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Constraints on the spectral index for the inflation models in the string landscape

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We conjecture that the inflation models with trans-Planckian excursions in the field space should be in the swampland. We check this conjecture in a few examples and investigate the constraints on the spectral index for the slow-roll inflation model in the string landscape where the variation of the inflaton during the period of inflation is less than the Planck scale M_p. A red primordial power spectrum with a lower bound on the spectral index is preferred. Both the tensor-scalar ratio and the running can be ignored.

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The inflation model [1–3] has been remarkably successful in not only explaining the large-scale homogeneity and isotropy of the universe, but also providing a natural mechanism to generate the observed magnitude of inhomogeneity. In the new inflation model, inflation may begin either in the false vacuum or in an unstable state at the top of the effective potential. Then the inflaton field slowly rolls down to the minimum of its effective potential. This picture relies on an application of low-energy effective field theory to inflation. However, the effective field theory can break down even in the region with low curvature.

It is clear that consistent theories of quantum gravity can be constructed in the context of string theory. The central problem in string theory is how to connect it with experiments. Recent developments for the flux compactifications [4,5] suggest that a vast number of at least semiclassically consistent string vacua emerge in string theory. It may or may not provide an opportunity for us to explore the specific low-energy phenomena in the experiments from the viewpoint of string theory. In fact, the vast series of semiclassically consistent effective field theories are actually inconsistent. We say that they are in the swampland [6]. The self-consistent landscape is surrounded by the swampland. In [7] gravity is conjectured as the weakest force on the validity of the effective field theories. This conjecture is supported by string theory and some evidence involving black holes and symmetries. In four dimensions an intrinsic UV cutoff for the U(1) gauge theory

$$\Lambda \leq g M_p \tag{1}$$

is suggested, where g is the gauge coupling and M_p is the Planck scale. Furthermore, an intrinsic UV cutoff for the scalar field theories with gravity is proposed in [8], e.g.

$$\Lambda \leq \lambda^{1/2} M_p \tag{2}$$

for $\lambda\phi^4$ theory. This conjecture provides some possible stringent constraints on the inflation model [8]. Some other related works on the weak gravity conjecture are [9–19].

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The gauge interactions are governed by the symmetry. However there is no such principle to constrain the interaction of scalar fields. One can construct thousands of inflation models corresponding to different shapes of the potential of the inflaton. Therefore it is difficult for us to work out some model-independent predictions for the inflation model.

In this paper, we collect several examples to support the fact that the variation of the inflaton for the inflation models in the string landscape should be less than the Planck scale. According to this observation, we figure out the constraints on the spectral index for the inflation model.

Inflation in the early universe is driven by the potential of the inflaton field ϕ. The equations of motion for an expanding, spatially flat universe containing a homogeneous scalar field take the form

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_p^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{3}$$

$$\ddot{\phi} + 3H\dot{\phi} = -V', \tag{4}$$

where $V(\phi)$ is the potential of the inflaton ϕ and the prime denotes the derivative with respect to ϕ. For simplicity, we define several slow-roll parameters as

$$\epsilon = \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2, \qquad \eta = M_p^2 \frac{V''}{V}, \qquad \xi = M_p^4 \frac{V'V'''}{V^2}. \tag{5}$$

If $\epsilon \ll 1$ and $|\eta| \ll 1$, the inflaton field slowly rolls down its potential and the equations of motion are simplified to be

$$H^2 = \frac{V}{3M_p^2}, \qquad 3H\dot{\phi} = -V'. \tag{6}$$

In this paper, we assume, without loss of generality, $\dot{\phi} < 0$, so that $V' > 0$. The number of e-folds N before the end of inflation is related to the vacuum expectation value (vev) of the inflaton by

$$dN = -Hdt = -\frac{H}{\dot{\phi}}d\phi = \frac{1}{\sqrt{2\epsilon}M_p}d\phi. \tag{7}$$

The slow-roll parameters also characterize the feature of the primordial power spectrum for the fluctuations: the amplitude of the scalar and tensor perturbations are, respectively [20],

$$\Delta_{\mathcal{R}}^2 = \frac{H^2/M_p^2}{8\pi^2\epsilon}, \qquad \Delta_T^2 = \frac{H^2/M_p^2}{\pi^2/2}. \tag{8}$$

The tensor-scalar ratio takes the form

$$r = \Delta_T^2/\Delta_{\mathcal{R}}^2 = 16\epsilon, \tag{9}$$

and the spectral index and its running are given by

$$n_s = 1 - 6\epsilon + 2\eta, \tag{10}$$

$$\alpha_s = -24\epsilon^2 + 16\epsilon\eta - 2\xi, \tag{11}$$

where we use

$$\frac{d\epsilon}{dN} = 2\epsilon(\eta - 2\epsilon), \qquad \frac{d\eta}{dN} = \xi - 2\epsilon\eta. \tag{12}$$

In [21] Lyth connects detectably large gravitational wave signals to the motion of the inflaton over Planckian distances in the field space. There is a long-term debate [21,22] on whether the classical evolution of the scalar field can probe the trans-Planckian region where the low-energy field theory is still an effective field theory. String theory gives us an opportunity to answer this question. In this note, we conjecture that the probing region of the scalar field is limited by the Planck scale M_p in the string landscape. A few examples to check our conjecture will be proposed as follows.

The first one is called "extra-natural inflation" [23]. Consider a U(1) gauge theory with gauge coupling g_5 in five dimensions. Compactifying this gauge theory on a circle with size R, we obtain four-dimensional gravity as well as a periodic scalar $\theta = \oint A_5 dx^5$ associated with the Wilson line around the circle. The effective Lagrangian for θ in four dimensions takes the form

$$\mathcal{L} = \frac{f^2}{2}(\partial\theta)^2 - \frac{1}{R^4}(1 - \cos\theta), \tag{13}$$

where $f^2 = \frac{1}{g_5^2 R} = \frac{1}{g^2 R^2}$ and g is the gauge coupling in four dimensions. The canonical scalar field ϕ is given by $\phi = f\theta$. The period of θ is 2π and the vev of ϕ takes the same order of magnitude as f. It is easily seen that f can be bigger than M_p for sufficiently small g and the slow-roll conditions are achieved. However, the weak gravity conjecture [7] says $\Lambda \sim 1/R \leq gM_p$ which implies $f = \frac{1}{gR} \leq M_p$. With the viewpoint of string theory, $g = g_s^{1/2}/\sqrt{M_s^6 V_6}$ and $M_p = M_s\sqrt{M_s^6 V_6}/g_s$, where M_s is the string scale and V_6 is the volume of the compactified space. Thus we have $f = \frac{1}{gR} = \frac{g_s^{1/2}}{M_s R} M_p < M_p$ in the perturbative region ($g_s < 1$), where we also require that the size of the compactified space is larger than the string length M_s^{-1}. In this case, the over-Planckian variation of the scalar field cannot be embedded into string theory and it is in the swampland.

The second is chaotic inflation [24]. For instance, we consider the $V(\phi) = \lambda\phi^4$ inflation model. The Hubble scale $H = \sqrt{\frac{V}{3M_p^2}} \sim \frac{\lambda^{1/2}\phi^2}{M_p}$ can be taken as the IR cutoff for the effective field theory. In [8] an upper bound on the UV cutoff (2) is proposed. Naturally the IR cutoff should be lower than the UV cutoff. Requiring $H < \Lambda$ yields $\phi < M_p$ [8]. Furthermore, we take into account the inflation model with potential $V = V_0 + \lambda\phi^4$. If the potential is dominated by the constant term V_0, it is a typical potential for hybrid inflation [25]. Since $H = \sqrt{\frac{V_0 + \lambda\phi^4}{3M_p^2}} > \frac{\lambda^{1/2}\phi^2}{M_p}$, requiring $H \leq \Lambda$ leads to $\phi < M_p$ as well. The trans-Planckian excursion in the field space cannot be achieved.

The third example is the inflation driven by the motion of a D3-brane in the warped background. The authors of [26] found the maximal variation of the canonical inflaton field as

$$|\Delta\phi| = \sqrt{T_3}R \leq \frac{2}{\sqrt{n_B}} M_p, \tag{14}$$

where R is the size of the throat and n_B is the number of the background D3 charge. Since $n_B \gg 1$ for the validity of the background geometry, the variation of the inflaton is not larger than the Planck scale.

Fourth, Ooguri and Vafa in [27] proposed several conjectures to limit the observable regions of moduli spaces. For a massless scalar field ϕ, the change of its vev is $|\Delta\phi| \sim |\frac{M_p}{3}\ln\varepsilon|$, where ε is the mass scale for the low-energy effective theory. There is an infinite tower of light particles at infinite distance from any point inside the moduli space, the effective field theory in the interior breaks down, and a new description takes over. This example also hints that the variation of the scalar field should be less than M_p in the string landscape.

In the following we will investigate the constraints on the spectral index by considering that the variation of the inflaton during the period of inflation is less than M_p. We reparametrize the slow-roll parameter ϵ in Eq. (7) as a function of N. Equation (7) becomes

$$\int_0^{N_{\rm tot}} \sqrt{2\epsilon(N)}dN = \int \frac{d\phi}{M_p} = \frac{|\Delta\phi|}{M_p} \leq 1. \tag{15}$$

We cannot really achieve a model-independent analysis, because the function $\epsilon(N)$ for the string landscape is unknown. Here we consider three typical parametrizations. Actually these parametrizations are quite general, and many well-known inflation models are included in them.

First, we assume ϵ is roughly a constant and then $\eta = 2\epsilon$. Equation (15) reads

$$\epsilon \leq \epsilon_m = \frac{1}{2N_{\rm tot}^2}. \tag{16}$$

Now the spectral index and the tensor-scalar ratio are

$$n_s = 1 - 2\epsilon \geq 1 - \frac{1}{N_{\rm tot}^2}, \qquad r = 16\epsilon \leq \frac{8}{N_{\rm tot}^2}. \tag{17}$$

Generically the total number of e-folds should be larger than 60 in order to solve the flatness and horizon problem. In this case, the scalar power spectrum is the scale-invariant HarrisonZel'dovich-Peebles (HZ) spectrum with tiny tensor perturbations $r \leq 0.002$. WMAP normalization is $\Delta_{\mathcal{R}}^2 = 2 \times 10^9$ [28]. Thus $\Delta_T^2 = r \cdot \Delta_{\mathcal{R}}^2 \leq 4 \times 10^{-12}$ and $V^{1/4} \leq 6.8 \times 10^{15}$ GeV which is lower than the grand unified theory (GUT) scale.

Second, we consider the case with

$$\epsilon(N) = \frac{c^2/2}{N^{2-2\beta}}, \tag{18}$$

where both c and β are constants. Since $\epsilon < 1$ for $N = 60$, it is reasonable to assume that the value of β is not larger than 1. Requiring that the integration on the left-hand side of Eq. (15) is finite yields $\beta > 0$.[1] Therefore the reasonable range for β is

$$0 < \beta \leq 1. \tag{19}$$

Using Eqs. (12) and (18), we obtain

$$\eta = 2\epsilon - \frac{1-\beta}{N}, \tag{20}$$

$$\xi = \frac{1-\beta}{N^2} - \frac{6(1-\beta)}{N}\epsilon + 4\epsilon^2. \tag{21}$$

The spectral index and its running and the tensor-scalar ratio are, respectively,

$$n_s = 1 - 2\epsilon - \frac{2(1-\beta)}{N}, \tag{22}$$

$$\alpha_s = -\frac{2(1-\beta)}{N^2} - \frac{4(1-\beta)}{N}\epsilon, \tag{23}$$

$$r = 16\epsilon. \tag{24}$$

Now Eq. (15) implies

$$c \leq \beta N_{\rm tot}^{-\beta}. \tag{25}$$

It comes back to the previous results for $\beta = 1$. Equation (25) leads to an upper bound on ϵ,

$$\epsilon \leq \frac{\beta^2}{2N^2}\left(\frac{N}{N_{\rm tot}}\right)^{2\beta}. \tag{26}$$

Since $0 \leq \beta \leq 1$ and $N \leq N_{\rm tot}$, $\epsilon \leq \frac{1}{2N^2} = 1.4 \times 10^{-4}$ for $N = 60$. The tensor-scalar ratio satisfies $r \leq 0.002$. Thus $\Delta_T^2 = r \cdot \Delta_{\mathcal{R}}^2 \leq 4 \times 10^{-12}$ and $V^{1/4} \leq 6.8 \times 10^{15}$ GeV. Since the maximum value of ϵ takes the order of magnitude 10^{-4}, we can ignore the terms with ϵ. Now the spectral index and its running become

$$n_s = 1 - \frac{2(1-\beta)}{N}, \qquad \alpha_s = -\frac{2(1-\beta)}{N^2}. \tag{27}$$

Since $\beta > 0$, there are lower bounds on the spectral index and its running:

$$1 - \frac{2}{N} \leq n_s < 1, \qquad -\frac{2}{N^2} \leq \alpha_s < 0. \tag{28}$$

A red tilted primordial power spectrum ($n_s < 1$) with tiny running and tensor perturbations arises in the string landscape. On the other hand, WMAP data [28] prefer a red tilted power spectrum:

$$n_s = 0.951 \pm 0.016, \qquad r \leq 0.65, \tag{29}$$

and the running can be ignored. We compare our constraints on the inflation models in the string landscape with WMAP in Fig. 1. Our analysis is consistent with observations.

Third, we consider

$$\epsilon(N) = \epsilon_0 + \frac{c^2/2}{N^{2-2\beta}}, \tag{30}$$

where ϵ_0, c, and β are constant. Here we assume $\epsilon_0 > 0$ and then the range of β is still $\beta \in [0, 1]$. In this case the constraints on ϵ_0 and c should be more stringent than those in the previous two cases, because both terms on the right-hand side of Eq. (30) are positive. For simplicity, we still take $\epsilon_0 \leq \epsilon_m$ and $c \leq \beta N_{\rm tot}^{-\beta}$, and thus the terms with ϵ can be ignored. Now the slow-roll parameters take the form

$$\eta = -\alpha\frac{1-\beta}{N}, \qquad \xi = \gamma\frac{1-\beta}{N^2}, \tag{31}$$

where $\alpha = 1/(1 + 2\epsilon_0 N^{2-2\beta}/c^2) \leq 1$ and $\gamma = 3 - 2\alpha^2 - 2(1-\alpha^2)\beta \leq 3$. Since $n_s = 1 - 2\eta = 1 - 2\alpha\frac{1-\beta}{N}$ and $\alpha \leq 1$, a more blue tilted power spectrum than the previous case with $\epsilon = c^2/(2N^{2-2\beta})$ is obtained. In this case the running of the spectral index can be ignored as well. The lower bound on the spectral index in Eq. (28) is still available.

The previous discussions are only valid for the single-field inflation model in the string landscape. For multifield inflation, the previous constraints may be released. For simplicity, we consider the assisted inflation [31] with potential $\sum_{i=1}^{n} V(\phi_i)$. In the assisted inflation, there is a unique late-time attractor with all the scalar fields equal, i.e. $\phi_1 = \phi_2 = \ldots = \phi_n$. With this ansatz, the equations of motion for the slow-roll assisted inflation are given by

$$H^2 = \frac{nV(\phi)}{3M_p^2}, \qquad 3H\dot{\phi} = -V', \tag{32}$$

where $\phi = \phi_i$, $i = 1, \ldots, n$. It is convenient for us to define a new slow-roll parameter ϵ_H as

[1]For example, brane inflation [29] (KKLMMT model [30]) takes $\beta = 1/6$.

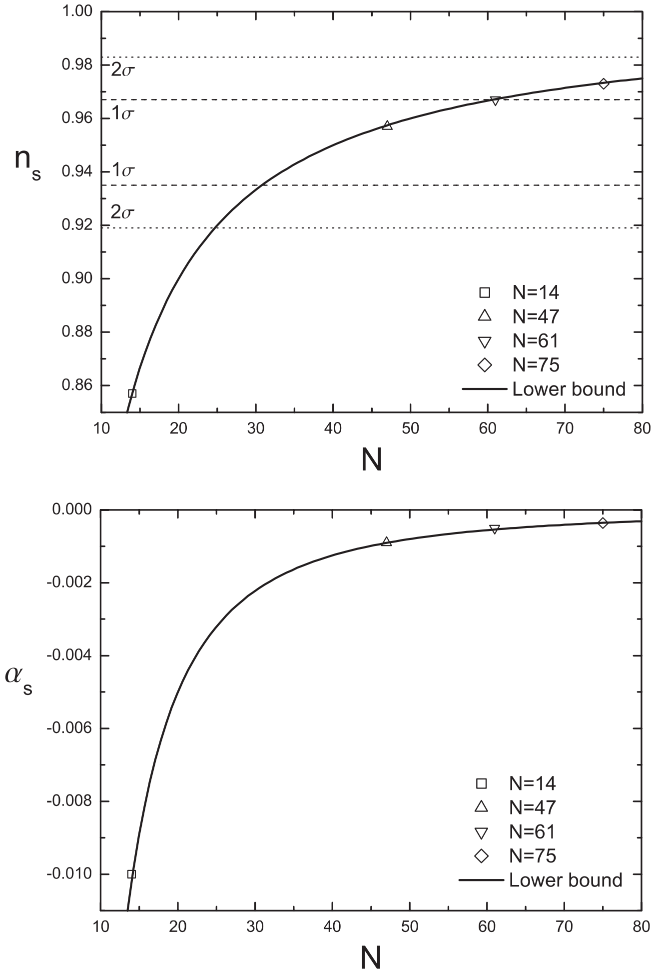

FIG. 1. The lower bounds on n_s and α_s are shown for the single-field inflation model with sub-Planckian excursion in the field space. With a high inflation scale, and radiation and/or matter domination between the end of inflation and nucleosynthesis, $47 \leq N \leq 61$. More generally, the range has to be $14 \leq N \leq 75$ [36].

$$\epsilon_H = -\frac{\dot{H}}{H^2}. \tag{33}$$

The slow-roll condition reads $\epsilon_H \ll 1$. Using Eq. (32), we find

$$\epsilon_H = \frac{1}{n}\frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2 = \frac{1}{n}\epsilon. \tag{34}$$

Because of the factor $1/n$ in the above equation, the slow-roll condition for the inflation model without a flat enough potential ($\epsilon \gg 1$) can be achieved if the number of inflatons is sufficiently large. Replacing ϵ in Eq. (15) with ϵ_H, we obtain

$$\int_0^{N_{\rm tot}} \sqrt{2\epsilon_H(N)}dN = \sqrt{n}\frac{|\Delta\phi|}{M_p}. \tag{35}$$

If we still have $|\Delta\phi| \leq M_p$ and

$$\epsilon_H(N) = \frac{c^2/2}{N^{2-2\beta}}, \tag{36}$$

the bound on c becomes

$$c \leq \sqrt{n}\beta N_{\rm tot}^{-\beta}. \tag{37}$$

The upper bound on the slow-roll parameter ϵ_H is given by

$$\epsilon_H \leq \frac{n\beta^2}{2N^2}\left(\frac{N}{N_{\rm tot}}\right)^{2\beta}. \tag{38}$$

If the number of inflatons n is large enough, we can get a larger slow-roll parameter ϵ_H, a larger tensor-scalar ratio $r = 16\epsilon_H$, and a more red tilted power spectrum. Before the end of this paragraph, we also want to reconsider an example in string theory: brane inflation in the warped background. If the number of probing D3-branes is n, which is just the number of inflatons, we have $\sqrt{n}|\Delta\phi| \leq 2\sqrt{\frac{n}{n_B}}M_p$. In order for the validity of the background geometry, $n < n_B$; otherwise the backreaction of the probing D3-branes will significantly change the background geometry. In this case, $\sqrt{n}|\Delta\phi| < M_p$. If this is the generic result for the inflation models in the string landscape, our previous results for the single-field inflation are recovered even for the multifield inflation models.

To summarize, the inflation model with over-Planckian variation in the scalar field space cannot be achieved in string theory. A red tilted primordial scalar power spectrum with a lower bound on the spectral index arises for the slow-roll inflation model in the string landscape due to the observation that the observable region in the scalar field space is limited by the Planck scale. The tensor fluctuations and the running of the spectral index can be ignored. Even though our analysis is not really model independent, the parametrizations in this paper are already quite general. In some sense, our results can be taken as the predictions of string theory. For the assisted inflation, the constraints on the spectral index might be released.

Finally, we also want to remind the reader that the chain inflation [32–35] may be generic in the string landscape. In this model, the universe tunneled rapidly through a series of metastable vacua with different vacuum energies. Since chain inflation is not really a slow-roll inflation model, it does not suffer from the constraints in this paper. A detectable gravitational wave fluctuation is still available in this model [35].

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