Cosmological perturbation theory, instantaneous gauges, and local inertial frames

Jiří Bičák

Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic and Max Planck Institute for Gravitational Physics, Albert Einstein Institute, Am Mühlenberg 1, D-14476 Golm, Germany

Joseph Katz

Racah Institute of Physics, Hebrew University, Jerusalem 91904, Israel

Donald Lynden-Bell

Institute of Astronomy, The Observatories, Madingley Road, Cambridge CB30HA, United Kingdom

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Linear perturbations of Friedmann-Robertson-Walker universes with any curvature and cosmological constant are studied in a general gauge without decomposition into harmonics. Desirable gauges are selected as those which embody best Mach's principle: in these gauges local inertial frames can be determined instantaneously via the perturbed Einstein field equations from the distributions of energy and momentum in the universe. The inertial frames are identified by their "accelerations and rotations" with respect to the cosmological frames associated with the "Machian gauges." In closed spherical universes, integral gauge conditions are imposed to eliminate motions generated by the conformal Killing vectors. The meaning of Traschen's integral-constraint vectors is thus elucidated. For all three types of Friedmann-Robertson-Walker universes the Machian gauges admit much less residual freedom than the synchronous or generalized harmonic gauge. Mach's principle is best exhibited in the Machian gauges in closed spherical universes. Independent of any Machian motivation, the general perturbation equations and discussion of gauges are useful for cosmological perturbation theory.

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I. INTRODUCTION

Einstein preferred a finite universe, bounded in space, over an infinite one because he wanted to avoid posing boundary conditions. What Einstein really disliked was that in open universes some of the motion of inertial frames is due to dragging by matter while the rest is due to the boundary conditions at infinity. Thirty-four years ago the authors of the acclaimed "Gravitation" [1] commented on the Einstein view in a footnote on p. 704: "Many workers in cosmology are skeptical of Einstein's boundary condition of closure of the universe, and will remain so until astronomical observations confirm it." The Wilkinson Microwave Anisotropy Probe has now provided data [2] which, among many other things, have constrained the present value of total mass-energy density parameter of the universe to be $\Omega_0 = 1.02 \pm 0.02$. With such a result, all three basic sets of standard Friedmann-Robertson-Walker (FRW) cosmological models (see, e.g., [1,3]) are compatible: the models with flat spatial sections (with curvature index $k = 0$, $\Omega_0 = 1$), positive spatial curvature models ($k = +1$, $\Omega_0 > 1$), as well as negative curvature models ($k = -1$, $\Omega_0 < 1$). Nevertheless, the WMAP data "marginally prefer" $k = +1$ (see, in particular, [4]), and, indeed, recently several authors studied closed models again in detail (see, e.g., [5,6]), after years of preference of flat universes which have been considered as natural outcomes of inflation. Even an idea going back to the Eddington-Lemaître cosmology has now been revived: if our universe is closed today, it was always closed, and perhaps inflation is "past-eternal"—the universe, dominated at early times by a single scalar field, could have started asymptotically from an initial Einstein static universe which enters an inflationary expanding phase, succeeded then by standard evolution (see [7] and references therein). The recent growing evidence for the existence of a cosmological constant Λ has been an inspiration for the reconsideration of spatially closed universes of de Sitter type [6].

In the present work we do not, technically, bestow a privilege to any value of spatial curvature. All three cases $k = 0, \pm 1$ are analyzed in equal detail, and we even discuss, albeit briefly, closed hyperbolic and closed flat universes with multiconnected topologies. From the physical (to some extent perhaps "philosophical") point of view, we adhere to the Einstein preference, i.e., to the closed universes with standard (spherical) topology, because our work on cosmological perturbation theory has been motivated by Mach's principle.

A. Mach's principle

Mach's principle has acquired certain unpopularity among some relativity and cosmology circles. The primary reason is perhaps the fact that under that name a range of meanings and interpretations, sometimes even mutually contradictory, has gradually accumulated. During the Prague conference in 1988 to celebrate the 150th anniversary of Mach's birth [8] and, in particular, at the Tübingen conference in 1993 devoted entirely to Mach's principle, numerous interpretations have been given (see the excellent book [9]). More recently, Bondi and Samuel [10] listed the "zeroth" plus ten other versions of Mach's principle and described within which theoretical framework a particular statement of the principle applies—see also [11], where the main formulations from [9,10] are summarized. A brief history of Mach's principle and its meaning in general relativity and cosmology is given in the Introduction to our first paper on the subject [12].

Despite a possible scepticism as regards the role of Mach's principle in contemporary cosmology, most of the standard treatises on the subject do include a discussion of the principle (see, e.g., [1,3,13,14]), and no one can deny that Mach's ideas have been a source of inspiration to many, not only Einstein. One of the purposes of the present work is to demonstrate that a search for a framework in which Mach's principle can be best embodied in the cosmological perturbation theory can lead to practical results, such as the formulation of the perturbation theory in a completely general gauge, followed by the selection of an advantageous "Machian gauge" for solving specific problems.

What then do we mean by Mach's principle? As in our previous work [12], as a starting point we adapt Bondi's original formulation from his classical book "Cosmology" [15]: "Local inertial frames are determined through the distributions of energy and momentum in the Universe by some weighted averages of the apparent motions." More specifically, we turn primarily to those among Einstein's equations for linear perturbations of the FRW models which represent the constraints, i.e., under suitable conditions partial differential equations of elliptic type, connecting the "initial values" for matter perturbations with the perturbations of the metric. In [12] we went quite a way in realizing the Machian program. We studied the frame-dragging effects due to slowly, rigidly rotating, but collapsing or expanding spheres in the (inhomogeneous) Lemaître-Tolman-Bondi universes, and we analyzed the dragging effects of vector perturbations of the FRW universes described in a special gauge such that three (momentum) constraint equations enabled us to determine instantaneously metric perturbations h^0_k ($k = 1, 2, 3$) in terms of energy-momentum perturbations δT^0_k. In the open universes, these are determined uniquely by requiring the perturbations to vanish at infinity—rotations are "absolute" in this sense. In closed universes a linear combination of six Killing vectors (three rotations plus three quasitranslations) may be added to the h^0_k. We still obtain the solutions of the three constraint equations when angular momenta corresponding to the three rotations and quasi-momenta corresponding to the three quasitranslations of the sources (determined by δT^0_k) are given. In this sense no absolute rotations exist in closed universe; only differences of rotation rates are determinable—in accord with Mach's ideas that "all motions are relative." If, however, the *velocities* of the bodies, described by perturbations of perfect fluid, are given, the metric perturbations h^0_k are determined uniquely.

The last result is related to the fundamental fact that six globally conserved quantities, corresponding to the six Killing vectors in a FRW universe, must all vanish if considered for the whole closed universe. The conserved quantities, being the derivatives of superpotentials, can be expressed as surface integrals like an electric charge by using Gauss' theorem. As the volume surrounded by the surface is expanded over all the universe at a given time, the surface must shrink to zero. It was, among others, an attempt to understand Mach's principle in cosmological perturbation theory, which inspired us to formulate conservation laws with respect to curved backgrounds [16]. The resulting "KBL superpotential," using the designation by Julia and Silva [17], was found, after applying certain natural criteria, to be unambiguous and most satisfactory in spacetimes with or without a cosmological constant, in any spacetime dimension $D \geq 3$ (see [17,18]). It also found applications in the recent studies of the causal generation of cosmological perturbations seeding large-scale structure formation and of the backreactions in slow-roll inflation (see [19,20], and references therein).

In the present paper we study *general* linear perturbations of the FRW universes from a "Machian perspective." This leads us to investigate both rotations and accelerations of local inertial frames in perturbed universes, and to develop *all* the perturbed Einstein equations in a general gauge "*ab initio*."

B. Cosmological perturbation theory

Observational evidence for isotropy and homogeneity of the Universe shows that it is broadly well described by a FRW model, but the clustering, the galaxies, and the stars constitute local perturbations from the idealized substratums of cosmological models. The other goal of cosmological perturbation theory is to link the physical conditions in the early universe with structures observed today. From the pioneering work of Lifshitz in 1946 (see review in [21]), there appeared numerous papers studying linear perturbations of FRW universes; for the more recent extensive reviews, see, for example, [22–26]. Recently, several authors have even found the impetus and skill to write down the complicated system of equations for the second-order perturbations of the FRW models—see [27] and references therein—but applications of these are yet to appear.

In any cosmological perturbation theory, two problems at once confront us: (i) What smooth cosmological model is best suited to our Universe, and (ii) how do we map the points of our inhomogeneous Universe onto a chosen cosmological model. Both these problems are, in relativity, connected with the gauge freedom that changes the apparent form of the perturbations by which our Universe differs from the smooth cosmological model adopted. Although the first problem is primarily connected with a difficult question of an appropriate averaging of an inhomogeneous universe, or so-called ‘‘fitting problem’’ (see, e.g., [28]), it is also related to the gauge problem because two different FRW universes can be close to one another, at least for some time, so one can be considered as a perturbation of another. When we have chosen an appropriate background, then we are confronted with the freedom to choose coordinates in the real Universe differently and so to remap the Universe onto the background model. This gives rise to the commonly discussed gauge freedom. Of course, what happens in the real world is independent of what background is used and how we map onto it. This is the motive behind gauge invariant perturbation theory, and this made the work of Bardeen [29] in 1980, in which gauge invariant quantities combined from cosmological perturbations were first introduced, so influential. The gauge problem is explained in technical terms from different (equivalent) perspectives in depth in the literature: one can consider a one-parameter family of 4-dimensional manifolds, with M_0 a background and M_ε a perturbed universe, embedded in a 5-dimensional manifold N and connected by a ‘‘point identification map’’ which is specified by a vector field X on N transverse to the manifolds M; the gauge transformation is then a change of X (see, e.g., [23,30]). Alternatively, in a more physical vein, in any chosen coordinate system in the real Universe one assigns to all physical quantities $Q(x^\alpha)$ also their background values $\bar{Q}(x^\alpha)$. These, in contrast to $Q(x^\alpha)$, do not change their functional dependence on coordinates under an infinitesimal coordinate transformation (see [24]). Mathematically, any of these approaches lead to the changes of physical quantities as they appear in the following (see, in particular, Sec. IV).

There exists a well-known lemma [30] stating that the linear perturbation of a quantity is gauge invariant only if the quantity vanishes on the background or is a constant (scalar or linear combination of products of Kronecker delta). The density perturbation $\delta\rho$, for example, is not gauge invariant since $\bar{\rho}$ is a time-dependent function in the FRW backgrounds. That is why, to obtain a gauge invariant quantity, one has to consider, e.g., the gradients of density perturbation or combine $\delta\rho$ with some other quantities. However, as in the black-hole perturbation theory, solving for gauge invariant quantities may not mean finding all quantities of interest. For example, in the problem of the motion of a charged black hole in a weak asymptotically uniform electric field, there is only one gauge invariant quantity. We need to fix the gauge at the end in order to find all perturbations of the metric and electromagnetic field to see how the hole accelerates [31]. It is advantageous at the start to have the possibility of a gauge choice according to the problem in hand. Selecting a gauge which implies a physically preferable coordinate system may eventually give both a better physical understanding and an easier mathematical procedure. After all, motions in the solar system can be described as seen from a frame that rotates to keep a planet ‘‘at rest,’’ but are much more readily comprehended in Newton’s inertial axes.

Last but not least, physical effects associated with Mach’s ideas like the dragging of inertial frames are of a global nature and they do require the introduction of suitable coordinate frames (the ‘‘gauges’’). A true understanding of inertia and inertial frames must involve specific frames or coordinates. To borrow Dieter Brill’s comment from Ref. [9], ‘‘Mach’s principle may point the way toward giving physical meaning to quantities usually considered frame-dependent.’’ We return to this issue in the concluding remarks where some of our other work on Machian effects [32,33] is summarized in the context of the present paper.

C. The goal

There exist many frameworks for treating linearized perturbations of FRW universes. The one which has been used most frequently involves the synchronous gauge, with all quantities decomposed into suitable harmonics in accordance with the spatial curvature. In what follows we make a general study of advantageous gauges without imposing *a priori* conditions on them and without decomposition into harmonics. *We identify desirable gauges as those which embody best Mach’s principle*. We find that these gauges are also motivated by the gauge choices used in full nonlinear general relativity. Most importantly, however, they are distinguished by the simplifications they bring both to the perturbed Einstein equations and to their physical interpretation.

What do we mean by Mach’s principle within this broader framework? We again start from Bondi’s formulation that ‘‘local inertial frames are determined through the distributions of energy and momentum in the Universe by some weighted averages.’’ However, to determine a local inertial frame in a general situation means to *find* both its ‘‘*rotation and acceleration*’’ from the distributions of energy and momentum, represented by δT^μ_ν. In a general situation we need to know the full spacetime metric in a neighborhood of a point in order to determine completely local inertial frames at that point; see, e.g. [1,3]. In Wheeler’s conception of Mach’s principle (e.g. [1,34]), we have to specify the initial data on a Cauchy spacelike surface like the conformal three-geometry and the mass-energy currents, solve for the spacetime geometry $g_{\mu\nu}$, and thus determine local inertial frames. In general, gravitational waves will globally also contribute to the dragging of the inertial frames but only when the waves are nonlinear perturbations of a FRW universe. However, limiting ourselves to the linear perturbations of the FRW universes, it is interesting to see what data are needed to determine the "accelerations" and "rotations" of local inertial frames with respect to what we call the *cosmological (observers') frames*. In a perturbed FRW universe, can a gauge be found such that the distribution of δT^{μ}_{ν} determines uniquely and instantaneously the rotations and accelerations of local inertial frames via Einstein's field equations?

D. The outline

After first reviewing the properties of a general congruence of timelike worldlines in a general spacetime (see, e.g., [35]), we consider the congruence of "cosmological observers" in a perturbed FRW universe with coordinates $\{x^{\mu}\}$ as a "perturbation" of the congruence of fundamental observers in the FRW background. We assume that in a "cosmological gauge" $\{x^{\mu}\}$ the cosmological observers move along $x^i =$ constants, but we describe the properties of their congruence by covariant expressions which can be calculated in any coordinates. The cosmological observers are equipped with their local frame vectors; the timelike ones are their 4-velocities, and the spacelike ones lie along their connecting vectors. Now a cosmological observer is, in general, *accelerated* with respect to a local freely falling inertial frame, in particular, the one which at a given spacetime point moves with the same 4-velocity. Expressing this acceleration in terms of the metric perturbations, we find that only certain components of the metric perturbations are needed. Next, we determine the *rotation* of the axes of the cosmological observer with respect to the nonrotating rigid orthogonal axes (gyroscopes held in their centers of mass) of the local inertial frame. Having the acceleration $\boldsymbol{\alpha}$ and the angular velocity $\boldsymbol{\omega}$ expressed, we have *determined the local inertial frame:* it accelerates and rotates with respect to the corresponding cosmological frame with just the opposite vectors, $-\boldsymbol{\alpha}$ and $-\boldsymbol{\omega}$. All these issues are analyzed in Sec. II.

Assuming a general congruence of cosmological observers, i.e., equivalently, a general gauge, we find that in order to determine the accelerations and (averaged) rotations of local inertial frames in the sense just described, we need to know the metric perturbations δg_{00} and δg_{0i} and their first derivatives. The main issue in Secs. III and IV is to find and study the gauges in which these quantities can be determined instantaneously from the knowledge of energy-momentum distributions δT^{μ}_{ν}. We give the perturbed Einstein equations for all three types of FRW universes with any value of Λ, in an arbitrary gauge. We first adopt the "relativists' attitude" and start from the perturbed FRW metric in the form

$$ds^2 = (\bar{g}_{\mu\nu} + h_{\mu\nu})dx^{\mu}dx^{\nu} = dt^2 - a^2(t)f_{kl}dx^k dx^l + h_{\mu\nu}dx^{\mu}dx^{\nu}, \quad (1.1)$$

where the spatial background metric is $f_{kl}(x^m)$, $k, l, m \ldots = 1, 2, 3$; t is the "cosmic time," so $\delta g_{\mu\nu} \equiv h_{\mu\nu}$. Perturbations $h_{\mu\nu}$ are small so that quadratic terms can be neglected. In one of the standard coordinate systems the background FRW metric $\bar{g}_{\mu\nu}$ reads

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \quad (1.2)$$

where in a positive curvature (closed) universe ($k = +1$)$r \in \langle 0, 1\rangle$, in flat ($k = 0$) and negative curvature ($k = -1$) open universes $r \in <0, \infty)$, $\theta \in \langle 0, \pi\rangle$, $\varphi \in <0, 2\pi)$. We shall also employ other common alternatives such as, e.g., hyperspherical coordinates,

$$ds^2 = dt^2 - a^2[d\chi^2 + \Sigma_k^2(d\theta^2 + \sin^2\theta d\varphi^2)], \quad (1.3)$$

with $\Sigma_k = \sin\chi, \chi, \sinh\chi$ for, respectively, $k = +1, 0, -1$. The perturbations δT^{μ}_{ν} are left general, but a perturbed perfect fluid is considered as an example. In Appendix A we give all the perturbed Einstein's equations and the Bianchi identities starting off from (1.1); in Sec. III we give them using conformal time η and metric perturbations defined as is usual in the cosmological literature, e.g., in [22,24,26]—again in a completely general gauge.

We do not decompose the perturbations in harmonics nor do we first separate them into the scalar, vector, and tensor parts (used, e.g., in [23]). Although both methods are very useful in cosmology, they involve nonlocal operations. In order to make Fourier-type analyses in the space variables, one needs to know quantities in the whole space, which is not "typical" in cosmology. The splitting of a local perturbation into some scalar, vector, and tensor perturbations is also nonlocal. Imagine a trivial (zero) perturbation in a given domain $\mathcal{O}$, and extend it to an annulus $\mathcal{A}$ so that it is nonvanishing there. Hence, in $\mathcal{O}$ the trivial perturbation will split into nontrivial (scalar, vector, tensor) pieces which depend on the extension into $\mathcal{A}$. Therefore, a perturbation which is the sum of scalar, vector, and tensor parts cannot be uniquely expressed in terms of the Bardeen gauge invariant variables [29] which are defined separately for each part. Without using harmonics or splittings, the perturbed Einstein field equations are in a form suitable for searching for solutions in terms of Green's functions. How the Green's function approach can reveal new aspects of cosmological perturbation theory has been recently indicated by Bashinsky and Bertschinger [36].

In Sec. IV, the main purpose is to motivate and describe geometrically several gauges in which the accelerations and rotations of the local inertial frames follow instantaneously from the field equations. We call these *Machian gauges*. We also clarify the residual gauge freedom that these gauges admit, and make a comparison with two typically non-Machian gauges—the synchronous gauge and the generalized Lorenz-de Donder ("harmonic") gauge. The Machian gauges turn out to admit much less residual freedom. The freedom represented by the gauge transformations generated by the conformal Killing vectors in closed (spherical) universes is removed by the *integral gauge conditions* which we impose. In closed hyperbolic universes our Machian local gauge conditions fix coordinates uniquely.

Finally, in Sec. V we give the field equations in the Mach 1 gauge and show how they can be solved to give the local inertial frames when the distribution of the matter energy-momentum is given. We also discuss Traschen's integral-constraint vectors [37,38] restricting possible δT^{μ}_{ν}. According to Traschen and others [37,39], their existence has implications for the Sachs-Wolfe effect and for microwave background anisotropies. Traschen considered these vectors in the synchronous gauge. By contrast, in the Mach 1 gauge, these constraints become a straightforward consequence of the constraint equations and acquire a simple, lucid meaning. We find integral constraints also on quantities not considered by Traschen. In closed universes these integral constraints are satisfied automatically as a consequence of our integral gauge conditions by which motions generated by the spatial conformal Killing vectors are eliminated. In Sec. V we also list all Green's functions known in the literature which solve the constraint equations needed for the determination of the local inertial frames; some are still unknown. We then review our recent work [40,41] on vorticity perturbations of FRW universes and study their effect on local inertial frames. As a second example, we consider perturbations of potential type for which the vorticity vanishes. At the end we analyze the "Machian" question of how uniquely local inertial frames are determined in perturbed universes.

In Concluding Remarks (Sec. VI) we briefly summarize the results and discuss global aspects of Mach's principle. In Appendix C the Killing and conformal Killing vectors in FRW universes are listed, and those harmonics which are needed in the main text are given. In Appendix D we discuss briefly the field equations in the other gauges considered in Sec. IV.

II. THE ACCELERATION AND ROTATION OF LOCAL INERTIAL FRAMES

A. The congruence of cosmological observers

Consider a *general* spacetime with coordinates $\{x^\mu\}$ in which a congruence of a timelike, nonintersecting worldline of "cosmological observers" is given by

$$x^\mu = x^\mu(y^i; p), \qquad i = 1, 2, 3. \tag{2.1}$$

The choice of fixed y^i determines the worldline of a particular observer; p is a parameter along the worldline, commonly chosen as either the cosmological time $x^0 = t$ or the observer's proper time τ. The cosmological observers use their 4-velocity as their normalized timelike frame vector,

$$\begin{aligned} \tau^\mu &= (\partial x^\mu/\partial\tau)_{y^i} = t^\mu/(g_{\alpha\beta}t^\alpha t^\beta)^{1/2}, \\ t^\mu &= (\partial x^\mu/\partial p)_{y^i}. \end{aligned} \tag{2.2}$$

For spatial frame vectors a cosmological observer naturally takes three independent vectors specified by δy^i pointing from him to three other observers of the congruence, orthogonal to τ^μ:

$$\delta x^\mu_\perp = P^\mu_\nu \delta x^\nu = (\delta^\mu_\nu - \tau^\mu\tau_\nu)(\partial x^\nu/\partial y^i)_p \delta y^i. \tag{2.3}$$

As a triad of spatial vectors $e^\mu_{(i)}$, any three linearly independent vectors proportional to $\delta x^\mu_{(i)\perp}$ can be taken. A triad based on the connecting vectors is given at a fixed spacetime point and can be extended along the observer's worldline because connecting vectors are Lie propagated (see, e.g., [42,43]) along the congruence. This gives

$$P^\lambda_\mu \delta x^\mu_{\perp;\nu}\tau^\nu = \tau^\lambda_{;\nu}\delta x^\nu_\perp. \tag{2.4}$$

Three independent connecting vectors define the triad of unit spacelike vectors $m^\mu_{(i)}$:

$$\delta x^\mu_{(i)\perp} = \delta l_{(i)} m^\mu_{(i)}, \qquad m_{(i)\mu} m^\mu_{(i)} = -1, \tag{2.5}$$

with no summation over index i. Equation (2.4) implies the propagation equations for scalar distances $\delta l_{(i)}$—the "generalized Hubble's law" (admitting a possibly anisotropic expansion) [35]—and the propagation equations for triad $m^\mu_{(i)}$. Decomposing the derivative of a 4-velocity in the standard manner (e.g. [1,35]),

$$\tau_{\mu;\nu} = \tau_\nu \alpha_\mu + \omega_{\mu\nu} + \sigma_{\mu\nu} + \tfrac{1}{3}\theta P_{\mu\nu}, \tag{2.6}$$

the acceleration α_μ, vorticity $\omega_{\mu\nu}$ (antisymmetric), shear $\sigma_{\mu\nu}$ (symmetric), and expansion θ are given, respectively, by

$$\alpha_\mu = \tau_{\mu;\nu}\tau^\nu, \tag{2.7}$$

$$\omega_{\mu\nu} = \tfrac{1}{2} P^\kappa_\mu P^\lambda_\nu (\tau_{\kappa;\lambda} - \tau_{\lambda;\kappa}), \tag{2.8}$$

$$\sigma_{\mu\nu} = \tfrac{1}{2} P^\kappa_\mu P^\lambda_\nu (\tau_{\kappa;\lambda} + \tau_{\lambda;\kappa}) - \tfrac{1}{3}\theta P_{\mu\nu}, \tag{2.9}$$

$$\theta = \tau^\nu_{;\nu}. \tag{2.10}$$

We obtain, successively,

$$P^\lambda_\mu \delta x^\mu_{\perp;\nu}\tau^\nu = (\omega^\lambda{}_\nu + \sigma^\lambda_\nu + \tfrac{1}{3}\theta P^\lambda_\nu)\delta x^\nu_\perp, \tag{2.11}$$

$$\frac{d}{d\tau}(\delta l_{(i)}) \Big/ \delta l_{(i)} = \left(\sigma_{\mu\nu} + \tfrac{1}{3}\theta P_{\mu\nu}\right) m^\mu_{(i)} m^\nu_{(i)}, \tag{2.12}$$

$$P^\lambda_\mu m^\mu_{(i);\nu}\tau^\nu = [\omega^\lambda{}_\nu + \sigma^\lambda_\nu + (\sigma_{\alpha\beta} m^\alpha_{(i)} m^\beta_{(i)}) P^\lambda_\nu] m^\nu_{(i)}. \tag{2.13}$$

B. Cosmological observers in a perturbed FRW universe: The frames

Consider first an unperturbed FRW model described by metric (1.1) with $h_{\mu\nu} = 0$. Fundamental (cosmological) observers move along the worldlines $x^i =$ constants with 4-velocity $\bar{\tau}^\mu = (1, 0, 0, 0)$. As the spatial triad, they take three independent vectors $\bar{e}^\mu_{(i)}$ perpendicular to $\bar{\tau}^\mu$. These need not be chosen to be necessarily mutually orthogonal if, for example, coordinates are used in which f_{kl} in (1.1) is not diagonal [as, e.g., in (C3)]. In standard coordinates in FRW backgrounds like in Eqs. (1.2) and (1.3), f_{kl} is diagonal and the vectors

$$\bar{e}^{\ \mu}_{(i)} = (0, \delta^m_i), \qquad \bar{e}_{(i)\mu} = (0, \bar{g}_{im}) \tag{2.14}$$

are orthogonal. It is easy to normalize them:

$$\bar{m}^{\ \mu}_{(i)} = (-\bar{g}_{ii})^{-1/2}[0, \delta^m_i], \qquad \bar{m}_{(i)\mu} = (-\bar{g}_{ii})^{-1/2}[0, \bar{g}_{im}], \tag{2.15}$$

again with no summation over i, index $m = 1, 2, 3$. The quantities (2.7), (2.8), (2.9), and (2.10) characterizing the congruence of the fundamental observers are well known: $\bar{\alpha}_\mu = \bar{\omega}_{\mu\nu} = \bar{\sigma}_{\mu\nu} = 0$, $\bar{\theta} = 3\dot{a}/a$, the dot is d/dt.

In a linearly perturbed FRW universe the metric is given by Eq. (1.1); the indices of the first-order quantities (including $h_{\mu\nu}$) are shifted by the background metrics $\bar{g}_{\mu\nu}$, respectively $\bar{g}^{\mu\nu}$. The congruence of cosmological observers will, in general coordinates, be given by (2.1). The frame vectors can be written in the form $\tau^\mu = \bar{\tau}^\mu + \delta\tau^\mu$, $e^\mu_{(i)} = \bar{e}^\mu_{(i)} + \delta e^\mu_{(i)}$ similarly for covariant components, and for α^μ, $\omega_{\mu\nu}$, $\sigma_{\mu\nu}$, and θ. In general coordinates these quantities can be found easily from the expressions given in Sec. II A. In the following we shall assume that coordinates $\{x^\mu\}$ represent a "cosmological gauge," in which the congruence of cosmological observers is given by $x^i = y^i =$ constants. We find τ^μ to be given by

$$\tau^\mu = \bar{\tau}^\mu + \delta\tau^\mu = (1 - \tfrac{1}{2}h_{00}, 0, 0, 0). \tag{2.16}$$

The spatial triad, determined by connecting vectors orthogonal to τ^μ and lying along coordinate lines, is

$$e^\mu_{(i)} = \bar{e}^\mu_{(i)} + \delta e^\mu_{(i)} = (-h_{i0}, \delta^m_i), \tag{2.17}$$

from which the corresponding unit spacelike vectors $m^\mu_{(i)}$ can be found:

$$m^\mu_{(i)} = \bar{m}^\mu_{(i)} + \delta m^\mu_{(i)} = (-\bar{g}_{ii})^{-1/2}[-h_{i0}, \delta^m_i(1 - \tfrac{1}{2}h_{ii}/\bar{g}_{ii})]. \tag{2.18}$$

We gave here both the background and perturbed frames for completeness. In the following we shall often use just the background frames because only these are needed when a small, first-order quantity is projected.

C. The acceleration of local inertial frames

We shall designate the local frame of a cosmological observer (CO) given by tetrad τ^μ, $e^\mu_{(i)}$, respectively $m^\mu_{(i)}$, by the COF—cosmological observer frame. This frame moving along $x^i =$ constants is, in general, *accelerated* with respect to local freely falling *inertial* frames. Among the inertial frames there is a frame which, moving at a given spacetime point with 4-velocity τ^μ, is momentarily at rest with respect to the COF; such a frame is called the LIF—local inertial frame.[1] The 4-acceleration of the COF with respect to the LIF is given by Eq. (2.7). Using τ^μ given by Eq. (2.16) and the perturbed metric (1.1), we find

$$\alpha^\lambda = (0, \alpha^l), \tag{2.19}$$

where

$$\alpha^l = \delta\Gamma^l_{00} = \bar{g}^{lm}(-\tfrac{1}{2}h_{00,m} + \dot{h}_{0m}). \tag{2.20}$$

We see that *only h_{00} and $\dot{h}_{0m}$ are needed in determining the acceleration of the COF with respect to the LIF or, equivalently, the acceleration of the LIF with respect to the COF* (which is $-\alpha^l$). Spatial metric perturbations do not even enter in the frame components of the acceleration because the unperturbed spatial triad is needed to the zeroth order only:

$$\alpha_{(i)} = e_{(i)\lambda}\alpha^\lambda \cong \bar{e}_{(i)\lambda}\alpha^\lambda \tag{2.21}$$

(similarly with projections on unit vectors $m^\lambda_{(i)}$). Although we calculated the acceleration in coordinates adapted to cosmological observers, it is given by a covariant expression (2.7) which can be expressed in *any* coordinates. The result is also invariant under gauge transformations (see Sec. IV) since in the background this acceleration vanishes.

D. The rotation of local inertial frames

Next we wish to determine the rotation of the axes of the COF with respect to the nonrotating rigid orthogonal axes (gyroscopes held in their centers of mass) of the LIF at a given point and thus, vice versa, the rotation of the LIF with respect to the COF.

First consider a cosmological observer carrying a gyroscope described by a spacelike vector W^μ, perpendicular to τ^μ. The gyroscope is transported along the observer's worldline by Fermi-Walker transport. Another gyroscope, carried by an inertial observer moving with the same τ^μ at a given point, does not rotate relative to W^μ. However, a vector S^μ, perpendicular to τ^μ, which is transported along the worldline of CO in a *general* manner, will rotate relative to W^μ by $(D_F S^\mu)\Delta\tau$, where $D_F S^\mu$ is the Fermi-

[1] There are of course infinitely many LIFs moving with the 4-velocity τ^μ at a given point. However, they differ just by purely spatial transformations or constant shifts of time. Among them, there is also such a LIF that its origin coincides with that of a corresponding COF and its acceleration is $-\alpha^l$.

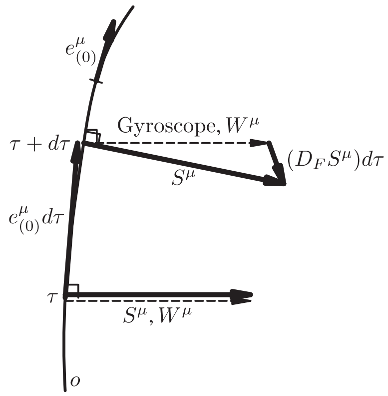

FIG. 1. The Fermi-Walker time derivative $D_F S^\mu$ (based in part on Fig. 2 in [44]). The cosmological observer o with four-velocity $e^\mu_{(0)}$ carries with himself a gyroscope, represented by the spatial vector W^μ (dashed arrow), and a spatial vector S^μ which are both perpendicular to $e^\mu_{(0)}$ and identical at the observer's proper time τ. After $d\tau$, both S^μ and W^μ remain perpendicular to $e^\mu_{(0)}$ but the generally propagating vector S^μ will differ from the nonrotating, Fermi-Walker transported gyroscope by the Fermi-Walker time derivative $(D_F S^\mu)d\tau$.

Walker time derivative defined by (see Fig. 1, and e.g. [44])

$$\frac{D_F S^\mu}{d\tau} \equiv P^\mu_\rho S^\rho_{;\nu}\tau^\nu = S^\mu_{;\nu}\tau^\nu + (\alpha_\nu S^\nu)\tau^\mu, \tag{2.22}$$

where α_ν is the acceleration (2.7) and $S^\mu \tau_\mu = 0$ was used. For the gyroscope, $D_F W^\mu / d\tau = 0$.

Now regarding Eq. (2.11) we see that the left-hand side (l.h.s.) is just equal to the Fermi-Walker derivative of the connecting vector so that

$$\frac{D_F \delta x^\mu_\perp}{d\tau} = \left(\omega^\mu{}_\nu + \sigma^\mu_\nu + \tfrac{1}{3}\theta P^\mu_\nu\right)\delta x^\nu_\perp. \tag{2.23}$$

Therefore, since the congruence of cosmological observers has, in general, a nonvanishing vorticity and shear, the connecting vectors rotate with respect to gyroscopes. The last term in Eq. (2.23) is proportional to $\delta x^\mu_\perp$ and represents only a dilation of the connecting vector due to the (isotropic) expansion of the congruence. Similarly, unit vectors $m^\mu_{(i)}$ of the COFs rotate with respect to gyroscopes according to Eq. (2.13):

$$\frac{D_F m^\mu_{(i)}}{d\tau} = [\omega^\mu{}_\nu + \sigma^\mu_\nu + (\sigma_{\alpha\beta} m^\alpha_{(i)} m^\beta_{(i)}) P^\mu_\nu] m^\nu_{(i)}. \tag{2.24}$$

Turning now to the perturbed FRW universes we find, using τ^μ from Eq. (2.16) and the perturbed metric (1.1), the vorticity (2.8) to have a simple form

$$\omega_{kl} = \delta\omega_{kl} = \tfrac{1}{2}(h_{0k,l} - h_{0l,k}), \qquad \omega_{0\alpha} = \delta\omega_{0\alpha} = 0, \tag{2.25}$$

the shear (2.9) turns out to be

$$\sigma_{kl} = \delta\sigma_{kl} = \frac{1}{2}\dot{h}_{kl} - \frac{1}{6}\dot{h}^m_m \bar{g}_{kl} - \frac{\dot{a}}{a} h_{kl}, \qquad \sigma_{0\alpha} = \delta\sigma_{0\alpha} = 0, \tag{2.26}$$

and the expansion (2.10) reads

$$\theta = \bar{\theta} + \delta\theta = \frac{3\dot{a}}{a} + \frac{1}{2}\left(\dot{h}^m_m - \frac{3\dot{a}}{a} h^0_0\right). \tag{2.27}$$

Since both $\omega^\mu{}_\nu$ and σ^μ_ν are of the first order in $h_{\mu\nu}$, on the right-hand side (r.h.s.) of Eq. (2.24) only $\bar{m}^\mu_{(i)}$ enters and the equation takes the form

$$\frac{D_F m^l_{(i)}}{d\tau} = [\omega^l{}_k + \sigma^l_k + (\sigma_{ab}\bar{m}^a_{(i)}\bar{m}^b_{(i)})\delta^l_k]\bar{m}^k_{(i)}, \tag{2.28}$$

where $\bar{m}^k_{(i)} = (-\bar{g}_{(ii)})^{-1/2}\delta^k_i$, and $\omega^l{}_k$, σ^l_k are given by Eqs. (2.25) and (2.26).

Clearly, the vector $m^l_{(i)}$ rotates relative to the gyroscopes and, hence, a gyroscope will rotate relative to the COF not only due to a nonvanishing vorticity but also due to the presence of a shear. A gyroscope will precess in a gravitational wave described by $\dot{h}_{kl}$ (cf. the discussion in [45]); it is *not* true, as sometimes stated ([26], p. 334) that a spin (a gyroscope) precesses relative to the cosmological frame at a rate given just by the vorticity ω_{kl}.

The axes of a LIF are determined by three orthogonal gyroscopes, while those of a COF are determined by three approximately orthogonal vectors $e^\mu_{(i)}$ or, after their normalization, by unit vectors $m^\mu_{(i)}$. As a consequence, *on average* the rotation of the COF relative to the LIF (moving with the same τ^μ at a given point) is determined just by vorticity ω_{kl}. Indeed, there is a significant difference between the terms on the r.h.s. of Eq. (2.28): ω_{kl} is antisymmetric while σ_{kl} is symmetric and traceless. If at a given instant a vector $m^\mu_{(i)}$ lies along a principal direction of σ_{kl}, its direction will be changed only by the vorticity. As in fluid kinematics [46], it is just the vorticity which describes the "effective angular velocity" of the fluid (see Fig. 2).

Hence, we conclude that, in order *to determine the averaged rotations of local inertial frames with respect to the cosmological frames in the perturbed FRW universes, it is sufficient to determine the vorticity tensor* (2.25), *i.e., spatial gradients of* h_{0k}.

Rather than by ω_{kl} the rotation is usually represented by the "cosmologist's vorticity vector"

$$\omega^\lambda = \tfrac{1}{2}\bar{\varepsilon}^{\lambda\sigma\mu\nu}\bar{\tau}_\sigma\omega_{\mu\nu}, \tag{2.29}$$

where

$$\bar{\varepsilon}^{\alpha\beta\gamma\delta} = (-\bar{g})^{-1/2}[\alpha\beta\gamma\delta], \qquad \bar{g} = \det(\bar{g}_{\mu\nu}), \tag{2.30}$$

and $[\alpha\beta\gamma\delta]$ is the permutation symbol. In our case we get

$$\omega^\lambda = (0, \omega^l), \qquad \omega^l = \tfrac{1}{2}\varepsilon^{lmn}h_{0n,m}, \tag{2.31}$$

$\varepsilon^{lmn} = (\bar{\gamma})^{-1/2}[lmn]$, $\bar{\gamma} = \det(-\bar{g}_{ik})$. Considering h_{0k} as a

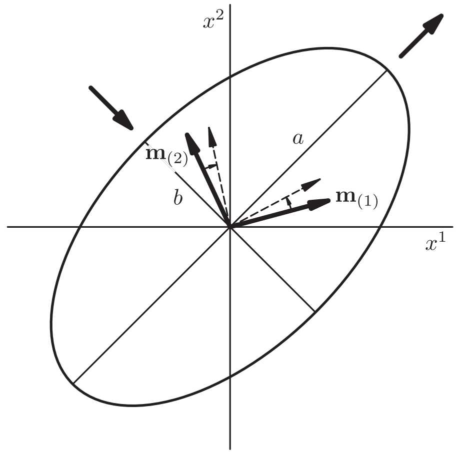

FIG. 2. Because of the shear with the principal axes a, b, almost orthogonal unit vectors $\mathbf{m}_{(1)}$, $\mathbf{m}_{(2)}$ change their directions but they do not, on average, rotate with respect to the fixed axes x^1, x^2.

3-dimensional velocity vector, the cosmologist's vorticity (2.29) yields $\frac{1}{2}$ of the standard vorticity, curl $\boldsymbol{v}$, in fluid dynamics. However, curl $\boldsymbol{v}$ represents twice the effective rigid local angular velocity of the fluid in an inertial frame [46]. Therefore, the averaged rigid angular velocity of COFs with respect to LIFs is determined exactly by $\boldsymbol{\omega}^l$ given by Eq. (2.31). Equivalently, LIFs rotate with respect to COFs with angular velocity $-\omega^l$.

E. A note on generalized backgrounds

When the background is not FRW but, say, a Lemaître-Tolman-Bondi or Bianchi model, the accelerations and the averaged rotations of the LIFs with respect to the COFs can still be determined from only perturbations h_{00} and h_{i0}. For example, if the coordinates can be chosen such that $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{0i} = 0$, $\bar{g}_{00}$ is an arbitrary function of time x^0, and $\bar{g}_{ik}$ are arbitrary functions of x^λ, then cosmological observers given by $x^i =$ constants have their accelerations with respect to LIFs equal to $\alpha^l = \bar{g}^{lm}(-\frac{1}{2}h_{00,m} + h_{0m,0} - \frac{1}{2}\bar{g}^{00}\bar{g}_{00}h_{m0})$, and their vorticity is $\bar{\omega}_{\mu\nu} = 0$, $\omega_{kl} = \delta\omega_{kl} = \frac{1}{2}(\bar{g}_{00})^{-1/2}(h_{0k,l} - h_{0l,k})$, $\omega_{0\alpha} = 0$.

F. Sources and their description in the cosmological frame

In the FRW universes the background energy-momentum tensor is commonly taken to be the perfect-fluid stress tensor, $\bar{T}^\nu_\mu = (\bar{\rho} + \bar{p})\bar{U}_\mu\bar{U}^\nu - \bar{p}\delta^\nu_\mu$, so that in the comoving coordinates

$$\bar{T}^{\,0}_{\,0} = \bar{\rho}, \qquad \bar{T}^i_j = -\bar{p}\delta^i_j, \qquad \bar{T}^0_j = \bar{T}^j_0 = 0. \tag{2.32}$$

The energy density $\bar{\rho}$ and the pressure $\bar{p}$ of the matter can describe a standard perfect fluid with a given equation of state. Alternatively, one may regard these expressions as the stress tensor components of a homogeneous time-dependent scalar field Φ of an inflationary model with the energy density $\bar{\rho} = \rho_\Phi$ and effective pressure $\bar{p} = p_\Phi$ (see, e.g., [47]). The special case with $\bar{\rho} + \bar{p} = 0$, $\bar{\rho} = -\bar{p} =$ constant, corresponds to the de Sitter vacuum spacetime with a cosmological constant $\Lambda = -\bar{p} = \bar{\rho}$, commonly interpreted as a vacuum energy. Any of these background matter contents can be considered in the present work. We shall thus not, in general, specify the form of the perturbations δT^ν_μ of the energy-momentum tensor. Employing the frame vectors $e^\mu_{(\alpha)}$ given by (2.17), we find the frame components of perturbations (indicated by $[e]$ and $[m]$) for a general energy-momentum tensor to be given by

$$\begin{aligned} \delta T^{(0)}_{[e](0)} &= \delta T^{(0)}_{[m](0)} = \delta T^0_0, \\ \delta T^{(0)}_{[e](i)} &= (-\bar{g}_{ii})^{1/2}\delta T^{(0)}_{[m](i)} = \delta T^0_i - (\bar{\rho} + \bar{p})h_{0i}, \end{aligned} \tag{2.33}$$

$$\begin{aligned} \delta T^{(i)}_{[e](0)} &= (-\bar{g}_{ii})^{-1/2}\delta T^{(i)}_{[m](0)} = \delta T^i_0, \\ \delta T^{(k)}_{[e](i)} &= (\bar{g}_{ii}/\bar{g}_{kk})^{1/2}\delta T^{(k)}_{[m](i)} = \delta T^k_i, \end{aligned} \tag{2.34}$$

with no summation over i, k. By employing the "mixed" tensorial coordinate components of perturbations, we see that their values, except for δT^0_i, coincide—up to the background "normalization" factors $\sim(-\bar{g}_{ii})^{\pm 1/2}$—with their frame scalar components.

In the case of a perfect fluid the coordinate components read

$$\begin{aligned} \delta T^0_0 &= \delta\rho, \qquad \delta T^0_i = (\bar{\rho} + \bar{p})(h_{i0} + V_i), \\ \delta T^i_0 &= (\bar{\rho} + \bar{p})V^i, \qquad \delta T^k_i = -\delta p\delta^k_i, \end{aligned} \tag{2.35}$$

where $\delta\rho$ and δp are perturbations of the matter density and pressure. The velocity

$$V^i = \frac{dx^i}{dt} \tag{2.36}$$

is the spatial part of the perturbation of the fluid's 4-velocity

$$U^\mu = \bar{U}^\mu + \delta U^\mu = (1 - \tfrac{1}{2}h_{00}, V^i). \tag{2.37}$$

It is easy to see that the 4-velocity is approximately the unit timelike vector since we assume $V^i \ll 1$, and terms proportional to V^2 and Vh can thus be neglected. The 4-acceleration of the fluid is defined by $A^\mu = U^\mu_{;\nu}U^\nu$. Since the background value $\bar{A}^\mu = 0$, A^μ is of the first order. The standard condition $A^\mu U_\mu = 0$ thus implies $A^\mu\bar{U}_\mu = 0$, and hence $A^0 = 0$. The calculation of the spatial components yields

$$\begin{aligned} A^i &= \dot{V}^i + 2\frac{\dot{a}}{a}V^i - \frac{1}{2}\bar{g}^{is}h_{00,s} + \bar{g}^{is}\dot{h}_{s0} \\ &= \dot{V}^i + 2\frac{\dot{a}}{a}V^i + \alpha^i, \end{aligned} \tag{2.38}$$

where α^i is the acceleration (2.20) of the cosmological frame with respect to the local inertial frame, or $-\alpha^i$ is the acceleration of the LIF with respect to the COF. The acceleration (2.38) is the fluid's acceleration with respect to the LIF, whereas $\dot{V}^i$ characterizes its acceleration with respect to the COF. If the fluid is momentarily at rest in the COF, $V^i = 0$, and the fluid has the same acceleration with respect to the COF as the LIF has, $\dot{V}^i = -\alpha^i$, then $A^i = 0$, as it should. Since the fluid's acceleration vanishes for the background, its frame components are just $A^{(i)}_{[e]} = (-\bar{g}_{ii})^{-1/2}A^{(i)}_{[m]} = A^i$. It is not difficult to check that the acceleration (2.38) satisfies the perturbed relativistic Euler's equations,

$$(\bar{\rho} + \bar{p})A^i = \delta(\Pi^{i\mu} p_{,\mu}), \qquad \Pi^{i\mu} = g^{i\mu} - U^i U^\mu, \tag{2.39}$$

where $\Pi^{i\mu}$ is the projection tensor into the 3-space orthogonal to U^μ. As we shall notice in Sec. III B, these are just the spatial parts of the perturbed Bianchi identities.

The vorticity of the fluid is defined by

$$\begin{aligned} \Omega^\alpha &= \tfrac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}U_\beta \Omega_{\gamma\delta}, \\ \Omega_{\gamma\delta} &= \tfrac{1}{2}(U_{\gamma;\mu}\Pi^\mu_\delta - U_{\delta;\mu}\Pi^\mu_\gamma). \end{aligned} \tag{2.40}$$

Since $\Omega^\alpha U_\alpha = U^\gamma \Omega_{\gamma\delta} = 0$ and Ω's are of the first order, we again get $\Omega^0 = 0 = \Omega_{0\alpha}$. The nonvanishing spatial parts turn out to be

$$\begin{aligned} \Omega_{kl} &= \tfrac{1}{2}[(V_k + h_{0k})_{,l} - (V_l + h_{0l})_{,k}], \\ \Omega^i &= \tfrac{1}{2}\varepsilon^{ikl}(V_l + h_{0l})_{,k} = \tfrac{1}{2}\varepsilon^{ikl}V_{l,k} + \omega^i; \end{aligned} \tag{2.41}$$

ω^i is the vorticity vector (2.31) of the cosmological frame. Again, this result is plausible in the following sense: Since the LIF rotates with respect to the COF with $-\omega^i$, then if the fluid rotates with respect to the COF with $\frac{1}{2}\varepsilon^{ikl}V_{l,k} = -\omega^i$, it does not rotate with respect to the LIF, $\Omega^i = 0$. As with the acceleration, the frame components are simply given by $\Omega^{(i)}_{[e]} = (-\bar{g}_{ii})^{-1/2}\Omega^{(i)}_{[m]} = \Omega^i$.

From Eqs. (2.33), (2.34), and (2.35) it is evident that to give the *frame* components of the source we need to know only the perturbations $\delta\rho$, δp, and V^i of the fluid. No metric perturbations are needed—in contrast to the coordinate components δT^0_i in which h_{0i} enters. This is important for our understanding of Mach's principle.

III. FIELD EQUATIONS

We have seen that the accelerations and rotations of LIFs with respect to the COFs are determined in a general gauge by h_{00} and h_{0l} components of the perturbations. We shall now write the perturbed Einstein equations for the FRW backgrounds in a general gauge.[2] We then shall see later which gauges will enable us to determine instantaneously perturbations h_{00} and h_{0l} (separately from h_{kl}) in terms of matter perturbations.

A straightforward way to express the perturbations of Einstein's equations is in terms of a physical cosmic time t and some convenient spatial coordinates x^l of the FRW background. However, there are advantages in using conformal time η, given by $a(\eta)d\eta = dt$. Both t and η are common in the literature and we shall thus give explicitly the perturbation equations in two forms—with t in Appendix A and with η in this section.

A. Perturbed field equations with a conformal time η

In terms of coordinates $\tilde{x}^\mu = (\eta, x^k)$ the metric of the background is

$$\begin{aligned} d\bar{s}^2 &= \tilde{\bar{g}}_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu = a^2 e_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu \\ &= a^2[d\eta^2 - f_{kl}dx^k dx^l], \end{aligned} \tag{3.1}$$

where we introduced the conformally related *static* background metric $e_{\mu\nu}$ by

$$e_{00} = 1, \qquad e_{0l} = 0, \qquad e_{kl} = -f_{kl}(x^i). \tag{3.2}$$

The components of a tensor, say $\tilde{W}^\nu_\mu$, are related to those of the tensor W^ν_μ in $x^\mu = (t, x^k)$ coordinates as follows:

$$\begin{aligned} \tilde{W}^0_0 &= W^0_0, \qquad \tilde{W}^0_l = a^{-1}W^0_l, \\ \tilde{W}^l_0 &= aW^l_0, \qquad \tilde{W}^l_k = W^l_k. \end{aligned} \tag{3.3}$$

Defining the dimensionless "relative Hubble constant" by $\mathcal{H} = \frac{1}{a}\frac{da}{d\eta} = \frac{a'}{a} = \dot{a} = aH$, we can write the nonvanishing background Christoffel symbols as

$$\begin{aligned} \tilde{\bar{\Gamma}}^0_{00} &= \mathcal{H}, \qquad \tilde{\bar{\Gamma}}^0_{kl} = \mathcal{H}f_{kl}, \\ \tilde{\bar{\Gamma}}^m_{0l} &= \mathcal{H}\delta^m_l, \qquad \tilde{\bar{\Gamma}}^m_{kl} = \bar{\Gamma}^m_{kl}, \end{aligned} \tag{3.4}$$

where $\bar{\Gamma}^m_{kl}$ is given in Appendix A. The prime hereafter denotes the derivative with respect to η. (Later, it will also be used to denote a coordinate change, but no confusion should arise.) The nonzero components of the background Einstein equations become

[2] Some of the equations presented here have been derived independently by Langlois (1994) in his Ph.D. thesis and by P. Uzan and N. Deruelle (private communication). The perturbed Ricci tensor components and the equations of motion have been written down by Bardeen (1980) [29] after a decomposition of the metric into scalar, vector, and tensor parts using his specific amplitudes.

$$\tilde{\bar{G}}{}^0_0 = \bar{G}^0_0 = \frac{3}{a^2}(k + \mathcal{H}^2) = \kappa\bar{p} + \Lambda,$$
$$\tilde{\bar{G}}{}^m_k = \bar{G}^m_k = \frac{1}{a^2}\delta^m_k(k + \mathcal{H}^2 + 2\mathcal{H}') = -(\kappa\bar{p} - \Lambda)\delta^m_k. \tag{3.5}$$

The linearly perturbed Einstein equations will be written in terms of the dimensionless perturbations $\tilde{h}_{\mu\nu}$ of $e_{\mu\nu}$,

$$\begin{aligned} ds^2 &= (\tilde{\bar{g}}_{\mu\nu} + \delta\tilde{g}_{\mu\nu})d\tilde{x}^\mu d\tilde{x}^\nu = a^2(e_{\mu\nu} + \tilde{h}_{\mu\nu})d\tilde{x}^\mu d\tilde{x}^\nu \\ &= a^2[(1 + \tilde{h}_{00})d\eta^2 + 2\tilde{h}_{0k}d\eta dx^k \\ &\quad - (f_{kl} - \tilde{h}_{kl})dx^k dx^l], \end{aligned} \tag{3.6}$$

which means that $\delta\tilde{g}_{\mu\nu} = a^2\tilde{h}_{\mu\nu}$. It is important to emphasize that, in contrast to tensors like in Eq. (3.3), $\tilde{h}_{\mu\nu}$'s are *not* components (in coordinates $\tilde{x}^\mu$) of $h_{\mu\nu}$ used in (1.1) and Appendix A; as seen from Eq. (3.6), they represent the perturbations of the static conformal metric $e_{\mu\nu}$, whereas $h_{\mu\nu}$'s represent perturbations of the physical background metric $\bar{g}_{\mu\nu}$. In a (1 + 3) decomposition, i.e., in quantities $\tilde{h}_{00}$, $\tilde{h}_{0l}$, $\tilde{h}_{kl}$, we do not raise the index 0 and we raise the spatial indices only with f^{kl}; thus $\tilde{h}^m_0 = f^{ml}\tilde{h}_{0l}$, $\tilde{h}^{mn} = f^{mk}f^{nl}\tilde{h}_{kl}$, etc. The explicit relations between $h_{\mu\nu}$, h^ν_μ, or $h^{\mu\nu}$ and $\tilde{h}_{\mu\nu}$, $\tilde{h}^\nu_\mu$, or $\tilde{h}^{\mu\nu}$ are given in Appendix A.

The perturbations of Einstein's equations in terms of $\tilde{h}_{\mu\nu}$ can be obtained from equations in Appendix A. We shall introduce two special symbols which not only simplify the equations but are also helpful in suggesting particularly useful gauge conditions. We set

$$\mathcal{K} = \tfrac{3}{2}\mathcal{H}\tilde{h}_{00} + \tfrac{1}{2}(\tilde{h}^n_n)' - \nabla_n\tilde{h}^n_0. \tag{3.7}$$

The second quantity we shall employ is defined with the traceless part of $\tilde{h}^l_k$,

$$\tilde{h}{}^l_{Tk} = \tilde{h}^l_k - \tfrac{1}{3}\delta^l_k\tilde{h}^n_n. \tag{3.8}$$

We set

$$\mathcal{T}_k = \nabla_l\tilde{h}^l_{Tk}. \tag{3.9}$$

Now we give Einstein's perturbation equations, separating $\delta\tilde{G}^l_{Tk}$, the traceless part of $\delta\tilde{G}^l_k$, from the trace $\delta\tilde{G}^n_n$ which we combine with $\delta\tilde{G}^0_0$ for a reason to be seen below. Thus, recalling that $\nabla^2 = f^{kl}\nabla_{kl}$, we have the following dimensionless equations:

$$a^2\kappa\delta\tilde{T}^0_0 = a^2\delta\tilde{G}^0_0 = \tfrac{1}{3}\nabla^2\tilde{h}^n_n + k\tilde{h}^n_n - 2\mathcal{H}\mathcal{K} - \tfrac{1}{2}\nabla_k\mathcal{T}^k, \tag{3.10}$$

$$\begin{aligned} a^2\kappa\delta\tilde{T}^0_k &= a^2\delta\tilde{G}^0_k \\ &= \tfrac{1}{2}\nabla^2\tilde{h}_{k0} + k\tilde{h}_{k0} + \tfrac{1}{6}\nabla_{kl}\tilde{h}^l_0 + \tfrac{2}{3}\nabla_k\mathcal{K} - \tfrac{1}{2}(\mathcal{T}_k)', \end{aligned} \tag{3.11}$$

$$\begin{aligned} a^2\kappa(\delta\tilde{T}^0_0 - \delta\tilde{T}^n_n) &= a^2(\delta\tilde{G}^0_0 - \delta\tilde{G}^n_n) \\ &= \nabla^2\tilde{h}_{00} + 3a\Big(\frac{1}{a}\mathcal{H}\Big)'\tilde{h}_{00} + \frac{2}{a}(a\mathcal{K})', \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} a^2\kappa\Big(\delta\tilde{T}^l_k - \frac{1}{3}\delta^l_k\delta\tilde{T}^n_n\Big) &= a^2\delta\tilde{G}^l_{Tk} \\ &= -\frac{1}{2}\nabla^2\tilde{h}{}^l_{Tk} + k\tilde{h}^l_{Tk} + \frac{1}{2a^2}[a^2(\tilde{h}^l_{Tk})']' + f^{lm}\Big(\nabla_{(m}\mathcal{T}_{k)} - \frac{1}{3}f_{mk}\nabla_n\mathcal{T}^n\Big) \\ &\quad - \frac{1}{a^2}f^{lm}\Big[a^2\Big(\nabla_{(m}\tilde{h}_{k)0} - \frac{1}{3}f_{mk}\nabla_n\tilde{h}^n_0\Big)\Big]' + \frac{1}{2}f^{lm}\Big(\nabla_{mk} - \frac{1}{3}f_{mk}\nabla^2\Big)\Big(\tilde{h}_{00} - \frac{1}{3}\tilde{h}^n_n\Big). \end{aligned} \tag{3.13}$$

For completeness we also write down the equation which can be derived from Eq. (3.11):

$$\begin{aligned} a^2\kappa\delta\tilde{T}^k_0 &= -\frac{1}{2}\nabla^2\tilde{h}^k_0 + \Big[k - 2a\Big(\frac{1}{a}\mathcal{H}\Big)'\Big]\tilde{h}^k_0 - \frac{1}{6}\nabla^k\nabla_l\tilde{h}^l_0 \\ &\quad - \frac{2}{3}\nabla^k\mathcal{K} + \frac{1}{2}(\mathcal{T}^k)'. \end{aligned} \tag{3.14}$$

This equation follows from Eq. (3.11) by using the relation

$$\delta\tilde{T}^0_k = -f_{kl}\delta\tilde{T}^l_0 - \frac{2}{\kappa a^2}\Big[-k + a\Big(\frac{\mathcal{H}}{a}\Big)'\Big]\tilde{h}_{0k}. \tag{3.15}$$

In the case of perfect-fluid perturbations we define the local coordinate velocity by

$$\tilde{V}^k = \frac{dx^k(\eta)}{d\eta}. \tag{3.16}$$

Notice that $\tilde{V}^k$ is not equal to V^k [defined in Eq. (2.36)] expressed in coordinates $\tilde{x}^\mu$ because $\tilde{V}^k$ is defined with respect to the conformal time. Since $\tilde{x}^k = x^k$, in both coordinates we have simple relations,

$$\begin{aligned} &\tilde{V}^k = aV^k, \\ &\tilde{V}_n = f_{nm}\tilde{V}^m = f_{nm}aV^m = -a^{-1}\bar{g}_{nm}V^m = -a^{-1}V_n. \end{aligned} \tag{3.17}$$

Nevertheless, the fluid's 4-velocity components (2.37) transform as a general tensor like (3.3).

In the case of perfect fluid the energy-momentum tensor perturbations become

$$\begin{aligned} a^2\kappa\delta\tilde{T}^0_0 &= a^2\kappa\delta\rho, \\ a^2\kappa\delta\tilde{T}^0_k &= 2(k+\mathcal{H}^2-\mathcal{H}')(-\tilde{V}_k+\tilde{h}_{k0}), \\ a^2\kappa\delta\tilde{T}^k_0 &= 2(k+\mathcal{H}^2-\mathcal{H}')\tilde{V}^k, \\ a^2\kappa\delta\tilde{T}^l_k &= -a^2\kappa\delta^l_k\delta p, \end{aligned} \tag{3.18}$$

so that the left-hand sides of Eqs. (3.12) and (3.13) are

$$\begin{aligned} a^2\kappa(\delta\tilde{T}^0_0-\delta\tilde{T}^n_n) &= a^2\kappa(\delta\rho+3\delta p), \\ a^2\kappa(\delta\tilde{T}^l_k-\tfrac{1}{3}\delta^l_k\delta\tilde{T}^n_n) &= 0. \end{aligned} \tag{3.19}$$

We combined Einstein's equations in such a way that Eqs. (3.10) and (3.12) contain scalars under the transformation of spatial coordinates, whereas Eq. (3.13) involves tensorial quantities only. In the perfect-fluid case, the "source" in Eq. (3.13) vanishes so that this equation represents propagation of a free gravitational field, i.e., of gravitational waves described by traceless quantities $\tilde{h}^l_{Tk}$. The first and the third terms on the r.h.s. of Eq. (3.13) combine into a d'Alembert wave operator modified by the time dependence of the expansion factor $a(\eta)$. More generally, however, the perturbed fluid could be an imperfect fluid which includes shear viscosity. This can be described by an additional term in δT^ν_μ, given by a symmetric shear tensor $\delta\Sigma^\nu_\mu$ which is traceless, $\delta\Sigma^\mu_\mu = 0$, and purely spatial in the fluid rest frame, $U^\nu\delta\Sigma^\nu_\mu = 0$ (see, e.g., [26]). Then the shear would appear as a source in Eq. (3.13).

B. Bianchi identities and conservation laws

The perturbed contracted Bianchi identities, $\nabla_\nu G^\nu_\mu = 0$ for $\mu = 0$ and $\mu = k$, imply

$$\frac{1}{a^2}(a^2\delta\tilde{G}^0_0)^{\cdot} + \frac{\dot{a}}{a}(\delta\tilde{G}^0_0-\delta\tilde{G}^n_n) - \frac{1}{a}\nabla^k\delta\tilde{G}^0_k + \frac{3}{2a}\kappa(\bar{\rho}+\bar{p})\left(\dot{a}\tilde{h}_{00}-\frac{2}{3}\mathcal{K}\right) = 0, \tag{3.20}$$

$$\frac{1}{a^3}(a^4\delta\tilde{G}^0_k)^{\cdot} + \nabla_m\delta\tilde{G}^m_k - \frac{1}{2}\kappa(\bar{\rho}+\bar{p})\nabla_k\tilde{h}_{00} = 0. \tag{3.21}$$

Replacing $\delta\tilde{G}^\mu_\nu$ by $\delta\tilde{T}^\mu_\nu$ from the field equations we get the conservation laws for $\delta\tilde{T}^\mu_\nu$:

$$(\delta\tilde{T}^0_0)^{\cdot} + \frac{\dot{a}}{a}(3\delta\tilde{T}^0_0-\delta\tilde{T}^k_k) - \frac{1}{a}\nabla^k\delta\tilde{T}^0_k + \frac{3}{2a}(\bar{\rho}+\bar{p})\left(\dot{a}\tilde{h}_{00}-\frac{2}{3}\mathcal{K}\right) = 0, \tag{3.22}$$

$$\frac{1}{a^3}(a^4\delta\tilde{T}^0_k)^{\cdot} + \nabla_m\delta\tilde{T}^m_k - \frac{1}{2}(\bar{\rho}+\bar{p})\nabla_k\tilde{h}_{00} = 0. \tag{3.23}$$

In the case of a perfect fluid the conservation laws become

$$(\delta\rho)^{\cdot} + \frac{3\dot{a}}{a}(\delta\rho+\delta p) + \frac{1}{a}(\bar{\rho}+\bar{p})\nabla_k(\tilde{V}^k-\tilde{h}^k_0) + \frac{3}{2a}(\bar{\rho}+\bar{p})\left(\dot{a}\tilde{h}_{00}-\frac{2}{3}\mathcal{K}\right) = 0, \tag{3.24}$$

$$\frac{1}{a^3}[a^4(\bar{\rho}+\bar{p})(\tilde{V}_k-\tilde{h}_{k0})]^{\cdot} + \nabla_k\delta p + \frac{1}{2}(\bar{\rho}+\bar{p})\nabla_k\tilde{h}_{00} = 0. \tag{3.25}$$

The first equation expresses the conservation of the mass energy $\delta\rho$. The second is the equation of motion; when the time-derivative term is negligible it represents the equilibrium condition between the gradients of pressure and gravitational potential $\frac{1}{2}\tilde{h}_{00}$, which would be much harder to see in the synchronous gauge with $\tilde{h}_{00} = 0$. Until now, all equations have been in an arbitrary gauge. The next section is devoted to the choice of "appropriate gauges."

IV. GAUGES

A change of the gauge can be regarded as an infinitesimal coordinate transformation $x^\mu \to x^{\mu'} = x^\mu + \zeta^\mu(x)$. Under the gauge transformations, the metric changes by the Lie derivative (e.g. [23]) as $\Delta g_{\mu\nu} = \mathcal{L}_\zeta g_{\mu\nu} \equiv \lim_{\zeta\to 0}[g_{\mu\nu}(x') - g'_{\mu\nu}(x')] = \zeta_{\mu;\nu} + \zeta_{\nu;\mu}$. The explicit formulas are given in Appendix B.

Since gauge transformations contain four arbitrary functions, we can impose four gauge conditions. Regarding the field equations (3.10), (3.11), (3.12), and (3.13) we see instantly that four gauge conditions $\mathcal{K} = 0 = \mathcal{T}_k$ decouple the first three equations from the rest. Equations (3.10), (3.11), and (3.12) determine directly the metric components $\tilde{h}_{00}$, $\tilde{h}_{k0}$, and $\tilde{h}^n_n$ from the *instantaneous* distribution of sources given by $\delta\tilde{T}^0_0$, $\delta\tilde{T}^0_k$, and $\delta\tilde{T}^0_0 - \delta\tilde{T}^n_n$; no time integration is needed. Accelerations and rotations of local inertial frames follow then from (2.20) and (2.25). Such an instantaneous determination of local inertial frames is also possible by employing other gauges. We call these gauges *Machian*.

The purpose of this section is to motivate and describe geometrically several Machian gauges, and to clarify what the residual gauge freedom is that these gauges admit. For a comparison we shall also consider two typically *non*-Machian gauges—the synchronous gauge and the generalized Lorenz-de Donder, or "harmonic" gauge. In the next section these gauges will be used to analyze the field equations and the way they can be solved to determine local inertial frames. In the Machian gauges we shall always restrict the spatial part of the metric by requiring the three gauge conditions $\mathcal{T}_k = 0$, where $\mathcal{T}_k$ is given by Eq. (3.9). These conditions will be motivated first.

A. Gauge conditions on the spatial metric

We start beyond the linear perturbation theory. Smarr and York (1978) [48], in treating full general relativity as an evolution from given initial Cauchy data on one spacelike slice to the next slice, studied the kinematics of the observers threading the slices. The evolution is represented in terms of coordinates attached to these "coordinate observers." Kinematical and dynamical effects can be suitably separated if a relative velocity of the coordinate observers, with respect to the (Eulerian) observers whose worldlines are perpendicular to the given slicing, is such that the shear of coordinates arising if one goes from one slice to the next is minimized. For a given slicing the relative velocity is determined by the shift vector and, therefore, Smarr and York require "the minimal-distortion" shift vector. This condition is equivalent to the equation

$$D^j \dot{\tilde{\gamma}}_{ij} = 0, \tag{4.1}$$

where $\tilde{\gamma}_{ij} = (\det\gamma)^{-1/3}\gamma_{ij}$ is the conformal 3-metric on a given slice, the dot denotes the time derivative, and D^j denotes the covariant derivative with respect to the spatial metric γ_{ij} induced on a given slicing by the 4-dimensional metric $g_{\mu\nu}$. The condition (4.1) is a natural choice from a number of points of view. We refer the reader to the original paper [48] for the details; here we wish to make just a few comments. In the weak-field limit in the wave zone, condition (4.1) generalizes and includes the well-known "transverse-traceless" gauges of Arnowitt, Deser, and Misner (1962) and of Dirac (1959) (see, e.g., [1]). In the linearized gravity in generally curved coordinates with $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{00} = 1$, $\bar{g}_{0i} = 0$, $\partial_t \bar{g}_{ij} = 0$, the condition (4.1) implies $\partial_t \bar{D}^j h_{Tij} = 0$, where $h_{Tij} = h_{ij} - \frac{1}{3}h\bar{g}_{ij}$, $h \equiv \bar{g}^{ij}h_{ij}$, and $\bar{D}^j$ is the spatial covariant derivative with respect to $\bar{g}_{ij}$. This is analogous to the radiation (or Coulomb) gauge condition in electrodynamics. In stationary spacetimes with a timelike Killing vector ξ^α the gauge condition (4.1) is satisfied if the slicing is carried into itself by the ξ^α isometry and ξ^α is tangent to coordinate observers. It is interesting to consider more general slicings. Choosing in Schwarzschild spacetime the slices orthogonal to the geodesics of particles freely falling from rest at infinity, one finds that the condition (4.1) implies

$$\begin{aligned} ds^2 = (1 - 2M/r)d\tau^2 - 2(2M/r)^{1/2}drd\tau - dr^2 \\ - r^2(d\theta^2 + \sin^2\theta\varphi^2), \end{aligned} \tag{4.2}$$

i.e., one obtains the time-independent but nondiagonal form (4.2) with the spatially flat metric on the slices.[3]

Now in our case of the perturbed FRW metric it is easy to see that the conformal 3-metric is $\tilde{\gamma}_{ij} = f^{-1/3}[f_{ij} - \tilde{h}_{Tij}]$, $f \equiv \det(f_{ij})$, and $\tilde{h}_{Tij}$ is given by (3.8). Therefore, the condition (4.1) implies $\dot{\mathcal{T}}_k = 0$, with $\mathcal{T}_k$ given by (3.9). It is the last condition which converts Eq. (3.11) into the equation for $\tilde{h}_{k0}$ without the terms depending on the traceless part of $\tilde{h}_{kl}$.

Motivated by the analysis above, we shall assume that, in fact, a slightly stronger condition,

$$\mathcal{T}_k = \nabla_l \tilde{h}^l_{Tk} = 0, \tag{4.3}$$

is satisfied. This just means that the spatial coordinates are restricted on an initial slice and this restriction is then maintained by the original condition. Notice that (4.3) is covariant under 3-dimensional coordinate transformations within chosen slices.

B. Gauge conditions on the time slicing

The three gauge conditions $\mathcal{T}_k = 0$, $k = 1, 2, 3$, do not restrict the time coordinate, i.e., the slicing by spatial hypersurfaces $x^0 =$ constant. We thus supplement them with the fourth gauge condition fixing the slices. In order to understand its geometrical meaning, we first calculate the geometrical quantities characterizing the slices. Using the perturbed FRW metrics in a general gauge, we find that the unit timelike vector field n^μ orthogonal to each slice is given by

$$\tilde{n}^\mu = \bar{\tilde{n}}^\mu + \delta\tilde{n}^\mu, \tag{4.4}$$

where $\bar{\tilde{n}}^\mu = (a^{-1}, 0, 0, 0)$, $\delta\tilde{n}^\mu = a^{-1}(-\frac{1}{2}\tilde{h}_{00}, \tilde{h}^j_0)$. Calculating the expansion $\theta = \tilde{n}^\mu_{;\mu}$ of the congruence of timelike curves that meet the slices orthogonally, we find

$$\begin{aligned} \theta &= \bar{\theta} + \delta\theta = 3\frac{a'}{a^2} - \frac{3}{2a}\left(\frac{a'}{a}\tilde{h}_{00} + \frac{1}{3}(\tilde{h}^n_n)' - \frac{2}{3}\mathcal{P}\right) \\ &= \bar{\theta} - \mathcal{K}, \end{aligned}$$

where $\mathcal{K}$ is given by Eq. (3.7), and

$$\mathcal{P} = \nabla_l \tilde{h}^l_0. \tag{4.5}$$

For the shear of the congruence we obtain $\tilde{\sigma}_{\alpha\beta} = \delta\tilde{\sigma}_{\alpha\beta}$, $\delta\tilde{\sigma}_{00} = 0$, $\delta\tilde{\sigma}_{0i} = 0$,

$$\begin{aligned} \delta\tilde{\sigma}_{ij} &= -a\nabla_{(i}\tilde{h}_{j)0} + \tfrac{1}{3}af_{ij}\nabla_m\tilde{h}^m_0 + \tfrac{1}{2}a\tilde{h}'_{Tij} \\ &= -a\nabla_{(i}\tilde{h}_{j)0} + \tfrac{1}{3}af_{ij}\mathcal{P} + \tfrac{1}{2}a\tilde{h}'_{Tij}. \end{aligned} \tag{4.6}$$

The *uniform-Hubble-expansion gauge*, introduced by Bardeen (1980) [29], but apparently not used much later (though see [50]), requires $\delta\theta = 0$, i.e.

$$\mathcal{K} = \frac{3}{2}\frac{a'}{a}\tilde{h}_{00} + \frac{1}{2}(\tilde{h}^n_n)' - \mathcal{P} = 0. \tag{4.7}$$

This gauge condition is again motivated by the popular choice of the "constant mean curvature slices" in the full theory (the trace of the extrinsic curvature tensor—the mean curvature—of a spacelike hypersurface with normal n^μ is $K = -n^\mu_{;\mu}$). The condition (4.7) thus means that we

[3] More recently, (4.2) was rediscovered as a technically suitable form of the Schwarzschild metric for describing the Hamiltonian dynamics for spherically symmetric gravitating shells by Kraus and Wilczek (1995) [49] without any geometrical argumentation.

choose such a time coordinate in the perturbed FRW universes that the extrinsic curvature of the $\eta =$ constant hypersurfaces is the same as in the unperturbed universe, i.e., it is constant along each hypersurface. Much work has been done on the existence and properties of such foliations (see [51] for the recent review and references).

The gauge condition (4.7) for time slicing combined with the gauge conditions (4.3) for the spatial part of the metric will be called the *Mach 1 gauge*. We have not found it in the literature, although the gauge conditions (4.3) and (4.7) were used separately.

Another basic geometrical object associated with a spacelike slice is its intrinsic curvature, the simplest measure of which is the intrinsic (3-dimensional) scalar curvature. In the perturbed FRW universes $\mathcal{R} = \bar{\mathcal{R}} + \delta\mathcal{R}$, $\bar{\mathcal{R}} = -\frac{6k}{a^2}$, and

$$\delta\mathcal{R} = -\frac{2}{3a^2}(\nabla^2\tilde{h}^n_n + 3k\tilde{h}^n_n) + \frac{1}{a^2}\nabla_n\mathcal{T}^n. \tag{4.8}$$

When the gauge condition (4.3) is combined with the "uniform-intrinsic-scalar curvature" condition $\delta\mathcal{R} = 0$, i.e.

$$\nabla^2\tilde{h}^n_n + 3k\tilde{h}^n_n = 0, \tag{4.9}$$

we speak about the *Mach 2 gauge*. A stronger version—the Mach 2* gauge—requires

$$\tilde{h}^n_n = 0. \tag{4.10}$$

Another possible condition for the choice of slicing is $\nabla_i\nabla_jK^{ij} = 0$, where K^{ij} is the extrinsic curvature tensor. Nothing appears to be known about this choice in the nonlinear context. In our formalism this condition reads [using Eq. (4.6) for the shear]

$$0 = \nabla^i\nabla^j\delta\tilde{\sigma}_{ij} = -\tfrac{2}{3}a(\nabla^2 + 3k)\nabla_l\tilde{h}^l_0 + \tfrac{1}{2}a\nabla^l\mathcal{T}'_l, \tag{4.11}$$

which justifies the name "minimal-shear hypersurface condition" suggested by Bardeen [29]. Combined with the conditions (4.3) the last equation implies the gauge condition

$$(\nabla^2 + 3k)\nabla_l\tilde{h}^l_0 = 0. \tag{4.12}$$

The *Mach 3 gauge* is defined by the gauge conditions (4.3) and (4.12). Its stronger version, the *Mach 3** gauge,

$$\mathcal{P} = \nabla_l\tilde{h}^l_0 = 0, \tag{4.13}$$

combined with (4.3), has been called the *Poisson gauge* by Bertschinger [26] in 1995. He analyzed its advantages for physical interpretation of cosmological perturbations, in particular, as compared with the synchronous gauge. The same gauge has already been proposed in 1994 by Bombelli, Couch, and Torrence [52] who called it the "cosmological gauge."

We also mention the standard *synchronous gauge*, still used most commonly in cosmology,

$$\tilde{h}_{00} = \tilde{h}_{0i} = 0, \tag{4.14}$$

and, in more detail, the generalized *Lorenz-de Donder gauge* (frequently also called the harmonic gauge—cf., e.g., [3], and recently [53]),

$$\bar{\nabla}_\mu\delta(\sqrt{-g}g^{\mu\nu}) = 0. \tag{4.15}$$

This has been extensively used in a number of problems, in particular, in weak-field approximations dealing with equations of motion and gravitational radiation (see, e.g., [54]), but not in cosmology. The gauge conditions (4.15) for $\mu = 0$ imply

$$-\nabla_l\tilde{h}^l_0 + \tfrac{1}{2}(\tilde{h}_{00} + \tilde{h}^n_n)' + \mathcal{H}(3\tilde{h}_{00} + \tilde{h}^n_n) = 0, \tag{4.16}$$

and for $\mu = k$ we get

$$-\nabla_l\tilde{h}^l_{Tk} + \tfrac{1}{6}\nabla_k(\tilde{h}^n_n - 3\tilde{h}_{00}) + \tilde{h}'_{0k} + 4\mathcal{H}\tilde{h}_{0k} = 0. \tag{4.17}$$

We shall now analyze the residual gauge freedom which the gauges introduced above admit.

C. Gauge fixing and residual gauge freedom

From relations (B3)–(B6) in Appendix B we readily obtain the changes of the geometrical quantities defined in (3.7), (3.9), (4.5), and (4.6) under gauge transformations:

$$\Delta\mathcal{T}_k = \Delta(\nabla_l\tilde{h}^l_{Tk}) = -(\nabla^2\zeta_k + 2k\zeta_k + \tfrac{1}{3}\nabla_k\nabla_l\zeta^l), \tag{4.18}$$

$$\Delta\mathcal{P} = \Delta(\nabla_l\tilde{h}^l_0) = \frac{1}{a}\nabla^2\zeta^0 - a\nabla_k\dot{\zeta}^k, \tag{4.19}$$

$$\Delta\mathcal{K} = -\frac{1}{a}\left[\nabla^2\zeta^0 + 3a^2\left(\frac{\dot{a}}{a}\right)^{\cdot}\zeta^0\right], \tag{4.20}$$

$$\Delta\mathcal{R} = \frac{4}{3a^2}(\nabla^2 + 3k)\left(\nabla_l\zeta^l + 3\frac{\dot{a}}{a}\zeta^0\right), \tag{4.21}$$

$$\Delta(\nabla^k\nabla^l\delta\tilde{\sigma}_{kl}) = -\tfrac{2}{3}(\nabla^2 + 3k)\nabla^2\zeta^0. \tag{4.22}$$

We shall discuss first *the minimal-distortion spatial gauge condition:* $\mathcal{T}_k = 0$.

Starting from a general gauge, we reach the required condition by purely spatial gauge transformations given by ζ_k, which satisfy the inhomogeneous equation with given l.h.s. $\Delta\mathcal{T}_k$. The residual gauge freedom is determined by ζ_k solving the homogeneous equation

$$\nabla^2\zeta_k + 2k\zeta_k + \tfrac{1}{3}\nabla_k\nabla_l\zeta^l = 0. \tag{4.23}$$

There are solutions of this equation given by linear combinations (with time-dependent coefficients) of the conformal Killing vectors in the constant-curvature spaces S^3, R^3,

H^3. To see this, recall that in 3-dimensional space a conformal Killing vector satisfies

$$\nabla_l \zeta_k + \nabla_k \zeta_l = \tfrac{2}{3} f_{kl} \nabla_n \zeta^n. \tag{4.24}$$

Since spaces of constant curvature are conformally flat, they admit ten linearly independent conformal Killing vectors as E^3 (see, e.g., [55]). Their explicit forms are given in Appendix C, where their relationship to the scalar and vector hyperspherical harmonics is also elucidated. Among the ten conformal Killing vectors, six are pure Killing vectors, $\xi_i^{(A)}$, $A = 1, 2, \dots, 6$; the remaining four $\psi_i^{(B)}$, $B = 1, \dots, 4$ do not reduce to the Killing vectors. In spaces of nonvanishing constant curvature, $\psi_i^{(B)}$ can be written as gradients of scalars:

$$k = \pm 1: \ \psi_i^{(B)} = \partial_i Q^B = \nabla_i Q^B, \qquad B = 1, \dots, 4. \tag{4.25}$$

The four scalar fields, Q^B, are equal, up to a multiplicative constant, to the following four scalar harmonics (see Appendix C for details):

$$k = +1: \ Q_{(L=1,l=0,m=0)} \simeq \cos\chi, \quad Q_{(L=1,l=1,m=-1,0,+1)} \simeq \sin\chi Y_{1m}(\theta, \varphi), \tag{4.26}$$

$$k = -1: \ Q_{(\lambda=2i,l=0,m=0)} \simeq \cosh\chi, \quad Q_{(\lambda=2i,l=1,m=-1,0,+1)} \simeq \sinh\chi Y_{1m}(\theta, \varphi). \tag{4.27}$$

Taking the divergence of Eq. (4.24) and using (A5) and (A6) in Appendix A to commute the derivatives, we obtain Eq. (4.23). Hence, any conformal Killing vector ζ_k solves Eq. (4.23). In open universes, all such solutions ζ_k diverge at infinity ($\chi \to \infty$), except for translations in a flat ($k = 0$) universe when ζ_k are constant in Cartesian-like coordinates. We now prove that *there exist no bounded solutions of Eq. (4.23) other than conformal Killing vectors in S^3 and translations in E^3*. To prove this we decompose ζ_k into a gradient of a scalar and a transverse vector:

$$\zeta_k = \nabla_k Z + \zeta_{Tk}, \qquad \nabla^k \zeta_{Tk} = 0. \tag{4.28}$$

This decomposition is unique up to $Z \to Z +$ constant, if for open universes we require ζ_k to decay asymptotically so that $\int \nabla^k \zeta_k dV$ converges [23]. Substituting then (4.28) into (4.24) and commuting the derivatives, we find

$$\nabla^2 \nabla^2 Z + 3k \nabla^2 Z = 0. \tag{4.29}$$

In S^3 the only smooth solutions of the equation

$$\nabla^2 Q + 3Q = 0 \tag{4.30}$$

are given by the linear combination of the four scalar harmonics (4.26), the gradients of which give the conformal Killing vectors. In closed space the solution of (4.29) is thus, with $\beta_B(t)$ arbitrary,

$$Z = \sum_{B=1}^{4} \beta_B(t) Q^B + Z_0(t). \tag{4.31}$$

In H^3 Eq. (4.29) becomes

$$\nabla^2 Q - 3Q = 0. \tag{4.32}$$

The solutions are four scalar harmonics given in (4.27)—these, however, diverge at infinity. In open universes the only solution of (4.32) leading to asymptotically well-behaved ζ_k is $Z = Z_0(t)$, the gradient of which does not contribute to ζ_k. Substituting now for ζ_k in Eq. (4.23) the decomposition (4.28), and regarding the above results for Z, we find that Eq. (4.23) reduces to the equation $\nabla^2 \zeta_{Tk} + 2k\zeta_{Tk} = 0$. In open universes this equation does not admit any asymptotically well-behaved solutions, except for $\zeta_{Tk} =$ constant for $k = 0$. In a closed universe the equation is equivalent to the Killing equation (Appendix C). Hence, under the assumption that the vector ζ^i is bounded, the condition (4.3) fixes the spatial coordinates uniquely in H^3; and in E^3 the remaining gauge freedom is just $\zeta^i = \sum_{A=1}^{3} f_A(t) \zeta_{\mathrm{tr}}^{(A)i}$, corresponding to a time-dependent linear combination of translations. In S^3 the residual gauge freedom is given by a linear combination of ten conformal Killing vectors (six Killing and four conformal Killing):

$$\zeta^i = \sum_{A=1}^{6} \alpha_A(t) \xi^{(A)i} + \sum_{B=1}^{4} \beta_B(t) \psi^{(B)i}. \tag{4.33}$$

We now discuss the three Machian gauges successively.

1. Mach 1: Uniform-Hubble-expansion gauge

From Eq. (4.20) we see that the residual freedom in ζ^0 is given by the solutions of

$$\nabla^2 \zeta^0 + 3a^2 \left(\frac{\dot a}{a}\right)^{\cdot} \zeta^0 = 0. \tag{4.34}$$

Multiplying Eq. (4.34) by ζ^0 and integrating by parts over a domain $\mathcal{D}$, we find

$$\int_{\mathcal{D}} \zeta^0 \nabla^2 \zeta^0 d^{(3)}V = \int_{\partial\mathcal{D}} \zeta^0 \nabla_k \zeta^0 dS^k - \int_{\mathcal{D}} f^{kl} \nabla_k \zeta^0 \nabla_l \zeta^0 d^{(3)}V = -\int_{\mathcal{D}} 3a^2 \left(\frac{\dot a}{a}\right)^{\cdot} (\zeta^0)^2 d^{(3)}V, \tag{4.35}$$

where $d^{(3)}V = \sqrt{f} d^3x$, $f = \det(f_{kl})$; $a_0^3 d^{(3)}V$, $a_0 = a(x^0)$, is the proper volume in a slice $x^0 =$ constant. Taking $\mathcal{D}$ to be all space, then the integral over the boundary vanishes in open spaces because of the boundary condition on ζ^0, and it is zero in closed spaces because there is no boundary. Therefore,

$$\int_{\mathcal{D}} f^{kl} \nabla_k \zeta^0 \nabla_l \zeta^0 d^{(3)}V = 3a^2 \left(\frac{\dot a}{a}\right)^{\cdot} \int_{\mathcal{D}} (\zeta^0)^2 d^{(3)}V. \tag{4.36}$$

The factor on the r.h.s. can be rewritten using the FRW background equations:

$$\mathcal{A}(t) \equiv 3a^2\Big(\frac{\dot a}{a}\Big)^{\cdot} = 3a^2\dot H = 3k - \frac{3}{2}a^2\kappa(\bar\rho + \bar p) = -3a^2\Big(H^2 - \frac{1}{3}\Lambda\Big) - \frac{1}{2}a^2\kappa(\bar\rho + 3\bar p). \tag{4.37}$$

In all standard models the strong energy condition $\bar\rho + 3\bar p > 0$ is valid so that $\mathcal{A} < 0$ ($H^2 - \frac{1}{3}\Lambda > 0$ is satisfied in realistic models). In inflationary universe models with $\bar\rho + \bar p = 0$ ($\Lambda = 0$), the function $\mathcal{A}(t) < 0$ for open universes. In all these cases the r.h.s. of Eq. (4.37) is nonpositive, whereas the l.h.s. is non-negative. Therefore, the only solution of Eq. (4.34) is $\zeta^0 = 0$. In the standard inflationary model with $k = 0$, $\bar\rho + \bar p = 0$, we have $\mathcal{A} = 0$, and $\zeta^0 = \zeta^0(t)$ is an admissible solution of Eq. (4.34) which is bounded and has a vanishing gradient (reflecting the higher symmetries of de Sitter space to which the FRW models reduce). If $k = +1$ *and* $\bar\rho + \bar p = 0$, the relation (4.37) turns Eq. (4.34) (for any Λ) into $\nabla^2\zeta^0 + 3\zeta^0 = 0$, which is Eq. (4.30), the solutions thus being

$$\zeta^0 = \sum_{B=1}^{4}\sigma_B(t)Q^B, \tag{4.38}$$

where 4 scalar harmonics Q^B are given in (4.26), and σ_B are arbitrary.

Let us summarize. Assuming ζ^μ bounded, the Mach 1 gauge fixes the coordinates uniquely in the open universes with $k = -1$, and for $k = 0$ it determines the spatial coordinates up to time-dependent translations, $\zeta^k(t)$, whereas the time slicing is unique if the background matter satisfies the strong energy condition; x^0 can be shifted by $\zeta^0(t)$ in the inflationary universe. In closed universes the spatial coordinates are determined up to the time-dependent motions (4.33) given by the Killing and conformal Killing vectors; the time slicing is unique in the standard backgrounds with the strong energy condition satisfied. In the inflationary backgrounds the time can be shifted by ζ^0 determined by Eq. (4.38).

2. Mach 2: Uniform-scalar-curvature gauge

Requiring the scalar 3-curvature of the time slices to be equal to the background values fixes the gauge up to the transformations satisfying [see Eq. (4.21)]

$$(\nabla^2 + 3k)\Big(\nabla_l\zeta^l + 3\frac{\dot a}{a}\zeta^0\Big) = 0. \tag{4.39}$$

Assuming again the condition (4.3), we restricted ζ^l already by Eq. (4.23), which implies the divergence $\nabla_l\zeta^l$ to satisfy $(\nabla^2 + 3k)\nabla_l\zeta^l = 0$. Equation (4.39) thus reduces to (assuming $\dot a \neq 0$) $(\nabla^2 + 3k)\zeta^0 = 0$. As discussed above [cf. (4.30) or (4.32)], the only bounded solutions are $\zeta^0 = 0$ if $k = -1$, $\zeta^0 = \zeta^0(t)$ if $k = 0$, and ζ^0 is given in terms of Q^B for $k = +1$.

Mach 2: The traceless gauge.*—The gauge condition $\tilde h^n_n = 0$ implies the previous one, and is stronger. Indeed, regarding Eq. (B6), we see that the residual gauge freedom is given by $\nabla_k\zeta^k + 3\frac{\dot a}{a}\zeta^0 = 0$, which determines ζ^0 in terms of ζ^k (assuming $\dot a \neq 0$). With the gauge conditions (4.3), $\zeta^0 = 0$ in open spaces, in S^3 the residual freedom in ζ^k is given by Eq. (4.33) which implies $\nabla_k\zeta^k = \sum_{B=1}^4\beta_B(t)\nabla^2 Q^B$ [see Eq. (4.25)], and thus leads to [using (4.30)]

$$\zeta^0 = (a/\dot a)\sum_{B=1}^{4}\beta_B(t)Q^B, \tag{4.40}$$

where $\beta_B(t)$ are arbitrary.

3. Mach 3: The minimal-shear gauge

As seen from Eq. (4.22) this gauge condition allows the transformations restricted by

$$(\nabla^2 + 3k)(\nabla^2\zeta^0 - a^2\nabla_k\dot\zeta^k) = 0, \tag{4.41}$$

which, using ζ^k that satisfy (4.3), reduces to $(\nabla^2 + 3k)\nabla^2\zeta^0 = 0$. This is the same as (4.29). In open spaces the only bounded solutions are $\zeta^0 = \zeta^0(t)$. In closed spaces

$$\zeta^0 = \sum_{B=1}^{4}\sigma_B(t)Q^B + \sigma^0(t), \tag{4.42}$$

where σ's are arbitrary, and Q^B is given by Eq. (4.26).

Mach 3: The Poisson gauge.*—The condition $\nabla_l h^l_0 = 0$ admits a smaller freedom. Equation (4.41) becomes

$$\nabla^2\zeta^0 - a^2\nabla_k\dot\zeta^k = 0, \tag{4.43}$$

which, after substituting for ζ^k from Eq. (4.33), for $k = +1$ gives

$$\zeta^0 = \sum_{B=1}^{4}a^2\dot\beta_B(t)Q^B + \sigma^0(t), \tag{4.44}$$

where σ^0 is arbitrary but $\beta_B(t)$ are the same functions as those in ζ^k in Eq. (4.33)—in contrast to Eq. (4.42) where σ's are independent of β's. In open universes the only residual freedom in the choice of time is given by arbitrary $\zeta^0(t)$.

Regarding the gauge freedom in $\mathcal{T}_k = 0$, we see that the Poisson gauge in the case $k = -1$ fixes the spatial coordinates uniquely; the time coordinate is fixed up to $\zeta^0(t)$. In the case $k = 0$ the spatial coordinates are fixed up to translations $\zeta^i(t)$ and time shifts $\zeta^0(t)$. In the closed case the freedom in spatial coordinates is determined by linear combinations of the Killing and conformal Killing vectors (4.33), whereas the time coordinate is determined by the combination (4.44) of scalar harmonics Q^B. Hence, in the closed case there are 11 arbitrary functions of time which represent the gauge freedom.

These results are at variance with Bertschinger's statement [26] that there is "an almost unique transformation from an arbitrary gauge to the Poisson gauge." Clearly, Bertschinger does not consider the possibility that his β solves equation $(\nabla^2 + 3k)\nabla^2\beta = 0$ [i.e. our Eq. (4.29)] which preserves the gauge condition $\nabla_l h^l_{Tk} = 0$. Solutions for β for $k = \pm 1$ are as in Eq. (4.31), where $Z_0(t)$ indeed has no effect but the terms containing Q^B do have an effect—not only on ζ_k but also on ζ^0 as described above.

Let us now mention the gauge freedom in two typical "non-Machian" gauges.

4. Synchronous gauge

From Eqs. (B3) and (B4) it is immediately seen that the synchronous gauge admits the well-known residual freedom given by transformations satisfying

$$\dot{\zeta}^0 = 0, \qquad \nabla_l\zeta^0 = a^2\dot{\zeta}_l, \tag{4.45}$$

which imply

$$\zeta^0 = \zeta^0(x^i), \qquad \zeta_l = \left[\int\frac{dt}{a^2(t)}\right]\nabla_l\zeta^0(x^i) + Z_l(x^i). \tag{4.46}$$

Functions $\zeta^0(x^i)$ and $Z_l(x^i)$ are arbitrary. The gauge freedom is the same for both open and closed universes.

5. The generalized Lorenz-de Donder gauge

Requiring the gauge conditions (4.16) and (4.17) to be satisfied, we can use relations (B3)–(B6) to find out the residual freedom in this gauge. It turns out to be restricted by

$$\nabla^2\zeta^0 - a^2\ddot{\zeta}^0 - 3a\dot{a}\dot{\zeta}^0 + 3(a\ddot{a} + \dot{a}^2)\zeta^0 - 2a\dot{a}\nabla_l\zeta^l = 0, \tag{4.47}$$

$$\nabla^2\zeta_k - a^2\ddot{\zeta}_k + 2k\zeta_k - 5\dot{a}a\dot{\zeta}_k + 2(\dot{a}/a)\nabla_k\zeta^0 = 0. \tag{4.48}$$

The only feasible way to solve this coupled system appears to be the use of harmonics, but here we shall just restrict ourselves to noticing that, for a slowly changing expansion factor ($\dot{a}$, $\ddot{a}$ small), the system turns just into two decoupled wave equations,

$$\frac{1}{a^2}\nabla^2\zeta^0 - \ddot{\zeta}^0 = 0, \qquad \frac{1}{a^2}\nabla^2\zeta_k - \ddot{\zeta}_k + \frac{2k}{a^2}\zeta_k = 0. \tag{4.49}$$

In the flat case these are just wave equations in flat space with coordinates ax^i (which give the proper lengths in $k = 0$ universes). The gauge freedom is thus analogous to the freedom of the Lorenz gauge in electrodynamics. Any solution of a wave equation can be characterized by its Cauchy values—here $\zeta^0(x^i)$, $\dot{\zeta}^0(x^i)$, $\zeta_k(x^i)$, and $\dot{\zeta}_k(x^i)$, i.e., by eight functions of spatial coordinates.

Summarizing, we find that the *Machian gauges are substantially more restrictive than the synchronous gauge and the generalized Lorenz-de Donder gauge*. The last two gauges admit transformations characterized by several (two and eight) *arbitrary functions of three variables*—of the spatial coordinates x^i. All the Machian gauges admit only several arbitrary functions of time. In some cases they fix the coordinates uniquely. An arbitrary additive function of time, $\zeta^0(t)$, represents just the changes of the units of time: $dt' = (1 + \dot{\zeta}^0)dt$. The spatially homogeneous changes of x^i by $\zeta^i(t)$ describe just the shifts of the origin of spatial coordinates. Otherwise, *all three Machian gauges fix the coordinates uniquely in the hyperbolic universes* H^3 as a consequence of boundary conditions at infinity. The remaining functions of time in spherical universes S^3 will be interpreted in the following.

D. Integral gauge conditions in closed universes with standard spherical topology

In the closed spherical spaces the spatial coordinates x^i are fixed up to the transformations $x^i \to x'^i = x^i + \zeta^i$, where ζ^i is given by a linear combination of six Killing and four (proper) conformal Killing vectors of S^3, in which the coefficients are arbitrary functions of time.

In order to acquire an insight into the effects such coordinate changes can produce, consider an unperturbed FRW universe with standard spherical topology ($k = +1$). Transform the metric in the hyperspherical coordinates (1.3) by a gauge transformation generated by one translational, one rotational, and one conformal Killing vector which have the simplest forms in the hyperspherical coordinates: $\zeta^i_{\rm tr} = (\cos\theta, -\cot\chi\sin\theta, 0)$, $\zeta^i_{\rm rot} = (0, 0, 1)$, $\zeta^i_{\rm conf} = (-\sin\chi, 0, 0)$ (cf. Appendix C). Admitting the time-dependent coefficients, the transformation has the form

$$\begin{aligned}\chi' &= \chi + \alpha(t)\cos\theta - \gamma(t)\sin\chi,\\ \theta' &= \theta - \alpha(t)\cot\chi\sin\theta, \qquad \varphi' = \varphi + \beta(t),\end{aligned} \tag{4.50}$$

which can easily be inverted since α, β, γ are small. In addition to the transformation (4.50) we consider a change of the time coordinate (time slicing) allowed by our Machian gauge conditions in closed universes.

Hence, we take ζ^0 of the form (4.42) because other possibilities (4.38), (4.40), and (4.44) are included in (4.42). However, since in (4.50) only the simplest conformal Killing vector enters, it is sufficient to take only those time transformations which are associated with this vector and with the shift of the time origin which is also allowed by Eq. (4.41):

$$t' = t + a^2[\delta(t)\cos\chi + \sigma(t)], \tag{4.51}$$

where for convenience the expansion factor is pulled out. Under the transformations (4.50) and (4.51) the standard FRW metric with $k = +1$ becomes

$$\begin{aligned} ds^2 = &[1 - (2a^2\delta)^{\cdot}\cos\chi' - (2a^2\sigma)^{\cdot}]dt'^2 \\ &- a^2[1 + 2(\gamma + a\dot{a}\delta)\cos\chi' + 2a\dot{a}\sigma] \\ &\times[d\chi'^2 + \sin^2\chi'(d\theta'^2 + \sin^2\theta' d\varphi'^2)] \\ &+ 2a^2\{[\dot{\alpha}\cos\theta' + (\delta - \dot{\gamma})\sin\chi']d\chi' \\ &- \dot{\alpha}\sin\chi'\cos\chi'\sin\theta' d\theta' + \dot{\beta}\sin^2\chi'\sin^2\theta' d\varphi'\}dt'. \end{aligned} \tag{4.52}$$

Since $\alpha, \ldots, \sigma$ are, in general, time-dependent and h'_{00}, h'_{0i} nonvanishing, the frames associated with χ', θ', φ' fixed are noninertial in general, and the inertial frames, momentarily at rest with respect to them, are seen to have the acceleration [cf. Eq. (2.20)]

$$\begin{aligned} \alpha'^{\chi} &= -(1/a^2)\{[2a^2(\delta - \dot{\gamma})]^{\cdot}\sin\chi' + (a^2\dot{\alpha})^{\cdot}\cos\theta'\}, \\ \alpha'^{\theta} &= -(1/a^2)(a^2\dot{\alpha})^{\cdot}\cot\chi'\sin\theta', \\ \alpha'^{\varphi} &= -(1/a^2)(a^2\dot{\beta})^{\cdot}, \end{aligned} \tag{4.53}$$

and to rotate with the vorticity [cf. Eq. (2.25)]

$$\begin{aligned} \omega'_{\chi\theta} &= a^2\dot{\alpha}\sin^2\chi'\sin\theta', \\ \omega'_{\chi\varphi} &= a^2\dot{\beta}\sin\chi'\cos\chi'\sin^2\theta', \\ \omega'_{\theta\varphi} &= -a^2\dot{\beta}\sin^2\chi'\sin\theta'\cos\theta', \end{aligned} \tag{4.54}$$

i.e., with the vorticity 3-vector (2.31) given by

$$\begin{aligned} \omega'^{\chi} &= (1/a)\dot{\beta}\cos\theta', \\ \omega'^{\theta} &= (1/a)\dot{\beta}\cot\chi'\sin\theta', \\ \omega'^{\varphi} &= (1/a)\dot{\alpha}. \end{aligned} \tag{4.55}$$

The above results are easily understood: time-dependent rotations in the φ direction, with $\chi = 0$, and the φ axis fixed, imply nonvanishing accelerations in this direction only, whereas the corresponding vorticity vector has no φ component. The translational Killing vector which for, say, $\theta = 0$ represents rotations in the χ direction (with $\chi = \pi/2$ fixed), leads to accelerations only in the χ direction, and the vorticity vector in the φ direction.

None of these acceleration or vorticity vectors can be compensated by an allowed change (4.51) of time slicing. As expected, the shift of the time origin, $\sigma(t)$, does not enter Eqs. (4.53), (4.54), and (4.55). However, the effect of the transformation generated by the conformal Killing vector, which appears only in the χ component of the acceleration, can be annulled by choosing $\delta = \dot{\gamma}$. Nevertheless, this condition does not remove the effect of both δ and γ in the conformal factor of the spatial background metric. The metric (4.52) implies a nonvanishing trace of the form (omitting the "time shift" σ)

$$h'^n_n = -\tilde{h}'^n_n = 6(\gamma + a\dot{a}\delta)\cos\chi'. \tag{4.56}$$

Therefore, the spatial metric differs from the canonical metric of a homogeneous and isotropic 3-sphere. This metric is preserved only by transformations representing real symmetries, i.e., those generated by the Killing vectors.[4]

We have described the particular effects of the gauge freedom corresponding to the Killing and (proper) conformal Killing vectors in order to show their different character. In a general, linearly perturbed, FRW universe, the metric will be much more complicated than that of Eq. (4.52). If it contains terms appearing in (4.52) (and those corresponding to other Killing and conformal Killing vectors), they can, of course, be removed by gauge transformations of the form (4.50) and (4.51). Since a natural goal in a relativistic perturbation theory is to fix the gauge at the end as uniquely as possible, we shall now require, in all Machian gauges, additional gauge conditions which exclude the freedom corresponding to the gauge transformations generated by (proper) conformal Killing vectors. However, we leave the freedom corresponding to the Killing vectors since these exhibit the symmetry of the background universe at any given time.

The trace (4.56) is, at a given time, proportional to $\cos\chi$ [we omit primes in metric (4.52)], i.e., just to the first of the scalar harmonics in Eq. (4.26). Neglecting the time shift $\sim\sigma(t)$, the perturbation $h_{00} = \tilde{h}_{00}$ in (4.52) is also proportional to this harmonic. The term in $h_{0i}(= a\tilde{h}_{0i})$ in (4.52) corresponding to the same conformal transformation of the spatial coordinates and of the time slicing is proportional to the gradient of $\cos\chi$ but the scalar $\mathcal{P} = \nabla_l \tilde{h}^l_0$ is again proportional to this harmonic.[5] The metric perturbations which arise or may be removed by gauge transformations generated by the conformal Killing vectors (i.e., the vectors of the form $\zeta^i = \sum_{B=1}^4 \beta_B \psi^{(B)i} = \sum_{B=1}^4 \beta_B \nabla^i Q^B$) will be eliminated by the following integral gauge conditions. These will be imposed at all times:

$$\int_{S^3} \tilde{h}^n_n Q^B d^{(3)}V = \int_{S^3} \tilde{h}_{00} Q^B d^{(3)}V = \int_{S^3} \mathcal{P} Q^B d^{(3)}V = 0, \tag{4.57}$$

where $\tilde{h}^n_n$, $\tilde{h}_{00}$, $\mathcal{P} = \nabla_l \tilde{h}^l_0$ are functions of all spacetime coordinates, and harmonics $Q^B(\chi, \theta, \varphi)$ are given in Eq. (4.26). The integral gauge conditions (4.57) require that spatial scalars $\tilde{h}^n_n$, $\tilde{h}_{00}$, and $\mathcal{P}$ are orthogonal to the 4-dimensional functional space spanned by Q^B, i.e., by harmonics which are eigenfunctions with zero eigenvalues of the operator $(\nabla^2 + 3)$ in S^3. In Sec. V we shall notice that conditions (4.57) are closely related to Traschen's integral constraints [37,38] which restrict perturbations of energy-momentum tensors representing sources. In this way we make sure that conditions (4.57) do not restrict physics.

[4]As in flat space, the canonical flat-space metric in Cartesian coordinates is preserved only by rigid translations and rotations, but not by dilatations.

[5]We do not consider the scalar $\nabla_l \nabla_k \tilde{h}^{lk}_T$ since the gauge condition (4.3) guarantees that it vanishes in all Machian gauges.

E. Machian gauges in closed spherical universes: Summary

After adopting the integral gauge conditions (4.57), the gauge freedom in all three Machian gauges becomes transparent and simple: It reflects the proper (Killing) symmetry of the background universe at any fixed time. The minimal-distortion shift gauge condition (4.3), together with integral gauge conditions (4.57), which we assume in all Machian gauges, fix the spatial coordinates uniquely up to transformations $x^i \to x'^i = x^i + \zeta^i$ with

$$\zeta^i = \sum_{A=1}^{6} \alpha_A(t)\xi^{(A)i}, \tag{4.58}$$

where $\xi^{(A)i}$ are six spatial Killing vectors—three (quasi)-rotations and three (quasi)translations.

In the Mach 1 gauge the time slicing is unique even without requiring integral gauge conditions (4.57) if the background matter satisfies the strong energy condition. After requiring (4.57), it is also unique in the inflationary backgrounds. In the uniform-scalar-curvature gauge (Mach 2), with (4.57) satisfied, the time coordinate is unique, the same being true for the special case of the traceless gauge (Mach 2*). Finally, in the minimal-shear gauge (Mach 3) and its special case of the Poisson gauge (Mach 3*), the adoption of the integral conditions (4.57) leaves the only freedom in Eqs. (4.42) and (4.44) to be $\zeta^0 = \sigma^0(t)$, i.e., the time coordinate is fixed up to trivial, ''universal'' shifts depending just on an arbitrary function of time. This, as noticed below Eqs. (4.53), (4.54), and (4.55), does not influence accelerations and rotations of local inertial frames. Therefore, all our Machian gauges fix coordinates uniquely up to the ''time-dependent'' Killing motions (4.58) of spatial coordinates. The Machian gauges with the integral gauge conditions (4.57) are thus determining coordinates both more restrictively and more plausibly than the synchronous and generalized Lorenz-de Donder gauge.

F. On closed hyperbolic and flat universes

Although in this work we generally assume the cosmological backgrounds with standard topologies only, and thus with geometries which are homogeneous and isotropic also globally, in this intermezzo we consider 3-dimensional backgrounds represented by closed flat ($k = 0$) and hyperbolic ($k = -1$) 3-manifolds. Finite universes with multi-connected topologies have become popular in recent years in the light of new theories extending general relativity, and in the view of a possibility (in principle) to determine the topology of our universe by means of cosmic microwave background observations, or from the distribution of distant sources. A comprehensive, nice review containing many references appeared recently [56].

Globally, these universes are different; in particular, they admit smaller families of continuous symmetries. Closed hyperbolic 3-manifolds do not have smooth Killing vectors at all [56] and do not possess nontrivial solutions of the equation $\nabla^2\phi + 3k\phi = 0$. Indeed, multiplying this equation by ϕ, integrating by parts over a 3-dimensional domain $\mathcal{D}$ [cf. Eq. (4.35)], we get

$$\int_{\partial\mathcal{D}} \phi\nabla^j\phi dS_j - \int_{\mathcal{D}} \nabla_j\phi\nabla^j\phi d^{(3)}V + 3k\int_{\mathcal{D}} \phi^2 d^{(3)}V = 0. \tag{4.59}$$

Taking $\mathcal{D}$ to be whole closed space, the first integral vanishes, because there is no boundary, and since both the second and the third integrals are non-negative, Eq. (4.59) for $k = -1$ can be satisfied only with $\phi = 0$. Hence, Eq. (4.32) has only solutions $Q = 0$, so Z in Eq. (4.28) does not contribute to ζ_k. Analogously, the equation for the transverse part of ζ_k, after being multiplied by $\zeta_{Tm}\gamma^{km}$, and integrated by parts, becomes

$$\int_{\partial\mathcal{D}} \gamma^{mk}\zeta_{Tm}\nabla^j\zeta_{Tk}dS_j - \int_{\mathcal{D}} \gamma^{mk}\nabla_j\zeta_{Tm}\nabla^j\zeta_{Tk}d^{(3)}V + 2k\int_{\mathcal{D}} \gamma^{mk}\zeta_{Tm}\zeta_{Tk}d^{(3)}V = 0. \tag{4.60}$$

Again, taking $\mathcal{D}$ to be all closed space, the first term is zero, and as the other integrals are non-negative, (4.60) for $k = -1$ is solved only by $\zeta_{Tk} = 0$. It is easily seen that, except for trivial shifts $\sigma(t)$ in the Mach 3 gauge, in *closed hyperbolic* universes our instantaneous *Machian local gauge conditions fix coordinates uniquely*, without integral gauge conditions being imposed.

For closed flat ($k = 0$) universes, Eq. (4.59) implies $\phi = \phi(t)$ and Eq. (4.60) gives $\nabla_j\zeta_{Tm} = 0$ so that $\zeta^i = \sum_{A=1}^{3} f_A(t)\xi_{\rm tr}^{(A)i}$, where $\xi_{\rm tr}^{(A)i}$, $A = 1, 2, 3$, are translation Killing vectors. Hence the condition (4.3) determines the spatial coordinates up to arbitrary time-dependent linear combinations of translations. That the rotational Killing vectors are globally ruled out can be well understood in the simplest example of a compact flat 3-manifold—a 3-torus T^3. All closed flat 3-manifolds are given in Fig. 26 in [56]. In fact, only the 3-torus admits globally a 3-parameter family of translational symmetries given in Cartesian coordinates by the 3 independent constant Killing vectors. Considering just T^3, we find that the gauge condition (4.3) fixes spatial coordinates up to

$$x^i \to x'^i = x^i + \zeta^i(t) = x^i + \sum_{A=1}^{3} f_A(t)\xi_{\rm tr}^{(A)i}, \tag{4.61}$$

where $\xi_{\rm tr}^{(A)i}$, $A = 1, 2, 3$, are 3 translation Killing vectors. In the Cartesian-type coordinates these can be chosen as $\zeta^i_{(1)} = (1, 0, 0)$, etc., so that the transformation generated by them is

$$x = x' - f_{(1)}(t), \qquad y = y' - f_{(2)}(t), \qquad z = z' - f_{(3)}(t). \tag{4.62}$$

This brings the FRW background metric with $k = 0$ into the form

$$ds^2 = dt^2 - a^2(dx'^2 + dy'^2 + dz'^2) + 2a^2(\dot{f}_{(1)}dtdx' + \dot{f}_{(2)}dtdy' + \dot{f}_{(3)}dtdz'). \tag{4.63}$$

The acceleration of the local inertial frames with respect to the frame with x', y', z' fixed is thus given by [cf. Eq. (2.20)]

$$\alpha'^x = -(1/a^2)(a^2\dot{f}_{(1)})^{\cdot},$$
$$\alpha'^y = -(1/a^2)(a^2\dot{f}_{(2)})^{\cdot}, \tag{4.64}$$
$$\alpha'^z = -(1/a^2)(a^2\dot{f}_{(3)})^{\cdot}.$$

Since the gradients $h'_{0k,l}$ are vanishing, local inertial frames do not rotate.

V. FIELD EQUATIONS, INTEGRAL CONSTRAINTS, SOLUTIONS, AND INERTIAL FRAMES

We now turn to the equations for perturbations in the Machian gauges. Here we pay attention to the Mach 1 gauge. Its choice of the constant mean curvature slices is most natural from the perspective of the full nonlinear theory. Moreover, the structure of the field equations for linear perturbations and their solutions do not differ significantly for the Mach gauges considered. The equations in Mach 2 and 3 gauges and in the generalized Lorenz-de Donder gauge are briefly discussed in Appendix D. Whenever solutions are known in terms of the Green's functions, we write them down. They can be used to determine the accelerations and rotations of local inertial frames. Alternatively, solutions in terms of harmonics [22,57] can be obtained by direct calculations, but they will not be studied in the present work.

A. Field equations in the Mach 1 gauge

The minimal-distortion shift condition (4.3) is combined with the constant mean external curvature condition, i.e., with [cf. Eq. (4.7)]

$$\tfrac{3}{2}\dot{a}\tilde{h}_{00} + \tfrac{1}{2}a\dot{\tilde{h}}^n_n - \mathcal{P} = 0, \qquad \mathcal{P} = \nabla_l\tilde{h}^l_0. \tag{5.1}$$

In addition, we impose integral gauge conditions (4.57). As a consequence of the differential gauge conditions (4.3) and (5.1), the field equations (3.10), (3.11), (3.12), (3.13), and (3.14) simplify considerably:

$$\nabla^2\tilde{h}^n_n + 3k\tilde{h}^n_n = 3a^2\kappa\delta\tilde{T}^0_0, \tag{5.2}$$

$$\nabla^2\tilde{h}_{k0} + 2k\tilde{h}_{k0} + \tfrac{1}{3}\nabla_k\mathcal{P} = 2a^2\kappa\delta\tilde{T}^0_k, \tag{5.3}$$

$$\nabla^2\tilde{h}_{00} + 3a^2\left(\frac{\dot{a}}{a}\right)^{\cdot}\tilde{h}_{00} = a^2\kappa(\delta\tilde{T}^0_0 - \delta\tilde{T}^n_n). \tag{5.4}$$

Instead of Eq. (5.3) we may, equivalently, consider the equation

$$\nabla^2\tilde{h}^k_0 - 2\left[k - 2a\left(\frac{1}{a}\mathcal{H}\right)'\right]\tilde{h}^k_0 + \frac{1}{3}\nabla^k\mathcal{P} = -2a^2\kappa\delta\tilde{T}^k_0, \tag{5.5}$$

in which $\delta\tilde{T}^k_0$ plays the role of a source. The field equation (3.13) can be written in the form

$$\nabla^2\tilde{h}^l_{Tk} - 2k\tilde{h}^l_{Tk} - \frac{1}{a}(a^3\dot{\tilde{h}}^l_{Tk})^{\cdot} + \mathcal{F}\{\tilde{h}_{00}, \tilde{h}^n_n, \tilde{h}_{0k}, \dot{\tilde{h}}_{0k}\} = -2a^2\kappa\delta\tilde{T}^l_{Tk}, \tag{5.6}$$

where $\mathcal{F}\{\ldots\}$ denotes terms linear in the quantities in the brackets and in their spatial derivatives. If $\mathcal{F}$ and $\delta\tilde{T}^l_{Tk}$ are known, the last equation is a wave-type equation for $\tilde{h}^l_{Tk}$.

It is remarkable that in this gauge none of Eqs. (5.2), (5.3), (5.4), and (5.5) contain any time derivative. All four of these equations are elliptic equations for $\tilde{h}^n_n$, $\tilde{h}_{k0}$, and $\tilde{h}_{00}$, when the right-hand sides are given. The first two are standard constraint equations, and the third became an elliptic equation for $\tilde{h}_{00}$ as a consequence of the gauge conditions. Another remarkable feature of (5.2), (5.3), (5.4), and (5.5) is that, with $\delta\tilde{T}^0_0$, $\delta\tilde{T}^0_k$, $\delta\tilde{T}^n_n$, $\delta\tilde{T}^k_0$ given, they represent a completely separated system of four equations for, subsequently, $\tilde{h}^n_n$, $\tilde{h}_{k0}$, and $\tilde{h}_{00}$. $\mathcal{P} = \nabla_l\tilde{h}^l_0$ in Eq. (5.3) is governed by a separate equation. Applying ∇^k on Eq. (3.11) and commuting derivatives, we obtain

$$\nabla^2\mathcal{P} + 3k\mathcal{P} + \nabla^2\mathcal{K} = \tfrac{3}{2}a^2\kappa\nabla^k\delta\tilde{T}^0_k, \tag{5.7}$$

which, in gauges for which $\mathcal{K} = 0$, turns into

$$\nabla^2\mathcal{P} + 3k\mathcal{P} = \tfrac{3}{2}a^2\kappa\nabla^k\delta\tilde{T}^0_k. \tag{5.8}$$

This has exactly the same form as (5.2) for $\tilde{h}^n_n$. With $\delta\tilde{T}^0_k$ given, we can solve (5.8) for $\mathcal{P}$ and substitute into Eq. (5.3), which can then be written as

$$\nabla^2\tilde{h}_{k0} + 2k\tilde{h}_{k0} = 2a^2\kappa\delta\tilde{T}^0_k - \tfrac{1}{3}\nabla_k\mathcal{P}, \tag{5.9}$$

where the "source" term on the r.h.s. is known. Considering, alternatively, Eq. (5.5), we get

$$\nabla^2\mathcal{P} + 3a\left(\frac{1}{a}\mathcal{H}\right)'\mathcal{P} = -\frac{3}{2}a^2\kappa\nabla_k\delta\tilde{T}^k_0, \tag{5.10}$$

$$\nabla^2\tilde{h}^k_0 - 2\left[k - 2a\left(\frac{1}{a}\mathcal{H}\right)'\right]\tilde{h}^k_0 = -2a^2\kappa\delta\tilde{T}^k_0 - \frac{1}{3}\nabla_k\mathcal{P}. \tag{5.11}$$

B. Global gauge conditions and integral-constraint vectors for spherical universes

Let us now consider the integral gauge conditions (4.57). We wish to elucidate their relation to Traschen's constraint vectors [37,38]. An integral-constraint vector V^μ is defined by the relation

$$\int_{\mathcal{D}} \delta T^{\alpha}_{\mu} V^{\mu} n_{\alpha} d^{(3)}V = \int_{\partial \mathcal{D}} d\Sigma_l B^l, \qquad (5.12)$$

in which $\mathcal{D}$ is (possibly a part of) a spacelike hypersurface, n^{α} its normal, $\partial \mathcal{D}$ its 2-dimensional boundary; B^l depends on $h_{\mu\nu}$ and its derivatives and it vanishes if these are zero on $\partial \mathcal{D}$; V^{μ} is gauge independent. Since V^{μ} does not depend on δT^{α}_{μ}, Eq. (5.12) represents simple constraints on source perturbations.

There exist 10 integral-constraint vectors in each of the FRW universes but 6 of them are just spatial Killing vectors. The other 4 are more interesting—Traschen and others considered their implications for microwave background anisotropies (see, e.g., [37,39]). The time components of the 4 Traschen vectors are proportional to the scalars Q^B [Eqs. (4.26) and (4.27)], the spatial parts—to $\nabla^i Q^B$. In a closed spherical universe

$$V^{\mu}_{(B)} = (Q^B, a^{-1}\dot{a}\nabla^i Q^B). \qquad (5.13)$$

Applying (5.12) to the whole closed universe, it takes the form

$$\int_{S^3} \left[Q^B \delta T^0_0 + \frac{\dot{a}}{a} \nabla^i Q^B \delta T^0_i \right] d^{(3)}V = 0. \qquad (5.14)$$

Integrating by parts in the second term, we obtain

$$\int_{S^3} \left[Q^B \delta T^0_0 - \frac{\dot{a}}{a} Q^B \nabla^i \delta T^0_i \right] d^{(3)}V = 0. \qquad (5.15)$$

In order to deduce the simplest constraints on the matter perturbations, Traschen *et al.* [37–39] consider the synchronous gauge and, in addition, restrict "physics" in assuming vanishing pressure so that the synchronous coordinates can be chosen to be comoving with the fluid (in fact dust) since the flow is irrotational. Then the fluid velocity $V_k = \bar{g}_{kl} V^l = \bar{g}_{kl} \delta U^l = 0$ and $\delta T^0_i = (\bar{\rho} + \bar{p}) \times (h_{i0} + V_i) = 0$ in the synchronous gauge. Since the second integral in (5.14) vanishes in this case, the constraints imply just [see, e.g., (14) in [39]]

$$\int_{S^3} Q^B \delta\rho d^{(3)}V = 0. \qquad (5.16)$$

In the Mach 1 gauge the constraints (5.14) and (5.15) have clear, simple consequences without any necessity to restrict physics. Since both the constraint equations (5.2) and (5.8) have on the l.h.s. the operator $\nabla^2 + 3$ which has eigenfunctions Q^B with zero eigenvalues (see Appendix C), it is evident that the sources on the r.h.s. must be orthogonal to the 4-dimensional function space spanned by 4 harmonics Q^B. Therefore, in closed universes the perturbations δT^{μ}_{ν} of any type of matter have to satisfy separately the constraints

$$\int_{S^3} Q^B \delta T^0_0 d^{(3)}V = 0 \qquad (5.17)$$

and

$$\int_{S^3} Q^B \nabla^i \delta T^0_i d^{(3)}V = 0, \qquad (5.18)$$

the last being equivalent to

$$\int_{S^3} \nabla^i Q^B \delta T^0_i d^{(3)}V = 0. \qquad (5.19)$$

The same is true for $\delta\tilde{T}^0_i = a^{-1}\delta T^0_i$ and $\delta\tilde{T}^0_0 = \delta T^0_0$. Hence, Traschen's constraints (5.12), resp. (5.14), are indeed satisfied—in such a way that, in fact, both integrals in the constraints have to vanish separately. The constraints now become a straightforward consequence of the Einstein equations. This is not the case in the synchronous gauge where the constraint equations are coupled and there are more complicated equations for $\tilde{h}^n_n$, $\mathcal{T}_k = \nabla_l \tilde{h}^l_{Tk}$ and their derivatives, as can be seen from Eqs. (3.9) and (3.10) with $\tilde{h}_{00} = \tilde{h}_{k0} = 0$.

The constraints (5.17) and (5.18) and the constraint field equations (5.2) and (5.8) also demonstrate, why our global gauge conditions (4.57) do not restrict physics. They just eliminate solutions of the homogeneous equations which, in any case, can be removed by gauge transformations generated by conformal Killing vectors. The gauge condition (5.1) implies that the same integral gauge constraint, satisfied for $\tilde{h}^n_n$ and $\mathcal{P}$, is valid also for the spatial scalar $\tilde{h}_{00}$, as is also required in (4.57). As a consequence, from the field equation (5.4) another constraint, which has not been discussed by Traschen *et al.*, follows:

$$\int_{S_3} Q^B \delta T^n_n d^{(3)}V = 0, \qquad (5.20)$$

the same for $\delta\tilde{T}^n_n (= \delta T^n_n)$. Hence, in the Mach 1 gauge the whole picture of the Traschen-type constraints and our global integral gauge conditions is nicely symmetrical: all scalar perturbations in both the metric and the energy-momentum tensor, $\tilde{h}^n_n$, $\tilde{h}^0_0$, $\mathcal{P} = \nabla_k \tilde{h}^k_0$, $\delta\tilde{T}^n_n$, $\delta\tilde{T}^0_0$, and $\nabla^k \delta\tilde{T}^0_k$, are orthogonal to the 4-dimensional space spanned by harmonics Q^B.

C. Solutions of the field equations and local inertial frames

We are interested in solutions for $\tilde{h}_{00}$, $\tilde{h}_{0i}$, and $\tilde{h}^n_n$ when the sources are given in terms of $\delta\tilde{T}^{\mu}_{\nu}$. These quantities determine local inertial frames. Hence, we wish to solve elliptic equations (5.2), (5.3), and (5.4). Solutions can be given in terms of harmonics but these will not be considered here. However, several of these equations have been solved in the literature in terms of the Green's functions. The Green's functions for the equation

$$\nabla^2 \Phi(x^i) + 3k\Phi(x^i) = -2P(x^i), \qquad (5.21)$$

where $x^i = \{\chi, \theta, \varphi\}$ are the hyperspherical coordinates, are [39]

$$G_{S^3}(x,x') = -(1/4\pi)\left[\frac{\cos 2\psi}{\sin\psi}\left(1-\frac{\psi}{\pi}\right) - \frac{1}{2\pi}\cos\psi\right], \tag{5.22}$$

$$G_{E^3}(x,x') = -(1/4\pi)\frac{1}{\tilde{l}}, \tag{5.23}$$

$$G_{H^3}(x,x') = -(1/4\pi)\left[\frac{\cosh 2\alpha}{\sinh\alpha} - 2\cosh\alpha\right], \tag{5.24}$$

where $\cos\psi = \cos\chi\cos\chi' + \sin\chi\sin\chi'\cos\gamma$ $(k=+1)$, $\tilde{l} = l^2 + l'^2 - 2ll'\cos\gamma$ $(k=0)$, $\cosh\alpha = \cosh\chi\cosh\chi' - \sinh\chi\sinh\chi'\cos\gamma$ $(k=-1)$, and $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$. Here α is the geodesic distance under the metric f_{ij} between the "source point" $x'^i = \{\chi', \theta', \varphi'\}$ and the "field point" $x^i = \{\chi, \theta, \varphi\}$. The Green's functions satisfy the equations

$$(\nabla^2 + 3K)G(x,x') = f^{-1/2}(x)\delta(x,x'), \tag{5.25}$$

where ∇^2 refers to the point x^i, $\delta(x,x')$ is the Dirac distribution, and $f = \det f_{ij}$. In terms of the Green's functions (5.22), (5.23), and (5.24) the solution to Eq. (5.21) is given by

$$\Phi(x) = -2\int G(x,x')P(x')d^{(3)}V'. \tag{5.26}$$

Hence, given $\delta\tilde{T}^0_0$ and $\nabla^k\delta\tilde{T}^0_k$ we can determine $\tilde{h}^n_n$ and $\mathcal{P}$ from Eqs. (5.2) and (5.8). $\tilde{h}_{00}$ can be determined from the gauge condition (5.1), or by solving (5.4) if $\delta\tilde{T}^n_n$ is known. Notice, however, that to determine $\tilde{h}_{00}$ from (5.1) we need to know also the time derivative $\dot{\tilde{h}}^n_n$. This can be found by taking the derivative of Eq. (5.2) and assuming that $\delta\dot{\tilde{T}}^0_0$ is known. [$\delta\dot{\tilde{T}}^0_0$ can be expressed from the Bianchi identity (3.22) in terms of $\delta\tilde{T}^\mu_\nu$ and $\tilde{h}_{00}$.] The solutions for $\dot{\tilde{h}}^n_n$ can then be constructed.

Knowing $\mathcal{P}$ from Eq. (5.8), we can determine $\tilde{h}_{k0}$ from Eq. (5.9) or (5.11). In general, we need to find a Green's function bitensor $G_a{}^{b'}(x,x')$ satisfying

$$f^{lm}\nabla_l\nabla_m G_a{}^{b'}(x,x') + 2kG_a{}^{b'}(x,x') = f^{-1/2}(x)\delta_a^{b'}\delta(x,x') \tag{5.27}$$

in the case of Eq. (5.9) [analogously for Eq. (5.11)]. Then the solution for $\tilde{h}_{0k}$ can be written as

$$\tilde{h}_{0k} = \int G_k{}^{b'}(x,x')\mathcal{S}^{(A)}_{b'}(x')d^{(3)}V'; \tag{5.28}$$

by the source $\mathcal{S}_b{}^{(A)}$, $A = I, II$, the r.h.s. of Eq. (5.9), respectively (5.11), is denoted.

In a spatially flat universe, the easiest way is to write Eqs. (5.9) and (5.11) in Cartesian coordinates. Then (5.9) decouples into three Poisson equations for each $\tilde{h}_{k0}$, the Green's functions are standard, and the solutions are given by Poisson integrals over the source:

$$\tilde{h}_{0k}(x^i,t) = \int \frac{\mathcal{S}^{(I)}_k(x')}{|x-x'|}d^3x', \qquad \mathcal{S}^{(I)}_k = 2a^2\kappa\delta\tilde{T}^0_k - \frac{1}{3}\partial_k\mathcal{P}. \tag{5.29}$$

Equation (5.11) turns into three equations of the Yukawa-type, as noticed recently by Schmid [58]. Indeed, the l.h.s. of Eq. (5.11) is of the form $\nabla^2\tilde{h}^k_0 - \lambda^2(\eta)\tilde{h}^k_0$, so two Green's functions are given by

$$G(x,x') = -\frac{1}{4\pi}\frac{e^{\mp\lambda|x-x'|}}{|x-x'|}, \tag{5.30}$$

where

$$\lambda^2(\eta) = -4a(\mathcal{H}/a)', \qquad \lambda^2(t) = -4a^2\dot{H}. \tag{5.31}$$

Usually $\dot{H} < 0$, so λ is real. The well-behaved solution of Eq. (5.11) is thus

$$\tilde{h}^k_0 = -\frac{1}{2\pi}\int \mathcal{S}^{(II)}_k(x')\frac{e^{-\lambda|x-x'|}}{|x-x'|}d^3x', \qquad \mathcal{S}^{(II)}_k = -2a^2\kappa\delta\tilde{T}^k_0 + \frac{1}{3}\partial_k\mathcal{P}. \tag{5.32}$$

For open universes, the properties of the Green's bitensor $G_a{}^{b'}$ solving Eq. (5.27) with $k=-1$ have been studied by d'Eath [59]. In particular, it can be shown that such $G_a{}^{b'}$ exists which satisfies the boundary conditions at the source points and decays as $\exp[-3d(x,x')]$ as $d(x,x') \to \infty$, with $d(x,x')$ being the geodesic distance between the points x, x' under the metric of an open universe. In hyperspherical coordinates, $d = \alpha$, where α is given below Eq. (5.24). In fact, it was d'Eath [59] who found the explicit form of the (scalar) Green's function (5.24), but the explicit form of the Green's bitensor for solving the equations for vector perturbations for $k=-1$ does not seem to be known. The same is the case with spherical universes where only the Green's function (5.22) for the scalar equation (5.21) with $k=+1$ is known. Nevertheless, we can find explicit solutions for quite general classes of the vector perturbations also in the case of $k=\pm1$.

1. Axisymmetric rotational perturbations

Recently we solved Eqs. (5.9) and (5.11) for all odd-parity vector perturbations, i.e. those, for example, corresponding to rotational perturbations with axial symmetry [40,41]. We decomposed perturbations in coordinates θ, φ on spheres only and assumed axial symmetry (spherical functions Y_{lm} having $m=0$). Since the backgrounds admit homogeneous, isotropic foliations, nonsymmetric perturbations can be found from the axisymmetric ones [41]. Thus, we write in spherical coordinates of Eq. (1.2)

$$\tilde{h}_{0\varphi} = \sum_{l=1}^{\infty}[\tilde{h}_{0\varphi}(\eta,r)]_l \sin\theta Y_{l0,\theta}, \tag{5.33}$$

$$\delta\tilde{T}^0_\varphi = \sum_{l=1}^{\infty}[\delta\tilde{T}^0_\varphi(\eta, r)]_l \sin\theta Y_{l0,\theta}, \tag{5.34}$$

where $Y_{l0,\theta} = \partial_\theta Y_{l0}$, and δT^μ_ν may represent any perturbation. In the case of perfect fluid, the fluid angular velocity [cf. Eq. (3.16)] is $\tilde{\Omega} = \tilde{V}^\varphi = d\varphi/d\eta = \Omega/a$, and we write

$$\tilde{V}_\varphi = -a^2r^2\sum_{l=1}^{\infty}\tilde{\Omega}_l(t, r)\sin\theta Y_{l0,\theta}. \tag{5.35}$$

Putting

$$[\tilde{h}_{0\varphi}]_l = a^2r^2\sin^2\theta\tilde{\omega}_l(t, r), \tag{5.36}$$

we have

$$[\delta\tilde{T}^0_\varphi]_l = a^2(\bar{\rho} + \bar{p})r^2\sin^2\theta(\tilde{\omega}_l - \tilde{\Omega}_l). \tag{5.37}$$

These perturbations are transverse: $\nabla_k\tilde{h}^k_0 = 0 = \nabla_k\tilde{T}^k_0$, $\mathcal{P} = 0$. Equations (5.9) and (5.11) become

$$\nabla^2\tilde{h}_{0\varphi} + 2k\tilde{h}_{0\varphi} = 2a^2\kappa\delta\tilde{T}^0_\varphi, \tag{5.38}$$

respectively,

$$\nabla^2\tilde{h}^\varphi_0 - 2[k - 2a(\mathcal{H}/a)']\tilde{h}^\varphi_0 = -2a^2\kappa\delta\tilde{T}^\varphi_0. \tag{5.39}$$

The relation (3.15) now implies

$$\delta\tilde{T}^0_\varphi = -r^2\sin^2\theta\left[\delta\tilde{T}^\varphi_0 - \frac{2}{\kappa a^2}\left[-k + a\left(\frac{\mathcal{H}}{a}\right)'\right]\tilde{h}^\varphi_0\right], \tag{5.40}$$

so Eq. (5.39) immediately follows from Eq. (5.38) and vice versa. Nevertheless, the equations differ in the sense that in Eq. (5.38) $\delta\tilde{T}^0_\varphi$ is considered as a source, whereas in (5.39) the source is given by $\delta\tilde{T}^\varphi_0$. $\delta\tilde{T}^0_\varphi$ determines (up to factor a^4) the density of the angular momentum—the perturbed Bianchi identities (3.23) imply the conservation law

$$(a^4\delta\tilde{T}^0_\varphi)^{\cdot} = 0. \tag{5.41}$$

On the other hand, $\delta\tilde{T}^\varphi_0$ determines the energy current. This is most apparent in the case of perfect fluid: $\delta\tilde{T}^0_\varphi$ is given by Eqs. (5.34) and (5.37), while

$$\begin{aligned}\delta\tilde{T}^\varphi_0 &= (\bar{\rho} + \bar{p})\tilde{V}^\varphi = (\bar{\rho} + \bar{p})\tilde{\Omega}\\ &= \sum_{l=1}^{\infty}[\delta\tilde{T}^\varphi_0]_l(\sin\theta)^{-1}Y_{l0,\theta},\end{aligned} \tag{5.42}$$

where $[\delta\tilde{T}^\varphi_0(\eta, r)]_l = (\bar{\rho} + \bar{p})\tilde{\Omega}_l$. Substituting the expansions into Eq. (5.38), and using the orthogonality of $\sin\theta Y_{l0,\theta}$ for different l's, we obtain the "radial" equation for each l:

$$\begin{aligned}&-\sqrt{1 - kr^2}\frac{1}{r^2}\frac{\partial}{\partial r}\left[\sqrt{1 - kr^2}\frac{\partial}{\partial r}(r^2\tilde{\omega}_l)\right] + \frac{l(l + 1)}{r^2}\tilde{\omega}_l\\ &\quad - 4k\tilde{\omega}_l = 2a^2\kappa(\bar{\rho} + \bar{p})(\tilde{\Omega}_l - \tilde{\omega}_l) \equiv \lambda^2(\tilde{\Omega}_l - \tilde{\omega}_l).\end{aligned} \tag{5.43}$$

For $l = 1$ the perturbations correspond to the "rigidly rotating spherical shells" in the FRW universes [12,40]. Each sphere rotates with no shear but it expands/contracts with the background so that its angular velocity changes. For $l \geq 2$ the motion of the fluid is "toroidal" [41]. In the case of closed universes the Legendre equation which follows from Eq. (5.43) requires a special treatment. For example, for $k = +1$, functions $\tilde{\omega}_l(t, r)$ determining $\tilde{h}_{0\varphi}$ by (5.33) and (5.36) turn out to be ($r = \sin\chi$)

$$\begin{aligned}\tilde{\omega}_l &= 2k(\sin\chi)^{-3/2}\left\{\tilde{\mathcal{P}}^l_2\int_0^\chi\frac{\tilde{\mathcal{Q}}^l_2}{W_l\sin^{1/2}\chi'}(\delta T^0_\varphi)_l d\chi'\right.\\ &\quad\left. + \tilde{\mathcal{Q}}^l_2\int_\chi^\pi\frac{\tilde{\mathcal{P}}^l_2}{W_l\sin^{1/2}\chi'}(\delta T^0_\varphi)_l d\chi'\right\},\end{aligned} \tag{5.44}$$

where W_l is the Wronskian of the functions $\tilde{\mathcal{P}}^l_2(\chi)$, $\tilde{\mathcal{Q}}^l_2(\chi)$ which are derived from the derivatives of the appropriate Legendre functions with respect to their degree [41].

The properties of the solutions of Eq. (5.43) differ significantly according to whether we consider the right-hand side of (5.43), i.e., the angular momentum density $\delta\tilde{T}^0_\varphi$ as the source of $\tilde{\omega}_l$, or we solve (5.43) for $\tilde{\omega}_l$ with $\tilde{\Omega}_l$ given, i.e., with the angular velocity as the source.

The rotation of inertial frames (2.30) is given by the angular velocity

$$-\omega^j = \frac{1}{2a}\left[\sum_{l=1}^{\infty}l(l + 1)\omega_l Y_{l0}, \sum_{l=1}^{\infty}\frac{1}{r^2}\frac{d}{dr}(r^2\omega_l)Y_{l0,\theta}, 0\right]. \tag{5.45}$$

The complete solutions for ω_l for both $\delta\tilde{T}^0_\varphi$ and $\delta\tilde{T}^\varphi_0$ given are determined in [41]. Roughly speaking, in flat and open universes the effects of toroidal motions beyond the cosmological horizons are exponentially damped when the angular velocity of matter is given. For flat universes, this was first noticed by Schmid [58]. We shall see it occurs also for accelerations. However, these dragging effects are not damped when angular momenta are given as sources. In [40] we give the physical explanation. Since Eqs. (5.9) and (5.11) are elliptic equations, in both cases the inertial influences of "distant matter" are expressed instantaneously.

We found toroidal perturbations to cause the rotation of local inertial frames by the angular velocity (5.44). Do they cause their acceleration? Since now only $\tilde{h}_{0\varphi} \neq 0$ among all $\tilde{h}_{\mu\nu}$, the only nonvanishing component of the acceleration (2.20) is

$$\alpha^{\varphi} = \frac{1}{r^2 \sin^2\theta}(a\tilde{h}_{0\varphi})^{\cdot}. \tag{5.46}$$

Substituting for $\tilde{h}_{0\varphi}$ from Eqs. (5.33) and (5.36), and expressing the acceleration in the "background" frame, we find

$$\alpha_{(\varphi)} = -r\sum_{l=1}^{\infty}\frac{1}{a}(a^2\omega_l)^{\cdot}Y_{l0,\theta}. \tag{5.47}$$

The acceleration vanishes at the axis of rotation. More interestingly, it vanishes everywhere in a static (Einstein) universe if the matter rotates uniformly. Indeed, the angular momentum density conservation law requires $[a^5(\rho + p)(\omega - \Omega)]^{\cdot} = 0$, which implies $\dot{\omega} = 0$ and hence $\alpha_{(\varphi)} = 0$ for time-independent Ω and $a =$ constant. In the FRW universes the acceleration (5.47) is nonvanishing, thus resembling the acceleration of the local inertial frames with respect to the static frames inside a collapsing, slowly rotating shell where particles at rest with respect to infinity experience the Euler acceleration, although the spacetime inside the shell is flat [32]. For all three types of FRW universes the solutions are of the form $\omega_l = g_l(r)a^{-3}(t)$, where $g_l(r)$ are explicitly given in terms of the integrals of the special functions mentioned above and the sources $[\delta T^0_{\varphi}]_l$. Hence, the accelerations (5.47) are of the form

$$\alpha_{(\varphi)} = -(H/a^2)r\sum_{l=1}^{\infty}g_l(r)Y_{l0,\theta}. \tag{5.48}$$

As an illustration, for $k = 0$ and $l = 1$ perturbation we get

$$\alpha_{(\varphi)} = 2(H/a^2)r\sin\theta\left(\frac{J(<r)}{r^3} + \int_r^{\infty}\frac{dJ}{dr'}r'^{-3}dr'\right), \tag{5.49}$$

where $J(<r)$ is the angular momentum inside r. With angular velocity Ω considered as a source, ω shows the exponential decline near the origin when the source is beyond the horizon [41]. As a consequence of Eq. (5.47) the acceleration behaves similarly.

2. Perturbations of potential type

In the example of toroidal perturbations, we had $\delta\tilde{T}^0_0 = \nabla_k\tilde{V}^k = \nabla_k\delta\tilde{T}^k_0 = \nabla_k\tilde{h}^k_0 = \tilde{h}_{00} = 0$. As a second example, consider briefly the case in which these quantities may be nonvanishing but $\tilde{V}_k$ and $\tilde{h}_{k0}$ have a vanishing transverse part so that $\tilde{h}_{k0} = \nabla_k h$ for some scalar h, and similarly for $\tilde{V}_k = \nabla_k w$. Physically, such perturbations describe a change in the matter density and a curl-free velocity field. No rotation of local inertial frames arises for such perturbations—the vorticity vector (2.29) vanishes for these "scalar perturbations."

In order to determine the acceleration, we can use the gauge condition (5.1). It enables us to find $\tilde{h}_{00}$ in terms of $\dot{\tilde{h}}^n_n$ and $\mathcal{P}$. Both $\tilde{h}^n_n$ and $\dot{\tilde{h}}^n_n$ can be determined from (5.2) and its time derivative. With $\delta\tilde{T}^0_0$ given, the Green's functions (5.22), (5.23), and (5.24) yield $\dot{\tilde{h}}^n_n$. In the simplest case of a flat universe,

$$\dot{\tilde{h}}^n_n = \kappa\int\frac{(a^2\delta\tilde{T}^0_0)^{\cdot}}{|x - x'|}d^3x'. \tag{5.50}$$

The scalar $\mathcal{P}$ can be obtained by solving either Eq. (5.8) or Eq. (5.10). As with toroidal perturbations, when the angular momentum $\delta\tilde{T}^0_k$ is prescribed, $\mathcal{P} = \nabla_k\tilde{h}^k_0$ will not be suppressed at the origin if $\delta\tilde{T}^0_k$ occurs beyond a horizon. A suppression takes place if the velocity, or the energy current $\delta\tilde{T}^k_0$, is prescribed, as it corresponds to solving Eq. (5.10). In the flat universe, for example, Eq. (5.10) reads

$$\nabla^2\mathcal{P} + 3a^2\left(\frac{\dot{a}}{a}\right)^{\cdot}\mathcal{P} = -\frac{3}{2}a^2\kappa\nabla_l\delta\tilde{T}^l_0, \tag{5.51}$$

which is a Yukawa-type equation with the solution $[\lambda^2(t) = -3a^2(\dot{a}/a)^{\cdot}]$

$$\mathcal{P} = -\frac{3a^2\kappa}{8\pi}\int(\nabla_l\delta\tilde{T}^l_0)(x')\frac{e^{-\lambda|x-x'|}}{|x - x'|}. \tag{5.52}$$

If we start from Eq. (5.8) with $\delta\tilde{T}^0_k$ given as a source, the solutions can be written in terms of the Green's functions (5.22), (5.23), and (5.24). Taking $\partial/\partial t$ of (5.8) and assuming $\delta\dot{\tilde{T}}^0_k$ given, the same Green's functions will yield $\dot{\mathcal{P}}$. Since $\tilde{h}_{0k} = \nabla_k h$, $\mathcal{P} = \nabla^2 h$, $\dot{\mathcal{P}} = \nabla^2\dot{h}$, we can find $\tilde{h}_{k0}$ and $\dot{\tilde{h}}_{k0}$ by solving Laplace equations for h and $\dot{h}$ with $\mathcal{P}$ and $\dot{\mathcal{P}}$ given. The solutions are unique up to an additive function of time which does not contribute to $\tilde{h}_{0k}$. Knowing $\dot{\tilde{h}}_{k0}$ and $\tilde{h}_{00} = \frac{2}{3}\dot{a}^{-1}(\mathcal{P} - \frac{1}{2}a\dot{\tilde{h}}^n_n)$ from the gauge condition, the acceleration (2.20) of the local inertial frames can be determined in terms of the sources.

Alternatively, we can start from Eq. (5.4) to determine directly $\tilde{h}_{00}$ in terms of the matter perturbations $\delta\tilde{T}^0_0$ and $\delta\tilde{T}^n_n$. Equation (5.4) for $k = 0$ becomes just a Yukawa-type equation exactly in the form of Eq. (5.52). $\tilde{h}_{00}$ exhibits an exponential suppression near the origin if the source $\delta\tilde{T}^0_0$, $\delta\tilde{T}^n_n$ occurs beyond the cosmological horizon. The suppression enters the formula (2.20) for the acceleration. However, the term $\dot{\tilde{h}}_{0k}$ will not be suppressed if the angular momentum $\delta\tilde{T}^0_k$ is prescribed. If the energy current $\delta\tilde{T}^k_0$ is considered as a source of $\dot{\tilde{h}}_{0k}$, the total acceleration will be exponentially suppressed. In both cases, however, the acceleration is determined instantaneously.

D. The determination of local inertial frames

We described how the accelerations and rotations of the local inertial frames can be determined explicitly in these specific examples in order to illustrate the general framework. Finally, let us discuss, within the Mach 1 gauge, a question of *the uniqueness of the solutions* of the field equations for general perturbations and of the resulting expressions for the acceleration and rotation of local inertial frames. The homogeneous equation corresponding to Eq. (5.3) for $\tilde{h}_{k0}$ is identical to Eq. (4.23), the well-behaved solutions of which were analyzed in detail between Eqs. (4.23) and (4.33). They do not exist in H^3; in E^3 they describe the time-dependent linear combination of translations, which can be eliminated by requiring $\tilde{h}_{k0}$ to decay at infinity. In S^3, they correspond to the time-dependent linear combinations of 10 conformal Killing vectors. However, by imposing our integral gauge conditions (4.57), we dispose of the four conformal Killing vectors which are not Killing.

Therefore, the complete general solution of Eq. (5.3) has for $k = 1$ the form

$$\tilde{h}_{0k} = \tilde{h}_{0k(\mathrm{inh})} + \sum_{A=1}^{6} f_A(t)\xi_k^{(A)}, \tag{5.53}$$

where $\tilde{h}_{0k(\mathrm{inh})}$ is a solution of the inhomogeneous equation (5.3) and $f_A(t)$ are arbitrary functions of time; $\xi_k^{(A)}$ are 6 Killing vectors of S^3 describing rotations and quasitranslations (see Appendix C). Owing to our integral gauge conditions, Eq. (5.2) admits a unique solution; $\mathcal{P} = \nabla^k \tilde{h}_{0k}$ is not affected by the Killing vectors in (5.53), and, hence, the Mach 1 gauge condition (5.1) determines a unique $\tilde{h}_{00}$. If we start from Eq. (5.4) to determine $\tilde{h}_{00}$ directly in terms of $\delta\tilde{T}_0^0$ and $\delta\tilde{T}_n^n$, we also arrive at a unique solution because the homogeneous equation corresponding to Eq. (5.4) coincides precisely with Eq. (4.38). Therefore, with $\delta\tilde{T}_0^0$, $\delta\tilde{T}_n^n$, and angular momenta $\delta\tilde{T}_k^0$ given, the accelerations and rotations of local inertial frames in closed universes are determined by formulas (2.20) and (2.31) only up to the freedom exhibited in Eq. (5.53). This freedom corresponds precisely to changing the coordinate system by the infinitesimal transformation in which $\zeta^0 = 0$, $\zeta^i = \sum_{A=1}^6 F_A(t)\xi^{(A)i}$. Then $\triangle\tilde{h}_{00} = \triangle\tilde{h}_n^n = \triangle\tilde{h}_{kl} = 0$ because ζ^i is a linear combination of the Killing vectors. The six spacelike Killing vectors generate motions which preserve the symmetries of the space. However,

$$\triangle\,\tilde{h}_{0k} = -a\dot{\zeta}_k = -a\sum_{A=1}^{6} \dot{F}_A(t)\xi_k^{(A)}, \tag{5.54}$$

which is equal to the additional term in Eq. (5.53) provided that $f_A(t) = -a(t)\dot{F}_A(t)$. The transformations

$$x'^i = x^i + \sum_{A=1}^{6} F_A(t)\xi^{(A)i} \tag{5.55}$$

with arbitrary coefficients $F_A(t)$ lead to mutually accelerated frames. The accelerations have special forms when regarded as functions in space. Putting $x^i =$ constant in (5.55) we get $d^2x'^i/dt^2 = \sum_{A=1}^6 \ddot{F}_A(t)\xi^{(A)i}$ where $\xi^{(A)i}$ are specific functions of x^i (cf. Appendix C). An example of accelerations and rotations generated by this type of transformation with one translational and one rotational Killing vector is given in Eqs. (4.53) and (4.54) above.

Consider now for simplicity just one rotational Killing vector and again the case of toroidal perturbations. The arbitrariness exhibited by the additional terms in Eq. (5.53) can then be seen distinctly. The homogeneous equation corresponding to Eq. (5.43) for $l = 1$ with the angular momentum given as a source is solved by $\tilde{\omega}_{(0)} = \tilde{\omega}_{(0)}(t)$, where $\tilde{\omega}_{(0)}$ is an arbitrary function of t. This implies [cf. Eq. (5.36)]

$$\tilde{h}_{0\varphi} = \tilde{\omega}_{(0)}a^2\sin^2\chi\sin^2\theta = \tilde{\omega}_{(0)}\xi_\varphi^{(\varphi)}, \tag{5.56}$$

where $\xi^{(\varphi)i} = (0, 0, 0, 1)$ is the rotational Killing vector, which is a special case of Eq. (5.54). The transformation $\varphi' = \varphi - \int\tilde{\omega}_{(0)}(t)dt$ [a special case of (5.55)] would make the term (5.56) vanish. Since, however, in closed universes such an arbitrary "integration constant" $\tilde{\omega}_{(0)}(t)$ cannot be eliminated by boundary conditions, all frames with different $\tilde{\omega}_{(0)}(t)$ are admitted. In this sense, only *relative* rotations of the local inertial frames can be determined if the angular momentum is considered as a source of their dragging.

In the case of *general* perturbations of the spherical universes with the distributions of energy $\delta\tilde{T}_0^0$ and angular momenta $\delta\tilde{T}_k^0$ given, the freedom described by $F_A(t)$ in Eq. (5.55) which preserves the symmetries of the space cannot be eliminated by boundary conditions. In this sense, only *relative rotations and accelerations of the local inertial frames can be determined.* The perturbations $\tilde{h}_{00}$ and $\tilde{h}_{0i}$, which imply these rotations and accelerations, are determined by the field equations and the Mach 1 gauge conditions *instantaneously* from appropriate averages over the distributions of $\delta\tilde{T}_0^0$ and $\delta\tilde{T}_k^0$. An explicit example is the expression (5.26) with the source term $P \sim \delta\tilde{T}_0^0$ and the Green's function given in S^3 by Eq. (5.22) determining $\tilde{h}_n^n$ and thus $\tilde{h}_{00}$. Other examples are the functions $\tilde{\omega}_l(t, r)$ in Eq. (5.44).

This instantaneous determination of the local inertial frames by such averages, up to global rotations and accelerations given by the symmetries of the space, is the crucial feature exhibiting the validity of Mach's principle in relativistic cosmology, at least in the first-order perturbation theory. The ability to describe the same physical situation using these differently rotating and accelerating frames is a consequence of the dynamics having a higher degree of symmetry than the realization of the world in terms of the positions of actual bodies. The frame in which we choose to describe the motions is not of importance; what matters is the relative motions of the bodies, not that of the frame relative to the bodies.

Finally, consider now energy currents $\delta\tilde{T}_0^k$ together with $\delta\tilde{T}_0^0$ as the sources. In case of the perfect-fluid perturbations we thus take the fluid velocity $\tilde{V}^k$ and $\delta\rho$ as the sources. When velocities and accelerations of "heavenly bodies'' are given, the rotations and accelerations of the local inertial frames are determined uniquely in spherical universes. As we have shown below Eq. (4.38), the homogeneous equation corresponding to Eq. (5.10) for $\mathcal{P} = \nabla_k \tilde{h}^k_0$ admits only the trivial solution $\mathcal{P} = 0$. Hence, the inhomogeneous equation (5.10) determines $\mathcal{P}$ uniquely when $\nabla_k \delta\tilde{T}^k_0$ is given. The same is true for (5.4) for $\tilde{h}_{00}$ because the homogeneous equation is the same. To determine perturbation $\tilde{h}_{00}$ we need to know both $\delta\rho$ and δp (resp. $\delta\tilde{T}^n_n$). The gauge condition (5.1) then implies $\dot{\tilde{h}}^n_n$. Alternatively, we can solve (5.2) for $\tilde{h}^n_n$ and $\dot{\tilde{h}}^n_n$ by giving $\delta\rho$ and $\delta\dot{\rho}$, and extract $\tilde{h}_{00}$ from the gauge condition. Solutions for $\tilde{h}^n_n$, $\dot{\tilde{h}}^n_n$ are unique due to our integral gauge conditions, so a unique $\tilde{h}_{00}$ can also be found. Finally, the homogeneous part of (5.11) for $k = 1$ with sources $\delta\tilde{T}^k_0$ and $\mathcal{P}$ given reads

$$\nabla^2 \tilde{h}^k_0 - 2\left[1 - 2a^2\left(\frac{\dot{a}}{a}\right)^{\cdot}\right]\tilde{h}^k_0 = 0. \tag{5.57}$$

This admits only $\tilde{h}^k_0 = 0$: multiply by $\tilde{h}_{k0}$ and integrate by parts over closed space,

$$-\int_D f^{ij} f^{kl} \nabla_i \tilde{h}_{k0} \nabla_j \tilde{h}_{l0} d^{(3)}V = 2\mathcal{A}(t) \int_D f^{kl} \tilde{h}_{k0} \tilde{h}_{l0} d^{(3)}V, \tag{5.58}$$

where $\mathcal{A}(t) = 1 - 2a^2(\dot{a}/a)^{\cdot} = a^2[\frac{1}{2}\kappa(\bar{\rho} + \bar{p}) - \dot{H}]$. Since the integrands on both sides are spatial scalars, we can calculate them at each point by using $f^{ij} = \mathrm{diag}(1, 1, 1)$. In this way we find that both are non-negative so that the only way to satisfy (5.58) for $\mathcal{A}(t) > 0$ is by putting $\tilde{h}_{k0} = 0$. For standard models $\dot{H} < 0$ and indeed $\mathcal{A}(t) > 0$. In the example of toroidal perturbations, the uniqueness of the solutions is reflected by the fact that Eq. (5.43) has unique solutions for given angular velocity $\tilde{\Omega}_l$ of matter. A purely time-dependent $\tilde{\omega}_{(0)l}(t)$ does not solve (5.43) when just $\tilde{\Omega}_l$ is prescribed and not the whole r.h.s. $\tilde{\Omega}_l - \tilde{\omega}_l$.

We thus arrive at another important *aspect of Mach's principle* in relativistic cosmology: *if the velocities, density and pressure perturbations of cosmic fluid are given, the (linearized) field equations in a closed universe provide a unique determination of the rotations and accelerations of the local inertial frames*.

VI. CONCLUDING REMARKS

Although this paper also includes items which have a review character, primarily it contains new developments: the analysis of accelerations and rotations of local inertial frames and of gyroscopes in perturbed FRW universes; the general forms of the perturbed Einstein field equations and Bianchi identities are formulated without gauge conditions, harmonics, or splittings; the motivation for and the analysis of the instantaneous, Machian-based gauges, including the integral gauge conditions and their relation to Traschen's integral constraints; and the manifestation of Mach's ideas in the framework of general linear cosmological perturbations of FRW universes. In particular, those who wish to study cosmological perturbation problems in position space, as advocated recently in Ref. [36], may find here useful relations not given before. Various specific perturbation problems can be attacked by applying the results presented here. We already used the formalism to investigate rotational and toroidal vector perturbations of FRW universes [40,41], as mentioned and applied in Sec. V; there, we also discussed vector perturbations of potential type.

For given distributions of energy-momentum and angular momentum of matter sources, the rotations and accelerations of local inertial frames are uniquely given in the Machian gauges in open universes under suitable boundary conditions, whereas in closed universes they are determined up to motions generated by the Killing vectors, i.e., by symmetries of the background. They are determined uniquely also in closed universes if velocities, density, and pressure perturbations of cosmic fluid are given. As a consequence of the constraint equations and the choice of gauges which imply suitable slicing of perturbed universes, these inertial properties are determined instantaneously. In this sense Mach's principle is embodied in the cosmological linear perturbation theory.

The dragging of inertial frames is an essentially *global effect* which, at least in linear perturbation theory, has to be seen as an instantaneous phenomenon. This was first demonstrated by Lindblom and Brill [60], who investigated rotational dragging by a slowly rotating, massive spherical shell freely falling under its own gravity. We reconsidered the problem and explored its electromagnetic analogue [32]. The need to introduce a suitable coordinate frame (the ''gauge'') to describe the dragging is well illustrated inside the shell. Spacetime is flat there; no local geometrical (gauge invariant) perturbations occur. The time-dependent rotation of inertial frames is exhibited by considering the congruence of static observers, i.e. those who are at rest with respect to static observers at infinity. They play the role analogous to that of the cosmological observers in the present paper. They experience acceleration, and the congruence of their worldlines twists. Both quantities, characterizing their congruence, can be expressed in a covariant manner as in formulas (2.7) and (2.8) for cosmological observers. Also, massive, slowly rotating shells immersed in FRW universes were analyzed [61,62], including their observational consequences on the appearance of sources behind the shells [11,62]. Slowly rotating heavy shells, as well as the perturbations of FRW universes, involve linear perturbations of fully relativistic (nonlinear) backgrounds. In [33] we considered *strong* rectilinear dragging using exact conformastatic solutions of the Einstein-Maxwell equations with charged dust.

A thorough nonlinear study of Mach's ideas within the framework of general relativity lies in the future. Quoting from the same source by which we started (see [1], p. 546), "Much must still be done to spell out the physics behind these equations [the initial-value equations] and to see this physics in action."

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APPENDIX A: PERTURBED FIELD EQUATIONS WITH COSMIC TIME t

We write the perturbed FRW metric in the form (1.1)

$$ds^2 = (\bar g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = dt^2 - a^2(t) f_{kl} dx^k dx^l + h_{\mu\nu} dx^\mu dx^\nu. \tag{A1}$$

The background Christoffel symbols are

$$\bar\Gamma^m_{0l} = H\delta^m_l, \qquad \bar\Gamma^0_{kl} = -H\bar g_{kl}, \qquad \bar\Gamma^m_{kl} = f^{mn}(\partial_{(k} f_{l)n} - \tfrac12 \partial_n f_{kl}); \tag{A2}$$

hereafter the symmetrization brackets ( ) include the factor $\frac12$, as do the antisymmetrization ones [ ]; $H = \dot a/a$ is the Hubble "constant." The nonvanishing components of the background Einstein equations, $\bar G^\nu_\mu = \bar R^\nu_\mu - \frac12 \delta^\nu_\mu \bar R = \kappa \bar T^\nu_\mu + \Lambda\delta^\nu_\mu$, read

$$\bar G^0_0 = 3\left(\frac{k}{a^2} + H^2\right) = \kappa\bar\rho + \Lambda, \qquad \bar G^l_k = \delta^l_k\left(\frac{k}{a^2} + 3H^2 + 2\dot H\right) = -(\kappa\bar p - \Lambda)\delta^l_k, \tag{A3}$$

with $\kappa = 8\pi G/c^4$, $k = 0, \pm1$ denoting the curvature index, and Λ the cosmological constant; the background energy-momentum tensor $\bar T^\nu_\mu$ of perfect fluid is given by (2.32). The indices of $h_{\mu\nu}$ are raised or lowered with $\bar g^{\mu\nu}$ and $\bar g_{\mu\nu}$; thus $h^0_0 = h_{00}$, $h^k_0 = \bar g^{kl} h_{0l} = -\frac{1}{a^2} f^{kl} h_{0l}$, etc. No spatial index is ever displaced with f_{kl} alone. We introduce the covariant derivative,

$$\nabla_k h^m_0 = \partial_k h^m_0 + \bar\Gamma^m_{kl} h^l_0. \tag{A4}$$

The background curvature tensor of spatial sections $t =$ constant is $\bar{\mathcal R}^r{}_{ksl} = k(\delta^r_s f_{kl} - \delta^r_l f_{ks})$ and the Ricci 3-tensor $\bar{\mathcal R}_{kl} = 2kf_{kl}$. Useful identities are ($\nabla_{kl} \equiv \nabla_k\nabla_l$, $\nabla^k = f^{kl}\nabla_l$, $\nabla^2 = f^{kl}\nabla_k\nabla_l$):

$$(\nabla_{kl} - \nabla_{lk})V^l = -2kf_{kl}V^l = \frac{2k}{a^2}\bar g_{kl}V^l = \frac{2k}{a^2}V_k, \tag{A5}$$

$$\nabla_k\nabla^2 V^k = \nabla^2(\nabla_k V^k) + 2k(\nabla_k V^k), \tag{A6}$$

$$(\nabla_{kl} - \nabla_{lk})h^l_0 = 2\frac{k}{a^2}h_{k0}, \tag{A7}$$

$$(\nabla_{kl} - \nabla_{lk})h^n_m = h^r_m \bar{\mathcal R}^n{}_{rkl} - h^n_r \bar{\mathcal R}^r{}_{mkl} = 2k(\delta^n_{[k} h^r_{l]} - h^n_{[k}\delta^r_{l]})f_{rm}. \tag{A8}$$

The perturbed Einstein equations, $\delta G^\nu_\mu = \kappa\delta T^\nu_\mu$, are expressed in terms of h^0_0, h^0_k, and h^l_k. In this "mixed" form Λ does not appear. The left-hand sides, δG^ν_μ, read as follows:

$$\delta G^0_0 = -\frac12\nabla_{rs}(\bar g^{rn}h^s_n - \bar g^{rs}h^n_n) - \frac{k}{a^2}h^n_n - 2H\left(\frac32 Hh^0_0 - \frac12\dot h^n_n + \nabla_n h^n_0\right), \tag{A9}$$

$$\delta G^0_k = \frac12\nabla_l(\dot h^l_k - \delta^l_k\dot h^n_n) + \bar g^{rs}(\nabla_{kr}h^0_s - \nabla_{r(k}h^0_{s)}) + H\nabla_k h^0_0, \tag{A10}$$

$$\delta G^k_0 = \bar g^{kl}\left[\delta G^0_l - 2\left(\frac{k}{a^2} - \dot H\right)h^0_l\right], \tag{A11}$$

$$\begin{aligned}\delta G^m_k = {}& -\frac12(\ddot h^m_k - \delta^m_k\ddot h^n_n) - \frac32 H(\dot h^m_k - \delta^m_k\dot h^n_n) - \frac{k}{a^2}h^m_k \\ &+ \frac12\nabla_{kl}(\bar g^{ln}h^m_n - \bar g^{lm}h^n_n) - \frac12\bar g^{rs}\nabla_{rs}(h^m_k - \delta^m_k h^n_n) \\ &+ \frac12\nabla_{rs}(\bar g^{mr}h^s_k - \delta^m_k h^{rs}) + \bar g^{ml}(\nabla_{(k}\dot h^0_{l)} + H\nabla_{(k}h^0_{l)}) \\ &- \delta^m_k\bar g^{rs}(\nabla_r\dot h^0_s + H\nabla_r h^0_s) - \frac12(\bar g^{ml}\nabla_{lk}h^0_0 \\ &- \delta^m_k\bar g^{rs}\nabla_{rs}h^0_0) - \delta^m_k[H\dot h^0_0 + (2\dot H + 3H^2)h^0_0].\end{aligned} \tag{A12}$$

If δT^ν_μ is a perfect-fluid perturbation then the right-hand side is given by [see Eqs. (2.35)]

$$\begin{aligned}\delta T^0_0 &= \delta\rho, \\ \delta T^0_k &= (\bar\rho + \bar p)(V_k + h^0_k) = \frac{2}{\kappa}\left(\frac{k}{a^2} - \dot H\right)(V_k + h^0_k), \\ \delta T^k_0 &= (\bar\rho + \bar p)V^k = \frac{2}{\kappa}\left(\frac{k}{a^2} - \dot H\right)V^k, \\ \delta T^l_k &= -\delta^l_k\delta p.\end{aligned} \tag{A13}$$

In the last equations we used the relation

$$(\bar{\rho}+\bar{p})=\frac{2}{\kappa}\left(\frac{k}{a^2}-\dot{H}\right), \tag{A14}$$

which follows from the background Einstein equations (A3) for all $\bar{\rho}$, $\bar{p}$, k, Λ.

The relations between various $h_{\mu\nu}$'s and $\tilde{h}_{\mu\nu}$'s used in the main text are

$$\begin{gathered} h_{00}=\tilde{h}_{00}, \qquad h_{0l}=a\tilde{h}_{0l}, \qquad h_{kl}=a^2\tilde{h}_{kl}, \\ h_0^0=\tilde{h}_{00}, \qquad h_0^l=-a^{-1}\tilde{h}_0^l, \qquad h_l^0=a\tilde{h}_{0l}, \\ h_k^l=-\tilde{h}_k^l, \qquad h^{00}=\tilde{h}_{00}, \\ h^{0l}=-a^{-1}\tilde{h}_0^l, \qquad h^{kl}=a^{-2}\tilde{h}^{kl}. \end{gathered} \tag{A15}$$

Equations (A4) and (A6) hold also for $\tilde{h}_0^m$ and $\tilde{V}^k$, but Eqs. (A5) and (A7) take the form

$$(\nabla_{kl}-\nabla_{lk})\tilde{V}^l=-2kf_{kl}\tilde{V}^l=-2k\tilde{V}_k, \tag{A16}$$

$$(\nabla_{kl}-\nabla_{lk})\tilde{h}_0^l=-2kf_{kl}\tilde{h}_0^l=-2k\tilde{h}_{0k}, \tag{A17}$$

and Eq. (A8) becomes

$$(\nabla_{kl}-\nabla_{lk})\tilde{h}_m^n=2k(\delta^n_{[k}\tilde{h}_{l]m}-\tilde{h}^n_{[k}f_{l]m}). \tag{A18}$$

APPENDIX B: GAUGE TRANSFORMATIONS OF PERTURBATIONS

As a consequence of infinitesimal transformations

$$x^0\to x^{0'}=x^0+\zeta^0(x), \tag{B1}$$

$$x^i\to x^{i'}=x^i+\zeta^i(x), \tag{B2}$$

we find the following changes of various metric components under the change of gauge (notice that $\Delta Q\equiv Q-Q'$ for any Q):

$$\Delta h_{00}=2\dot{\zeta}^0=\Delta\tilde{h}_{00}, \tag{B3}$$

$$\Delta h_{0l}=\partial_l\zeta^0-a^2\dot{\zeta}_l=a\Delta\tilde{h}_{0l}, \qquad (\zeta_l=f_{lk}\zeta^k), \tag{B4}$$

$$\Delta h_{kl}=-2a^2\left[\nabla_{(k}\zeta_{l)}+\frac{\dot{a}}{a}f_{kl}\zeta^0\right]=a^2\Delta\tilde{h}_{kl}, \tag{B5}$$

$$\Delta h_n^n=2a^2\left[\nabla_n\zeta^n+3\frac{\dot{a}}{a}\zeta^0\right]=-\Delta\tilde{h}_n^n. \tag{B6}$$

Similarly, the perturbations of the energy-momentum tensor components change under the transformations (B1) and (B2) as follows:

$$\Delta\delta T_0^0=\dot{\bar{\rho}}\zeta^0=\Delta\delta\tilde{T}_0^0, \tag{B7}$$

$$\Delta\delta T_k^0=(\bar{\rho}+\bar{p})\partial_k\zeta^0=a\Delta\delta\tilde{T}_k^0, \tag{B8}$$

$$\Delta\delta T_0^k=-(\bar{\rho}+\bar{p})\dot{\zeta}^k=a^{-1}\Delta\delta\tilde{T}_0^k, \tag{B9}$$

$$\Delta\delta T_k^l=-\dot{\bar{p}}\zeta^0\delta_k^l=\Delta\delta\tilde{T}_k^l, \tag{B10}$$

where we substituted from Eq. (2.32) for the background values of $\bar{T}_\mu^\nu$. In particular, in the fluid case

$$\Delta\delta\rho=\dot{\bar{\rho}}\zeta^0, \qquad \Delta\delta p=\dot{\bar{p}}\zeta^0, \tag{B11}$$

$$\begin{gathered}\Delta\delta U^0=-\dot{\zeta}^0=-\Delta\delta U^0, \qquad \Delta\delta U^m=-\dot{\zeta}^m, \\ \Delta\delta U_m=\partial_m\zeta^0,\end{gathered} \tag{B12}$$

$$\Delta V^m=-\dot{\zeta}^m=a^{-1}\Delta\tilde{V}^m, \qquad \Delta V_m=a^2\dot{\zeta}_m=-a\Delta\tilde{V}_m. \tag{B13}$$

Let us emphasize that the above results for the changes of both $h_{\mu\nu}$'s and $\tilde{h}_{\mu\nu}$'s are expressed in $x^\mu=(t,x^i)$ coordinates. In $\tilde{x}^\mu=(\eta,x^i)$ coordinates we find, for example,

$$\begin{aligned}\tilde{h}'_{00}&=\tilde{h}_{00}-\Delta\tilde{h}_{00}=\tilde{h}_{00}-2\dot{\zeta}^0=\tilde{h}_{00}-2\frac{d(a\tilde{\zeta}^0)}{d\eta}a^{-1}\\ &=\tilde{h}_{00}-2\mathcal{H}\tilde{\zeta}^0-2\frac{d\tilde{\zeta}^0}{d\eta}.\end{aligned} \tag{B14}$$

APPENDIX C: KILLING AND CONFORMAL KILLING VECTORS ON THE FRW BACKGROUNDS

Killing vectors are also conformal Killing vectors but here we call "conformal Killing vectors"—sometimes more explicitly "proper" conformal Killing vectors—those which are not Killing vectors. All these vectors on the FRW backgrounds are well known. Since, however, we did not find all of them listed in a transparent manner in one place, we give them here. Their relation to the scalar and vector harmonics will also be elucidated. The 3-dimensional spatial vectors are frequently used in the main text. There exists extensive literature on the harmonics in S^3 and H^3; see, e.g., [57,63].

1. Killing and conformal Killing 3-vectors in $E^3(k=0)$, $S^3(k=1)$, and $H^3(k=-1)$

The standard Killing equation

$$\nabla_k\xi_i+\nabla_i\xi_k=0 \tag{C1}$$

in the FRW 3-backgrounds with curvature tensor (A8) can be written in an equivalent form,

$$\nabla^2\xi_i+2k\xi_i=0. \tag{C2}$$

The Killing vectors have their simplest form in the coordinates x^m in which the metric is (see e.g. [3], Ch. 13),

$$ds^2=dt^2-a^2(t)\left[\delta_{kl}+\frac{kx^kx^l}{1-kr^2}\right]dx^kdx^l, \tag{C3}$$

where

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2. \quad \text{(C4)}$$

The 3 (quasi)translational Killing vectors are given by

$$\xi^{(J)i} = \sqrt{1 - kr^2}\delta^i_J, \qquad J = 1, 2, 3, \quad \text{(C5)}$$

and the 3 rotational Killing vectors by

$$\xi^{(J)i} = \varepsilon^{iJ}{}_l x^l. \quad \text{(C6)}$$

Here ε is the usual permutation symbol, $\varepsilon_{123} = +1$, with indices moved by δ_{ik}, resp. δ^{ik}.

The 4 conformal Killing vectors are also simple in the x^i coordinates:

$$\psi^i = \sqrt{1 - kr^2}x^i, \quad \text{(C7)}$$

$$\psi^{(J)i} = \delta^{iJ} - kx^i x^J, \qquad k = \pm 1, \quad \text{(C8)}$$

$$\psi^{(J)i} = \tfrac{1}{2}\delta^{iJ} r^2 - x^i x^J, \qquad k = 0. \quad \text{(C9)}$$

The same Killing and conformal Killing vectors in hyperspherical coordinates are more complicated but are directly connected with better known forms of (hyper)-spherical harmonics. Here we denote $r = \sin\chi$ ($k = +1$), $r = \chi$ ($k = 0$), $r = \sinh\chi$ ($k = -1$), and, correspondingly, $r' = \cos\chi$, 1, $\cosh\chi$. As in the main text, we denote the six Killing vectors by $\xi^{(A)i}$, $A = 1, \ldots, 6$, and the four conformal Killing vectors by $\psi^{(A)i}$, $A = 1, \ldots, 4$. The 3 (quasi)-translational Killing vectors read

$$\begin{aligned}\xi^{(1)i} &= (\sin\theta\cos\varphi, r'r^{-1}\cos\theta\cos\varphi, -r'r^{-1}\sin\varphi/\sin\theta),\\ \xi^{(2)i} &= (\sin\theta\sin\varphi, r'r^{-1}\cos\theta\sin\varphi, r'r^{-1}\cos\varphi/\sin\theta),\\ \xi^{(3)i} &= (\cos\theta, -r'r^{-1}\sin\theta, 0). \end{aligned} \quad \text{(C10)}$$

The 3 rotational Killing vectors (C6) turn into the same forms independent of k:

$$\begin{aligned}\xi^{(4)i} &= (0, -\sin\varphi, -\cot\theta\cos\varphi),\\ \xi^{(5)i} &= (0, \cos\varphi, -\cot\theta\sin\varphi),\\ \xi^{(6)i} &= (0, 0, 1).\end{aligned} \quad \text{(C11)}$$

The "dilatation" conformal Killing vector (C7) is, for all k, simply given by

$$\psi^{(1)i} = (r, 0, 0), \quad \text{(C12)}$$

whereas the other three conformal Killing vectors read, for $k = \pm 1$,

$$\begin{aligned}\psi^{(2)i} &= (r'\sin\theta\cos\varphi, r^{-1}\cos\theta\cos\varphi, -r^{-1}\sin\varphi/\sin\theta),\\ \psi^{(3)i} &= (r'\sin\theta\sin\varphi, r^{-1}\cos\theta\sin\varphi, r^{-1}\cos\varphi/\sin\theta),\\ \psi^{(4)i} &= (r'\sin\theta, -r^{-1}\sin\theta, 0),\end{aligned} \quad \text{(C13)}$$

and for $k = 0$,

$$\begin{aligned}\psi^{(2)i} &= \tfrac{1}{2}r^2(-\sin\theta\cos\varphi, r^{-1}\cos\theta\cos\varphi, -r^{-1}\sin\varphi/\sin\theta),\\ \psi^{(3)i} &= \tfrac{1}{2}r^2(-\sin\theta\sin\varphi, r^{-1}\cos\theta\sin\varphi, r^{-1}\cos\varphi/\sin\theta),\\ \psi^{(4)i} &= \tfrac{1}{2}r^2(-\cos\theta, -r^{-1}\sin\theta, 0).\end{aligned} \quad \text{(C14)}$$

2. Scalar harmonics in S^3 and H^3

In S^3, the scalar harmonics Q_{Llm} with $L \ge l \ge 0$, L, l integers, $m = -l, \ldots, +l$, satisfy

$$\nabla^2 Q_{Llm} + L(L + 2)Q_{Llm} = 0. \quad \text{(C15)}$$

In the normalized form they read

$$Q_{Llm} = \sqrt{N_{Ll}}\frac{1}{\sqrt{\sin\chi}}P^{-(l+(1/2))}_{L+(1/2)}(\cos\chi)Y_{lm}(\theta, \varphi), \quad \text{(C16)}$$

where $N_{Ll} = \frac{(L+1)(L+l+1)!}{(L-l)!}$, $P^{-\mu}_{\nu}$ are Legendre functions of the first kind, and Y_{lm} are the usual spherical harmonics. In H^3, the harmonics are $Q_{\lambda lm}$, where $\lambda \ge 0$ is continuous and must be real for square integrability; they satisfy

$$\nabla^2 Q_{\lambda lm} + (\lambda^2 + 1)Q_{\lambda lm} = 0. \quad \text{(C17)}$$

The normalized form is

$$\begin{aligned}Q_{\lambda lm} &= \sqrt{N_{\lambda l}}\frac{1}{\sqrt{\sinh\chi}}P^{-(l+(1/2))}_{-(1/2)+i\lambda}(\cosh\chi)Y_{lm}(\theta, \varphi),\\ N_{\lambda l} &= \lambda^2(\lambda^2 + 1)(\lambda^2 + 2^2)\ldots(\lambda^2 + l^2).\end{aligned} \quad \text{(C18)}$$

It can easily be seen that, for $L = 1$, $l = 0$, $m = 0$ and $L = 1$, $l = 1$, $m = -1, 0, +1$, the expression (C16) leads, up to multiplicative constants, to the four functions given in Eq. (4.26), whereas nonintegrable harmonics (C18) for $\lambda = 2i$, $l = 0$, $m = 0$ and $\lambda = 2i$, $l = 1$, $m = -1, 0, +1$ imply Eq. (4.27). Their gradients (4.25) yield the covariant components of the conformal Killing vectors (C12) and (C13).

Let us remark that for $k = \pm 1$ the translational Killing vectors are not gradients of scalars (as they are for $k = 0$). They are proportional to the vector spherical harmonics with even parity and $L = 1$, $l = 1$ (for $k = \pm 1$) and $\lambda = 2i$, $l = 1$ (for $k = -1$). The rotational Killing vectors are proportional to vector harmonics with odd parity and $L = 1$, $l = 1$, respectively, $\lambda = 2i$, $l = 1$.

APPENDIX D: FIELD EQUATIONS AND SOLUTIONS IN OTHER GAUGES

1. Mach 2 gauge

The gauge conditions $\mathcal{T}_k = \nabla_l \tilde{h}^l_{Tk} = 0$ and $\nabla^2 \tilde{h}^n_n + 3k\tilde{h}^n_n = 0$ [cf. (4.3) and (4.9)] simplify the field equation (3.10) into the relation

$$-2\mathcal{H}\mathcal{K} = a^2\kappa\delta\tilde{T}^0_0, \quad \text{(D1)}$$

from which $\mathcal{K}$ can be expressed and $\nabla^2\mathcal{K}$ which is needed in the following step can easily be calculated. Applying ∇^k to Eq. (3.11) we obtain the equation for $\mathcal{P} = \nabla_l \tilde{h}^l_0$,

$$\nabla^2 \mathcal{P} + 3k\mathcal{P} = \frac{3}{2}a^2\kappa\left(\nabla^k \delta\tilde{T}^0_k + \frac{1}{3\mathcal{H}}\nabla^2\delta\tilde{T}^0_0\right). \quad \text{(D2)}$$

Solving for $\mathcal{P}$ and substituting back into Eq. (3.10) we get the elliptic equation for $\tilde{h}_{0k}$:

$$\nabla^2\tilde{h}_{0k} + 2k\tilde{h}_{0k} = 2a^2\kappa\left(\delta\tilde{T}^0_k + \frac{1}{3\mathcal{H}}\nabla_k\delta\tilde{T}^0_0\right) - \frac{1}{3}\nabla_k\mathcal{P}. \quad \text{(D3)}$$

From Eq. (3.12) $\tilde{h}_{00}$ can be determined. $\mathcal{K}$ can be expressed in terms of $\delta\tilde{T}^0_0$ from relation (D1), and the last term on the r.h.s. involving the time derivative $(a^2\mathcal{H}^{-1}\delta\tilde{T}^0_0)'$ can be calculated by employing the perturbed Bianchi identities, Eq. (3.22). Equation (3.12) becomes

$$\nabla^2\tilde{h}_{00} + 3k\tilde{h}_{00} = \frac{a^2\kappa}{\mathcal{H}}\left(\nabla^k\delta\tilde{T}^0_k - \frac{k}{\mathcal{H}}\delta\tilde{T}^0_0\right). \quad \text{(D4)}$$

From the elliptic equations (D2)–(D4) the metric perturbations $\tilde{h}_{00}$, $\tilde{h}_{0k}$ follow instantaneously if the sources $\delta\tilde{T}^0_0$ and $\delta\tilde{T}^0_k$ (resp. $\delta\tilde{T}^k_0$) are given.

2. Mach 3 gauge

Together with $\mathcal{T}_k = \nabla_l\tilde{h}^l_{Tk} = 0$ it is now assumed that $\nabla^2\mathcal{P} + 3k\mathcal{P} = 0$, or simply $\mathcal{P} = \nabla_l\tilde{h}^l_0 = 0$. Applying ∇^k to Eq. (3.11), one gets

$$\nabla^2\mathcal{K} = \tfrac{3}{2}a^2\kappa\nabla^k\delta\tilde{T}^0_k. \quad \text{(D5)}$$

With $\mathcal{K}$ known we obtain the elliptic equation for $\tilde{h}^n_n$ from Eq. (3.10):

$$\nabla^2\tilde{h}^n_n + 3k\tilde{h}^n_n = 3a^2\kappa\delta\tilde{T}^0_0 + 6\mathcal{H}\mathcal{K}. \quad \text{(D6)}$$

Next, we make the time derivative of (D5) and substitute for $\nabla^k\delta\tilde{T}^{0\prime}_k$ from the perturbed Bianchi identities. Applying then ∇^2 to Eq. (3.12) we arrive at the elliptic equation for $\tilde{h}_{00}$:

$$\nabla^2(\nabla^2\tilde{h}_{00} + 3k\tilde{h}_{00}) = a^2\kappa[\nabla^2(\delta\tilde{T}^0_0 - \delta\tilde{T}^n_n) + 3\mathcal{H}\nabla^k\delta\tilde{T}^0_k + 3\nabla^k\nabla_m\delta\tilde{T}^m_k]. \quad \text{(D7)}$$

There is another simple elliptic equation satisfied by the quantity $\chi = \tilde{h}_{00} - \frac{1}{3}\tilde{h}^n_n$. Taking ∇^2 of Eq. (D6) and regarding Eq. (D5), we combine it with Eq. (D7) to obtain

$$\nabla^2(\nabla^2\chi + 3k\chi) = 3a^2\kappa(\nabla^k\nabla_l\delta\tilde{T}^l_k - \tfrac{1}{3}\nabla^2\delta\tilde{T}^n_n). \quad \text{(D8)}$$

The quantity χ appears directly also in Eq. (3.13) for the spatial components $\delta\tilde{G}^l_{Tk}$. Applying $\nabla^k\nabla_l$ on this equation, one arrives again at Eq. (D8) above.

With $\mathcal{P}$ known ($\mathcal{P} = 0$ in the simplest choice of the Mach 3* gauge) and $\mathcal{K}$ determined from Eq. (D5), the constraint equation (3.11) becomes a simple elliptic equation for $\tilde{h}_{k0}$:

$$\nabla^2\tilde{h}_{k0} + 2k\tilde{h}_{k0} = 2a^2\kappa\delta\tilde{T}^0_k - \tfrac{1}{3}\nabla_k\mathcal{P} - \tfrac{4}{3}\nabla_k\mathcal{K}. \quad \text{(D9)}$$

The equations for $\tilde{h}_{00}$, $\tilde{h}_{0k}$ in both the Mach 2 and 3 gauges are elliptic. Their form is very similar to the equations in the Mach 1 gauge. We can solve them by the same methods.

3. Generalized Lorenz-de Donder gauge

We start from the gauge conditions (4.15), expressed explicitly in Eqs. (4.16) and (4.17). Now, in general, $\mathcal{T}_k = \nabla_l\tilde{h}^l_{Tk} \neq 0$, and also $\mathcal{P} = \nabla_l\tilde{h}^l_0$ and $\mathcal{K}$ are nonvanishing. Nevertheless, the field equations (3.10), (3.11), (3.12), and (3.13) can be rewritten into a quite telling form. Denoting $\tilde{h}_{00} = \varphi$, $\frac{1}{3}\tilde{h}^n_n = \psi$, $\mathcal{P} = \nabla_l\tilde{h}^l_0$, we arrive at the following system:

$$\nabla^2\varphi - \varphi'' - 2\mathcal{H}a^3\left(\frac{\varphi}{a^3}\right)' - 6a\left(\frac{\mathcal{H}}{a}\right)'\psi - 4\mathcal{H}\mathcal{P} = a^2\kappa(\delta\tilde{T}^0_0 - \delta\tilde{T}^n_n), \quad \text{(D10)}$$

$$\nabla^2\psi - \psi'' + 4k\psi - 2\mathcal{H}a^3\left(\frac{\psi}{a^3}\right)' - 2a\left(\frac{\mathcal{H}}{a}\right)'\varphi - \frac{4}{3}\mathcal{H}\mathcal{P} = a^2\kappa\left(\delta\tilde{T}^0_0 + \frac{1}{3}\delta\tilde{T}^n_n\right), \quad \text{(D11)}$$

$$\nabla^2\tilde{h}_{0k} - \tilde{h}''_{0k} + 2k\tilde{h}_{0k} - 4(\mathcal{H}\tilde{h}_{0k})' - \mathcal{H}\nabla_k(\varphi + 3\psi) = 2a^2\kappa\delta\tilde{T}^0_k, \quad \text{(D12)}$$

$$\nabla^2\mathcal{P} - \mathcal{P}'' + 4k\mathcal{P} - 4(\mathcal{H}\mathcal{P})' - \mathcal{H}\nabla^2(\varphi + 3\psi) = 2a^2\kappa\nabla^k\delta\tilde{T}^0_k, \quad \text{(D13)}$$

$$-\nabla^2\tilde{h}^l_{Tk} + \tilde{h}^{l\prime\prime}_{Tk} + 2k\tilde{h}^l_{Tk} + 2\mathcal{H}\tilde{h}^{l\prime}_{Tk} + 4\mathcal{H}(f^{lm}\nabla_{(m}\tilde{h}_{k)0} - \tfrac{1}{3}\delta^l_k\nabla_n\tilde{h}^n_0) = 2a^2\kappa(\delta\tilde{T}^l_k - \tfrac{1}{3}\delta^l_k\delta\tilde{T}^n_n). \quad \text{(D14)}$$

All equations now have the character of hyperbolic generalized wave equations. The metric perturbations are *not* determined instantaneously in terms of the sources $\delta\tilde{T}^\mu_\nu$. Although the main parts of the equations are given by the standard wave operators $\nabla^2 - d^2/d\eta^2$, there are terms involving lower derivatives of the metric perturbations which make the system coupled. These equations may turn out to be useful in cosmology as the standard harmonic gauge is in the post-Minkowskian approximations to general relativity.

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