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Thermal derivation of the Coleman-De Luccia tunneling prescription

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We derive the rate for transitions between de Sitter vacua by treating the field theory on the static patch as a thermal system. This reproduces the Coleman-De Luccia formalism for calculating the rate, but leads to a modified interpretation of the bounce solution and a different prediction for the evolution of the system after tunneling. The bounce is seen to correspond to a sequence of configurations interpolating between initial and final configurations on either side of the tunneling barrier, all of which are restricted to the static patch. The final configuration, which gives the initial data on the static patch for evolution after tunneling, is obtained from one-half of a slice through the center of the bounce, while the other half gives the configuration before tunneling. The formalism makes no statement about the fields beyond the horizon. This approach resolves several puzzling aspects and interpretational issues concerning the Coleman-De Luccia and Hawking-Moss bounces. We work in the limit where the backreaction of matter on metric can be ignored, but argue that the qualitative aspects remain in the more general case. The extension to tunneling between anti-de Sitter vacua is discussed.

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I. INTRODUCTION

Although it has long been a subject of considerable interest, the problem of transitions between field theory vacua in curved spacetime has received renewed attention in recent years, inspired in part by the possibility of a string landscape containing a vast number of metastable vacua.

The problem was first addressed by Coleman and De Luccia (CDL) [1], who generalized the flat space Euclidean bounce formalism [2,3] for calculating the rate at which true vacuum bubbles nucleate within a false vacuum. When the relevant mass scales are much smaller than the Planck mass and the flat spacetime bubble size is small compared to the spacetime curvature, the CDL formalism gives a small gravitational correction to the bubble nucleation rate, as might be expected. However, as gravitational effects become larger, some unexpected features appear that give one pause. If the initial false vacuum has positive vacuum energy, then the matter field within the bounce is nowhere equal to its false vacuum value, and the bounce itself is completely insensitive to the shape of the scalar field potential near the false vacuum. Further, the formalism appears to involve the fields in regions beyond the de Sitter horizon and to describe nucleation processes that take place on a complete spacelike slice of de Sitter spacetime, even though it is generally understood that this "somehow" cannot quite be so. In addition, one finds a greater variety of bounce solutions than in the flat spacetime case, including both the homogeneous Hawking-Moss (HM) solution [4] and families of "oscillating bounces" [5,6]; yet, there are also examples of theories with no CDL bounce at all.

One can view the puzzles associated with the CDL formalism as being due to the fact that this formalism was originally proposed by arguing from analogy with the flat spacetime case, rather than by being explicitly derived. As a result, even though the formalism may yield a correct answer, it is not quite clear what question it is answering. It is our goal in this paper to rectify this situation. We will do this by formulating a well-defined question, and then showing that its solution is given by the CDL formalism. In the course of doing so, we will be led to explanations of some of the unusual features of the formalism, while other troubling aspects will be seen to disappear when the solutions are properly interpreted.

It should be stressed that this is not simply a matter of putting the formalism on a firmer footing. We will find that the generally accepted scheme for obtaining from the bounce the initial conditions for the Lorentzian evolution after vacuum decay [1,7] is based on an incorrect understanding of the relation between the Euclidean and Lorentzian spacetimes and must be modified. The correct procedure gives these initial conditions only within the horizon and leads, in some cases, to rather different results.

Gravity affects vacuum decay in two distinct ways. First, it requires that the dynamics of the matter field be worked out in a spacetime that is curved and that, if the original state is a de Sitter vacuum, possesses horizons. Second, the dynamics of the gravitational field itself must be considered. In this paper we will, as an initial step, consider only the former aspect. To be able to do this in a self-consistent manner, we consider the theory of a single scalar field governed by a potential, such as that shown in Fig. 1, with two unequal minima—a metastable "false vacuum" and a stable "true vacuum"—but under the assumption that the variation of the potential in the region between the minima is small compared to its absolute value; i.e., that

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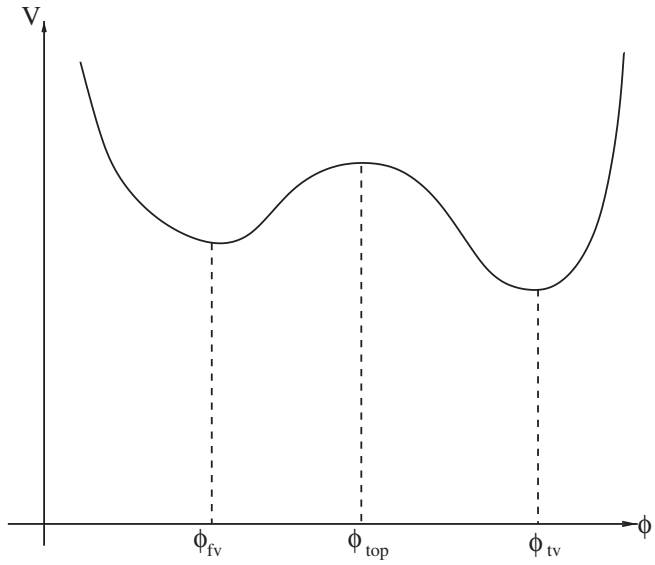

FIG. 1. A potential with two minima.

$$V(\phi_{\rm top}) - V(\phi_{\rm tv}) \ll V(\phi_{\rm tv}). \tag{1.1}$$

With this assumption, we can, to a first approximation, freeze the metric degrees of freedom and treat the geometry as being a fixed de Sitter spacetime with a horizon distance

$$H^{-1} = \Lambda = \sqrt{\frac{3M_{\rm Pl}^2}{8\pi V(\phi_{\rm fv})}}. \tag{1.2}$$

We will see that many of the troubling aspects of the CDL prescription are already present in this fixed-background limit, and can be understood without considering dynamical gravity. The extension of our methods to the more general case will be discussed in Sec. VI. Although we have not yet been able to extend the technical details of our derivation, we will see that some of the qualitative features, including the resolution of some interpretational issues, are readily generalized.

It is possible to describe a portion of de Sitter spacetime, the causal diamond or static patch, by the time-independent metric[1]

$$ds^2 = -\left(1 - \frac{r^2}{\Lambda^2}\right)dt^2 + \left(1 - \frac{r^2}{\Lambda^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.3}$$

Here θ and ϕ are the usual angular variables on the two-sphere and t ranges over all real values, but r is restricted to the range $0 \le r < \Lambda$, with the hypersurface $r = \Lambda$ being the horizon. With this fixed static background geometry, the field theory on the static patch is a finite volume system with a well-defined time-independent Hamiltonian. However, because the space is curved, the Hamiltonian density contains position-dependent factors that would be absent in flat spacetime.

In addition, the existence of a horizon leads to a characteristic temperature [8]

$$T_{\rm dS} = \frac{1}{2\pi\Lambda}. \tag{1.4}$$

To be a bit more precise, if the field on de Sitter space is in the Bunch-Davies state [9] corresponding to the false vacuum, then within the static patch one will appear to have a thermal mixed state with $T = T_{\rm dS}$. However, the converse need not be true—the existence of such a thermal state on the static patch does not necessarily imply anything about the system beyond the horizon.

In this paper we will consider vacuum decay within the framework of this system. In other words, we will treat the scalar field on the static patch as a thermal system with $T = T_{\rm dS}$. The flat spacetime, finite temperature WKB formalism is then readily adapted to the problem. This leads to an algorithm for calculating the rate of vacuum transitions that reproduces the CDL result, but with some critical changes in interpretation and a new prediction for the evolution after tunneling.

A number of previous authors have studied this problem by applying WKB methods to the calculation of the wave functional of the false vacuum [10–13]. Our approach differs from these in some important aspects. First, we very explicitly restrict our consideration to the degrees of freedom that lie within the horizon. The remarkable fact that the CDL formalism can be recovered from such an approach lends strong support for our modification of the initial conditions for post-tunneling evolution. Second, our approach is based on a thermal analysis of the curved spacetime field theory. This leads to a clear physical interpretation of both the CDL and HM bounces, and the relation between the two, and also provides an explanation for some of the most troubling features of the CDL method.

We begin, in Sec. II, by reviewing the treatment of vacuum decay in the absence of gravity, at both zero and nonzero temperature. Next, in Sec. III, we apply these methods to the field theory on the static patch and show how the CDL formalism emerges. In Sec. IV we examine a variety of characteristic bounces in light of this approach. Next, in Sec. V, we discuss the evolution of the system in the classically allowed regime after the tunneling process has taken place. Finally, in Sec. VI, we summarize our results and make some final comments.

II. TUNNELING IN FLAT SPACETIME

The bounce formalism for treating tunneling in the context of a flat spacetime field theory was developed by Coleman [2] using an approach, based on a multidimensional WKB approximation [14,15], that yields the exponent in the tunneling rate. The leading subexponential prefactor terms were obtained by Callan and Coleman

[1]Our conventions are such that $ds^2 < 0$ for timelike intervals.

[3], using a path integral approach to the calculation of the imaginary part of the energy of the false vacuum. Because finite temperature field theory can be formulated in terms of a periodic path integral, the path integral formalism is a natural route for the extension of the tunneling calculation to finite temperature [16,17]. However, because we are interested in extending these results to curved spacetime, it will turn out to be more convenient to use a WKB approach to finite temperature tunneling; we will see that this leads to the same periodicity requirement on the bounce as the path integral. The difficulty with proceeding via the path integral is not so much the technical issues involved in calculating the determinant factors (which are ameliorated considerably when working in a fixed-background metric) as the fact that the dilute gas approximation used by Callan and Coleman cannot be applied when the size of the bounce becomes comparable to the horizon size. In addition, the WKB approach has the added bonus of giving a much clearer physical interpretation of the bounce.

A. Zero temperature

Consider first a particle in one dimension with dynamics defined by the Lagrangian

$$L = \tfrac{1}{2}M\dot{q}^2 - U(q), \tag{2.1}$$

with $U(q)$ having two unequal minima. An elementary result in quantum mechanics is that if the particle has an energy E that is less than the height of the potential energy barrier separating the minima, the rate for it to tunnel through that barrier is proportional to $e^{-B(E)}$, where

$$B(E) = 2\int_{q^{(1)}}^{q^{(2)}} dq\sqrt{2M[U(q) - E]} \tag{2.2}$$

and $q^{(1)}$ and $q^{(2)}$ are the turning points where $U(q) = E$.

This WKB approximation can be extended to a system with $N > 1$ degrees of freedom and

$$L = \frac{1}{2}\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j - U(q), \tag{2.3}$$

where $U(q)$ again has two minima. The potential energy barrier separating the two minima now exists in an N-dimensional configuration space, so we are faced with a multidimensional tunneling problem. To find the WKB approximation to the tunneling rate, one considers all possible paths that start at a point $q_j^{(1)}$ on one side of the barrier and end at a point $q_j^{(2)}$ on the other side, with the requirement that $U(q^{(1)}) = U(q^{(2)}) = E$. For each such path P, one can calculate a tunneling exponent

$$B(E, q^{(1)}, q^{(2)}, P) = 2\int_{s_1}^{s_2} ds\sqrt{2(U[q(s)] - E)}, \tag{2.4}$$

where the parameter s along the path is defined so that $ds^2 = \sum_{ij} M_{ij}dq_i dq_j$ with $q(s_1) = q^{(1)}$ and $q(s_2) = q^{(2)}$. The leading approximation to the tunneling rate is obtained from the path and endpoints that minimize B. Taking over a standard result in classical mechanics and inserting a few sign changes, one readily shows that this minimization problem is equivalent to finding a solution of the Euler-Lagrange equations that follow from the Euclidean Lagrangian

$$L_E = \frac{1}{2}\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j + U(q). \tag{2.5}$$

The Euclidean time τ is simply a reparametrization of the path, with $q_j(\tau_1) = q_j^{(1)}$ and $q_j(\tau_2) = q_j^{(2)}$. Because $dq_j/d\tau = 0$ at the end points of the path, the continuation of the solution beyond τ_2 gives a τ-reversed version of the original path. The solution obtained by continuing this back to the starting point $q_j^{(1)}$ at τ_1' is known as the "bounce." In particular, if $q_j(\tau_1)$ is taken to be the false vacuum minimum itself, one finds that $\tau_1 = -\infty$, and hence $\tau_1' = \infty$. Furthermore, the value of B for the optimal path is related to the Euclidean action

$$S_E = \int_{\tau_1}^{\tau_1'} d\tau L_E \tag{2.6}$$

of the bounce by

$$\begin{aligned} B(E) &= S_E(q_{\text{bounce}}) - (\tau_1' - \tau_1)U(q^{(1)}) \\ &= S_E(q_{\text{bounce}}) - S_E(q_{\text{init}}). \end{aligned} \tag{2.7}$$

Finally, we turn to the case of tunneling within the context of a quantum field theory. We consider a theory of a single scalar field, with Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{2.8}$$

and $V(\phi)$ of the form shown in Fig. 1. It is crucial to remember that the potential energy is not $V(\phi)$, but rather the functional

$$U[\phi(\mathbf{x})] = \int d^3x\left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]. \tag{2.9}$$

The decay of the homogeneous false vacuum proceeds by a tunneling process in which true vacuum bubbles are nucleated. However, it is not a tunneling through the barrier in $V(\phi)$ (which would correspond to a transition from a homogeneous false vacuum to a homogeneous field configuration on the true vacuum side of the barrier), but rather a tunneling through the barrier in the infinite-dimensional configuration space defined by $U[\phi(\mathbf{x})]$, with one end of the tunneling path being the homogeneous false vacuum and the other end being a configuration with a bubble of approximate true vacuum embedded in a false vacuum background.

It is a straightforward process to take over the previous results, with the role of the discrete coordinates q_j now

played by the values of the field at each point in space. For tunneling from an initial configuration $\phi_{\rm init}(\mathbf{x})$, with $U[\phi_{\rm init}(\mathbf{x})] = E$, the tunneling exponent $B(E)$ is obtained by solving the Euclidean field equation

$$0 = \left(\frac{\partial^2}{\partial\tau^2} + \nabla^2\right)\phi - \frac{dV}{d\phi} \tag{2.10}$$

subject to the conditions that

$$\phi(\mathbf{x}, \tau_1) = \phi(\mathbf{x}, \tau_1') = \phi_{\rm init}(\mathbf{x}),$$
$$\frac{\partial\phi(\mathbf{x}, \tau)}{\partial\tau}\bigg|_{\tau_1} = \frac{\partial\phi(\mathbf{x}, \tau)}{\partial\tau}\bigg|_{\tau_1'} = 0 \tag{2.11}$$

for some choice of τ_1 and τ_1'. A slice through the middle of the bounce, along the hypersurface $\tau_2 = (\tau_1' + \tau_1)/2$, gives the optimal point for emerging from the potential energy barrier; i.e., the form of the optimal bubble. Thus, $\phi(\mathbf{x}, \tau_2)$ gives the initial conditions for the classical evolution of the bubble after nucleation.

If the initial configuration is the false vacuum itself, then τ_1 and τ_1' are $\pm\infty$, and $\phi(\mathbf{x}, \tau = \pm\infty) = \phi_{\rm fv}$. Because the intermediate configurations must have the same potential energy as the initial configuration, the field at spatial infinity must also approach the false vacuum; i.e., $\phi(|\mathbf{x}| = \infty, \tau) = \phi_{\rm fv}$. The tunneling exponent is then

$$B = S_E(\phi_{\rm bounce}) - S_E(\phi_{\rm fv}). \tag{2.12}$$

B. Nonzero temperature

Let us now turn to the case of nonzero temperature, beginning again with a particle with one degree of freedom. We will assume that the temperature T is much less than the height of the potential energy barrier, so that a metastable false vacuum state is possible. Since this is a thermal system, the initial state is a mixed state.

In this thermal system, the particle need not tunnel from the bottom of potential well, but can instead tunnel from some thermally excited higher energy state, as illustrated in Fig. 2. The rate for this thermally assisted tunneling is obtained by taking a thermal average of the energy-dependent tunneling rates [18]; i.e.,

$$\Gamma_{\rm tunn} \sim \int_{E_{\rm fv}}^{E_{\rm top}} dE e^{-\beta(E-E_{\rm fv})} e^{-B(E)} \sim e^{-\beta(E_*-E_{\rm fv})} e^{-B(E_*)}, \tag{2.13}$$

where E_* is the value of the energy that maximizes the integrand; i.e., the value for which $\beta(E - E_{\rm fv}) + B(E)$ is a minimum.

More generally, with many degrees of freedom, we must minimize

$$\beta(E - E_{\rm fv}) + 2\int_{s_1}^{s_2} ds\sqrt{2(U[q(s)] - E)} \tag{2.14}$$

with respect to path, endpoints, and energy. The variations

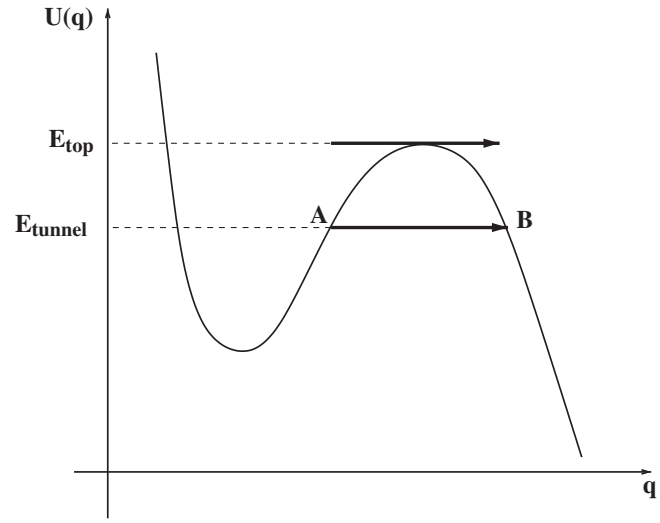

FIG. 2. A cross section through the potential energy barrier, illustrating the two modes of escaping from the false vacuum: thermal excitation to A, followed by tunneling to B; and thermal excitation to the top of the barrier.

with respect to the endpoints vanish because the integrand vanishes at both of these. From our previous discussion, we know that minimization with respect to the path is achieved by requiring that the path correspond to a solution of the Euclidean Euler-Lagrange equations. Thus, it only remains to determine the optimal energy by solving

$$\beta = -2\frac{d}{dE}\int_{s_1}^{s_2} ds\sqrt{2(U[q(s)] - E)} \tag{2.15}$$

with $q_j(s)$ understood to be along a classical path. The integral on the right-hand side depends on the energy through the explicit appearance of E in the square root and through the implicit energy dependences of the end points and path. Neither of the implicit dependences contribute here: as noted above, the variations with respect to $q_j^{(1)}$ and $q_j^{(2)}$ vanish because the integrand is zero at the end points; the variation with respect to the path vanishes once the path is chosen to be a classical solution. We therefore have

$$\beta = 2\int_{s_1}^{s_2} ds\frac{1}{\sqrt{2(U[q(s)] - E)}} = 2\int_{s_1}^{s_2} ds\frac{1}{\sqrt{M_{ij}\frac{dq_i}{d\tau}\frac{dq_j}{d\tau}}}$$
$$= 2\int_{s_1}^{s_2} ds\frac{d\tau}{ds} = 2(\tau_1 - \tau_2), \tag{2.16}$$

where the second equality uses the fact that we are working with a solution of the Euclidean equations.

Equation (2.16) tells us that the passage through the barrier must take a Euclidean "time" $\Delta\tau = \beta/2$. We also know that continuation of the solution past the endpoint gives a τ-reversed solution back toward the starting point. With this continuation, we have a solution that is periodic in τ with period β. Thus, the prescription for

calculating the rate of thermally assisted tunneling is based on finding a solution of the Euclidean equations with period β that has the additional property that on at least two τ-slices (which are conveniently taken to be $\tau = 0$ and $\tau = \beta/2$) the $dq_j/d\tau$ all vanish. The tunneling exponent is then obtained by integrating the Euclidean action for this solution over one full period, so that

$$\Gamma_{\rm tunn} \sim e^{-[S_E({\rm bounce}) - S_E({\rm fv})]}. \tag{2.17}$$

However, there is a second possible mode for the transition. Instead of being thermally excited part of the way up the barrier and then tunneling, the particle can be thermally excited all the way to the top of the barrier. Up to pre-exponential factors, the rate for this process is proportional to the Boltzmann factor,

$$\Gamma_{\rm therm} \sim e^{-\beta(E_{\rm top} - E_{\rm fv})}. \tag{2.18}$$

When there is more than one degree of freedom, there are many possible paths over the potential energy barrier. The lowest of these dominates, with the rate governed by the energy $E_{\rm saddle}$ of the saddle point on this path.[2] This saddle point is a stationary point of the potential energy and is thus a time-independent solution of the ordinary equations of motion or, equivalently, a τ-independent solution of the Euclidean equations of motion. Viewing it this way, we can write

$$\beta(E_{\rm saddle} - E_{\rm fv}) = S_E({\rm saddle}) - S_E({\rm fv}), \tag{2.19}$$

where the actions are understood to be calculated over a τ interval equal to β.

Since a τ-independent solution can be viewed as being periodic with any period, the prescription to seek a Euclidean solution with period β actually covers both transition modes, with the dominant mode being determined by the value of the Euclidean action. Figure 3 illustrates the two types of solutions in the field theory setting, for the case where β is somewhat larger than the characteristic size of the zero-temperature bounce. The bounce for thermally assisted tunneling, shown in Fig. 3(a), is a somewhat deformed version of the zero-temperature bounce. Note, in particular, that, in contrast with the zero-temperature case, the initial configuration, given by the $\tau = 0$ hypersurface, is not identically equal to $\phi_{\rm fv}$. Figure 3(b) shows the saddle point solution; a slice along any hypersurface of constant τ gives the configuration of a critical bubble.

It can be shown [2] that a bounce solution always exists for the zero-temperature case; the same arguments can be easily modified to show that a critical bubble solution of the sort shown in Fig. 3(b) always exists. However, there is no guarantee that the τ-dependent bounce of Fig. 3(a) will persist for all values of β. The absence of such a bounce would correspond to a situation in which the expression in Eq. (2.14) had no minimum, but instead was monotonically decreasing as E varied from $E_{\rm fv}$ to $E_{\rm saddle}$.

[2]Note that all that is required here is that the path be a local minimum among paths across the barrier. There may be higher saddle points that are also relevant because they lead to different final states.

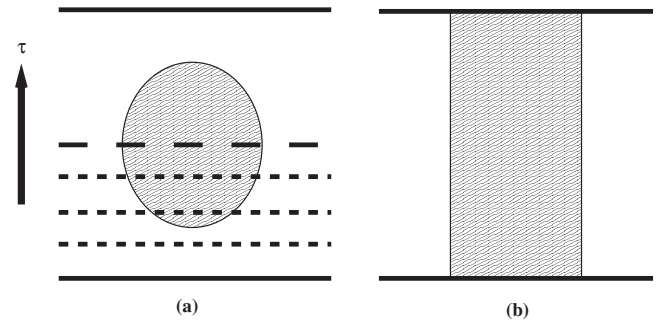

FIG. 3. The two types of bounces at finite temperature for flat spacetime. The shaded areas denote regions where the field is on the true vacuum side of the barrier. In both cases the imaginary time τ runs vertically, while the horizontal direction represents the three spatial directions. In each diagram the top and bottom solid lines are identified, making τ compact. The bounce in (a) corresponds to thermally assisted tunneling from the approximately false vacuum configuration on the τ slice represented by the solid lines to the configuration on the τ slice indicated by the dashed line. These two configurations are connected by a series of intermediate configurations, corresponding to the dotted lines. The τ-independent bounce in (b) corresponds to thermal excitation over the barrier. A constant-τ slice through this bounce gives a critical bubble configuration, which is a saddle point on the potential energy barrier.

III. THERMAL TUNNELING ON THE STATIC PATCH

We now turn to the field theory on the static patch of de Sitter spacetime. In terms of the coordinates and metric of Eq. (1.3), the action for our scalar field theory takes the form

$$S = \int d^4x\sqrt{-\det g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{3.1}$$

where the spatial integral is restricted to the region $r < \Lambda$. If we write the three-dimensional spatial metric as $h_{ij} = g_{ij}$, and define

$$-g_{tt} = 1 - \frac{r^2}{\Lambda^2} = A(r), \tag{3.2}$$

we can rewrite this action as

$$S = \int dt \int d^3x\sqrt{\det h}\left[\frac{1}{2\sqrt{A(r)}}\left(\frac{d\phi}{dt}\right)^2 - \frac{1}{2}\sqrt{A(r)}h^{ij}\partial_i\phi\partial_j\phi - \sqrt{A(r)}V(\phi)\right] = \int dtL. \tag{3.3}$$

Putting aside for the moment the origins of this expression in a curved spacetime, we can choose to view the

Lagrangian L as describing a field theory, on a curved three-dimensional space, whose interactions happen to have an extra position dependence arising from the various factors of $\sqrt{A}$. The energy functional for this theory is

$$E = \int d^3x\sqrt{\det h}\left[\frac{1}{2\sqrt{A(r)}}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}\sqrt{A(r)}h^{ij}\partial_i\phi\partial_j\phi + \sqrt{A(r)}V(\phi)\right]. \tag{3.4}$$

We can now immediately carry over the results of the previous section. In particular, to study vacuum transitions at $T = T_{\rm dS} = 1/(2\pi\Lambda)$, we look for periodic solutions to the Euler-Lagrange equations of the Euclidean action

$$S_E = \int_{-\pi\Lambda}^{\pi\Lambda} d\tau \int d^3x\sqrt{\det h}\left[\frac{1}{2\sqrt{A(r)}}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}\sqrt{A(r)}h^{ij}\partial_i\phi\partial_j\phi + \sqrt{A(r)}V(\phi)\right]. \tag{3.5}$$

Note that, for later convenience, we have chosen the integral over the periodic variable τ to run from $-\beta/2 = -\pi\Lambda$ to $\beta/2 = \pi\Lambda$ rather than from 0 to β. We will use the convention that the hypersurface $\tau = -\pi\Lambda \sim \tau = \pi\Lambda$ is taken to give the (approximately false vacuum) configuration before tunneling, with the hypersurface half a period away, at $\tau = 0$, giving the configuration after tunneling, and thus the initial condition for the subsequent classical evolution.

We now restore A to its role as a metric factor, but now as part of a Euclidean metric. Thus, we define

$$\begin{aligned}\tilde{g}_{ab}dx^a dx^b &= Ad\tau^2 + h_{ij}dx^i dx^j \\ &= \left(1 - \frac{r^2}{\Lambda^2}\right)d\tau^2 + \left(1 - \frac{r^2}{\Lambda^2}\right)^{-1}dr^2 \\ &\quad + r^2(d\theta^2 + \sin^2\theta d\phi^2).\end{aligned} \tag{3.6}$$

With $\tau = -\pi\Lambda$ and $\tau = \pi\Lambda$ identified and r ranging between 0 and Λ, this is the round metric for a four-sphere.[3] More explicitly, with the identifications

$$\begin{aligned} y^1 &= r\sin\theta\cos\phi, \\ y^2 &= r\sin\theta\sin\phi, \\ y^3 &= r\cos\theta, \\ y^4 &= \sqrt{\Lambda^2 - r^2}\cos(\tau/\Lambda), \\ y^5 &= \sqrt{\Lambda^2 - r^2}\sin(\tau/\Lambda) \end{aligned} \tag{3.7}$$

this is the metric on a four-sphere of radius Λ embedded in five-dimensional Euclidean space. Our Euclidean action can now be written as

$$S_E = \int d^4x\sqrt{\det\tilde{g}}\left[\frac{1}{2}\tilde{g}^{ab}\partial_a\phi\partial_b\phi + V(\phi)\right] \tag{3.8}$$

and the rate for vacuum decay is

$$\Gamma \sim e^{-[S_E(\text{bounce}) - S_E(\text{fv})]}. \tag{3.9}$$

Let us compare this with the CDL prescription. CDL instruct us to solve the Euclidean equations for coupled matter and gravity. If $V(\phi)$ satisfies Eq. (1.1), then to leading order one can ignore the effects of the variation of ϕ on the metric, which becomes that of a four-sphere of radius Λ. With the background gravity thus fixed, the equations for ϕ are precisely those following from the action of Eq. (3.8).

In our thermal tunneling picture we have the additional requirement, as noted above Eq. (2.17), that $d\phi/d\tau$ vanish identically on the hypersurfaces $\tau = 0$ and $\tau = \pi\Lambda$. In terms of the y_a defined above, the union of these two hypersurfaces is the three-sphere formed by the intersection of the $y^4 = 0$ hyperplane with the four-sphere of radius Λ. Although the existence of such a three-sphere with vanishing τ derivatives is not explicitly stated in the CDL prescription, it follows from their assumption that the bounce has O(4) symmetry.

Once the bounce has been found, CDL determine the tunneling rate from the combined gravity plus matter Euclidean action. In the fixed-background approximation the contributions of the Einstein-Hilbert term cancel between the bounce and the homogeneous false vacuum, leaving precisely the result in Eq. (3.9).

Thus, by treating the field on the static patch as a thermal system, we have arrived at precisely the CDL prescription for the vacuum decay rate. However, our approach leads to a radically different interpretation of the bounce solution itself. In our approach, as illustrated in Fig. 4, the hypersurfaces of constant τ provide a foliation of the four-sphere

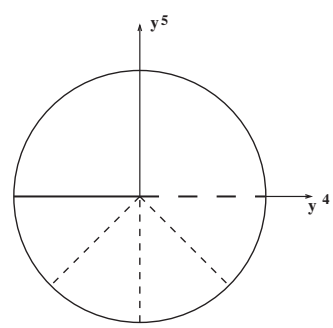

FIG. 4. Slices of constant τ projected onto the y^4-y^5 plane. The correspondence with those in the flat spacetime bounce of Fig. 3(a) is indicated by the form of the lines. The initial and final configurations of the tunneling path correspond to the solid and dashed lines, respectively, while intermediate configurations are obtained from slices along the dotted lines.

[3]Any other choice of the temperature would have led to a Euclidean manifold with a conical singularity at $r = \Lambda$. At least within the framework of our fixed-background approximation, we see no inconsistency in such a situation; it just happens not to be the one that is relevant for the thermal state encountered in vacuum tunneling.

corresponding to a tunneling path (traversed in both directions) through configuration space, with the endpoints of the path given by the hypersurfaces $\tau = 0$ and $\tau = \pi\Lambda$. For each value of τ, the configuration is specified only on the region within the horizon; no reference is ever made to quantities beyond the horizon. In the CDL description, there is no hypersurface corresponding to the initial configuration, and the entire hypersurface bisecting the bounce—the union of our $\tau = 0$ and $\tau = \pi\Lambda$ hypersurfaces—is taken to define the final configuration after emergence from the barrier, and thus the initial data for the subsequent classical evolution. This hypersurface is interpreted as giving initial data on an entire spatial slice of de Sitter spacetime, including the region outside the horizon.

We can now also understand what is perhaps the most puzzling aspect of the CDL formalism, the fact that ϕ never achieves its false vacuum value on the bounce, with the result that the bounce solution is independent of the shape of the potential in a region near $\phi_{\rm fv}$. This is because tunneling from a thermally excited state is always preferable to tunneling from the false vacuum. As illustrated in Fig. 2, we can think of this thermally assisted tunneling as a two-step process in which the field is first thermally excited to a preferred starting configuration $\phi_A(\mathbf{x})$, indicated by A in the figure, and then tunnels quantum mechanically through the barrier to a configuration $\phi_B(\mathbf{x})$. The first step depends (up to pre-exponential factors) only on the energy difference between the false vacuum and A, but not on any other details of the potential energy, including the shape of $V(\phi)$. The second step depends on the configurations that interpolate between $\phi_A(\mathbf{x})$ and $\phi_B(\mathbf{x})$; because ϕ is nowhere equal to $\phi_{\rm fv}$ on any of these configurations, the details of the potential near there never enter.[4]

IV. A MISCELLANY OF BOUNCES

It may be helpful to examine a variety of characteristic bounces in the light of our approach. In all cases we will assume that the bounce has O(4) symmetry, and in all but the last will take the solution to be oriented so that this symmetry corresponds to invariance under all rotations that leave the y^4-axis invariant; the solutions will then only depend on y^4. There are then two distinguished poles, at $y^4 = \Lambda$ ($r = \tau = 0$) and at $y^4 = -\Lambda$ ($r = 0$, $\tau = \pi\Lambda$).

(1) *Small CDL bounces*: When the mass scales in $V(\phi)$ are much less than the Planck mass and the radius of the flat space bounce is much less than Λ, the CDL bounce has a region of approximate true vacuum, centered about the pole at $y^4 = \Lambda$, that is very similar to the corresponding flat space bounce [see Fig. 5(a)]. Outside this region ϕ rapidly tends toward—but never quite reaches—its false vacuum value, with $|\phi - \phi_{\rm fv}|$ being smallest at the antipodal point, $y^4 = -\Lambda$. The initial and final configurations of the tunneling path, $\phi_A(\mathbf{x})$ and $\phi_B(\mathbf{x})$, are given by the three-dimensional slices $\tau = \pi\Lambda$ (i.e., $y^4 \leq 0$ and $y^5 = 0$) and $\tau = 0$ ($y^4 \geq 0$ and $y^5 = 0$), respectively. Note that these configurations, as well as all of the interpolating configurations, overlap at the two-sphere $x^4 = x^5 = 0$, which is just the horizon, $r = \Lambda$. The initial configuration differs only very slightly from the homogeneous false vacuum (with, curiously, the deviation being greatest at the horizon), while the final configuration has a true vacuum bubble embedded in a background of approximate false vacuum. Apart from the fact that the bounce corresponds to only a finite region of space, the situation is similar to that in flat spacetime tunneling.

(2) *Large thin-wall bounces*: As the mass scales increase, one possibility is that the true vacuum region of the bounce occupies a larger fraction (although always less than half) of the four-sphere, while the wall separating the true vacuum and false vacuum regions remains thin. It was noted some time ago that this bounce can be interpreted either as mediating decay of the false vacuum via the nucleation of true vacuum bubbles, or as mediating the creation of a false vacuum region inside a true vacuum background [20]. We can now sharpen this interpretation. In the thin-wall limit, we can assume that away from the wall the field is exponentially close to one vacuum or the other. The $\tau = \pi\Lambda$ slice is then a horizon volume within which $\phi \approx \phi_{\rm fv}$ everywhere, while the $\tau = 0$ slice is a horizon volume in which there is a true vacuum bubble surrounded by a false

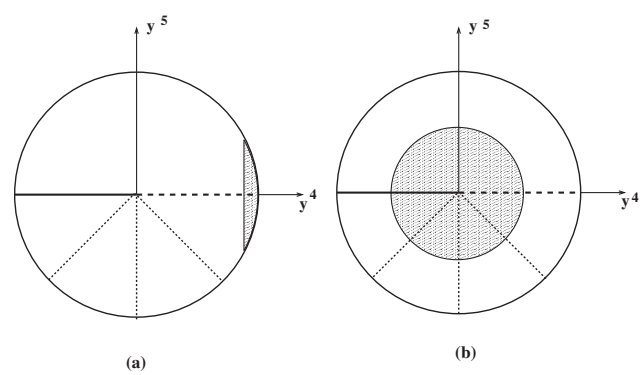

FIG. 5. Two possible orientations of a CDL bounce. In (a), the bounce is centered about the pole at $y^4 = \Lambda$, and corresponds to nucleation via tunneling of a bubble in the center of the static patch. In (b), the location of the bounce has been rotated by 90 degrees. The bounce now corresponds to the purely thermal creation of a true vacuum region that intersects the horizon.

[4]For an illuminating related discussion in the context of a toy model, see [19].

vacuum background. Taking the former to be the initial configuration and the latter to be the final one corresponds to the standard nucleation of a true vacuum bubble. Reversing the roles of the two slices gives a process in which a horizon volume of true vacuum is first thermally excited to a configuration where the region near the horizon is in the false vacuum and that in the center in the true vacuum, and then tunnels through the potential barrier to a configuration of homogeneous false vacuum. Note that the final configuration does not contain a bubble wall. This does not signify that the false vacuum region covers all of space, but merely that it fills the horizon volume and that no statement can be made concerning the interface (beyond the horizon) between it and other regions.

(3) *Thick-wall bounces*: An alternative possibility as the mass scales increases, one which is typically associated with the case where $V(\phi)$ is relatively flat near its maximum, is that the wall separating the two phases becomes wide enough to occupy a significant fraction of the four-sphere. This eventually leads to a situation in which at neither of the poles is the field near a vacuum value. Let the values of the field at the poles be ϕ_a and ϕ_b and its value at the horizon (and indeed everywhere on the $\tau = \pi\Lambda/2$ slice) be ϕ_H, with $\phi_{\rm fv} < \phi_a < \phi_H < \phi_b < \phi_{\rm tv}$. Starting from a false vacuum state, say, the field thermally fluctuates to a configuration with the field equal ϕ_H at the horizon and ϕ_a at the center of the horizon volume, and then tunnels through the potential energy barrier to a configuration where the field is still ϕ_H at the horizon, but now equal to ϕ_b at the center. As ϕ_a and ϕ_b approach the local maximum of V at $\phi_{\rm top}$, the transition increasingly becomes predominantly a thermal excitation process rather than one of quantum tunneling.

(4) *Hawking-Moss bounce*: The limiting case of the thick-wall bounce is the homogeneous HM solution. Although the original paper [4] refers to a transition of the field over the whole universe, it has long been understood that this cannot be the case, and that the HM solution must really correspond to a transition over a region of roughly horizon size. Our formalism makes this precise, in that it is explicitly restricted to precisely a horizon volume. The HM solution is a τ-independent solution that happens also to be spatially constant over the horizon volume (we will shortly encounter τ-independent solutions that are not spatially homogeneous), and so corresponds to a thermal fluctuation[5] to the top of the barrier in $V(\phi)$ (or, more properly, to a saddle point of the barrier in the potential energy $U[\phi(\mathbf{x})]$). It is a curved-space analogue of the bounce illustrated in Fig. 3(b).

(5) *Remote Hawking-Moss bounces*: If the potential has more than one local maximum, then there will be HM bounce solutions corresponding to each. The standard expression for the transition rate, which is unchanged in our approach, depends only on the values of $V(\phi)$ at the false vacuum and at the local maximum, and not on the separation in field space between the two. If this were correct, then in a system with many such local maxima (e.g., a string landscape with an exponentially large number of vacua) the lifetime of any de Sitter vacuum could be quite short. In Ref. [24] it was argued, from a path integral point of view, that these bounces do not contribute. The same conclusion can be reached from the WKB approach we are using here, since it is clear that the only τ-independent bounces that are relevant are those corresponding to saddle points in the barrier that immediately surrounds the initial metastable state.

(6) *Oscillating bounces*: For some choices of potential the CDL bounce equations admit solutions in which the field oscillates back and forth across the top of the barrier in $V(\phi)$ as one moves from one pole of the four-sphere to the other. Examined more closely, these are typically found to have two regions of approximate vacuum (either true or false), one about each of the poles, separated by a region in which the field oscillates with a relatively small amplitude about the top of the barrier [5,6]. If the initial data after tunneling were obtained from a complete slice through the bounce, these would correspond to transitions that created not one but two vacuum bubbles, centered on antipodal poles of the de Sitter space and separated by a region of spatially oscillating field. Once it is recognized that this slice must be divided into two parts, one for the configuration before tunneling and one for the configuration after, a much more natural picture emerges. The oscillating bounce corresponds to a tunneling process from an excited horizon volume configuration containing a vacuum region surrounded by a region with the field near the top of the barrier, through a series of intermediate configurations with the field everywhere near the top of the barrier, and finally to a configuration that is qualitatively like the original one except that the final vacuum region might (or might not) be the opposite one from the original. We will have a bit more to say about the contribution of these bounces in Sec. VI.

(7) *No CDL bounce*: If $V(\phi)$ is sufficiently flat near its maximum, it can happen [4,25] that there is no CDL

[5] In the thermal approach to the problem that we are taking, the stochastic interpretation of the HM bounce [21–23] may be viewed as one mechanism by which a thermal distribution of configurations is established.

bounce at all, but only a HM solution.[6] This simply corresponds to a situation, like that discussed at the end of Sec. II, in which the rate for thermally assisted tunneling is a monotonically increasing function of E.

(8) *Rotated bounces*: In all of the previous cases we oriented the bounce so that it was symmetric about the y^4-axis. This implied that the configurations before and after tunneling were symmetric about the center of the horizon volume. However, this need not be the case. The only condition on the symmetry axis is that it be such that the τ-derivatives of the field vanish on the $\tau = 0$ and $\tau = \pi\Lambda$ hypersurfaces. This only requires that this axis be perpendicular to the y^5-axis. Rotating the symmetry axis away from the y^4-axis gives a bounce that is not centered within the horizon volume of interest, a possibility that one should expect. The surprise comes when the symmetry axis is taken to be perpendicular to the y^4-axis (e.g., along the y^3-axis), as shown in Fig. 5(b). In this case, the bounce has no τ-dependence at all. It is then analogous to the critical bubble solution of Fig. 3(b), and corresponds to a saddle point of the static patch potential energy. Thus, by simply rotating the bounce we have gone from a tunneling transition to a purely thermal one. In contrast with the flat spacetime case, there is no sharp distinction between the two.

V. CLASSICAL EVOLUTION AFTER TRANSITION

An important feature of the bounce formalism is that it not only yields a tunneling rate, but also gives initial conditions for the classical evolution after tunneling. In addition, via a rotation from Euclidean to Lorentzian spacetime (supplemented in some regions by analytic continuation), it yields an actual solution of the Lorentzian field equations.

In flat spacetime the relation between the Euclidean and Lorentzian solutions is relatively simple. Let us describe these by Cartesian coordinates (x, y, z, τ) and (x, y, z, t), respectively. Now suppose that one is given a solution $\phi_E(\mathbf{x}, \tau)$ of the Euclidean field equation with the property that $\partial\phi_E/\partial\tau = 0$ everywhere on the hypersurface $\tau = 0$. One can then define initial data for the Lorentzian equation by taking $\phi_L(\mathbf{x}, 0) = \phi_E(\mathbf{x}, 0)$ and $\partial\phi_L/\partial t = 0$ everywhere on the hypersurface $t = 0$.

Comparing the Euclidean and Lorentzian field equations, one immediately sees that $\phi_L(\mathbf{x}, t)$ is the analytic continuation of $\phi_E(\mathbf{x}, \tau)$, with $t = i\tau$. Since one does not usually have a closed-form expression for the Euclidean solution, the actual implementation of this continuation over all of the Lorentzian spacetime will, in general, require an explicit solution of the field equations. However, there may be regions of spacetime where the analytic continuation implies equalities between the Euclidean and Lorentzian solutions. For example, if the Euclidean solution is a function only of $s_E = \sqrt{\mathbf{x}^2 + \tau^2}$, then the Lorentzian solution is a function only of $s_L = \sqrt{\mathbf{x}^2 - t^2}$, and its values in the part of spacetime lying outside the light cone of the origin can be read off directly from the Euclidean solution.

It should be stressed that the above are merely mathematical statements about solutions of differential equations. Their physical relevance depends on an additional fact, namely, that $\phi_L(\mathbf{x}, 0) = \phi_E(\mathbf{x}, 0)$ actually is the configuration from which the classical post-tunneling system evolves.

There are some complicating factors when we generalize this procedure to de Sitter spacetime. First, in contrast with the flat Minkowski case, where the Euclidean space and Lorentzian spacetime are both topologically R^4, the Euclidean space and Lorentzian spacetime now have different topologies. Second, and more importantly, there are a number of "natural" ways of putting coordinates on the Euclidean space and then continuing them to the Lorentzian spacetime.

To see the first, recall that de Sitter spacetime may be defined as the hyperboloid

$$\Lambda^2 = (y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 - (y^5)^2 \tag{5.1}$$

in a five-dimensional spacetime with Minkowskian metric

$$ds^2 = (dy^1)^2 + (dy^2)^2 + (dy^3)^2 + (dy^4)^2 - (dy^5)^2. \tag{5.2}$$

Its natural Euclidean counterpart is the four-sphere of radius Λ in five-dimensional Euclidean space.

Of the various coordinate systems that can be put on the four-sphere, two are of particular relevance to us. The first is the generalized Hopf coordinates, defined by Eq. (3.7), in terms of which the metric on the four-sphere is given by the expression in Eq. (3.6) or, if we define $\eta = \sin^{-1}(r/\Lambda)$, by

$$ds^2 = \cos^2\eta d\tau^2 + \Lambda^2[d\eta^2 + \sin^2\eta(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{5.3}$$

These coordinates give an $S^2 \times S^1$ foliation of the four-sphere. Making the replacement $\tau \to it$ gives

$$ds^2 = -\cos^2\eta dt^2 + \Lambda^2[d\eta^2 + \sin^2\eta(d\theta^2 + \sin^2\theta d\phi^2)], \tag{5.4}$$

which is equivalent to the metric of Eq. (1.3). With coor-

[6]Note that flatness at the maximum is a necessary, but not a sufficient condition for the nonexistence of a CDL bounce. For examples of flat potentials with CDL bounces, see Refs. [6,26].

dinates restricted to real values, this covers only a portion of de Sitter spacetime, the causal diamond of the point $y^1 = y^2 = y^3 = y^5 = 0$, $y^4 = \Lambda$.

The second is the hyperspherical coordinates defined by

$$\begin{aligned} y^1 &= \Lambda \sin\xi \cos\tau' \sin\theta \cos\phi, \\ y^2 &= \Lambda \sin\xi \cos\tau' \sin\theta \sin\phi, \\ y^3 &= \Lambda \sin\xi \cos\tau' \cos\theta, \\ y^4 &= \Lambda \cos\xi, \\ y^5 &= \Lambda \sin\xi \sin\tau'. \end{aligned} \tag{5.5}$$

The corresponding metric is

$$ds^2 = \Lambda^2 d\xi^2 + \Lambda^2 \sin^2\xi [d\tau'^2 + \cos^2\tau'(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{5.6}$$

These coordinates foliate the four-sphere with three-spheres. The replacement $\tau' \to it'$ takes this to the Lorentzian metric

$$ds^2 = \Lambda^2 d\xi^2 + \Lambda^2 \sin^2\xi [-dt'^2 + \cosh^2 t'(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{5.7}$$

Again, real values of the coordinates cover only a portion of the de Sitter spacetime,[7] in this case the region where $|y^4| \leq \Lambda$. These are the coordinates used by CDL. They are particularly convenient for studying O(4)-symmetric bounces, because (with appropriate orientation of axes) this symmetry is equivalent to the statement that the fields depend only on ξ.

These two sets of coordinates define different hypersurfaces of constant Euclidean time. For the former, these are half three-spheres (or, equivalently, three-dimensional balls bounded by two-spheres). For the latter, these hypersurfaces are full three-spheres. Visualized in two fewer dimensions, the former are semicircles of fixed longitude, while the latter are circles of fixed latitude, with the defining poles rotated 90° in going from one case to the other. The constant τ and constant τ' hypersurfaces do not generally coincide. The one exception is the hypersurface $\tau' = 0$ which, as was noted previously, is the union of the hypersurfaces $\tau = 0$ and $\tau = \pi\Lambda$.

In the CDL bounce, $\partial\phi/\partial\tau'$ vanishes everywhere on the $\tau' = 0$ hypersurface. Mapping this hypersurface onto the three-sphere $t' = 0$ in de Sitter spacetime and continuing this solution via $\tau' \to it'$ certainly gives a solution of the Lorentzian field equations. However, this solution is not the physically relevant one, because the configuration on emerging from tunneling is specified by the $\tau = 0$ slice of the bounce. This maps onto the de Sitter $t = 0$ hypersurface, which is only half of the $t' = 0$ hypersurface, namely, the portion lying within a horizon radius of the point P with coordinates $y^1 = y^2 = y^3 = y^5 = 0$, $y^4 = \Lambda$. The data outside the horizon, on the remainder of the hypersurface, is not specified. Hence, the future evolution of ϕ cannot be determined everywhere, but only in the causal diamond of P, which happens to be precisely the region covered by the continuation of the Hopf coordinates. It is only this restriction of the continued CDL solution that is physically meaningful.

Because the field on the $\tau = \pi\Lambda$ slice is just the continuation of the bounce solution from the $\tau = 0$ slice, it might seem that by keeping only the latter we are needlessly discarding initial data on half of the de Sitter space. This is not so. Considered in isolation, the data on any spacelike hypersurface are completely unconstrained. The fact that the data on one part of the hypersurface are related to an interesting Euclidean solution implies nothing at all about the data on the remainder of the hypersurface. As our derivation of the CDL decay rate makes clear, the bounce solution only gives information about the fields within the static patch. Outside this patch, *any* data are possible.

When written in terms of the hyperspherical coordinates of Eq. (5.5), O(4)-symmetric bounces that lie at the center of the causal patch (i.e., that are centered about the point P) are functions solely of ξ, and thus are determined (in any coordinate system) solely by the value of y^4. For treating bounces that are not centered about the point P, it is more convenient to use a rotated set of hyperspherical coordinates, such as those defined by

$$\begin{aligned} y^1 &= \Lambda \sin\tilde{\xi} \cos\tilde{\tau}' \sin\tilde{\theta} \cos\tilde{\phi}, \\ y^2 &= \Lambda \sin\tilde{\xi} \cos\tilde{\tau}' \sin\tilde{\theta} \sin\tilde{\phi}, \\ y^3 &= \Lambda(\cos\alpha \sin\tilde{\xi} \cos\tilde{\tau}' \cos\tilde{\theta} - \sin\alpha \cos\tilde{\xi}), \\ y^4 &= \Lambda(\sin\alpha \sin\tilde{\xi} \cos\tilde{\tau}' \cos\tilde{\theta} + \cos\alpha \cos\tilde{\xi}), \\ y^5 &= \Lambda \sin\tilde{\xi} \sin\tilde{\tau}' \end{aligned} \tag{5.8}$$

(with α some fixed angle), so that the field still depends only on the value of a single variable, in this case $\tilde{\xi}$. In particular, the choice $\alpha = \pi/2$ corresponds to the τ-independent bounces discussed at the end of Sec. IV, as can be confirmed by comparison with Eq. (3.7). These depend only on y^3, and so after rotation to Lorentzian space are independent of t. In other words, they are static (but unstable) solutions when viewed in static de Sitter coordinates, in agreement with their interpretation as saddle

[7] Although not especially relevant for our purposes, there are coordinate systems on the four-sphere that cover the full de Sitter spacetime after the rotation to Lorentzian time. One such is obtained from the hyperspherical coordinates above by interchanging y^4 and y^5, shifting ξ to $\xi + \pi/2$, and then interpreting ξ, and not τ', as the Euclidean time.

points of the potential energy. In other coordinate systems, on the other hand, these solutions are time dependent.

VI. CONCLUDING REMARKS

In this paper we have provided the question for CDL's answer. We have shown that their prescription for calculating transition rates between de Sitter vacua can be obtained by considering only the field degrees of freedom lying within the horizon-sized static patch. Indeed, because no reference is made to data outside the causal diamond, it is not necessary to assume that spacetime is globally de Sitter (which it certainly is not) or, in fact, to make any assumption about conditions beyond the horizon.

Our approach clarifies the meaning of the bounce solution itself. We see that, completely in parallel with the flat spacetime case, the bounce gives a sequence of spatial slices interpolating between two configurations that are turning points on opposite sides of the potential energy barrier. As a result, although we recover precisely the CDL result for the transition rate, we differ from them, and from other previous treatments (including Ref. [7]), concerning the extraction of the post-tunneling initial conditions from the bounce. As we have seen, these are specified only within the static patch, and not over an entire spacelike hypersurface of de Sitter space. In particular, our approach gives a crisp and unambiguous explanation of why the HM solution only refers to a horizon volume.

The thermal context of our derivation shows that the CDL bounce should be understood as a process of thermally assisted tunneling in which the tunneling takes place, not from the ground state itself, but from a thermally excited state. This explains both why ϕ never achieves its false vacuum value on the bounce, and why the bounce solution and its action are independent of the details of the $V(\phi)$ near $\phi_{\rm fv}$. It also leads to a natural understanding of why some potentials admit no CDL bounces at all.

Our results also clarify the meaning of the oscillating bounce solutions which, under the previous interpretation, would seem to correspond to the emergence of a configuration with two vacuum bubbles. We now see that these bounces actually specify a path though configuration space that connects two thermally excited horizon volume configurations, each of which contains a single vacuum bubble. But, at the same time, we now have a clear argument, based on the counting of negative eigenvalues, for discounting these. For the flat spacetime problem, the path integral approach depends on the fact that the fluctuations about the bounce include one mode with a negative eigenvalue, to provide the factor of i needed to give the energy of the false vacuum an imaginary part. Although the factors of i would also come out correctly if there were $4n + 1$ negative eigenvalues, the WKB approach shows that bounces with more than one negative mode must be discarded because they correspond to tunneling paths that are only saddle points for the tunneling exponent B; i.e., there is a linear combination of these modes that gives a continuous variation of the tunneling path that lowers[8] B. Now that we have an extension of the tunneling path approach to the curved spacetime problem, we see unambiguously that bounces with multiple negative eigenvalues should not be included. Since oscillating bounces always have multiple negative eigenvalues [32,33], they must be excluded.

We can also see that similar remarks apply to the HM solution when it has more than one negative eigenvalue [i.e., when $V''(\phi) > 4\Lambda^{-2}$ at the top of the barrier]. When it has only one negative eigenvalue, the HM solution corresponds to a local minimum in the barrier surrounding the false vacuum. Any additional negative eigenvalues denote the existence of directions in which the barrier decreases, thus indicating that there must be some lower saddle point along the top of the barrier. This saddle point is a spatially inhomogeneous static solution corresponding to a rotated bounce of the sort described in Sec. IV. This explains why the appearance of multiple negative modes about the HM bounce is always accompanied by the appearance of a new CDL solution [6]. (Note, however, that the existence of a CDL solution does not necessarily imply that the HM solution has multiple negative eigenvalues. There are potentials for which a HM solution with just one negative eigenvalue coexists with a CDL bounce, so that there are two competing modes for completing the transition.)

Our approach can be applied to transitions in anti-de Sitter spacetime, again under the assumption that the variation in the potential between the two vacua is small compared to its absolute value. Because there is no horizon, the tunneling problem in the fixed-background limit is analogous to that in flat spacetime, with the Euclidean time taking on all real values and no periodicity condition. As is well known, for certain choices of potentials there is no bounce solution connecting the true and false vacua [1,34]. This can be understood by recalling that at zero temperature the tunneling path must connect the initial vacuum with a configuration of equal energy. In flat spacetime this is always possible, because a bubble of true vacuum can always be made large enough that the energy gain from converting false vacuum to true compensates for the energy in the bubble wall. This is not always so in anti-de Sitter space, because for a radius much greater than the curvature length the volume and area grow at the same rate. Although there is a spatially homogeneous configuration on the true vacuum side of the barrier that has the same energy as the homogeneous false vacuum, it is inaccessible because the two are separated by an infinite potential energy barrier, just as are degenerate field theory vacua in flat spacetime.

Clearly the primary issue to be addressed is the extension of the method to the more general case, where the

[8] This is the essence of the argument demonstrating [27] that in flat spacetime the bounce of lowest action has precisely one negative eigenvalue, a result that has been extended to the case with gravity [28–31].

approximation of a fixed background geometry is not applicable. Doing this will require the resolution of some significant technical issues. However, we have seen that the problematic aspects of the CDL formalism that were described in the introduction can be understood even when the gravitational background is taken to be fixed, and we expect these qualitative features to persist when the gravitational degrees of freedom are included. In particular, the configuration after tunneling should still be determined only on a causal region, and should still be obtained from a partial, not a full, slice through the CDL bounce. We hope to return to this in a future publication.

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[1] S. Coleman and F. De Luccia, Phys. Rev. D **21**, 3305 (1980).
[2] S. Coleman, Phys. Rev. D **15**, 2929 (1977); **16**, 1248(E) (1977).
[3] C. G. Callan and S. Coleman, Phys. Rev. D **16**, 1762 (1977).
[4] S. W. Hawking and I. G. Moss, Phys. Lett. B **110**, 35 (1982).
[5] T. Banks, arXiv:hep-th/0211160.
[6] J. C. Hackworth and E. J. Weinberg, Phys. Rev. D **71**, 044014 (2005).
[7] A. H. Guth and E. J. Weinberg, Nucl. Phys. **B212**, 321 (1983).
[8] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).
[9] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. A **360**, 117 (1978).
[10] N. Deruelle, Mod. Phys. Lett. A **4**, 1297 (1989).
[11] W. Fischler, D. Morgan, and J. Polchinski, Phys. Rev. D **42**, 4042 (1990).
[12] V. A. Rubakov and S. M. Sibiryakov, Teor. Mat. Fiz. **120**, 451 (1999) [Theor. Math. Phys. (Engl. Transl.) **120**, 1194 (1999)].
[13] U. Gen and M. Sasaki, Phys. Rev. D **61**, 103508 (2000).
[14] T. Banks, C. M. Bender, and T. T. Wu, Phys. Rev. D **8**, 3346 (1973).
[15] T. Banks and C. M. Bender, Phys. Rev. D **8**, 3366 (1973).
[16] J. S. Langer, Ann. Phys. (N.Y.) **41**, 108 (1967).
[17] A. D. Linde, Nucl. Phys. **B216**, 421 (1983); **B223**, 544(E) (1983).
[18] I. Affleck, Phys. Rev. Lett. **46**, 388 (1981).
[19] P. Batra and M. Kleban, arXiv:hep-th/0612083.
[20] K. Lee and E. J. Weinberg, Phys. Rev. D **36**, 1088 (1987).
[21] A. A. Starobinsky, in *Field Theory, Quantum Gravity and Strings*, edited by H. J. De Vega and N. Sanchez (Springer, New York, 1986), p. 107.
[22] A. S. Goncharov, A. D. Linde, and V. F. Mukhanov, Int. J. Mod. Phys. A **2**, 561 (1987).
[23] A. D. Linde, Nucl. Phys. **B372**, 421 (1992).
[24] E. J. Weinberg, Phys. Rev. Lett. **98**, 251303 (2007).
[25] L. G. Jensen and P. J. Steinhardt, Nucl. Phys. **B237**, 176 (1984).
[26] L. G. Jensen and P. J. Steinhardt, Nucl. Phys. **B317**, 693 (1989).
[27] S. Coleman, Nucl. Phys. **B298**, 178 (1988).
[28] T. Tanaka, Nucl. Phys. **B556**, 373 (1999).
[29] A. Khvedelidze, G. V. Lavrelashvili, and T. Tanaka, Phys. Rev. D **62**, 083501 (2000).
[30] G. V. Lavrelashvili, Nucl. Phys. B, Proc. Suppl. **88**, 75 (2000).
[31] S. Gratton and N. Turok, Phys. Rev. D **63**, 123514 (2001).
[32] J. C. Hackworth, Ph.D. thesis, Columbia University, 2006.
[33] G. Lavrelashvili, Phys. Rev. D **73**, 083513 (2006).
[34] S. J. Parke, Phys. Lett. B **121**, 313 (1983).