireux, S.J. Gates, Jr., and Q.-Han Park, *Phys. Lett. B* **224**, 364 (1989); E. Witten, *Commun. Math. Phys.* **144**, 189 (1992); E.M.C. Abreu, R. Banerjee, and C. Wotzasek, *Nucl. Phys.* **B509**, 519 (1998); R. Amorim, A. Das, and C. Wotzasek, *Phys. Rev. D* **53**, 5810 (1996).
  - [25] E.M.C. Abreu and C. Wotzasek, *Phys. Rev. D* **58**, 101701(R) (1998).
  - [26] C. Wotzasek, arXiv:hep-th/9806005; E.M.C. Abreu and C. Wotzasek, in *Progress in Boson Research* (Nova Scientific, New York, 2007), p. 259–318.

- [27] R. Banerjee and C. Wotzasek, Nucl. Phys. **B527**, 402 (1998).
- [28] S. Ghosh, Phys. Lett. B **579**, 377 (2004).
- [29] E.M.C. Abreu, R. Menezes, and C. Wotzasek, Phys. Rev. D **71**, 065004 (2005).
- [30] R. Banerjee and S. Ghosh, Phys. Lett. B **482**, 302 (2000).
- [31] E.M.C. Abreu, Int. J. Mod. Phys. A **19**, 1823 (2004).
- [32] C.M. Hull, Phys. Lett. B **206**, 234 (1988); **212**, 437 (1988).