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Cosmological expansion and local physics

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The interplay between cosmological expansion and local attraction in a gravitationally bound system is revisited in various regimes. First, weakly gravitating Newtonian systems are considered, followed by various exact solutions describing a relativistic central object embedded in a Friedmann universe. It is shown that the ''all or nothing'' behavior recently discovered (i.e., weakly coupled systems are comoving while strongly coupled ones resist the cosmic expansion) is limited to the de Sitter background. New exact solutions are presented which describe black holes perfectly comoving with a generic Friedmann universe. The possibility of violating cosmic censorship for a black hole approaching the big rip is also discussed.

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I. INTRODUCTION

The issue of whether a planet, a star, or a galaxy expands following the rest of the universe is a problem of principle in general relativity that still awaits a definitive answer. The effect of the cosmological expansion on local systems such as the Solar System has a long history dating back to the 1933 paper by McVittie [1] introducing a spacetime metric that represents a point mass embedded in a Friedmann-Lemaitre-Robertson-Walker (FLRW) universe. Later work by Einstein and Straus [2] introduced the Swiss-cheese model which is, however, unable to describe the Solar System [3,4] and is unlikely to be extended to nonspherical systems [5,6]. Many papers in the following years [3,6–24] presented contradictory results, casting a shadow of ambiguity on the problem (see Refs. [4,25] for brief reviews). The most popular model consists of a test particle in a quasicircular orbit around a Newtonian central object. It appears that many of the contradictory results are simply due to the use of different or unphysical coordinates [13]. Moreover, a quantitative answer to the problem of how much the cosmic expansion affects local dynamics differs according to the type of local system considered. If the FLRW metric is an adequate model of spacetime down to small scales, then weakly gravitating systems of size small in comparison to the Hubble radius H_0^{-1} do participate in the expansion, but the effect is so small to be completely negligible for practical purposes. When the size of the weakly gravitating system becomes a larger fraction of the Hubble radius, the cosmic expansion plays a significant role in the dynamics; this is the situation of large scale structures [8,9,17,23].

A recent paper by Price [22] studies a classical atom in a de Sitter background, with arbitrary strength of the coupling between an electron and the central charge. A new result is the ''all or nothing'' behavior: if the coupling is weak the ''atom'' is comoving with the rest of the universe, while if the coupling is very strong the atom is only slightly perturbed by a transient and does not expand [22]. This work breaks free of the standard assumption of previous literature that the coupling (of a gravitationally, instead of electrically, bound system) is weak. However, it has two fundamental limitations: first, the cosmological background is restricted to be de Sitter space, which is very special: in fact, the de Sitter metric can be put in static form, which may explain why strongly bound systems in this background do not expand. Second, the classical atom is not an adequate model when the local energy density and stresses of the central charge grow and induce local deviations from the cosmological metric. In these situations exact solutions of the Einstein equations are needed to describe both the relativistic central object with a strong local field and the surrounding universe. Solutions of this kind could also be useful in studying the evolution of primordial black holes [26] regarded as probes of the early universe [27,28]. New and existing exact solutions of this kind are studied in Secs. III and IV.

There are also more modern and perhaps more compelling motivations to study the effect of the cosmological expansion on local physics. It is now well known from the observations of supernovae of type Ia that the expansion of the universe is accelerated [29]. Marginal evidence for an equation of state parameter $w \equiv \frac{P}{\rho} < -1$ (where ρ and P are the energy density and pressure of the cosmic fluid, respectively) has led theorists to take seriously into account the possibility of a big rip singularity at a finite time in the future [30]. Various authors have studied how local systems (clusters, galaxies, stars, etc.) are teared apart as the big rip is approached [21,30]. In this situation the catastrophic cosmological expansion is not merely a perturbation of the local dynamics, but dominates it.

The current inability to explain away the Pioneer anomaly has led some authors to attribute it to the effect of the cosmological expansion, although this possibility seems to be ruled out [20]. Finally, independent motivation for studying the interplay between local and cosmological dynamics comes from another problem of principle in general relativity. If a universe dominated by phantom

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dark energy, which violates all the energy conditions and causes $w \equiv P/\rho$ to be less than -1, is heading toward the big rip and all bound systems are gradually ripped apart [30], it is legitimate to ask what is the fate of the most strongly bound local object, namely, a black hole. Does the strong local field resist the expansion (as suggested by extrapolating Price's work [22]), or does the horizon expand and disappear, exposing the central singularity before the big rip is reached? This would entail violation of cosmic censorship in its cosmological formulation [31] and would constitute a (further) argument against phantom energy. An answer comes from Ref. [32], in which accretion of a phantom test fluid onto a Schwarzschild black hole is studied. The results are extrapolated to a gravitating fluid and the conclusion is reached that the black hole decreases its mass due to the fact that the gravitating energy density accreted $P + \rho$ is negative. Accretion proceeds until the horizon disappears together with the central singularity before the big rip is reached [32]. Although the extrapolation from a test to a gravitating fluid is quite plausible, it would be preferable to base the conclusion on an exact solution of the Einstein equations displaying accretion of cosmic fluid. A step in this direction is taken with new solutions presented here.

In this paper we examine various exact solutions representing strong field objects in a cosmological background: they include the McVittie metric [1], the Schwarzschild-de Sitter black hole, the Nolan interior solution [33], a solution found recently by Sultana and Dyer [19], and new exact solutions that are perfectly comoving. It turns out that the "all or nothing" behavior discovered by Price [22] persists in the Schwarzschild-de Sitter black hole but it is a peculiarity of the de Sitter background adopted and more general FLRW backgrounds do not allow for it. Participation of a local object, even strongly bound, to the cosmological expansion seems to be the general rule, a conclusion supported by various exact solutions.

The plan of this paper is as follows: Sec. II studies a Newtonian quasicircular orbit and the effect of the cosmic expansion upon it, making clear the peculiarity of de Sitter space even for this kind of systems. Sec. III examines known and new exact solutions, while Sec. IV contains a discussion and the conclusions. We adopt the notations of Ref. [34].

II. A NEWTONIAN OBJECT EMBEDDED IN A FLRW UNIVERSE: QUASI-CIRCULAR ORBITS

In the literature, the most common line of approach to the problem of the effect of the cosmic expansion on local systems is to consider a spherical Newtonian object of mass M and a test particle in a circular orbit of radius r around it, and then "switch on" the cosmological dynamics as a small perturbation of this two-body problem (elliptical orbits were considered in Ref. [14]). Motivated by the current model of our universe (and by simplicity), and following most authors, we consider as the background a spatially flat FLRW universe described by the line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{1}$$

in comoving coordinates. The evolution equation for the physical radial coordinate of the otherwise circular orbit is

$$\ddot{r} = \frac{\ddot{a}}{a} r - \frac{GM}{r^2} + \frac{L^2}{r^3}, \tag{2}$$

where L is the (constant) angular momentum per unit mass of the test particle and an overdot denotes differentiation with respect to comoving time. This equation is derived in several ways in the many papers on this subject: they range from heuristic derivations (e.g., Refs. [7,22]) to calculations using the geodesic deviation equation in a locally inertial frame of the cosmological metric (1) [13,24,35]. This derivation uses Fermi normal coordinates, regarded as the physical coordinates connected to a freely falling observer in the cosmological gravitational field. Other derivations of Eq. (2) [8,21] use various approximations to equations for timelike geodesics in the McVittie metric [1], in the limit in which a central object produces only a small deviation from the cosmological background. If the current era in the history of the universe is considered, the term $\ddot{a}r/a$ on the right hand side of Eq. (2) is a small perturbation when $r \ll H_0^{-1}$, where H_0 is the present value of the Hubble parameter $H \equiv \dot{a}/a$. This correction becomes increasingly important as the size of the orbit increases in comparison with the Hubble radius H_0^{-1}, for example, going to galaxy clusters and superclusters, for which it is certainly not negligible [8,9,17,23].

Note that, for a decelerating universe, the term $\frac{\ddot{a}}{a} r$ is negative and is considered as such in pre-1998 literature, while it gives a positive contribution to $\ddot{r}$ in an accelerated universe, which is the case of the present epoch, as is now well known from the study of supernovae of type Ia at high redshift [29].

Whether $\ddot{a}r/a$ can be treated as a perturbation or not, both the central object and the FLRW background are assumed to be spherically symmetric, which yields conservation of the angular momentum per unit mass L of the test particle,

$$r^2 \dot{\varphi} = L \tag{3}$$

in spherical coordinates. This equation holds true in later sections in which we consider exact solutions of the Einstein equations describing a strongly gravitating, spherically symmetric, central object embedded in a FLRW universe. It follows that, if $r(t)$ increases with time, the angular velocity $\varphi(t)$ decreases. The test particle has to cover a larger linear distance to attempt to close its orbit, which would be circular in the absence of the cosmological perturbation.

A different point of view is the one, found in recent literature, in which it is assumed that the universe is dominated by phantom energy with equation of state parameter $w \equiv P/\rho < -1$ [36], and is heading toward a big rip in which the scale factor $a(t)$ diverges at a finite time $t_{\rm rip}$.

A long time before the big rip, the term $\frac{\ddot{a}}{a}r$ can be treated as a small perturbation. In flat space, at the Newtonian level, this term is absent, Kepler's third law yields $\dot{\varphi}^2 r^3 = GM$, and the two terms $-GM/r^2$ and L^2/r^3 in Eq. (2) cancel each other, leading to circular orbits $r=$ const.. When the perturbation $\frac{\ddot{a}}{a}r$ is introduced, this is no longer true and this perturbation changes the (otherwise circular) orbit. When the big rip is approached as $t \rightarrow t_{\rm rip}^-$ and the central object is weakly gravitating (Newtonian), the term $\frac{\ddot{a}}{a}r$ dominates over the terms in r^{-2} and r^{-3}, leaving

$$\ddot{r} = \frac{\ddot{a}}{a} r \tag{4}$$

as the asymptotic evolution equation for the physical radius of the orbit. The solution of Eq. (4) is $r \propto a(t)$ or, the orbit becomes comoving with the cosmic substratum. For simplicity, assume that the phantom energy dominating the cosmic dynamics and causing the big rip has constant equation of state $P = w\rho$, where $w < -1$. Then, the scale factor is

$$a(t) = a_0 (t_{\rm rip} - t)^\alpha, \qquad \alpha = \frac{2}{3(w+1)} < 0, \tag{5}$$

and Eq. (4) reduces to

$$\ddot{r} = \frac{\alpha(\alpha - 1)}{(t_{\rm rip} - t)^2} r, \tag{6}$$

which has the general solution

$$r(t) = A(t_{\rm rip} - t)^\alpha + B(t_{\rm rip} - t)^{1-\alpha}, \tag{7}$$

where A and B are integration constants. Since $1 - \alpha = \frac{3w+1}{3(w+1)} > 0$, the second term on the right hand side of Eq. (7) becomes negligible with respect to the first one as the big rip is approached. Therefore, the solution (7) reduces to $r(t) \propto a(t)$ as $t \rightarrow t_{\rm rip}^-$, i.e., the putative circular orbit becomes comoving.

The angular motion is obtained by integrating Eq. (3), which yields

$$\begin{aligned}\varphi(t) &= \int dt \frac{L}{r^2} \\ &= \frac{3L(w+1)}{1-3w}[A^2(t_{\rm rip} - t)^{(1-3w)/3(w+1)} + AB]^{-1} \\ &\quad + \varphi_0 \\ &\simeq \frac{3(w+1)}{1-3w}\frac{L}{A^2}(t_{\rm rip} - t)^{(3w-1)/3(w+1)} + \varphi_0. \end{aligned} \tag{8}$$

Then $\varphi(t) \rightarrow \varphi_0$ as $t \rightarrow t_{\rm rip}^-$: as the big rip is approached and r grows without bound (but comoving), the angular motion slows down and freezes. Strictly speaking, the orbit is never "disrupted" before the spacelike singularity is reached; it just participates in the cosmic expansion that is accelerating catastrophically. In this sense, it is not true that bound systems become unbound: this is a rather misleading sentence often echoed by the media. If one wants to insist on the use of this terminology, the meaning of "bound" and "unbound" system should be clearly defined. For example, one may think of (arbitrarily) setting the threshold between bound and unbound when, in Eq. (2), the cosmological term $\ddot{a}r/a$ becomes of the order of the other two terms $-GM/r^2$ and L^2/r^3. The condition $\ddot{a}r/a \approx GM/r^2$ can be expressed by saying that the time scale $\tau \sim \sqrt{\frac{a}{\ddot{a}}}$ is of the order of the free fall time scale of a fictitious region of radius r containing the mass M, i.e., $\frac{r}{v_E} \sim \sqrt{\frac{r^3}{GM}}$, where v_E is the escape velocity. Or, in other words, the energy density $\sim M/r^3$ of this fictitious region equals the cosmological density $\sim \ddot{a}/a$ which grows in a phantom-dominated universe. A more precise characterization of when a bound system is disrupted is given by Nesseris and Perivolaropoulos [21]. These authors consider the effective potential $V(t, r)$ of the one-dimensional equation of motion of the test particle and determine when its minimum disappears by solving numerically the equation $\partial V/\partial r = 0$ (the location of this minimum depends on time). This procedure corrects our order of magnitude estimate by a factor ~ 3. The effective potential used is the subject of the next subsection.

A. Effective potential, Lagrangian, and Hamiltonian

By rewriting Eq. (2) as

$$\ddot{r} = -\frac{GM}{r^2} + \frac{L^2}{r^3} + \frac{\ddot{a}}{a} r = -\frac{\partial V}{\partial r}, \tag{9}$$

integrating with respect to r, and setting to zero an arbitrary integration function of time, one obtains the effective potential

$$V(t, r) = -\frac{\ddot{a}}{2a} r^2 - \frac{GM}{r} + \frac{L^2}{2r^2}. \tag{10}$$

In a general FLRW spacetime this effective potential for the bound system test particle-central object depends on time because of the cosmological term $-\frac{\ddot{a}}{2a}r^2$ and one can say that this system exchanges energy with the cosmological background. However, in the special case of a de Sitter background described by the scale factor $a(t) = a_0 \exp(H_0 t)$ with constant H_0, this term is time-independent and reduces to $-H_0^2 r^2/2$—no energy is then exchanged between the two-body system and the cosmological background, and the energy of the former is constant. This simplification was noted by Price [22], who restricted his study of the effect of cosmological dynamics on local systems to a de Sitter background.

Given the effective potential (10), it is straightforward to deduce the effective Lagrangian

$$\mathcal{L}(t, r, \dot{r}) = \frac{\dot{r}^2}{2} + \frac{GM}{r} - \frac{L^2}{2r^2} + \frac{\ddot{a}}{2a} r^2, \qquad (11)$$

which reproduces Eq. (2) through the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0, \qquad (12)$$

and the effective Hamiltonian

$$\mathcal{H}(t, r, p_r) = \frac{p_r^2}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{\ddot{a}}{2a} r^2, \qquad (13)$$

where $p_r = \partial \mathcal{L}/\partial \dot{r} = \dot{r}$ and

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r}, \qquad \dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r}. \qquad (14)$$

This Hamiltonian is, of course, time-dependent for any FLRW background that is not a de Sitter space:

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial V}{\partial t}. \qquad (15)$$

For a de Sitter background $a = a_0 e^{H_0 t}$ with $H_0 = \sqrt{\Lambda/3}$ (where $\Lambda > 0$ is the cosmological constant), the effective potential (10) is consistent with the weak-field limit of the Schwarzschild-de Sitter (or Köttler) metric, given by the line element

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2, \qquad (16)$$

in static coordinates, where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on the unit two sphere. The analogue of the Newtonian potential Φ_N for a test particle can be read off the (0, 0) component of the Köttler metric $g_{00} = -[1 + 2\Phi_N(r)]$, which yields

$$\Phi_N(r) = -\frac{GM}{r} - \frac{\Lambda r^2}{6}. \qquad (17)$$

On the other hand, the cosmological part of the potential (10) in the equation of motion (2) of the test particle in the de Sitter background with $H_0 = \sqrt{\Lambda/3}$ is

$$-\frac{\ddot{a}}{2a} r^2 = -\frac{H_0^2 r^2}{2} = -\frac{\Lambda r^2}{6}, \qquad (18)$$

which is consistent with Eq. (17). This is not a coincidence because Eq. (2) can be derived as a special approximation of the timelike geodesic equation of the McVittie metric [1], which reduces to the Köttler metric for a de Sitter background (see Sec. III C).

By adding the centrifugal potential term $\frac{L^2}{2r^2}$ one recovers the effective potential (10) for the equivalent one-dimensional problem and $V(r) = \Phi_N(r)$.

B. Orbits of constant radial coordinate

It is time to comment on the physical meaning of the radial coordinate r. As shown in Ref. [13], this coincides with the proper radius when $rH_0 \ll 1$ and receives corrections of higher order when r is larger and larger with respect to the Hubble radius H_0^{-1}. When radii $r \sim H_0^{-1}$ are considered, r assumes the meaning of comoving radial coordinate in the FLRW metric (1). However, when the field of the central object is strong, neither of the above describes precisely the meaning of r for $r \ll H_0^{-1}$.

The question of whether orbits of constant radial coordinate r exist is of some interest. Orbits of constant radius are found by Bonnor [3] who, similarly to Price [22], considers a classical atom embedded in a cosmological background. Two situations can be distinguished.

(i) The cosmological background is *not* a de Sitter space. Then, $\ddot{a}/a$ depends on time and, imposing $r = r_0 \equiv \text{const}$, one obtains

$$\frac{\ddot{a}}{a} r_0 - \frac{GM}{r_0^2} + \frac{L^2}{r_0^3} = 0. \qquad (19)$$

The first term on the left hand side depends explicitly on time, while the remaining terms are time-independent, hence this equation can not be satisfied and orbits of constant r do not exist in this case.

(ii) The cosmological background is de Sitter space. Then imposing $r = r_0 \equiv$ constant yields the quartic equation for r_0

$$H_0^2 r_0^4 - GM r_0 + L^2 = 0. \qquad (20)$$

This equation can, in principle, have real solutions under conditions which are discussed in the next subsection.

C. Test particle around a Newtonian central object in a de Sitter background: phase plane analysis

The equation of motion of an electron in a classical atom embedded in a de Sitter background is analogous to our Eq. (2) and is solved numerically by Price [22]. Contrary to previous authors, Price does not restrict himself to considering weak couplings of the electron, which is the equivalent of the situation considered so far in our paper, of a test particle in the field of a Newtonian, weakly gravitating central object embedded in a cosmological background. Price considers instead arbitrarily strong coupling of the electron to the central charge and discovers an "all or nothing" behavior: if the coupling is weak the electron trajectory becomes comoving with the de Sitter substratum while, if the coupling is strong, the evolution of the orbit exhibits a transient after which it is essentially unperturbed. A critical value of the angular momentum separates these two behaviours. This "all or nothing" feature went undetected in the (abundant) previous literature which did not break free of the weak coupling assumption. In our formalism, allowing for an arbitrarily strong gravitational field due to the central object means making this object relativistic and one must leave the regime in which the latter is merely a perturbation of a cosmic substratum, and move to a fully relativistic regime by studying exact solutions describing a strongly gravitating central object embedded in a FLRW background. This will be done in the next section. Here we limit ourselves to weak coupling and we provide a phase space analysis of the equation of motion analyzed numerically by Price [22], in the analogous situation of a two-body system completely ruled by gravity. This phase space analysis is still missing in the literature.

By introducing variables $x \equiv r$ and $y \equiv \dot{r}$ [not to be confused with Cartesian coordinates in the de Sitter background (1)], Eq. (2) is written as the autonomous dynamical system

$$\dot{x} = y, \tag{21}$$

$$\dot{y} = H_0^2 x - \frac{GM}{x^2} + \frac{L^2}{x^3}, \tag{22}$$

for $x > 0$ and any real value of y. We assume $L \neq 0$ (the case $L = 0$ will be discussed later). Equilibrium points, if they exist, correspond to orbits of constant radius $(x, y) = (r_0, 0)$ of the kind discovered by Bonnor [3]. The search for these fixed points is equivalent to solving Eq. (20), or

$$\psi(x) \equiv H_0^2 x^4 - GMx + L^2 = 0. \tag{23}$$

The function $\psi(x)$ is represented by a quartic parabola with its concavity facing upward, therefore there can be solutions of Eq. (23) only if the minimum $\psi_{\min}$ of $\psi(x)$ is nonpositive. If $\psi_{\min} = 0$ there are two coincident roots, while if $\psi_{\min} < 0$ there are two real distinct roots, and no real roots exist if $\psi_{\min} > 0$. The study of the first derivative $d\psi/dx$ establishes that $\psi(x)$ has the absolute minimum

$$\psi_{\min} \equiv \psi(x_{\min}) = L^2 - \frac{3z}{H_0^2} \tag{24}$$

at

$$x_{\min} = \left(\frac{GM}{4H_0^2}\right)^{1/3}, \tag{25}$$

where

$$z \equiv \left(\frac{GMH_0}{4}\right)^{4/3} \tag{26}$$

is a dimensionless variable. It follows that $\psi_{\min} > 0$, hence there are no orbits of constant r if

$$L > \sqrt{3}\left(\frac{GM}{4\sqrt{H_0}}\right)^{2/3} \equiv L_c. \tag{27}$$

There is a single orbit of constant $r = x_{\min}$ if $L = L_c$, and there are two orbits of constant radii $r_{1,2}$ with $r_1 < x_{\min} < r_2$ when $L < L_c$. These are all the fixed points $(x_0, 0)$ of the dynamical system (21) and (22). In order to assess the stability of these fixed points let us consider perturbations described by $x(t) = x_0 + \delta x(t)$, $y(t) = \delta y(t)$. Equations (21) and (22) yield the evolution equation for the orbital radius

$$\delta\ddot{r} + \omega^2 \delta r \simeq 0, \qquad \omega^2 = \frac{L^2}{r^4} - 3H_0^2. \tag{28}$$

Linear stability corresponds to $\omega^2 \geq 0$. Orbits of constant r exist when $L \leq L_c$, with the equality corresponding to a single orbit of radius r_0 and the strict inequality to two orbits of radii $r_{1,2}$ with $r_1 < x_{\min} < r_2$. The stability condition of an orbit of constant radius r_* is equivalent to $r_* \leq \sqrt{\frac{L}{\sqrt{3}H_0}}$. In the case of a single orbit it is $L = L_c$ and $r_* = r_0 = x_{\min} \leq x_{\min}$ and the stability condition is satisfied. In the case of two distinct orbits it is $r_1 < x_{\min} < r_2$ and therefore the outer orbit is unstable. The inner orbit is stable only if $r_1 \leq \sqrt{\frac{L}{\sqrt{3}H_0}}$. Since $\psi(x = \sqrt{\frac{L}{\sqrt{3}H_0}}) = \frac{4L^2}{3} - GM\sqrt{\frac{L}{\sqrt{3}H_0}} < 0$ when $L > L_c$ and $\psi(x)$ is a decreasing function between $x = 0$ and $x_{\min}$, while $\psi(r_1) = 0$, it must be $r_1 < \sqrt{\frac{L}{\sqrt{3}H_0}}$ and therefore the inner orbit is stable.

At large values of x, Eq. (22) reduces to the asymptotic equation $\dot{y} \simeq H_0^2 x$, which has the solution

$$(x(t), y(t)) = (x_* e^{H_0 t}, H_0 x_* e^{H_0 t}) \tag{29}$$

represented by the line $y = H_0 x$ in the (x, y) plane. This solution is an attractor for large values of x. In fact, by perturbing the solution as described by $x(t) = x_* e^{H_0 t} + \delta x(t)$ and $y(t) = H_0 x_* e^{H_0 t} + \delta y(t)$ one obtains, to first order in the perturbations, $\delta\ddot{x} = H_0^2 \delta x + \ldots$ with solution $\delta x(t) = \delta_0 e^{H_0 t}$ (with δ_0 a constant). Therefore, the ratios

$$\frac{\delta x(t)}{x_* e^{H_0 t}} = \frac{\delta_0}{x_0}, \qquad \frac{\delta y(t)}{H_0 x_* e^{H_0 t}} = \frac{\delta_0}{x_0}, \tag{30}$$

stay small if they start small ($|\frac{\delta_0}{x_0}| \ll 1$).

The qualitative picture of the phase space can be completed as follows. The system (21) and (22) has the first integral

$$E = \frac{y^2}{2} - \frac{H_0^2 x^2}{2} + \frac{L^2}{2x^2} - \frac{GM}{x} \tag{31}$$

corresponding to the energy of the test particle, which is conserved in a de Sitter background.

(i) If $L > L_c$ there are no fixed points and $y = H_0 x$ is an attractor for large values of x. Then $\dot{y} = \psi(x)/x^3 > 0$ (the equation $\psi = 0$ has no real roots for $L > L_c$). Hence $y = \dot{x}$ always increases and either $y \to +\infty$ or $y(t)$ has a horizontal asymptote with $y(t) \to y_0$ as $t \to +\infty$. If $y = \dot{x} \to +\infty$, then either $x(t) \to +\infty$ or $x(t)$ has a vertical asymptote with $x \to x_0$ and $\dot{x} \to +\infty$. There are, in principle, four possibilities.

(1) $x(t) \to +\infty$ and $y(t) \to +\infty$; then the orbit of the solution is necessarily captured in the

attraction basin of the attractor $y = H_0 x$.

(2) $x(t) \to +\infty$ and $y(t) \to y_0$; this is not possible because as $x \to +\infty$ it is $y = H_0 x \to +\infty$.

(3) $x(t) \to x_0$ and $y(t) \to +\infty$ ($x(t)$ has a vertical asymptote $\dot{x} \to +\infty$); then the energy E given by Eq. (31) diverges at late times, which is not possible because E is constant.

(4) $x(t) \to x_0$ and $y(t) \to y_0$; this means that there is an attractor point, then $\dot{y} = H_0^2 x - GMx^{-2} + L^2 x^{-3} \to 0$ as $t \to +\infty$ then x_0 must be a root of the equation $\dot{y} = 0$ but this is not possible because this equation has no real roots when $L > L_c$. This case is not possible. In summary, for $L = L_c$ all the trajectories of the solutions in phase space go to the attractor $y = H_0 x$ as $t \to +\infty$, i.e., all physical orbits become comoving.

(ii) If $L = L_c$ there are the attractor $y = H_0 x$ as $x \to +\infty$ and the fixed point $x_0 = x_{\min} = (\frac{GM}{4H_0^2})^{1/3}$, $y_0 = 0$ which is the only real root of the equation $\dot{y} = 0$. Since there is only one such root, *a priori* either $\dot{y} > 0$ or $\dot{y} < 0$ along any orbit that does not coincide with the fixed point, without the possibility of changing sign during the evolution. However, we know that $\dot{y} = \psi(x)/x^3 > 0 \ \forall \ x \neq x_0$ then $y = \dot{x}$ is always increasing for $x \neq x_0$. By repeating the reasoning of the case $L > L_c$, we have now two possibilities:

(1) $x(t) \to +\infty$ and $y(t) \to +\infty$, in which case the orbit of the solution in phase space gets captured by the attractor $y = H_0 x$ at infinity.

(2) $(x(t), y(t)) \to (x_0, 0)$ with $y(t) \to 0^-$ so $y = \dot{x} < 0$ and $x(t)$ is always decreasing to x_0. Then, $\ddot{x} > 0$ with $\dot{x} < 0$; the point $(x_0, 0)$ is an attractor point. Since $\dot{y} > 0 \ \forall \ (x, y) \neq (x_0, y_0)$ there are no periodic orbits. The cases in which $(x(t), y(t)) \to (+\infty, y_0)$ or $(x(t), y(t)) \to (x_0, +\infty)$ are excluded as previously discussed for $L > L_c$. In summary, for $L = L_c$ the orbits of the solutions either go to the attractor point or to the attractor at infinity $y = H_0 x$.

(iii) If $0 < L < L_c$ there are the equilibrium points $(r_{1,2}, 0)$ and the attractor at infinity $y = H_0 x$ with $r_1 < x_{\min} < r_2$. There are two real distinct roots of the equation $\dot{y} = \psi(x)/x^3 = 0$ and it is $\dot{y} > 0$ for $x < r_1$ and for $x > r_2$, while it is $\dot{y} < 0$ for $r_1 < x < r_2$.

For $x < r_1$ the function $y(t) = \dot{x}$ is increasing and either $y \to +\infty$ or $y(t) \to y_0^-$ (horizontal asymptote of $y(t)$). If $y \to +\infty$ it must be because $x \to 0$, the familiar situation in which the Newtonian centrifugal potential $L^2 x^{-3}$ dominates and repels the particle to larger x. If $y(t)$ has a horizontal asymptote, $y \to y_0$, then $\dot{y} \to 0$ and the attractor point $(r_1, 0)$ is approached.

For $r_1 < x < r_2$ it is $\dot{y} = \psi(x)/x^3 < 0$ so $y(t)$ is always decreasing and either $y(t) \to -\infty$ or $y(t)$ has a horizontal asymptote $y \to y_0^-$. The first situation is not possible because it would imply that the energy E diverges, while it is instead forced to be constant. Therefore, $y(t)$ must have a horizontal asymptote and $\dot{y} \to 0$. Then the attractor point (zero of the function $\psi(x)$) is approached.

For $x > r_2$ it is $\dot{y} > 0$ and either $y(t) \to +\infty$ or $y(t) \to y_0^+$ (horizontal asymptote). If $y \to +\infty$ there are, in principle, two possibilities:

(1) $x(t) \to +\infty$ and $y(t) \to +\infty$; then the orbits of the solutions are captured by the attraction basin of $y = H_0 x$.

(2) $x(t)$ stays finite and $y(t) \to +\infty$. Again, this would give $E \to +\infty$, which is excluded.

1. Zero angular momentum

Finally, let us consider the special case $L = 0$ in which the radial coordinate of the test particle satisfies

$$\ddot{r} = H_0^2 r - \frac{GM}{r^2}. \tag{32}$$

Only one "orbit" of constant physical radius $r_0 = (GM/H_0^2)^{1/3}$ exists in this case. This situation is not possible in flat space and it corresponds to the cosmological "force" $H_0^2 r$ exactly compensating the attractive force of the central object $-GM/r^2$. As is clear from the physical point of view, this position of equilibrium is unstable because an arbitrarily small radial displacement will move the test particle to a region where one of the two forces is dominant. The value $r_0 = (GM/H_0^2)^{1/3}$ corresponds to a maximum of the effective potential

$$V(r) = -\frac{H_0^2 r^2}{2} - \frac{GM}{r}. \tag{33}$$

In fact, setting $dV/dr = 0$ reproduces this value of r and $d^2V/dr^2|_{r_0} = -3H_0^2 < 0$. This simple solution is apparently missed in previous literature.

D. Purely radial motion in a general FLRW background

One can generalize to any FLRW background the search for solutions with purely radial motion satisfying

$$\ddot{r} = \frac{\ddot{a}}{a} r - \frac{GM}{r^2}. \tag{34}$$

Orbits of constant r are impossible if the background is not de Sitter space. We consider universes with scale factor given by a power-law $a(t) = a_0 t^p$ (with p a constant). Then, it is easy to find the power-law solution

$$r_*(t) = \left[\frac{GM}{p(p-1) + \frac{2}{9}}\right]^{1/3} t^{2/3} \equiv r_c t^{2/3}. \tag{35}$$

This solution exists only for values of the constant p

satisfying $p < 1/3$ or $p > 2/3$, which follows from the requirement that the denominator of the constant r_c in Eq. (35) be positive. The solution (35) is unstable with respect to small radial perturbations, In fact, if $r(t) = r_*(t) + \delta r(t)$, using Eq. (35) one finds the evolution equation for the linear perturbations δr

$$\delta \ddot{r} - \left[3p(p-1) + \frac{4}{9}\right]\frac{\delta r}{t^2} = 0, \tag{36}$$

which has solutions $\delta r(t) = \delta_0 t^\beta$ with δ_0 a constant and

$$\beta_\pm = \frac{1}{2}\left(1 \pm \sqrt{12p(p-1) + \frac{25}{9}}\right). \tag{37}$$

In the allowed range of values of p, the discriminant $\Delta = 12p(p-1) + \frac{25}{9}$ satisfies $\sqrt{\Delta} > 1/3$, which yields $\beta_+ = (1 + \sqrt{\Delta})/2 > 2/3$ and the corresponding mode $\delta r_+(t)$ satisfies

$$\frac{\delta r_+(t)}{r_*(t)} \simeq t^{\beta_+ - (2/3)}. \tag{38}$$

Because of the positive exponent $\beta_+ - 2/3$, this ratio grows when $t \longrightarrow +\infty$ and therefore the solution (35) is unstable.

To conclude, one recognizes that the de Sitter universe, although simpler to study than a general FLRW background, is a very special case. Furthermore, the simple problem of a test particle in a circular orbit is extremely simplified and perhaps is not the best physical system in which to study the competition between local attraction and cosmological expansion. The next step would be to consider the expansion of a Newtonian star embedded in a FLRW universe. We skip this step and, in the next section, we consider instead a *relativistic* star embedded in a FLRW universe, and its Newtonian limit.

III. A STRONGLY GRAVITATING OBJECT IN A FLRW BACKGROUND

The study of a classical atom in a de Sitter background by Price [22] shows that systems that are strongly bound "resist" the cosmological expansion and are only perturbed a little by it. This leads one to believe that this situation would carry over to the case of a gravitationally bound system and that a strongly gravitating central object, such as a black hole, would not be perturbed at all by the cosmic expansion. Whether this is true or not is the first question addressed in this section. Second, one would like to know whether this time-independent behavior of a strongly gravitating central object (if it really carries over from Price's classical atom to the gravitating system) is peculiar of de Sitter space, or if it is valid in *general* FLRW backgrounds. After all, de Sitter space (other than the trivial Minkowski space with $a \equiv 1$) is the closest to a static space, and (a portion of) it can be expressed in static coordinates; therefore, it could be too special to derive general conclusions. Third, it is claimed in the literature that in a universe approaching the big rip all bound objects (galaxies, stars, atoms, *etc.*) are ripped apart before the singularity is reached [30]. What about a black hole horizon? If the expansion to a big rip tears apart a black hole horizon and a naked singularity appears, the cosmological version of cosmic censorship [31] is violated, the implications for black hole thermodynamics are nontrivial, and the phantom energy causing the big rip would be questioned further from the point of view of fundamental physics. Regarding this problem, it is claimed that a black hole accreting phantom energy with $P < -\rho$ in a universe approaching the big rip *decreases* its mass and the horizon disappears together with the central singularity before the big rip is reached [32]. This phenomenon avoids the violation of cosmic censorship by eliminating the central singularity altogether. New exact solutions can help clarifying this issue.

In all the situations mentioned above one can heuristically see the "all or nothing behavior" as resulting from the competition between two strong fields, the cosmological one and the local one due to the central object. It is conceivable that one of the two is locally stronger than the other and "wins" but, on the other hand, many examples of black holes are known which, placed in a strong external gravitational, electric, or magnetic field, have their horizons stretched [11,37–52].

In this section we want to give a more precise meaning to these naive considerations, and a general picture of these phenomena that is not limited to a de Sitter background space. Exact solutions of the Einstein equations representing strongly gravitating objects embedded in a FLRW universe are needed. We neglect semiclassical Hawking radiation in the following.

A. The McVittie solution

An obvious starting point is the McVittie metric introduced in 1933 with the explicit purpose of investigating the effects of the cosmic expansion on local systems [1]. The line element is

$$ds^2 = -\frac{(1 - \frac{m(t)}{2\bar{r}})^2}{(1 + \frac{m(t)}{2\bar{r}})^2} dt^2 + a^2(t)\left(1 + \frac{m(t)}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\Omega^2) \tag{39}$$

in isotropic coordinates, where the function $m(t)$ of the comoving time satisfies the equation [1]

$$\frac{\dot{m}}{m} = -\frac{\dot{a}}{a}, \tag{40}$$

which yields

$$m(t) = \frac{m_H}{a(t)}, \tag{41}$$

where m_H is a constant representing the Hawking-

Hayward quasilocal mass [53,54] (see Refs. [6,15] for a discussion). Equation (40) was imposed by McVittie to explicitly forbid the accretion of cosmic fluid onto the central object (assumption *e*) of Ref. [1]). It corresponds to $G_0^1 = 0$, which in turn implies that the stress-energy tensor component $T_0^1 = 0$ and there is no radial flow. In modern language, Eq. (40) corresponds to the constancy of the Hawking-Hayward mass, $\dot{m}_H = 0$. It is important to recognize m_H as the physically relevant mass (eventually related to the physical size of the horizon or of the central object) in order to avoid making coordinate-dependent statements on the mass and size (cf., e.g., Refs. [18,55]), or temperature [28] of the central object. $m(t)$ is just a metric coefficient in a particular coordinate system.

While there is little doubt that the McVittie metric represents some kind of strongly gravitating central object, its physical interpretation is not completely clear and is still debated today [10,12,15,16]. This metric reduces to the Schwarzschild solution in isotropic coordinates when $a \equiv 1$ and to the FLRW metric if $m \equiv 0$. However, in general, the metric (39) can not be interpreted as describing a black hole embedded in a FLRW universe because it is singular on the 2-sphere $\bar{r} = m/2$ (which reduces to the Schwarzschild horizon if $a \equiv 1$) [10,12,15] and this singularity is spacelike [15,16]. It is claimed that the McVittie metric describes a point mass located at $\bar{r} = 0$ (which is another, expected, singularity of the metric) embedded in a FLRW universe. However, this point mass is, in general, surrounded by the singularity at $\bar{r} = m/2$, which is difficult to interpret. This singularity was studied in Refs. [10,12,15]. Nolan [6] showed that this is a weak singularity in the sense that the volume of an object falling onto the $\bar{r} = m/2$ surface is not crushed to zero, and therefore the energy density of the surrounding fluid is finite. However, the pressure

$$P = -\frac{1}{8\pi G}\left[3H^2 + \frac{2\dot{H}(1+\frac{m}{2\bar{r}})}{1-\frac{m}{2\bar{r}}}\right] \tag{42}$$

diverges at $\bar{r} = m/2$ together with the Ricci scalar $R = 8\pi G(3P - \rho)$ [10,12,15,55]. Notwithstanding this, there is a situation in which the singularity disappears and the surface $\bar{r} = m/2$ describes a true black hole horizon: this happens when the background FLRW is de Sitter space. Then $\dot{H} = 0$ and the second term on the right hand side of the expression (42)—the only one causing P to diverge—is absent (this point was also noted in Ref. [15]). A possible reason for the disappearance of the singular surface in the de Sitter case is discussed below. Similar problems affect the charged McVittie metric [18] and the solutions of Thakurta [56], Vaidya [57], Patel and Trivedi [44] representing rotating black holelike objects in a cosmological background ([55]—see also Refs. [4,58]).

B. The Nolan interior solution: a relativistic star in a FLRW universe and its Newtonian limit

It is of interest to study the behavior of a relativistic star embedded in a FLRW background with respect to the problem of local physics versus cosmological expansion. The Nolan interior solution [33] describes a relativistic star of uniform density in such a background. The metric is

$$ds^2 = -\left[\frac{1-\frac{m}{\bar{r}_0}+\frac{m\bar{r}^2}{\bar{r}_0^3}(1-\frac{m}{4\bar{r}_0})}{(1+\frac{m}{2\bar{r}_0})(1+\frac{m\bar{r}^2}{2\bar{r}_0^3})}\right]^2 dt^2 + a^2(t)\frac{(1+\frac{m}{2\bar{r}_0})^6}{(1+\frac{m\bar{r}^2}{2\bar{r}_0^3})^2}(d\bar{r}^2 + \bar{r}^2 d\Omega^2) \tag{43}$$

in isotropic coordinates, where $\bar{r}_0$ is the star radius, $\frac{\dot{m}}{m} = -\frac{\dot{a}}{a}$ (the condition forbidding accretion onto the star surface), and $0 \leq \bar{r} \leq \bar{r}_0$. The interior metric is regular at the center and is matched to the exterior McVittie metric at $\bar{r} = \bar{r}_0$ by imposing the Darmois-Israel junction conditions. The energy density is uniform and discontinuous at the surface $\bar{r} = \bar{r}_0$, while the pressure is continuous. These quantities are given by [33]

$$\rho(t) = \frac{1}{8\pi G}\left[3H^2 + \frac{6m}{a^2\bar{r}_0^3(1+\frac{m}{2\bar{r}_0})^6}\right], \tag{44}$$

$$P(t,\bar{r}) = \frac{1}{8\pi G}\left[-3H^2 - 2\dot{H}\frac{(1+\frac{m}{2\bar{r}_0})(1+\frac{m\bar{r}^2}{2\bar{r}_0^3})}{1-\frac{m}{\bar{r}_0}+(1-\frac{m}{4\bar{r}_0})\frac{m\bar{r}^2}{\bar{r}_0^3}} + \frac{1}{a^2}\frac{\frac{3m^2}{\bar{r}_0^4}(1-\frac{\bar{r}^2}{\bar{r}_0^2})}{(1+\frac{m}{2\bar{r}_0})^6[1-\frac{m}{\bar{r}_0}+(1-\frac{m}{4\bar{r}_0})\frac{m\bar{r}^2}{\bar{r}_0^3}]}\right]. \tag{45}$$

The Nolan interior solution is a special member of Kustaanheimo's family of shear-free solutions [59] that generalizes the Schwarzschild interior solution of uniform constant density to the case of a time-dependent cosmological background. By setting $a \equiv 1$ one recovers familiar expressions for the Schwarzschild interior solution [34]. The energy density is always positive and the condition $P \geq 0$ imposed by Nolan coincides with $\ddot{a} + 3\dot{a}^2/2 < 0$ [33].

Let $\Sigma_0(t) = \{(t, \bar{r}, \theta, \varphi): \bar{r} = \bar{r}_0\}$ be the surface of the star at time t; by construction, the metric on this 2-sphere coincides with the McVittie metric

$$ds^2|_{\Sigma_0} = -\frac{(1-\frac{m(t)}{2\bar{r}_0})^2}{(1+\frac{m(t)}{2\bar{r}_0})^2}dt^2 + a^2(t)\left(1+\frac{m(t)}{2\bar{r}_0}\right)^4\bar{r}_0^2 d\Omega^2. \tag{46}$$

The proper area of Σ_0 is

$$\mathcal{A}_{\Sigma_0}(t) = \iint_{\Sigma_0} d\theta d\varphi \sqrt{g_{\Sigma_0}} = 4\pi a^2(t)\bar{r}_0^2\left(1 + \frac{m(t)}{2\bar{r}_0}\right)^4, \tag{47}$$

where $g_{ab}|_{(\Sigma_0)}$ is the metric on Σ_0 at a fixed time t and g_{Σ_0} is its determinant. By using the Schwarzschild curvature coordinate $r \equiv \bar{r}(1 + \frac{m}{2\bar{r}})^2$, one has

$$\mathcal{A}_{\Sigma_0}(t) = 4\pi a^2(t) r_0^2. \tag{48}$$

The star surface is comoving with the cosmic substratum and the proper curvature radius of the star is $r_{\text{phys}}(t) = a(t)\bar{r}_0(1 + \frac{m}{2\bar{r}_0})^2$. Therefore, we have a local relativistic object with strong field which is perfectly comoving at all times: in this case the cosmic expansion wins over the local dynamics.

It is interesting to compute the generalized Tolman-Oppenheimer-Volkoff equation [34] valid for this crude star model and to derive the first order correction to the Newtonian equation of hydrostatic equilibrium. For a perfect fluid described by the stress-energy tensor $T_{ab} = (P + \rho)u_a u_b + P g_{ab}$, the covariant conservation equation $\nabla^b T_{ab} = 0$ splits into the two equations [34]

$$u^c \nabla_c \rho + (P + \rho)\nabla^c u_c = 0, \tag{49}$$

$$h^c{}_a \partial_c P + (P + \rho) u^c \nabla_c u_a = 0, \tag{50}$$

where u^a is the fluid four-velocity and $h_{ab} \equiv g_{ab} + u_a u_b$ defines the projector $h_a{}^c$ onto the three-space orthogonal to u^a. In comoving coordinates it is $u^c \propto \delta^{0c}$ and the normalization $u^c u_c = -1$ yields

$$u^c = u\delta^{0c} = \frac{(1 + \frac{m}{2\bar{r}_0})(1 + \frac{m\bar{r}^2}{2\bar{r}_0^3})}{1 - \frac{m}{\bar{r}_0} + (1 - \frac{m}{4\bar{r}_0})\frac{m\bar{r}^2}{\bar{r}_0^3}} \cdot (1, 0, 0, 0), \tag{51}$$

or $u = |g_{00}|^{-1/2}$. Equations (40) and (49) then yield

$$\frac{\partial \rho}{\partial t} + 3H(P + \rho)\left\{1 - \frac{m}{\bar{r}_0}\left[\frac{3}{2(1 + \frac{m}{2\bar{r}_0})} - \frac{\bar{r}^2}{2\bar{r}_0^2}\left(1 + \frac{m\bar{r}^2}{2\bar{r}_0^3}\right)^{-1}\right]\right\} = 0, \tag{52}$$

which generalizes the well-known conservation equation $\dot{\rho} + 3H(P + \rho) = 0$ valid in a FLRW universe, to which it reduces in the limit $m \to 0$. For the Schwarzschild interior solution there is no equation analogous to (52) because $H = 0$ for the static Minkowski background and ρ is static.

Equation (50) can be rewritten as

$$\partial_c P + u_c u^b \partial_b P + (P + \rho) u^b \nabla_b u_c = 0. \tag{53}$$

By setting the index $c = 1$ and computing the covariant derivative one obtains

$$\frac{\partial P}{\partial r} + (P + \rho)\frac{m\bar{r}}{\bar{r}_0^3}\frac{(1 + \frac{m}{2\bar{r}_0})}{(1 + \frac{m\bar{r}^2}{2\bar{r}_0^3})[1 - \frac{m}{\bar{r}_0} + \frac{m\bar{r}^2}{\bar{r}_0^3}(1 - \frac{m}{4\bar{r}_0})]} = 0. \tag{54}$$

In the Newtonian limit $m/\bar{r}, m/\bar{r}_0 \ll 1$, $P \ll \rho$, $r \simeq \bar{r}$, this equation reduces to

$$\frac{\partial P}{\partial r} + \frac{d\Phi_N}{dr}\rho = 0, \tag{55}$$

where $\rho = m(\frac{4\pi}{3} r_0^3)^{-1}$ and $\Phi_N = \frac{m\bar{r}^2}{2\bar{r}_0^3}$ is the Newtonian potential. This expression for the density can also be obtained from Eq. (44) by setting $a \equiv 1$ and using the curvature radius. The first order correction to the equation of hydrostatic equilibrium for a spherically symmetric, uniform density star, is given by

$$\frac{dP}{dr} + \frac{d\Phi_N}{dr}\rho\left\{1 - \frac{3}{2}[\Phi_N(r) - \Phi_N(r_0)]\right\} = 0. \tag{56}$$

C. The Schwarzschild-de Sitter black hole

When the de Sitter space is chosen as the background, it is well known that the McVittie metric reduces to the Schwarzschild-de Sitter (Köttler) metric, which can be put into the static form (16). There is little doubt that in this case the central object described by this metric is a black hole. The Schwarzschild-de Sitter black hole has been the subject of much literature, mainly devoted to study the thermodynamical properties of dynamical horizons, which are interesting because of the simultaneous presence of a Schwarzschild and a Rindler horizon. The inner horizon at $\bar{r} = m/2$ has area

$$\mathcal{A} = \iint d\theta d\varphi \sqrt{g_\Sigma}, \tag{57}$$

where $g_{ab}|_\Sigma$ is the restriction of the metric tensor to this 2-sphere and g_Σ is its determinant. Equation (39) yields

$$\mathcal{A} = 16\pi a^2 m^2 \tag{58}$$

or, upon use of Eq. (41),

$$\mathcal{A} = 16\pi m_H^2. \tag{59}$$

This area does not depend on time because $\dot{m}_H = 0$ for all McVittie solutions as a consequence of Eq. (40). The Schwarzschild radial coordinate corresponding to the horizon is the curvature coordinate

$$r = \sqrt{\frac{\mathcal{A}}{4\pi}} = 2m_H \tag{60}$$

and is also time-independent. Therefore, the horizon area of the Schwarzschild-de Sitter black hole does not increase: the horizon does not expand, "resisting" the cosmic (accelerated) expansion. Cosmic censorship (in its version for nonasymptotically flat spaces [31]) is not violated in this case; the central singularity remains forever and is always surrounded by a horizon.

It is not true that *all* physical systems in *any* expanding universe participate in the cosmic expansion; in a de Sitter universe this is true only for weakly coupled systems. This result agrees with the "all or nothing" behavior discovered by Price [22]. In the next section we argue that this phenomenon, however, is peculiar to the de Sitter background.

IV. BLACK HOLES IN ARBITRARY FLRW BACKGROUNDS: OLD AND NEW SOLUTIONS

Although the Schwarzschild-de Sitter black hole does not participate in the cosmic expansion, in more general FLRW spacetimes black holes or other strongly gravitating objects can expand under suitable conditions, as is shown below.

The first step toward this discussion consists of identifying exact inhomogeneous solutions of the Einstein equations that are suitable for describing strongly gravitating objects embedded in a FLRW universe that is not a de Sitter space. In this case the McVittie metric (39) can not be interpreted as describing black holes [10,12,15,55], however, it does describe some kind of singular central object. The computation of the area of the singular surface $\bar{r} = m/2$ and of its curvature radius proceeds exactly as in the Schwarzschild-de Sitter case leading again to the time-independent area (59) and radius (60). This applies also to the Reissner-Nordstrom generalization of McVittie's metric found by Gao and Zhang [18]. The issue is whether this central object is a realistic one. It is tempting to interpret the singularity at $\bar{r} = m/2$ as an artificial one created by explicitly forbidding accretion of the cosmic fluid onto the central object, much like the axial singularity in cylindrically symmetric Bach-Weyl solutions [60]. In the latter, two massive particle are in static equilibrium at a finite distance and an axial singularity is interpreted as a strut holding them apart [61]. One could think that the $\bar{r} = m/2$ surface in the McVittie metric similarly acts as a wall to keep out the cosmic fluid. This interpretation seems to be corroborated by the following observation: the accretion rate for spherical accretion of a *test fluid* with energy density ρ and pressure P onto a Schwarzschild black hole of mass μ is [62]

$$\dot{\mu} = 4\pi D\mu^2(P_\infty + \rho_\infty), \tag{61}$$

where ρ_∞ and P_∞ are the energy density and pressure at spatial infinity, respectively, and D is a constant corresponding to a first integral of motion [32,62]. The accretion rate (61) vanishes for a fluid satisfying the quantum vacuum equation of state $P = -\rho = \frac{-\Lambda}{8\pi G}$. Extrapolating this result to a gravitating fluid, (spherical) accretion onto a Schwarzschild-de Sitter black hole is automatically prevented, the hole's mass does not change, and the condition (40) is satisfied naturally—it does no longer enforce the presence of a spherical wall to stop cosmic matter from falling onto the black hole.

This interpretation of the two-sphere singularity turns out to be at least partially erroneous. Below, some new exact solutions are presented in which the no-accretion condition is removed but the pressure is still singular on the surface $\bar{r} = m/2$ unless the black hole is exactly comoving.

A. The Sultana-Dyer solution

Recently Sultana and Dyer [19] found an exact solution of Petrov type D describing a black hole embedded in a spatially flat FLRW universe with scale factor $a(t) \propto t^{2/3}$ (in comoving time). The technique used to generate this solution consists of conformally transforming the Schwarzschild metric $g_{ab}^{(S)} \rightarrow \Omega^2 g_{ab}^{(S)}$ with the goal of changing the Schwarzschild global timelike Killing field ξ^c into a conformal Killing field for $\xi^c \nabla_c \Omega \neq 0$, generating the conformal Killing horizon (which differs from Hayward's future outer trapping horizons [63] and from dynamical horizons [64]). The metric obtained by choosing $\Omega = a(t) = a_0 t^{2/3}$—the scale factor of a dust-filled $k = 0$ FLRW universe—is

$$ds^2 = -\frac{(1 - \frac{m_0}{2\bar{r}})^2}{(1 + \frac{m_0}{2\bar{r}})^2} dt^2 + a^2(t)\left(1 + \frac{m_0}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\Omega^2); \tag{62}$$

contrary to Ref. [19] we use isotropic radius $\bar{r}$ and comoving time t. This is formally the same as the McVittie solution (39) but with the important difference that the metric coefficient m is now a constant m_0. To relate to our previous discussion, this implies that the Hawking quasilocal mass is $m_H(t) = m_0 a(t)$ and it *increases* with time. The condition (40) is violated and an accretion flow of cosmic fluid onto the central object is present and is responsible for the increase in the gravitating mass.

The source for this metric is a combination of two non-interacting fluids: $T_{ab} = T_{ab}^{(I)} + T_{ab}^{(II)}$, where $T_{ab}^{(I)} = \rho u_a u_b$ describes an ordinary (massive) dust and $T_{ab}^{(II)} = \rho_n k_a k_b$ describes a null dust with density ρ_n and $k^c k_c = 0$ [19]. The surface area of the conformal Killing horizon $\bar{r} = m_0/2$ is

$$\mathcal{A}(t) = \iint d\theta d\varphi \sqrt{g_\Sigma} = 16\pi m_0^2 a^2(t) \tag{63}$$

and its physical (curvature coordinate) radius is simply

$$r_{\text{phys}}(t) = \sqrt{\frac{\mathcal{A}}{4\pi}} = 2m_0 a(t), \tag{64}$$

which coincides with the familiar Schwarzschild radius $r = 2m$ multiplied by the scale factor, as customary in FLRW cosmology. Thus, the physical radius of the horizon and the quasilocal mass are comoving with the cosmic background.

The Sultana-Dyer solution is nonsingular at the surface $\bar{r} = m/2$ and can be interpreted as describing a black hole embedded in a two-fluid universe. We see that removing the McVittie no-accretion condition (40) can indeed remove the singularity here, allowing the black hole to become comoving with the rest of the universe. There are, however, problems with the Sultana-Dyer solution: the cosmological fluid becomes tachyonic (negative energy density) at late times near the horizon [19]. Moreover, it is desirable to have cosmological matter described by a single fluid composed of particles following timelike geodesics, and to drop the restriction to the special choice of the scale factor $a \propto t^{2/3}$.

McClure and Dyer [55] found another solution for a black holelike object embedded in a radiation-dominated universe with a heat current, which satisfies the energy conditions everywhere and is perfectly comoving. However, the energy density and pressure are singular at $\bar{r} = m/2$. A similar solution in a dust-dominated universe has singular energy density [55].

It is not clear at this point whether solutions exist describing black holes in arbitrary FLRW backgrounds, which are free of singularities at $\bar{r} = m/2$ and satisfy the energy conditions everywhere. This is the subject of the next subsection.

B. New exact solutions

We look for solutions described by a generalized McVittie metric of the form (39), but without imposing the no-accretion restriction (40), and with arbitrary scale factor $a(t)$. The line element is written in the form

$$ds^2 = -\frac{B^2(t,\bar{r})}{A^2(t,\bar{r})}dt^2 + a^2(t)A^4(t,\bar{r})(d\bar{r}^2 + \bar{r}^2 d\Omega^2), \quad (65)$$

where

$$A(t,\bar{r}) = 1 + \frac{m(t)}{2\bar{r}}, \qquad B(t,\bar{r}) = 1 - \frac{m(t)}{2\bar{r}}. \quad (66)$$

The only nonvanishing components of the mixed Einstein tensor are

$$G_0^0 = -\frac{3A^2}{B^2}\left(\frac{\dot{a}}{a} + \frac{\dot{m}}{\bar{r}A}\right)^2, \quad (67)$$

$$G_0^1 = \frac{2m}{\bar{r}^2 a^2 A^5 B}\left(\frac{\dot{m}}{m} + \frac{\dot{a}}{a}\right), \quad (68)$$

$$G_1^1 = G_2^2 = G_3^3 = -\frac{A^2}{B^2}\left\{2\frac{d}{dt}\left(\frac{\dot{a}}{a} + \frac{\dot{m}}{\bar{r}A}\right) + \left(\frac{\dot{a}}{a} + \frac{\dot{m}}{\bar{r}A}\right)\cdot\left[3\left(\frac{\dot{a}}{a} + \frac{\dot{m}}{\bar{r}A}\right) + \frac{2\dot{m}}{\bar{r}AB}\right]\right\}. \quad (69)$$

It is convenient to introduce the quantity

$$C \equiv \frac{\dot{a}}{a} + \frac{\dot{m}}{\bar{r}A} = \frac{\dot{m}_H}{m_H} - \frac{\dot{m}}{m}\frac{B}{A}, \quad (70)$$

which reduces to $\dot{m}_H/m_H$ for Sultana-Dyer-type solutions with $m = \text{const.}$. For any choice of the function $m(t)$ the quantity C reduces to

$$C_\Sigma = \frac{\dot{a}}{a} + \frac{\dot{m}}{m} = \frac{\dot{m}_H}{m_H} \quad (71)$$

on the surface $\bar{r} = m/2$. McVittie solutions have $C_\Sigma = 0$ while Sultana-Dyer-type solutions have $C = C_\Sigma = H$ everywhere.

The Ricci curvature is

$$R = -g^{ab}G_{ab} = \frac{3A^2}{B^2}\left(2\dot{C} + 4C^2 + \frac{2\dot{m}C}{\bar{r}AB}\right) \quad (72)$$

and is singular on the surface $\bar{r} = m/2$ if the pressure is singular there. We now specialize the discussion to different forms of matter.

1. Perfect fluid

If we assume that matter is described by a single perfect fluid with stress-energy tensor of the form

$$T_{ab} = (P + \rho)u_a u_b + P g_{ab} \quad (73)$$

and allow for a radial energy flow described by the fluid four-velocity $u^c = (u^0, u, 0, 0)$, it is easy to see that the only possible solution is the Schwarzschild–de Sitter black hole already considered. In fact, the normalization $u^c u_c = -1$ yields

$$u^0 = \frac{A}{B}\sqrt{1 + a^2A^4u^2} \quad (74)$$

and the Einstein equations give, using Eqs. (67)–(69)

$$\dot{m}_H = -GB^2au(P + \rho)\mathcal{A}\sqrt{1 + a^2A^4u^2}, \quad (75)$$

where $\mathcal{A} = \iint d\theta d\varphi\sqrt{g_\Sigma} = 4\pi a^2A^4\bar{r}^2$ is the area of a spherical surface of isotropic radius $\bar{r}$,

$$3\left(\frac{AC}{B}\right)^2 = 8\pi G[(P + \rho)a^2A^4u^2 + \rho], \quad (76)$$

$$-\left(\frac{A}{B}\right)^2\left(2\dot{C} + 3C^2 + 2\frac{\dot{m}C}{\bar{r}AB}\right) = 8\pi G[(P + \rho)a^2A^4u^2 + P], \quad (77)$$

$$-\left(\frac{A}{B}\right)^2\left(2\dot{C} + 3C^2 + 2\frac{\dot{m}C}{\bar{r}AB}\right) = 8\pi GP. \quad (78)$$

By adding Eqs. (77) and (78) one obtains $P = -\rho$, i.e., only the de Sitter equation of state is allowed. Equation (75) accordingly yields $\dot{m}_H = 0$. With the exception of the nonaccreting Schwarzschild–de Sitter black hole, a single perfect fluid can not source solutions representing spherically symmetric black holes embedded in a cosmological background. A mixture of two perfect fluids still constitutes a potential source, as exemplified by the Sultana-Dyer solution [19].

2. Imperfect fluid and no radial mass flow

The following solutions describe perfectly comoving black holes. We assume now that cosmological matter is described by the imperfect fluid stress-energy tensor

$$T_{ab} = (P + \rho)u_a u_b + P g_{ab} + q_a u_b + q_b u_a, \tag{79}$$

where the purely spatial vector q^c describes a radial energy flow,

$$u^a = \left(\frac{A}{B}, 0, 0, 0\right), \qquad q^b = (0, q, 0, 0), \qquad q^c u_c = 0, \tag{80}$$

and $u^c u_c = -1$. The (0, 1) component of the Einstein equations $G_{ab} = 8\pi G T_{ab}$ yields

$$\frac{\dot{m}}{m} + \frac{\dot{a}}{a} = -\frac{4\pi G}{m}\bar{r}^2 a^2 A^4 B^2 q. \tag{81}$$

Since the Hawking mass is $m_H = m(t)a(t)$, it is

$$\frac{\dot{m}_H}{m_H} = \frac{\dot{m}}{m} + \frac{\dot{a}}{a} \tag{82}$$

and the area of a spherical surface of isotropic radius $\bar{r}$ is $\mathcal{A} = \iint d\theta d\varphi \sqrt{g_\Sigma} = 4\pi a^2 A^4 \bar{r}^2$, yielding the relation between energy flow, area $\mathcal{A}$, and accretion rate (Hawking mass added per unit time)

$$\dot{m}_H(t) = -GaB^2\mathcal{A}q. \tag{83}$$

On a sphere of radius $\bar{r} \gg m$ this can be written as (taking into account the fact that radial inflow corresponds to $q < 0$)

$$\dot{m}_H \simeq Ga\mathcal{A}|q|. \tag{84}$$

Hence, the quasilocal mass increases for inflow of matter.

The (0, 0) and (1, 1) [or (2, 2) or (3, 3)] components of the Einstein equations yield the energy density and pressure, respectively,

$$\rho(t, \bar{r}) = \frac{1}{8\pi G}\frac{3A^2}{B^2}\left(\frac{\dot{a}}{a} + \frac{\dot{m}}{\bar{r}A}\right)^2, \tag{85}$$

$$P(t, \bar{r}) = \frac{-1}{8\pi G}\frac{A^2}{B^2}\left\{2\frac{d}{dt}\left(\frac{\dot{a}}{a} + \frac{\dot{m}}{\bar{r}A}\right) + \left(\frac{\dot{a}}{a} + \frac{\dot{m}}{\bar{r}A}\right) \times \left[3\left(\frac{\dot{a}}{a} + \frac{\dot{m}}{\bar{r}A}\right) + \frac{2\dot{m}}{\bar{r}AB}\right]\right\}, \tag{86}$$

from which it is clear that the energy density is always nonnegative. The expansion scalar is $3C$ and, in terms of this quantity, Eq. (86) becomes the generalized Raychaudhuri equation

$$\dot{C} = -\frac{3C^2}{2} - \frac{\dot{m}}{\bar{r}AB}C - 4\pi G\frac{B^2}{A^2}P. \tag{87}$$

In the limit $m \to 0$ this reduces to the well-known Raychaudhuri equation of FLRW cosmology $\dot{H} = -\frac{3H^2}{2} - 4\pi GP$, for which the Hamiltonian constraint $H^2 = 8\pi G\rho/3$ then yields

$$\dot{H} = -4\pi G(P + \rho). \tag{88}$$

Similarly, in the case $m \neq 0$, Eq. (85) yields

$$\dot{C} = -4\pi G\frac{B^2}{A^2}(P + \rho) - \frac{\dot{m}C}{\bar{r}AB}, \tag{89}$$

which generalizes Eq. (88).

It can be noted that, due to the factor B in the denominators, the energy density and pressure (85) and (86) appear to be singular on the surface $\bar{r} = m/2$ where B vanishes. The situation is ameliorated by using the proper time τ defined by $d\tau = \frac{B}{A}dt$ instead of the comoving time t to absorb a factor B. This corresponds to using proper time instead of coordinate Schwarzschild time to offset an infinite redshift on the horizon of an ordinary Schwarzschild black hole, and yields

$$\frac{A}{B}\left(\frac{\dot{a}}{a} + \frac{\dot{m}}{\bar{r}A}\right) \to \frac{a_\tau}{a} + \frac{m_\tau}{m} \quad \text{as } \bar{r} \to m/2, \tag{90}$$

so that

$$8\pi G\rho = 3\left(\frac{a_\tau}{a} + \frac{m_\tau}{\bar{r}A}\right)^2 \tag{91}$$

and

$$8\pi GP = -2\left(\frac{a_\tau}{a} + \frac{m_\tau}{\bar{r}A}\right)_\tau - 3\left(\frac{a_\tau}{a} + \frac{m_\tau}{\bar{r}A}\right)^2. \tag{92}$$

The pressure, the energy density, and the Ricci scalar $R = -\rho + 3P$ appear to be finite on the surface $\bar{r} = m/2$. This is true for any form of the function $m(t)$ and contrasts with the singularities in the solutions of Ref. [55].

Sultana-Dyer-type solutions with $\dot{m} = 0$ have a conformal Killing horizon describing a cosmological black hole, as in the Sultana-Dyer [19] solution. By design, this black hole is perfectly comoving: also in this case the cosmological expansion wins over the local strong field of the black hole. To keep $\dot{m}_H > 0$ at $\bar{r} = m/2$, where $B = 0$, one needs $q \to -\infty$ there. The Sultana-Dyer solution also suffers from a similar problem [19]. This unphysical situation is due to the unrealistically simplified model of accretion.

3. Imperfect fluid and radial mass flow

We now consider an imperfect fluid with stress-energy tensor of the form (79) and both radial mass flow and energy current described by

$$u^a = \left(\frac{A}{B}\sqrt{1 + a^2A^4u^2}, u, 0, \right), \qquad q^c = (0, q, 0, 0). \tag{93}$$

By using the components (67)–(69) of the Einstein tensor, the field equations become

$$\dot{m}_H = -GaB^2\mathcal{A}\sqrt{1 + a^2A^4u^2}[(P + \rho)u + q], \tag{94}$$

$$-3\left(\frac{AC}{B}\right)^2 = -8\pi G[(P + \rho)a^2A^4u^2 + \rho], \tag{95}$$

$$-\left(\frac{A}{B}\right)^2\left(2\dot{C} + 3C^2 + 2\frac{\dot{m}C}{\bar{r}AB}\right) = 8\pi G[(P + \rho)a^2A^4u^2 + P + 2a^2A^4qu], \tag{96}$$

$$-\left(\frac{A}{B}\right)^2\left(2\dot{C} + 3C^2 + 2\frac{\dot{m}C}{\bar{r}AB}\right) = 8\pi GP. \tag{97}$$

Adding the last two equations yields

$$q = -(P + \rho)\frac{u}{2}, \tag{98}$$

i.e., to an ingoing radial flow of mass there corresponds an outgoing radial heat current if $P > -\rho$. By substituting Eq. (98) into Eq. (94), one obtains the accretion rate

$$\dot{m}_H = -\frac{G}{2}aB^2\sqrt{1 + a^2A^4u^2}(P + \rho)\mathcal{A}u, \tag{99}$$

where $(P + \rho)\mathcal{A}u$ can be seen as the flux of gravitating energy through the surface of area $\mathcal{A}$. Since $u < 0$, the mass m_H increases if $P + \rho > 0$, stays constant in a de Sitter background, and decreases if phantom energy with $P < -\rho$ is accreted. This lends support to the conclusions of Ref. [32] on the fate of a black hole in a phantom-dominated universe.

Moreover, the energy density is given by

$$8\pi G\rho = \frac{A^2}{B^2}\left[3C^2 + \left(\dot{C} + \frac{\dot{m}C}{\bar{r}AB}\right)\frac{2a^2A^4u^2}{1 + a^2A^4u^2}\right]. \tag{100}$$

For Sultana-Dyer-type solutions with $m = m_0 =$ const. the energy density reduces to

$$8\pi G\rho = \frac{A^2}{B^2}\left[3H^2 + \frac{2\dot{H}a^2A^4u^2}{1 + a^2A^4u^2}\right] \tag{101}$$

and is positive-definite in a superaccelerating universe with $\dot{H} > 0$, which is necessarily phantom-dominated [65]. Moreover, one can solve for the velocity of the fluid obtaining

$$u = -\left\{\frac{\sqrt{1 + \frac{4m_0^2H^2a^2A^4}{G^2B^4\mathcal{A}^2(P+\rho)^2}} - 1}{2a^2A^4}\right\}^{1/2}. \tag{102}$$

The motion of the fluid becomes superluminal as $\bar{r} \to m_0/2$, where $B \to 0$. In a realistic model the flow becomes supersonic at a certain distance from this surface. This fact can only be taken into account in a more realistic model of accretion, which will be studied elsewhere.

V. DISCUSSION AND CONCLUSIONS

It is well known that the effect of the cosmological expansion on weakly gravitating Newtonian systems of small size r (i.e., $rH_0 \ll 1$) is completely negligible for practical purposes, even though these systems do participate in the expansion. In larger systems such as voids, filaments, and large scale structures, the cosmic expansion plays a significant role [8,9,17,23]. But the size of the local system is not the only relevant factor. The study by Price [22] is the first to focus on the strength with which the local system is bound. The "all or nothing" behavior discovered constitutes an important step in the understanding of the process. However, Price's discussion is limited to a de Sitter background which is too special to draw general conclusions. Moreover, the classical nonrelativistic atom considered can not describe arbitrarily strong binding of the local system because, when the energy density and stresses involved become very large, they distort spacetime causing the metric to substantially deviate from a cosmological one. It is preferable to study exact solutions describing a local object with a strong local gravitational field—e.g., a black hole—embedded in an expanding universe.

Independent motivation for our study arises from the recent realization that if the current acceleration of the universe is dominated by phantom dark energy with $P < -\rho$, then the universe may be running into a big rip at a finite time in the future [30]. Recent literature has focused on the way this catastrophic accelerated expansion of the universe comes to dominate the local dynamics of bound systems (clusters, galaxies, stars, *etc.*) and tears these systems apart. In this context, it is interesting to pose the question of whether the big rip can destroy a black hole horizon and expose the central singularity, thus violating cosmic censorship [31]. A partial answer comes from Ref. [32]: these authors analyze spherical accretion of a phantom test fluid onto a Schwarzschild black hole and, extrapolating the results to a gravitating fluid, reach the conclusion that the horizon disappears before the big rip together with the central singularity, without violating cosmic censorship. This result is quite plausible, however it is desirable to have a study of the accretion process for a gravitating fluid, which brings us again to the realm of exact solutions.

The fully relativistic systems considered in the present paper provide at least a partial answer to the questions above. The McVittie solution (39) and (40) [1] is accretion-free, describes a general FLRW background universe, and does not expand. However, it has a mild singularity on the surface $\bar{r} = m/2$ (the putative horizon) [10,12,15] and therefore it does not describe a black hole. There is an important exception, the Schwarzschild-de Sitter black hole (a special case of the McVittie metric) which does not suffer from this singularity and describes a true black hole horizon which resists the cosmic expansion. This feature is consistent with Price's study of the classical atom in a de Sitter background [22] and with our phase space picture of Sec. II C.

The overall picture emerging is that the consideration of a de Sitter background is rather misleading, even though it considerably simplifies the calculations: the "all or nothing" behavior is not generic of FLRW space but is limited to a de Sitter background. The next system considered, the Nolan interior solution [33], does not suffer from the singularity problem of McVittie's metric because the surface $\bar{r} = m/2$ is covered by the matter composing the star. This solution, which can be thought of as providing a source for McVittie's metric to which it is matched, does not accrete either and can be embedded in a general FLRW background: it is comoving. Removing the limiting assumption of a de Sitter background allows the strongly gravitating central object to expand.

Another exact solution describing a black hole embedded in a non–de Sitter universe is the one of Sultana and Dyer [19] in which, contrary to the McVittie metric, accretion onto the black hole does occur and the conformal Killing horizon (the black hole horizon) is, by design, comoving. The peculiarity of the de Sitter cosmos is thus further put into evidence. The Sultana-Dyer solution, however, suffers from the following limitations: (i) it is restricted to the special form of the scale factor $a(t) = a_0 t^{2/3}$ in comoving time; (ii) the matter source is not a simple fluid but a mixture of two noninteracting fluids, one of which is a null dust; and (iii) the energy density becomes negative at late times near the horizon. It is well known that, in general, matching black hole and cosmological metric produces stress-energy tensors that violate the energy conditions in some regions of the spacetime manifold [55].

To overcome these difficulties we have studied new alternative solutions of the Einstein equations which generalize the McVittie metric by allowing radial accretion onto the central object. We have presented new solutions for which the surface $\bar{r} = m/2$ is nonsingular and is perfectly comoving. These new solutions do not always suffer from the limitations i)–iii) above—in particular, the energy density is everywhere positive at all times for one of the solutions, but the accretion flow generally becomes superluminal, an artifact of the simplified "rigid" model of accretion used, in which matter everywhere in the universe moves toward the central black hole (albeit its radial speed becomes zero far away from it). Moreover, these solutions support the result of Ref. [32] that a black hole embedded in a phantom-dominated universe disappears, respecting cosmic censorship.

It appears, therefore, that the strong external cosmological gravitational field can distort a black hole horizon, or anyway stretch an object dominated by a strong local field. de Sitter-like expansion does not, but this should be seen as an exception to the rule due to the scale-invariant nature of the exponential scale factor (in fact, the de Sitter metric can be put into static form). *A posteriori*, the fact that a black hole horizon expands with the cosmic substratum should not be regarded as surprising: many exact solutions are known in which a black hole horizon is distorted by its surroundings, due to an external gravitational or acceleration field [11,37–39,41–45], an electric [46] or magnetic [47–50] field, or combinations of them [51]. A substantial amount of literature has been devoted to horizon deformation (Ref. [52] and references therein).

Our results do not constitute the last word on the issue of cosmological expansion and local systems. Future endeavours include the search for more general and more realistic exact solutions of the Einstein equations describing accreting black holes in arbitrary FLRW or Bianchi backgrounds.

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[1] G. C. McVittie, Mon. Not. R. Astron. Soc. **93**, 325 (1933).
[2] A. Einstein and E. G. Straus, Rev. Mod. Phys. **17**, 120 (1945); **18**, 148 (1946).
[3] W. B. Bonnor, Classical Quantum Gravity **16**, 1313 (1999).
[4] A. Krasinski, *Inhomogeneous Cosmological Models* (CUP, Cambridge, 1997).
[5] J. M. M. Senovilla and R. Vera, Phys. Rev. Lett. **78**, 2284 (1997); M. Mars, Phys. Rev. D **57**, 3389 (1998).
[6] B. C. Nolan, Phys. Rev. D **58**, 064006 (1998).
[7] J. Pachner, Phys. Rev. **132**, 1837 (1963); Phys. Rev. **137**, B1379 (1965); W. M. Irvine, Ann. Phys. (N.Y.) **32**, 322 (1965); R. H. Dicke and P. J. E. Peebles, Phys. Rev. Lett. **12**, 435 (1964); C. Callan, R. H. Dicke, and P. J. E. Peebles, Am. J. Phys. **33**, 105 (1965); P. D'Eath, Phys. Rev. D **11**, 1387 (1975); R. P. A. Newman and G. C. McVittie, Gen. Relativ. Gravit. **14**, 591 (1982); R. Gautreau, Phys. Rev. D **29**, 198 (1984); P. A. Hogan, Astrophys. J. **360**, 315 (1990); J. L. Anderson, Phys. Rev. Lett. **75**, 3602 (1995); W. B. Bonnor, Mon. Not. R. Astron. Soc. **282**, 1467 (1996); A. Feinstein, J. Ibanez, and R. Lazkoz, Astrophys. J. **495**, 131 (1998); K. R. Nayak, M. A. H. MacCallum, and C. V. Vishveshwara, Phys. Rev. D **63**, 024020 (2000); V. Guruprasad, arXiv:gr-qc/0005090; arXiv:gr-qc/0005014; G. A. Baker, Jr., arXiv:astro-ph/0003152; A. Dominguez and J. Gaite,

Europhys. Lett. **55**, 458 (2001); T.M. Davis and C.H. Lineweaver, *Cosmology and Particle Physics, Verbier, Switzerland, 2000*, edited by R. Durrer, J. Garcia-Bellido, and M. Shaposhnikov, AIP Conf. Proc. No. 555 (American Institute of Physics, New York, 2001), p. 348; G.F.R. Ellis, arXiv:gr-qc/0102017; C. Stornaiolo, Gen. Relativ. Gravit. **34**, 2089 (2002); T.M. Davis, C.H. Lineweaver, and J.K. Webb, Am. J. Phys. **71**, 358 (2003); L. Lindegren and D. Dravins, Astron. Astrophys. **401**, 1185 (2003); C.J. Gao, Classical Quantum Gravity **21**, 4805 (2004); T.M. Davis and C.H. Lineweaver, Publ. Astron. Soc. Pac. **21**, 97 (2004); T.M. Davis, Ph.D. thesis, Univ. of New South Wales [arXiv:astro-ph/0402278]; D.P. Sheehan and V.G. Kriss, arXiv:astro-ph/0411299; W.J. Clavering, arXiv:astro-ph/0511709; Z.-H. Li and A. Wang, arXiv:astro-ph/0607554; O. Gron and O. Elgaroy, arXiv:astro-ph/0603162; L.A. Barnes, M.J. Francis, J.B. James, and G.F. Lewis, Mon. Not. R. Astron. Soc. **373**, 382 (2006); R. Lieu and D.A. Gregory, arXiv:astro-ph/0605611; P.K.F. Kuhfittig, arXiv:gr-qc/0608120; G. Adkins, J. McDonnell, and R. Fell, Phys. Rev. D **75**, 064011 (2007); D.L. Wiltshire, arXiv:gr-qc/0702082; M. Sereno and P. Jetzer, Phys. Rev. D **75**, 064031 (2007); D.J. Shaw and J.D. Barrow, Phys. Lett. B **639**, 596 (2006); arXiv:gr-qc/0607132; Phys. Rev. D **73**, 123506 (2006); **73**, 123505 (2006); T. Clifton, D.F. Mota, and J.D. Barrow, Mon. Not. R. Astron. Soc. **358**, 601 (2005); arXiv:gr-qc/0406001; D. Mota and J.D. Barrow, Mon. Not. R. Astron. Soc. **349**, 291 (2004); Phys. Lett. B **581**, 141 (2004); Classical Quantum Gravity **20**, 4837 (2003); **19**, 6197 (2002).

[8] P.D. Noerdlinger and V. Petrosian, Astrophys. J. **168**, 1 (1971).

[9] H. Sato and K. Maeda, Prog. Theor. Phys. **70**, 119 (1983).

[10] R. Sussman, Gen. Relativ. Gravit. **17**, 251 (1985).

[11] P.D. D'Eath, Phys. Rev. D **11**, 1387 (1975).

[12] M. Ferraris, M. Francaviglia, and A. Spallicci, Nuovo Cimento **111B**, 1031 (1996).

[13] F.I. Cooperstock, V. Faraoni, and D.N. Vollick, Astrophys. J. **503**, 61 (1998).

[14] B. Bolen, L. Bombelli, and R. Puzio, Classical Quantum Gravity **18**, 1173 (2001).

[15] B.C. Nolan, Classical Quantum Gravity **16**, 1227 (1999).

[16] B.C. Nolan, Classical Quantum Gravity **16**, 3183 (1999).

[17] M.T. Busha, F.C. Adams, R.H. Wechsler, and A.E. Evrard, Astrophys. J. **596**, 713 (2003).

[18] C.J. Gao and S.N. Zhang, Phys. Lett. B **595**, 28 (2004).

[19] J. Sultana and C.C. Dyer, Gen. Relativ. Gravit. **37**, 1349 (2005).

[20] M. Mizony and M. Lachièze-Rey, arXiv:gr-qc/0412084; D. Izzo and A. Rathke, arXiv:astro-ph/0504634; F.J. Oliveira, arXiv:gr-qc/0610029; M. Lachièze-Rey, arXiv:gr-qc/0701021; J. Rosales and J. Sanchez-Gomez, arXiv:gr-qc/9810085; M. Carrera and D. Giulini, arXiv:gr-qc/0602098; arXiv:gr-qc/0605078; C. Lammerzhal and O. Preuss, arXiv:gr-qc/0604052; S.G. Turyshev and J.G. Williams, arXiv:gr-qc/0611095; H.-J. Fahr and M. Siewert, arXiv:gr-qc/0610034; J.G. Williams, S.G. Turyshev, and D.H. Boggs, Phys. Rev. Lett. **98**, 059002 (2007); Y.V. Dumin, arXiv:gr-qc/0610035.

[21] S. Nesseris and L. Perivolaropoulos, Phys. Rev. D **70**, 123529 (2004).

[22] R.H. Price, arXiv:gr-qc/0508052.

[23] A. Balaguera-Antolinez and M. Nowakowski, arXiv:0704.1871.

[24] B. Mashhoon, N. Mobed, and D. Singh, arXiv:0705.1312.

[25] J.M. Senovilla, M. Mars, and R. Vera, Phys. World 20 (1999); W.B. Bonnor, Gen. Relativ. Gravit. **32**, 1005 (2000).

[26] Y.B. Zel'dovich and I.D. Novikov, Sov. Astr. A. J. **10**, 602 (1967); S.W. Hawking, Phys. Rev. Lett. **26**, 1344 (1971); Mon. Not. R. Astron. Soc. **152**, 75 (1971); B.J. Carr and S.W. Hawking, Mon. Not. R. Astron. Soc. **168**, 399 (1974); N. Sakai and J.D. Barrow, Classical Quantum Gravity **18**, 4717 (2001).

[27] T. Harada and B.J. Carr, Phys. Rev. D **71**, 104009 (2005); **71**, 104010 (2005); **72**, 044021 (2005); T. Harada, H. Maeda, and B.J. Carr, Phys. Rev. D **74**, 024024 (2006); B.J. Carr, arXiv:astro-ph/0511743.

[28] H. Saida, T. Harada, and H. Maeda, arXiv:0705.4012.

[29] A.G. Riess *et al.*, Astron. J. **116**, 1009 (1998); **118**, 2668 (1999); Astrophys. J. **560**, 49 (2001); **607**, 665 (2004); S. Perlmutter *et al.*, Nature (London) **391**, 51 (1998); Astrophys. J. **517**, 565 (1999); J.L. Tonry *et al.*, Astrophys. J. **594**, 1 (2003); R. Knop *et al.*, Astrophys. J. **598**, 102 (2003); B. Barris *et al.*, Astrophys. J. **602**, 571 (2004); A.G. Riess *et al.*, arXiv:astro-ph/0611572.

[30] R.R. Caldwell, M. Kamionkowski, and N.N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003).

[31] B.J. Carr and S.W. Hawking, Mon. Not. R. Astron. Soc. **168**, 399 (1974); G.W. Gibbons and S.W. Hawking, Phys. Rev. D **15**, 2738 (1977); R. Penrose, in *IAU Symposium* vol. 64 (Reidel, Dordrecht, 1974); in *Theoretical Principles in Astrophysics and Relativity*, edited by N.R. Lebovitz, W.H. Reidel, and P.O. Vandecoord (University of Chicago Press, Chicago, IL, 1978); A. Krolak, Gen. Relativ. Gravit. **16**, 121 (1984).

[32] E. Babichev, V. Dokuchaev, and Yu. Eroshenko, Phys. Rev. Lett. **93**, 021102 (2004).

[33] B.C. Nolan, J. Math. Phys. (N.Y.) **34**, 178 (1993).

[34] R.M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).

[35] The authors of Ref. [13] do not include the centrifugal term L^2/r^3 which is, however, straightforward to reintroduce.

[36] S. Capozziello, S. Nojiri, and S.D. Odintsov, Phys. Lett. B **632**, 597 (2006); S. Nojiri and S.D. Odintsov, arXiv:hep-th/0506212; Phys. Rev. D **72**, 023003 (2005); V. Faraoni, Classical Quantum Gravity **22**, 3235 (2005); W. Fang *et al.*, Int. J. Mod. Phys. D **15**, 199 (2006); M.G. Brown, K. Freese, and W.H. Kinney, arXiv:astro-ph/0405353; E. Elizalde, S. Nojiri, and S.D. Odintsov, Phys. Rev. D **70**, 043539 (2004); Phys. Lett. B **574**, 1 (2003); Phys. Rev. D **70**, 043539 (2004); J.-G. Hao and X.-Z. Li, Phys. Lett. B **606**, 7 (2005); Phys. Rev. D **68**, 043501 (2003); **69**, 107303 (2004); J.M. Aguirregabiria, L.P. Chimento, and R. Lazkoz, Phys. Rev. D **70**, 023509 (2004); Y.-S. Piao and Y.-Z. Zhang, Phys. Rev. D **70**, 063513 (2004); H.Q. Lu, Int. J. Mod. Phys. D **14**, 355 (2005); V.B. Johri, Phys. Rev. D **70**, 041303 (2004); H. Stefancić, Phys. Lett. B **586**, 5 (2004); D.J. Liu and X.-Z. Li, Phys. Rev. D **68**, 067301 (2003); J.G. Hao and X.-Z.

Li, Phys. Rev. D **69**, 107303 (2004); M. P. Dabrowski, T. Stachowiak, and M. Szydlowski, Phys. Rev. D **68**, 103519 (2003); Z. K. Guo, Y. S. Piao, and Y. Z. Zhang, Phys. Lett. B **594**, 247 (2004); J. M. Cline, S. Jeon, and G. D. Moore, Phys. Rev. D **70**, 043543 (2004); S. Nojiri and S. D. Odintsov, Phys. Lett. B **562**, 147 (2003); L. Mersini, M. Bastero-Gil, and P. Kanti, Phys. Rev. D **64**, 043508 (2001); M. Bastero-Gil, P. H. Frampton, and L. Mersini, Phys. Rev. D **65**, 106002 (2002); P. H. Frampton, Phys. Lett. B **555**, 139 (2003).

[37] R. M. Kerns and W. J. Wild, Gen. Relativ. Gravit. **14**, 1 (1982).

[38] H. Faroosh and R. L. Zimmerman, Phys. Rev. D **21**, 317 (1980).

[39] D. Bini, C. Cherubini, and B. Mashhoon, arXiv:gr-qc/0410098.

[40] S. V. Dhurandhar and N. Dadhich, Phys. Rev. D **29**, 2712 (1984); **30**, 1625 (1984).

[41] E. Poisson, Phys. Rev. Lett. **94**, 161103 (2005);

[42] P. Anninos, D. Hobill, E. Seidel, L. Smarr, and W.-M. Suen, Phys. Rev. Lett. **71**, 2851 (1993).

[43] N. Breton, A. A. Garcia, V. S. Manko, and T. E. Denisova, Phys. Rev. D **57**, 3382 (1998).

[44] L. K. Patel and H. B. Trivedi, J. Astrophys. Astron. **3**, 63 (1982).

[45] A. G. Doroshkevich, Ya. B. Zel'dovich, and I. D. Novikov, Zh. Eksp. Teor. Fiz. **49**, 170 (1965).

[46] F. J. Ernst, J. Math. Phys. (N.Y.) **17**, 515 (1976); **17**, 54 (1976).

[47] F. J. Ernst and W. J. Wild, J. Math. Phys. (N.Y.) **17**, 182 (1976).

[48] W. J. Wild and R. M. Kerns, Phys. Rev. D **21**, 332 (1980).

[49] R. Kulkarni and N. Dadhich, Phys. Rev. D **33**, 2780 (1986).

[50] K. D. Krori, J. Math. Phys. (N.Y.) **25**, 607 (1984).

[51] J. Bicak, Proc. R. Soc. A **371**, 429 (1980).

[52] V. P. Frolov and I. D. Novikov, *Black Hole Physics* (Kluwer Academic, Dordrecht, 1998).

[53] S. W. Hawking, J. Math. Phys. (N.Y.) **9**, 598 (1968).

[54] S. A. Hayward, Phys. Rev. D **49**, 831 (1994).

[55] M. L. McClure and C. C. Dyer, Classical Quantum Gravity **23**, 1971 (2006).

[56] S. N. G. Thakurta, Indian J. Phys. B **55**, 304 (1981).

[57] P. C. Vaidya, Pramana **8**, 512 (1977).

[58] M. L. McClure and C. C. Dyer, Gen. Relativ. Gravit. **38**, 1347 (2006).

[59] P. Kustaanheimo, Comment. Phys. Math. Helsingf. **13**, 8 (1947).

[60] R. Bach and H. Weyl, Math. Z. **13**, 134 (1922).

[61] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (CUP, Cambridge, 2003), 2nd ed..

[62] F. C. Michel, Astrophys. Space Sci. **15**, 153 (1972); B. J. Carr and S. W. Hawking, Mon. Not. R. Astron. Soc. **168**, 399 (1974); M. C. Begelman, Astron. Astrophys. **70**, 583 (1978); D. Ray, Astron. Astrophys. **82**, 368 (1980); K. S. Thorne, R. A. Flammang, and A. N. Zytkow, Mon. Not. R. Astron. Soc. **194**, 475; E. Bettwieser and W. Glatzel, Astron. Astrophys. **94**, 306 (1981); K. M. Chang, Astron. Astrophys. **142**, 212 (1985); U. S. Pandey, Astrophys. Space Sci. **136**, 195 (1987); L. I. Petrich, S. L. Shapiro, and S. A. Teukolsky, Phys. Rev. Lett. **60**, 1781 (1988).

[63] S. Hayward, Phys. Rev. D **49**, 6467 (1994).

[64] A. Ashtekar and B. Krishnan, Phys. Rev. Lett. **89**, 261101 (2002).

[65] V. Faraoni, Int. J. Mod. Phys. D **11**, 471 (2002).