Relativistic BCS-BEC crossover at finite temperature and its application to color superconductivity

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The nonrelativistic G_0G formalism of BCS-BEC crossover at finite temperature is extended to relativistic fermion systems. The uncondensed pairs contribute a pseudogap to the fermion excitations. The theory recovers the BCS mean field approximation at zero temperature and the nonrelativistic results in a proper limit. For massive fermions, when the coupling strength increases, there exist two crossovers from the weak coupling BCS superfluid to the nonrelativistic BEC state and then to the relativistic BEC state. For color superconductivity at moderate baryon density, the matter is in the BCS-BEC crossover region, and the behavior of the pseudogap is quite similar to that found in high temperature superconductors.

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I. INTRODUCTION

It is well known that, by adjusting the attractive coupling strength among the constituents, a fermion system may undergo a smooth crossover from the Bardeen-Cooper-Shriffer (BCS) superfluidity/superconductivity in degenerate fermion gas to the Bose-Einstein condensation (BEC) of composite molecules. Such a BCS-BEC crossover is theoretically due to the fact that the wave functions of BCS and BEC ground states are essentially the same [1,2]. The BCS-BEC crossover is expected to be realized in high temperature superconductor and atomic fermion gas [3–8] via using an external magnetic field to change the s-wave scattering length [9].

The superconductivity in quantum chromodynamics (QCD), i.e., the color superconductivity [10], is naturally considered as a system to study the relativistic BCS-BEC crossover. Because of the asymptotic property of QCD, there may exist a crossover from the BCS superconductivity with weakly bound quark pairs at high baryon density to the BEC state of composite hadrons at low baryon density [11]. Such a BCS-BEC crossover in QCD may also be realized in chiral condensed matter [12,13] and in pion superfluid [11]. At moderate baryon density, while a diquark BEC state may not be realized due to the chiral symmetry restoration, the attractive coupling strength is obviously not located in the weak coupling region. It is shown in many effective QCD models that the quark energy gap at moderate baryon density is about 100 MeV [14] which is already of the order of the Fermi energy. The strong coupling in this case may induce a so-called pseudogap effect, which has been investigated in two-flavor color superconductivity above the critical temperature [15]. A natural question is how the pseudogap modifies the critical temperature and thermodynamics of the color superconductor. To answer this question, one needs to construct a relativistic theory at finite temperature which can describe the pseudogap and possible BCS-BEC crossover.

The BCS-BEC crossover in relativistic fermion systems was recently investigated in the Nozieres–Schmitt-Rink (NSR) theory above the critical temperature [16,17], the boson-fermion model [18], and the BCS-Leggett mean field theory at zero temperature [19]. It is shown that, not only the BCS superfluidity and the nonrelativistic BEC (NBEC) of heavy molecules but also the NBEC and the relativistic BEC (RBEC) of nearly massless molecules can be smoothly connected. In the RBEC state, antifermion pairs (antibosons) are excited and become nearly degenerate with fermion pairs (bosons). From the NSR theory at $T \geq T_c$, where T_c is the critical temperature, the difference between the NBEC [20] and RBEC [21,22] states is significant [16,17].

It is widely known that, at zero temperature, the mean field theory is a good approximation to describe the BCS-BEC crossover [23], and the pair fluctuations can be safely neglected even at strong coupling. Only around the unitary limit, i.e., the infinite scattering length limit, are the pair fluctuations somewhat important to obtain a proper value of the universal constant [7]. In our previous paper [19], we investigated the generalization from nonrelativistic to relativistic BCS-BEC crossover at zero temperature in the BCS-Leggett mean field theory. At finite temperature, however, the condensed pairs with zero momentum can be thermally excited, and one should go beyond the mean field approximation to treat properly the uncondensed pairs [6].

There exist many methods to treat pair fluctuations at finite temperature. In the NSR theory, which is also called G_0G_0 theory, the pair fluctuations enter only the number equation, and the fermion loops which appear in the pair propagator are constructed by bare Green function G_0. As a consequence, such a theory is, in principle, not self-consistent and is valid only at $T \geq T_c$. For the study of BCS-BEC crossover, one needs a theory which is valid not only above the critical temperature but also in the symmetry breaking phase. While such a strict theory has not been reached so far, some T matrix approaches were recently developed; see, for instance, [6,23]. Among them, the asymmetric pair approximation or the so-called G_0G scheme [6,24] is a competitive one. The effect of the pair fluctuations in the G_0G method is treated as a fermion pseudogap which has been widely discussed in high temperature superconductivity. In contrast to the NSR theory (G_0G_0 scheme), the G_0G scheme is self-consistent and keeps the Ward identity [6].

In the study of color superconductivity at moderate density, the color condensed phase is of great interest. The NSR theory [16,17], which seems valid in the normal phase, can only predict the transition temperature of color superconductivity. A necessary task in this field of research is to develop a relativistic BCS-BEC crossover theory in the symmetry breaking phase. In this paper, we will generalize the G_0G scheme to relativistic fermion systems. A necessary requirement for such a generalization is to recover the nonrelativistic limit [6] and mean field limit [4] properly. With this theory, we can calculate the critical temperature T_c for arbitrary coupling and describe the BCS-NBEC-RBEC crossover at finite temperature. As an application, it can be used to study the pseudogap effect on color superconductivity.

The paper is organized as follows. In Sec. II we review the BCS mean field theory for relativistic superfluidity/superconductivity. In the framework of the G_0G scheme, we include in Sec. III the contribution from the uncondensed pairs and construct coupled equations for the superfluid order parameter and pseudogap. In Sec. IV, we apply the theory to massive fermions and study the BCS-NBEC-RBEC crossover at finite temperature. In Sec. V, we apply the theory to color superconducting quark matter. We will calculate the transition temperature and the quark pseudogap and show the significance of the fluctuations at moderate baryon density. We summarize in Sec. VI.

II. BCS MEAN FIELD THEORY

We consider a model with only fermions as elementary blocks. The Lagrangian density can be written as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \mathcal{L}_I, \tag{1}$$

where ψ, $\bar{\psi}$ denote the Dirac fermion fields with mass m, and $\mathcal{L}_I$ indicates the attractive interaction among fermions. Since the dominant interaction is the $J^P = 0^+$ scalar channel, the interaction for the pairing between different spins can take the form [16,19]

$$\mathcal{L}_I = \frac{g}{4}(\bar{\psi}i\gamma_5 C\bar{\psi}^T)(\psi^T Ci\gamma_5\psi), \tag{2}$$

where g is the attractive coupling constant, and $C = i\gamma_0\gamma_2$ is the charge conjugation matrix. Generally, by adjusting the coupling strength, the crossover from condensation of spin-zero Cooper pairs with large size at weak coupling to the Bose-Einstein condensation of deeply bound bosons at strong coupling can be realized. In our model, only fermions are elementary particles. Another type of model which is used to discuss the BCS-BEC crossover in high temperature superconductors and atomic Fermi gases is the so-called boson-fermion model where both fermions and bosons are considered as elementary blocks. Such a model was recently generalized to study the relativistic BCS-BEC crossover [18].

In order to develop a finite temperature theory including pair fluctuations in the symmetry breaking phase, we first review in this section the BCS mean field theory in the functional integral approach and G_0G formalism.

A. Functional integral approach

In the functional integral approach, we start the calculation from the partition function in imaginary time formalism,

$$Z = \int D\bar{\psi}D\psi e^{\int_0^\beta d\tau \int d^3\mathbf{x}(\mathcal{L}+\mu\psi^\dagger\psi)} \tag{3}$$

where β is the inverse temperature, $\beta = 1/T$, and μ is the chemical potential corresponding to the net charge density $\psi^\dagger\psi$ and determined by the charge conservation. Performing a Hubbard-Stratonovich transformation which introduces an auxiliary pair field $\Delta(x) = g\psi^T(x)Ci\gamma_5\psi(x)/2$, and then integrating out the fermions, we derive the partition function

$$Z = \int D\Delta D\Delta^* e^{-S_{\mathrm{eff}}[\Delta,\Delta^*]} \tag{4}$$

with the effective boson action

$$S_{\mathrm{eff}} = \int_0^\beta d\tau \int d^3\mathbf{x}\left[\frac{|\Delta(x)|^2}{g} - \frac{1}{2\beta}\mathrm{Tr}\ln[\beta\mathbf{G}^{-1}]\right] \tag{5}$$

in terms of the inverse Nambu-Gorkov propagator

$$\mathbf{G}^{-1} = i\gamma^\mu\partial_\mu - m + \mu\gamma_0\sigma_3 + i\gamma_5\Delta\sigma_+ + i\gamma_5\Delta^*\sigma_-, \tag{6}$$

where $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$ are defined in the Nambu-Gorkov space with $\sigma_i(i = 1, 2, 3)$ being the Pauli matrices.

The mean field theory is a good approximation to describe the BCS-BEC crossover at low enough temperature, namely $T \ll T_c$, since the dominant contribution of fluctuations to the effective potential is from the Goldstone mode and is proportional to T^4 [5]. In the mean field approximation, we consider a uniform static saddle point $\Delta(x) = \Delta_{\mathrm{sc}}$ which satisfies the stationary condition $\delta S_{\mathrm{eff}}[\Delta_{\mathrm{sc}}]/\delta\Delta_{\mathrm{sc}} = 0$. The thermodynamic potential $\Omega_{\mathrm{mf}} = S_{\mathrm{eff}}[\Delta_{\mathrm{sc}}]/(\beta V)$ at the saddle point can be evaluated as

$$\Omega_{\rm mf} = \frac{\Delta_{\rm sc}^2}{g} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\bigg[(E_{\mathbf{k}}^+ + E_{\mathbf{k}}^- - \xi_{\mathbf{k}}^+ - \xi_{\mathbf{k}}^-) - \frac{1}{\beta}(\ln(1+e^{-\beta E_{\mathbf{k}}^+}) + \ln(1+e^{-\beta E_{\mathbf{k}}^-}))\bigg], \quad (7)$$

where we have defined the quasiparticle energies $E_{\mathbf{k}}^{\pm} = \sqrt{(\xi_{\mathbf{k}}^{\pm})^2 + \Delta_{\rm sc}^2}$ with $\xi_{\mathbf{k}}^{\pm} = \epsilon_{\mathbf{k}} \pm \mu$ and $\epsilon_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. Minimizing $\Omega_{\rm mf}$, we get the gap equation to determine the order parameter $\Delta_{\rm sc}$ in the symmetry breaking phase,

$$\frac{1}{g} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\bigg[\frac{1-2f(E_{\mathbf{k}}^-)}{2E_{\mathbf{k}}^-} + \frac{1-2f(E_{\mathbf{k}}^+)}{2E_{\mathbf{k}}^+}\bigg], \quad (8)$$

where $f(x) = 1/(e^{\beta x}+1)$ is the Fermi-Dirac distribution function. In the study of BCS-BEC crossover, people often consider the thermodynamics in canonical ensemble with fixed fermion density n by fixing the Fermi momentum k_f through the relation $n = k_f^3/(3\pi^2)$ at zero temperature. At finite temperature, the density can be obtained from the first order derivative of the thermodynamic potential with respect to the chemical potential,

$$n = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\bigg[\bigg(1 - \frac{\xi_{\mathbf{k}}^-}{E_{\mathbf{k}}^-}(1-2f(E_{\mathbf{k}}^-))\bigg) - \bigg(1 - \frac{\xi_{\mathbf{k}}^+}{E_{\mathbf{k}}^+}(1-2f(E_{\mathbf{k}}^+))\bigg)\bigg]. \quad (9)$$

The first and second terms in the square brackets on the right-hand side of Eqs. (8) and (9) correspond, respectively, to fermion and antifermion degrees of freedom.

B. G_0G formalism

Now we reexpress the BCS mean field theory in the G_0G formalism [6,7,25]. Such a formalism is convenient for us to go beyond the BCS and include uncondensed pairs at finite temperature. Let us start from the fermion propagator $\mathcal{S}$ in the symmetry breaking phase. The inverse propagator reads

$$\mathcal{S}^{-1}(k) = \begin{pmatrix} \mathcal{G}_0^{-1}(k,\mu) & i\gamma_5\Delta_{\rm sc} \\ i\gamma_5\Delta_{\rm sc} & \mathcal{G}_0^{-1}(k,-\mu)\end{pmatrix} \quad (10)$$

with the inverse free propagator

$$\mathcal{G}_0^{-1}(k,\mu) = (i\omega_n + \mu)\gamma_0 - \gamma\cdot\mathbf{k} - m, \quad (11)$$

where $k = (i\omega_n, \mathbf{k})$ is the fermion four-momentum at finite temperature with ω_n being the fermion frequency $\omega_n = (2n+1)\pi T$ $(n = 0, \pm1, \pm2, \ldots)$. The propagator can be formally expressed as

$$\mathcal{S}(k) = \begin{pmatrix} \mathcal{G}(k,\mu) & \mathcal{F}(k,\mu) \\ \mathcal{F}(k,-\mu) & \mathcal{G}(k,-\mu)\end{pmatrix} \quad (12)$$

with the diagonal and off-diagonal elements

$$\mathcal{G}(k,\mu) = [\mathcal{G}_0^{-1}(k,\mu) - \Sigma_{\rm sc}(k)]^{-1}, \qquad \mathcal{F}(k,\mu) = -\mathcal{G}(k,\mu)i\gamma_5\Delta_{\rm sc}\mathcal{G}_0(k,-\mu), \quad (13)$$

where the fermion self-energy $\Sigma_{\rm sc}$ is defined as

$$\Sigma_{\rm sc}(k) = i\gamma_5\Delta_{\rm sc}\mathcal{G}_0(k,-\mu)i\gamma_5\Delta_{\rm sc} = -\Delta_{\rm sc}^2\mathcal{G}_0(-k,\mu). \quad (14)$$

With the help of the energy projectors

$$\Lambda_{\pm}(\mathbf{k}) = \frac{1}{2}\bigg[1 \pm \frac{\gamma_0(\vec{\gamma}\cdot\mathbf{k} + m)}{\epsilon_{\mathbf{k}}}\bigg], \quad (15)$$

the propagator elements can be explicitly evaluated as

$$\mathcal{G}(k,\mu) = \frac{(i\omega_n + \xi_{\mathbf{k}}^-)\Lambda_+\gamma_0}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2} + \frac{(i\omega_n - \xi_{\mathbf{k}}^+)\Lambda_-\gamma_0}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2}, \qquad \mathcal{F}(k,\mu) = \frac{i\Delta_{\rm sc}\Lambda_+\gamma_5}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2} + \frac{i\Delta_{\rm sc}\Lambda_-\gamma_5}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2}. \quad (16)$$

The gap equation for the order parameter $\Delta_{\rm sc}$ is related to the off-diagonal element,

$$\Delta_{\rm sc} = -i\frac{g}{2}\sum_k \mathrm{Tr}[i\gamma_5\mathcal{F}(k,\mu)] = -i\frac{g}{2}\Delta_{\rm sc}\sum_k \mathrm{Tr}[\mathcal{G}(k,\mu)\mathcal{G}_0(-k,\mu)], \quad (17)$$

and the fermion number is controlled by the diagonal element,

$$n = -i\sum_k \mathrm{Tr}[\gamma_0\mathcal{G}(k,\mu)] \quad (18)$$

with the four-momentum integration $\sum_k = iT\sum_n \int d^3\mathbf{k}/(2\pi)^3$ at finite temperature. Completing the Matsubara frequency summation, we can reobtain the gap equation (8) and number equation (9).

In the BCS mean field theory, fermion-fermion pairs and antifermion-antifermion pairs explicitly enter the system below T_c only through the condensate $\Delta_{\rm sc}$. In the G_0G formalism, the fermion self-energy can equivalently be expressed as

$$\Sigma_{\rm sc}(k) = \sum_q t_{\rm sc}(q)\mathcal{G}_0(q-k,\mu) \quad (19)$$

associated with a condensed-pair propagator given by

$$t_{\rm sc}(q) = i\frac{\Delta_{\rm sc}^2}{T}\delta(q), \quad (20)$$

where $q = (i\nu_n, \mathbf{q})$ is the boson four-momentum with boson frequency $\nu_n = 2n\pi T$.

The BCS theory can be related to a specific pair susceptibility χ defined by

$$\chi_{\rm BCS}(q) = -\frac{i}{2}\sum_k \mathrm{Tr}[\mathcal{G}(k,\mu)\mathcal{G}_0(q-k,\mu)], \quad (21)$$

with which the gap equation for the condensate $\Delta_{\rm sc}$ can be written as

$$1 - g\chi_{\rm BCS}(0) = 0. \tag{22}$$

This implies that the uncondensed-pair propagator should be of the form

$$t(q) = \frac{ig}{1 - g\chi_{\rm BCS}(q)}, \tag{23}$$

and $t^{-1}(q = 0)$ is proportional to the pair chemical potential $\mu_{\rm pair}$. Therefore, the fact that in the symmetry breaking phase the pair chemical potential is zero leads to the BEC-like condition

$$t^{-1}(q = 0) = 0. \tag{24}$$

While the uncondensed pairs do not play any real role in the BCS mean field theory, such a specific choice of the pair susceptibility and the BEC-like condition tells us how to go beyond the BCS mean field theory and include the effect of uncondensed pairs.

III. BEYOND MEAN FIELD THEORY

While the uncondensed pairs can be safely neglected at weak coupling, they should be included for a self-consistent theory at arbitrary coupling and at finite temperature. We now go beyond the BCS mean field approximation and include the uncondensed pairs in the G_0G formalism. It is clear that, in the BCS mean field approximation, the fermion self-energy $\Sigma_{\rm sc}$ includes contributions only from the condensed pairs. At finite temperature, the condensed pairs with zero total momentum can be thermally excited, and the total propagator should contain both the condensed (sc) and uncondensed or "pseudogap"-associated (pg) contributions,

$$\begin{aligned} t(q) &= t_{\rm sc}(q) + t_{\rm pg}(q), \qquad t_{\rm sc}(q) = i\frac{\Delta_{\rm sc}^2}{T}\delta(q), \\ t_{\rm pg}(q) &= \frac{ig}{1 - g\chi(q)}, \qquad q \neq 0. \end{aligned} \tag{25}$$

Now the total fermion self-energy becomes

$$\Sigma(k) = \sum_q t(q)\mathcal{G}_0(q - k, \mu) = \Sigma_{\rm sc}(k) + \Sigma_{\rm pg}(k), \tag{26}$$

with the mean field part

$$\Sigma_{\rm sc}(k) = \sum_q t_{\rm sc}(q)\mathcal{G}_0(q - k, \mu) \tag{27}$$

and the pseudogap related part

$$\Sigma_{\rm pg}(k) = \sum_q t_{\rm pg}(q)\mathcal{G}_0(q - k, \mu). \tag{28}$$

With the full propagator

$$\mathcal{G}(k, \mu) = [\mathcal{G}_0^{-1}(k, \mu) - \Sigma(k)]^{-1} \tag{29}$$

in terms of the total self-energy, the pair susceptibility is still given by

$$\chi(q) = -\frac{i}{2}\sum_k {\rm Tr}\mathcal{G}(k, \mu)\mathcal{G}_0(q - k, \mu). \tag{30}$$

The G_0G formalism used here is diagrammatically illustrated in Fig. 1.

Note that the feedback of the pair fluctuations on the order parameter $\Delta_{\rm sc}$ is included, and it and the chemical potential μ are, in principle, determined by the BEC condition $t_{\rm pg}^{-1}(0) = 0$ and the number equation $n = -i\sum_k {\rm Tr}[\gamma_0\mathcal{G}(k, \mu)]$.

The above equations are hard to handle analytically. In the symmetry breaking phase with $T \leq T_c$, the BEC condition $t_{\rm pg}^{-1}(0) = 0$ implies that $t_{\rm pg}(q)$ is peaked at $q = 0$. This allows us to approximate

$$\Sigma(k) \simeq -\Delta^2\mathcal{G}_0(-k, \mu), \tag{31}$$

where Δ^2 contains contributions from the condensed and uncondensed pairs,

$$\Delta^2 = \Delta_{\rm sc}^2 + \Delta_{\rm pg}^2 \tag{32}$$

with the pseudogap $\Delta_{\rm pg}$ defined as

$$\Delta_{\rm pg}^2 = -\sum_{q\neq 0} t_{\rm pg}(q). \tag{33}$$

It is necessary to point out that, above the critical temperature T_c, such an approximation is no longer good, since the BEC condition is not valid in the normal phase.

Since the pseudogap $\Delta_{\rm pg}$ looks similar to the condensate $\Delta_{\rm sc}$, a natural question is whether a finite $\Delta_{\rm pg}$ breaks the symmetry of the system. If yes, $\Delta_{\rm sc}$ will no longer be considered as the order parameter of the phase transition. By omitting a term of the order of $O(\Delta_{\rm sc}^2/\Lambda^2)$, where Λ is a momentum cutoff, the inverse fermion propagator including the feedback of the pair fluctuations can be written as

$$\mathcal{S}^{-1}(k) = \begin{pmatrix} \mathcal{G}_0^{-1}(k, \mu) - \Sigma_{\rm pg}(k) & i\gamma_5\Delta_{\rm sc} \\ i\gamma_5\Delta_{\rm sc} & \mathcal{G}_0^{-1}(k, -\mu) - \Sigma'_{\rm pg}(k) \end{pmatrix} \tag{34}$$

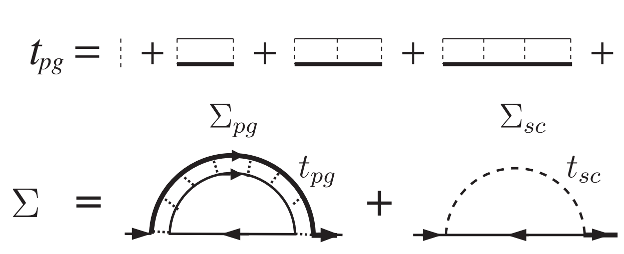

FIG. 1. Diagrammatic representation of the propagator $t_{\rm pg}$ for the uncondensed pairs and the fermion self-energy. The total fermion self-energy contains contributions from condensed ($\Sigma_{\rm sc}$) and uncondensed ($\Sigma_{\rm pg}$) pairs. The dashed, thin solid, and thick solid lines in $t_{\rm pg}$ represent, respectively, the coupling constant $g/2$, bare propagator $\mathcal{G}_0$, and full propagator $\mathcal{G}$. This diagram is taken from [29].

where $\Sigma'_{\rm pg} = \Sigma_{\rm pg}(\mu \to -\mu)$. It is now clear that the pseudogap appears in the diagonal elements of the Nambu-Gorkov propagator and does not break the symmetry of the system. On the other hand, parallel to the discussion in nonrelativistic theory [6,7,24], we can show that $\Delta^2_{\rm pg}$ is just the fluctuation of the order parameter field $\Delta(x)$,

$$\Delta^2_{\rm pg} = \langle |\Delta|^2 \rangle - \langle |\Delta| \rangle^2, \tag{35}$$

and hence it does not break the symmetry.

Under the approximation (31), all the equations in the mean field theory are still valid; the only change is the replacement of $E^\pm_{\mathbf{k}} = \sqrt{(\xi^\pm_{\mathbf{k}})^2 + \Delta^2_{\rm sc}}$ by $E^\pm_{\mathbf{k}} = \sqrt{(\xi^\pm_{\mathbf{k}})^2 + \Delta^2}$. For instance, the diagonal element $\mathcal{G}$ of the full propagator, the fermion number n, and the gap equation for Δ take exactly their mean field forms (8), (9), and (16). Equations (8), (9), and (33) determine self-consistently the order parameter $\Delta_{\rm sc}$, the pseudogap $\Delta_{\rm pg}$, and the chemical potential μ as functions of temperature T. Note that the pair fluctuation effect is self-consistently included in the coupled equations through the pseudogap $\Delta_{\rm pg}$. It is necessary to point out that the G_0G approach we used is quite different from the NSR theory. In the NSR theory, the pair fluctuations enter only the number equation via adding a molecule number term [16,17].

However, solving such a coupled set of equations is still rather complicated. Fortunately, the BEC condition allows us to do further approximations for the pair propagator $t_{\rm pg}(q)$. Using the BEC condition $1 - g\chi(0) = 0$, the T matrix can be written as

$$t_{\rm pg}(q) = \frac{-i}{\chi(q) - \chi(0)}. \tag{36}$$

Since the pseudogap is dominated by the gapless pair dispersion in the long-wavelength limit, we can expand the susceptibility around $q = 0$ in this limit,

$$t_{\rm pg}(q) \simeq \frac{-i}{Z_1 q_0 + Z_2 q_0^2 - \xi^2 \mathbf{q}^2 + i\Gamma(q)}, \tag{37}$$

where the coefficients Z_1, Z_2, and ξ^2 are defined as

$$Z_1 = \frac{\partial \chi}{\partial q_0}\bigg|_{q=0}, \qquad Z_2 = \frac{1}{2}\frac{\partial^2 \chi}{\partial q_0^2}\bigg|_{q=0}, \qquad \xi^2 = -\frac{1}{2}\frac{\partial^2 \chi}{\partial \mathbf{q}^2}\bigg|_{q=0}, \tag{38}$$

and we have considered the fact that the susceptibility depends only on $\mathbf{q}^2$. The explicit expressions for Z_1, Z_2, and ξ^2 are listed in the Appendix.

In the symmetry breaking phase where the temperature is low, it is believed that the pairs are long-lived and we can neglect their width Γ. With the expansion for the pair propagator, Eq. (33) now takes a simple form,

$$\Delta^2_{\rm pg} = \frac{1}{Z_2}\int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1 + b(\omega_{\mathbf{q}} - \nu) + b(\omega_{\mathbf{q}} + \nu)}{2\omega_{\mathbf{q}}}, \tag{39}$$

where $b(x) = 1/(e^{\beta x} - 1)$ is the Bose-Einstein distribution function and $\omega_{\mathbf{q}}$ and ν are defined as

$$\omega_{\mathbf{q}} = \sqrt{\nu^2 + c^2\mathbf{q}^2}, \qquad \nu = \frac{Z_1}{2Z_2}, \qquad c^2 = \frac{\xi^2}{Z_2}. \tag{40}$$

The first term on the right-hand side of (39) suffers ultraviolet divergence, but it can be dropped out via renormalization [6,7].

Let us first discuss some conclusions from the above equations without detailed numerical calculations.

(1) At zero temperature, the pseudogap $\Delta_{\rm pg}$ vanishes automatically and the theory is reduced to the BCS mean field approximation [19].

(2) If the coupling is not so strong that the molecule binding energy E_b satisfies $E_b \ll 2m$, the theory is reduced to its nonrelativistic version [6] for systems with $k_f \ll m$ or $n \ll m^3$.
If Z_1 dominates the propagator $t_{\rm pg}$, the pair dispersion is quadratic in $|\mathbf{q}|$, and the pseudogap $\Delta_{\rm pg}$ can be analytically integrated out and is proportional to $T^{3/4}$ at low temperature. On the other hand, if Z_2 is the dominant term, the pair dispersion is linear in $|\mathbf{q}|$ and $\Delta_{\rm pg}$ becomes proportional to T at low temperature. In the next section, we will show that the first case happens in the NBEC region and the second case occurs in the RBEC region.

(3) From the explicit expression of Z_1 shown in the Appendix,

$$Z_1 = \frac{1}{\Delta^2}\left[\frac{n}{2} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}(f(\xi^-_{\mathbf{k}}) - f(\xi^+_{\mathbf{k}}))\right], \tag{41}$$

the quantity in the square brackets is just the total number density $n_{\rm B}$ of the bound pairs (bosons),

$$n_{\rm B} = Z_1 \Delta^2. \tag{42}$$

From the relation $\Delta^2 = \Delta^2_{\rm sc} + \Delta^2_{\rm pg}$, $n_{\rm B}$ can be decomposed into the condensed-pair number $n_{\rm sc}$ and the uncondensed-pair number $n_{\rm pg}$,

$$n_{\rm sc} = Z_1 \Delta^2_{\rm sc}, \qquad n_{\rm pg} = Z_1 \Delta^2_{\rm pg}. \tag{43}$$

The fraction of the condensed pairs can be defined by

$$P_c = \frac{n_{\rm sc}}{n/2} = \frac{2Z_1\Delta^2_{\rm sc}}{n}. \tag{44}$$

(4) In the weak coupling BCS region, we expect the fermion number density

$$n \simeq 2\int \frac{d^3\mathbf{k}}{(2\pi)^3}(f(\xi^-_{\mathbf{k}}) - f(\xi^+_{\mathbf{k}})) \tag{45}$$

which leads to $n_B = 0$ in this region. In the deep BEC region, however, almost all the fermions form two-body bound states which results in $n_B \simeq n/2$. At zero temperature, we have $\Delta_{pg} = 0$, $n_B = n_{sc}$, and $P_c = 1$, while at the critical temperature T_c, the order parameter Δ_{sc} disappears, and the uncondensed-pair number n_{pg} becomes dominant and is approximately equal to $n/2$.

Numerically, the transition temperature T_c can be calculated from (39) and the generalized equations (8) and (9) by setting $\Delta_{sc} = 0$. Usually, at and above T_c where the order parameter Δ_{sc} disappears, the pseudogap Δ_{pg} does not vanish. We can define a limit temperature T^* where the pseudogap starts to disappear. Between the two temperatures T_c and T^* is the so-called pseudogap phase. While the present G_0G formalism is likely valid only in the symmetry breaking phase with $T \leq T_c$, it can be generalized to the region above T_c by introducing a nonvanishing pair chemical potential μ_{pair} [7]. We will do such a generalization, but the numerical results in the following will be presented mainly at $T \leq T_c$.

Above the critical temperature T_c, the order parameter Δ_{sc} vanishes, and the BEC condition is no longer valid, $1 - g\chi(0) \neq 0$. As a consequence, the propagator of the pair takes the form

$$t_{pg}(q) = \frac{-i}{\chi(q) - \chi(0) - Z_0} \tag{46}$$

with $Z_0 = 1/g - \chi(0)$. As an estimation of Δ_{pg} and T^*, we still perform the expansion for the susceptibility around $q = 0$,

$$t_{pg}(q) \simeq \frac{-i}{Z_1 q_0 + Z_2 q_0^2 - \xi^2 |\mathbf{q}|^2 - Z_0 + i\Gamma(q)}. \tag{47}$$

Now the pseudogap equation becomes

$$\Delta_{pg}^2 = \frac{1}{Z_2} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{b(\omega'_{\mathbf{q}} - \nu) + b(\omega'_{\mathbf{q}} + \nu)}{2\omega'_{\mathbf{q}}} \tag{48}$$

with the definition

$$\omega'_{\mathbf{q}} = \sqrt{\nu^2 + \lambda^2 + c^2 \mathbf{q}^2}, \qquad \lambda^2 = Z_0/Z_2. \tag{49}$$

Equation (48) together with the number equation (9) determines the pseudogap Δ_{pg} and chemical potential μ above T_c. Since the pair dispersion is now no longer gapless in the long-wavelength limit, and Z_0 will generally increase with temperature, we expect that Δ_{pg} will drop down and approach zero at the dissociation temperature T^*.

In the end of this section, we discuss the thermodynamics of the system. The naive BCS mean field theory does not include the contribution from the uncondensed bosons which, however, dominate the thermodynamics at strong coupling. In the present theory, considering the uncondensed pairs, the total thermodynamic potential Ω contains both the fermion and boson contributions,

$$\Omega = \Omega_{cond} + \Omega_{fermion} + \Omega_{boson}, \tag{50}$$

where Ω_{cond} is from the condensed pairs,

$$\Omega_{cond} = \frac{\Delta_{sc}^2}{g}, \tag{51}$$

$\Omega_{fermion}$ from the fermion excitations,

$$\Omega_{fermion} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[(\xi_{\mathbf{k}}^+ + \xi_{\mathbf{k}}^- - E_{\mathbf{k}}^+ - E_{\mathbf{k}}^-) - \frac{1}{\beta}(\ln(1 + e^{-\beta E_{\mathbf{k}}^+}) + \ln(1 + e^{-\beta E_{\mathbf{k}}^-})) \Big], \tag{52}$$

and Ω_{boson} from the uncondensed pairs,

$$\Omega_{boson} = \sum_q \ln[1 - g\chi(q)]. \tag{53}$$

Under the approximation (37) for the pair propagator, the boson part in the symmetry breaking phase can be evaluated as

$$\Omega_{boson} = \frac{1}{\beta} \int \frac{d^3\mathbf{q}}{(2\pi)^3} [\ln(1 - e^{-\beta\omega_{\mathbf{q}}^+}) + \ln(1 - e^{-\beta\omega_{\mathbf{q}}^-})] \tag{54}$$

with $\omega_{\mathbf{q}}^{\pm} = \omega_{\mathbf{q}} \pm \nu$.

There exist two limiting cases for the boson contribution. If Z_1 dominates the pair propagator, the pair dispersion is quadratic in $\mathbf{q}$, and Ω_{boson} recovers the thermodynamic potential of a nonrelativistic boson gas,

$$\Omega_{boson}^{NR} = \frac{1}{\beta} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \ln(1 - e^{-\beta\mathbf{q}^2/(2m_B)}). \tag{55}$$

On the other hand, if Z_2 dominates the pair properties, the pair dispersion is linear in $|\mathbf{q}|$ and we obtain the thermodynamic potential for an ultrarelativistic boson gas

$$\Omega_{boson}^{UR} = \frac{2}{\beta} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \ln(1 - e^{-\beta c|\mathbf{q}|}). \tag{56}$$

As we will show below, the former and the latter correspond to the NBEC and RBEC regions, respectively. The bosons and fermions behave very differently in thermodynamics. As is well known, the specific heat C of an ideal boson gas is proportional to T^α with $\alpha = 3/2$ and 3 corresponding to nonrelativistic and ultrarelativistic systems, but the naive BCS mean field theory predicts an exponential law $C \propto e^{-\Delta_0/T}$, where Δ_0 is the gap at zero temperature, $\Delta_0 = \Delta(T = 0)$.

IV. BCS-NBEC-RBEC CROSSOVER WITH MASSIVE FERMIONS

In this section, we study the BCS-BEC crossover when the coupling constant g increases. Since our model is nonrenormalizable, a proper regularization is needed. In the case with massive fermions, we employ the often used nonrelativistic regularization, namely, to replace the bare coupling g by a renormalized coupling U [17,19],

$$-\frac{1}{U} = \frac{1}{g} - \frac{1}{2}\int_{|\mathbf{k}|\leq\Lambda}\frac{d^3\mathbf{k}}{(2\pi)^3}\left(\frac{1}{\epsilon_{\mathbf{k}} - m} + \frac{1}{\epsilon_{\mathbf{k}} + m}\right). \quad (57)$$

The effective s-wave scattering length a_s can be related to U by $U = 4\pi a_s/m$. While this is a natural extension of the nonrelativistic regularization to relativistic systems, the ultraviolet divergence cannot be completely removed, and a cutoff Λ still exists in the theory. In this regularization, the solution of the coupled equations depends on three dimensionless parameters: the effective coupling constant $\eta = 1/(k_f a_s)$, the quantity $\zeta = k_f/m$ which reflects the fermion number density, and the cutoff Λ/m.

We assume in this section that the fermion density n is not very high and satisfies the relation $n < m^3$ or $\zeta < 1$. In this case the system is not ultrarelativistic and can even be treated nonrelativistically in some parameter region. From the study in NSR theory above T_c and in the BCS-Leggett theory at $T = 0$, if the dimensionless coupling η varies from $-\infty$ to $+\infty$, the system will undergo two crossovers [16–19], the crossover from the BCS state to the NBEC state around $\eta \sim 0$ and the crossover from the NBEC state to the RBEC state around $\eta \sim \zeta^{-1}$. The NBEC state and the RBEC state are characterized by the molecule binding energy E_b. We have $E_b \ll 2m$ in the NBEC state and $E_b \sim 2m$ in the RBEC state.

(1) *BCS region.*—In the weak coupling BCS region, there exist no bound pairs in the system. In this case, Z_1 is sufficiently small and Z_2 dominates the pair dispersion [7], and we have $\Delta_{\rm pg}^2 \propto 1/(Z_2 c^3)$ after a simple algebra. Since Δ should be small in the weak coupling region and c can be proven to be approximately equal to the Fermi velocity [7], the pseudogap $\Delta_{\rm pg}$ is very small and can be safely neglected in this region, as we expected. Therefore, the BCS mean field approximation is good enough at any temperature, and the critical temperature satisfies the well-known relation $T_c \simeq 0.57\Delta_0$. For example, in the nonrelativistic limit with $k_f \ll m$, the antifermion degrees of freedom can be ignored and the pair susceptibility recovers its nonrelativistic version [6]; see the result in the Appendix. The critical temperature can be expressed as [6]

$$T_c = \frac{8e^{\gamma-2}}{\pi}\epsilon_f e^{2\eta/\pi}, \quad (58)$$

where γ is the Euler constant and $\epsilon_f = k_f^2/(2m)$ is the Fermi kinetic energy. In this region, even though Z_2 dominates the pair dispersion, we can show that $c \propto \Delta$ is vanishingly small due to the weak coupling. Since the boson contribution to thermodynamics can be neglected, the specific heat at low temperature takes the well-known form $C \propto e^{-\Delta_0/T}$.

(2) *NBEC region.*—In the nonrelativistic BEC region with $\eta > 1$ but $\eta \ll \zeta^{-1}$ [19], the molecule binding energy E_b is much less than $2m$ (namely, $|\mu - m| \ll m$), the boson mass is approximately $2m$, and the system can be regarded as a nonrelativistical boson gas, if k_f/m is small enough. Assuming $k_f \ll m$, the antifermion degrees of freedom can be neglected, and we can recover the nonrelativistic result [6]. In this region, the gap Δ becomes as large as the Fermi kinetic energy ϵ_f. From $Z_1 \propto 1/\Delta^2$ and $Z_2 \propto 1/\Delta^4$, Z_1 is the dominant one and the pair dispersion becomes quadratic in $|\mathbf{q}|$. In this case, the propagator of the uncondensed pairs can be approximated by

$$t_{\rm pg}(q) \simeq \frac{-iZ_1^{-1}}{q_0 - |\mathbf{q}|^2/(2m_{\rm B})}, \quad (59)$$

where the pair mass $m_{\rm B}$ is defined by $m_{\rm B} = Z_1/2\xi^2$, and we have the simple relation

$$Z_1\Delta_{\rm pg}^2 = \int\frac{d^3\mathbf{q}}{(2\pi)^3}b\left(\frac{|\mathbf{q}|^2}{2m_{\rm B}}\right) = \left(\frac{m_{\rm B}T}{2\pi}\right)^{3/2}\zeta\left(\frac{3}{2}\right). \quad (60)$$

Since $Z_1\Delta_{\rm pg}^2$ is equal to the total boson density $n_{\rm B}$ at $T = T_c$, we arrive at the standard critical temperature for Bose-Einstein condensation in nonrelativistic boson gas [20],

$$T_c = \frac{2\pi}{m_{\rm B}}\left(\frac{n_{\rm B}}{\zeta(\frac{3}{2})}\right)^{2/3}. \quad (61)$$

The boson mass $m_{\rm B}$ is generally expected to be equal to the boson chemical potential $\mu_{\rm B} = 2\mu$. In the nonrelativistic limit $k_f \ll m$, we can show $m_{\rm B} \simeq 2m$ and $Z_1\Delta_{\rm pg}^2 \simeq n/2$ at $T = T_c$; the critical temperature becomes $T_c = 0.218\epsilon_f$. Since Z_1 dominates the pair dispersion, the pseudogap is proportional to $T^{3/4}$ and the specific heat is proportional to $T^{3/2}$ at low temperature.

(3) *RBEC region.*—In this region we have the molecule binding energy $E_b \to 2m$ and chemical potential $\mu \to 0$. In this case, the nonrelativistic limit cannot be reached even for $k_f \ll m$ [19]. Since the bosons with mass $m_{\rm B} = 2\mu$ become nearly massless in this region, the antibosons and antifermions can be easily excited, and the system contains both bosons and antibosons. From the relation

$$n_{\rm B} = n_{\rm b} - n_{\bar{\rm b}} = Z_1\Delta_{\rm pg}^2 \quad (62)$$

at $T = T_c$, where $n_{\rm b}$ and $n_{\bar{\rm b}}$ are the boson and antiboson numbers, while $n_{\rm b}$ and $n_{\bar{\rm b}}$ are both very large, their difference produces a small, pure boson density $n_{\rm B} \simeq n/2$. On the other hand, for $\mu \to 0$ we can expand Z_1 in powers of the chemical potential μ,

$$Z_1 \simeq R\mu + O(\mu^3) = \frac{R}{2} m_B + O(\mu^3), \tag{63}$$

and hence Z_2 dominates the pair dispersion, which means that the pseudogap is proportional to T at low temperature. In this case, the propagator of the uncondensed pairs can be approximated by

$$t_{pg}(q) \simeq \frac{-iZ_2^{-1}}{q_0^2 - c^2|\mathbf{q}|^2}, \tag{64}$$

which leads to the relation

$$Z_2\Delta_{pg}^2 \simeq \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{b(c|\mathbf{q}|)}{c|\mathbf{q}|} = \frac{T^2}{12c^3}. \tag{65}$$

Combining the above equations, we find

$$T_c = \left(\frac{24c^3 Z_2}{R} \frac{n_B}{m_B}\right)^{1/2}. \tag{66}$$

In the RBEC limit $\mu \to 0$, we can approach the standard critical temperature for ultrarelativistic Bose-Einstein condensation [21,22],

$$T_c = \left(\frac{3n_B}{m_B}\right)^{1/2}. \tag{67}$$

Since n_B is almost fixed and $m_B \to 0$, T_c would approach infinity in the RBEC limit. In the ultrarelativistic boson gas, the specific heat at low temperature is proportional to T^3.

We now turn to numerical calculations. From the coupled equations (8), (9), and (39), we can solve the critical temperature T_c, chemical potential $\mu(T_c)$, and pseudogap $\Delta_{pg}(T_c)$ at T_c as functions of the coupling η at fixed k_f/m. In Fig. 2 we show the numerical results with the parameters $\Lambda/m = 10$ and $k_f/m = 0.5$. The BCS-NBEC-RBEC crossover can be seen directly from the behavior of the chemical potential μ. In the BCS region with $-\infty < \eta < 0.5$, μ is larger than the fermion mass and approaches the Fermi energy E_f in the weak coupling limit $\eta \to -\infty$. The NBEC region is located around $-0.5 < \eta < 4$ and the RBEC region is at about $\eta > 4$. The critical coupling $\eta \simeq 4$ for the RBEC state is consistent with our analytical result

$$\eta_c = \frac{2}{\pi}\left(\frac{k_f}{m}\right)^{-1} \ln(\Lambda/m + \sqrt{(\Lambda/m)^2 + 1}) \tag{68}$$

derived in [19]. The difference between NBEC and RBEC states is that the chemical potential μ is of the order of m in the NBEC region but approaches zero in the RBEC region.

The critical temperature, plotted as the solid line in Fig. 2(a), shows significant change from the weak to strong coupling. To compare it with the standard critical temperature for the ideal boson gas, we solve the equation [21]

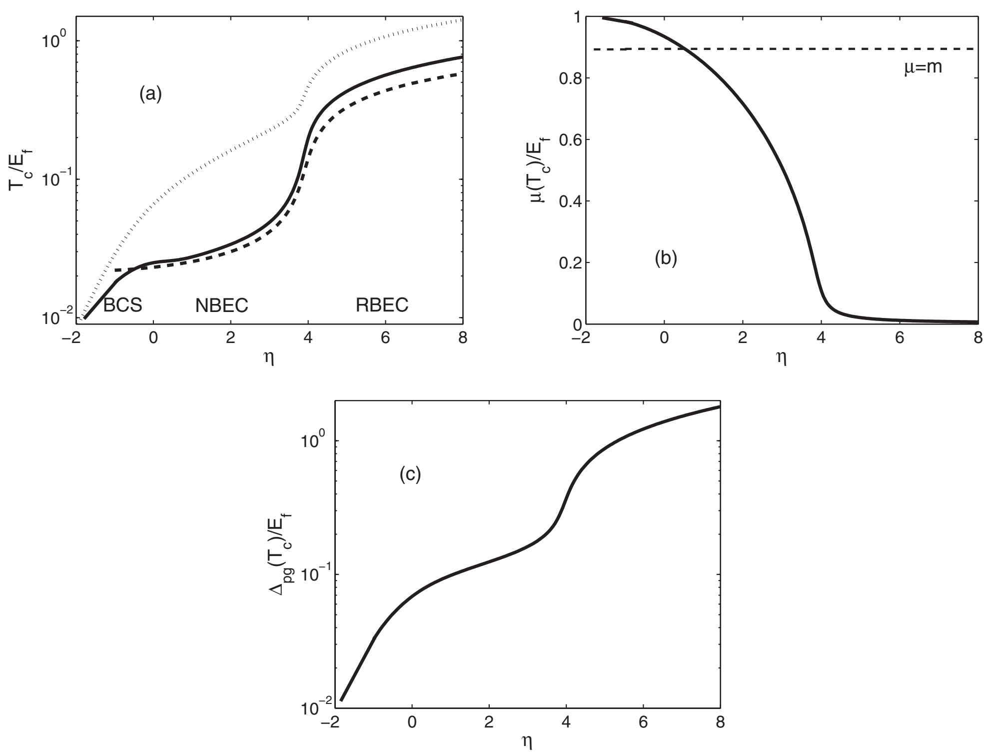

FIG. 2. The critical temperature T_c (a), chemical potential $\mu(T_c)$ (b), and pseudogap $\Delta_{pg}(T_c)$ (c) as functions of coupling η at $\Lambda/m = 10$ and $k_f/m = 0.5$. T_c, μ and Δ_{pg} are all scaled by the Fermi energy E_f. The dashed line is the standard critical temperature for the ideal boson gas in (a) and stands for the position $\mu = m$ in (b); the dotted line in (a) is the limit temperature T^* where the pseudogap starts to disappear.

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3}[b(\epsilon_{\mathbf{q}}^{\mathrm{B}} - \mu_{\mathrm{B}}) - b(\epsilon_{\mathbf{q}}^{\mathrm{B}} + \mu_{\mathrm{B}})]|_{\mu_{\mathrm{B}}=m_{\mathrm{B}}} = n_{\mathrm{B}} \quad (69)$$

with $\epsilon_{\mathbf{q}}^{\mathrm{B}} = \sqrt{\mathbf{q}^2 + m_{\mathrm{B}}^2}$, boson mass $m_{\mathrm{B}} = 2\mu$, and boson number $n_{\mathrm{B}} = n/2$, and show the obtained critical temperature as a dashed line in Fig. 2(a). In the weak coupling region T_c is very small and agrees with the BCS theory. In the NBEC region T_c changes smoothly and there is no remarkable difference between the solid and dashed lines. Around the coupling $\eta_c = 4$, T_c increases rapidly and then varies smoothly again. In the RBEC region, the critical temperature from our calculation deviates significantly from the standard critical temperature for ideal boson gas. Note that T_c is of the order of the Fermi kinetic energy $\epsilon_f \simeq k_f^2/(2m)$ in the NBEC region but becomes as large as the Fermi energy E_f in the RBEC region. The pseudogap Δ_{pg} at $T = T_c$, shown in Fig. 2(c), behaves similarly as the critical temperature. To see clearly the pseudogap region, we present in Fig. 2(a) the limit temperature T^* as a dotted line. The pseudogap exists between the solid and dotted lines and begins to vanish on the dotted line.

To explain why the critical temperature in the RBEC region deviates remarkably from the standard one for ideal boson gas, we calculate the boson number fraction $r_{\mathrm{B}} = n_{\mathrm{B}}/(n/2)$ and the fermion number fraction $r_{\mathrm{F}} = 1 - r_{\mathrm{B}}$ and show them as functions of the coupling η in Fig. 3. While there are only bosons at T_c in the NBEC region, r_{B} is obviously less than 1 in the RBEC region. This conclusion is consistent with the results from the NSR theory [16,17]. In the NBEC region, the binding energy of the molecules is $E_b \simeq 1/ma_s^2 = 2\eta^2\epsilon_f$, which is much larger than the critical temperature $T_c \simeq 0.2\epsilon_f$, and the molecules can be safely regarded as point bosons at temperatures near T_c. However, the critical temperature in the RBEC region is as large as the Fermi energy E_f, which is of the order of the molecule binding energy $E_b \simeq 2m$. Because of the competition between the condensation and dissociation of composite bosons in a hot medium, the molecules cannot be regarded as point bosons and the critical temperature should deviate from the result for ideal boson gas. This may be a general characteristic of a composite boson system, when the condensation temperature T_c is of the order of the molecule binding energy. The phenomenon can be explained by the competition between free energy and entropy [17]: In terms of entropy, a two-fermion state is more favorable than a one-boson state, but in terms of free energy it is less favorable. Since the condensation temperature T_c in the RBEC region is of the order of $(n_{\mathrm{B}}/m_{\mathrm{B}})^{1/2} \sim (n/\mu)^{1/2}$, we conclude that, only for a system with a sufficiently small value of k_f/m, the standard RBEC critical temperature can be reached and is much smaller than $2m$.

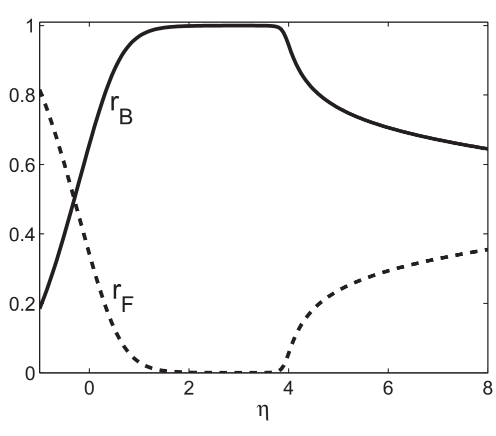

FIG. 3. The boson number fraction r_{B} and the fermion number fraction r_{F} at the critical temperature T_c as functions of the coupling η at $\Lambda/m = 10$ and $k_f/m = 0.5$.

V. APPLICATION TO MASSLESS FERMIONS: COLOR SUPERCONDUCTIVITY

As a natural application of the relativistic G_0G formalism, we calculate in this section the transition temperature and pseudogap in two-flavor color superconductivity at moderate baryon density. The two-flavor color superconducting quark matter corresponds to the ultrarelativistic case with $n \gg m^3$, where m is the current quark mass. At moderate baryon density, the quark energy gap due to color superconductivity is of the order of 100 MeV, which is not located in the weak coupling region. As a result, the pseudogap effect is expected to be significantly important near the critical temperature. To apply the present theory directly, we employ the generalized Nambu–Jona-Lasinio (NJL) model with the scalar diquark channel, which has been widely used to study color superconductivity at moderate baryon density. The Lagrangian density is defined as

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + G_s[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2] \\ &\quad + G_d\sum_{a=2,5,7}(\bar{\psi}i\gamma^5\tau_2\lambda_a C\bar{\psi}^T)(\psi^T Ci\gamma^5\tau_2\lambda_a\psi), \quad (70)\end{aligned}$$

where ψ and $\bar{\psi}$ denote the quark fields with two flavors ($N_f = 2$) and three colors ($N_c = 3$), $\tau_i (i = 1, 2, 3)$ are the Pauli matrices in flavor space, $\lambda_a (a = 1, 2, \ldots, 8)$ are the Gell-Mann matrices in color space, and G_s and G_d are coupling constants in meson and diquark channels.

At moderate baryon density, the chiral symmetry has already been restored and we need not consider the chiral condensate $\langle\bar{\psi}\psi\rangle$. Since the current quark mass m is about 5 MeV, the quarks are nearly massless. The order parameter field for color superconductivity is defined as

$$\Phi_a = -2G_d\psi^T Ci\gamma^5\tau_2\lambda_a\psi. \quad (71)$$

To simplify the calculation, one usually considers a spontaneous color breaking from the SU(3) symmetry to an SU(2) subgroup. Because of the residual color SU(2) symmetry, the effective potential in the mean field approximation should depend only on the combination

$\Delta_2^2 + \Delta_5^2 + \Delta_7^2$ with $\Delta_a = \langle\Phi_a\rangle$, and we can choose a specific gauge $\Delta_{\rm sc} = \Delta_2 \neq 0$, $\Delta_5 = \Delta_7 = 0$ without loss of generality. In this gauge, the red and green quarks participate in the condensation, but the blue one does not.

The detailed formalism of the G_0G theory in the NJL model is similar to what we show in Secs. II and III but becomes somewhat complicated due to the presence of color and flavor degrees of freedom. Comparing the quark propagator in the NJL model with the one shown in the above sections, the relativistic G_0G scheme can be directly applied to the study of color superconductivity, provided that we consider carefully the difference between the pairing including a blue quark and the pairing with only red and green quarks. The dispersion for red and green quarks is identical to the one obtained in the toy model; their excitation gap is $\Delta = (\Delta_{\rm sc}^2 + \Delta_{\rm pg}^2)^{1/2}$, and the pair susceptibility $\chi(q)$ should be multiplied by a factor $N_f(N_c - 1)$ where N_f and N_c are flavor and color numbers of quarks. The new feature is that a gapless blue quark in the naive BCS mean field theory obtains a gap $\Delta_{\rm pg}$ in the G_0G scheme. This can be understood by the fact that the color symmetry is controlled only by the order parameters themselves, and fluctuations of any order parameter field Φ_a do not change it. At and above the critical temperature, $\Delta_{\rm sc} = 0$, the color symmetry is restored, all colors become degenerate, and their gaps are just the pseudogap.

The two-flavor quark matter may exist in the region of $\mu = 350$–500 MeV, where the strange quarks are not yet excited. Unlike the study in the above sections in the canonical ensemble with fixed fermion number, people usually investigate color superconductivity in the grand canonical ensemble with a fixed quark chemical potential. In this case, the quark number is not directly coupled to the calculation of the order parameter $\Delta_{\rm sc}$ and pseudogap $\Delta_{\rm pg}$. For numerical calculations, we take the current quark mass $m = 5$ MeV, the often used quark momentum cutoff $\Lambda = 650$ MeV, and a fixed quark chemical potential $\mu = 400$ MeV. We have checked that a reasonable change in the value of μ does not bring qualitative difference. As is conventionally considered in the literature, we use the pairing gap Δ_0 at zero temperature to reflect the strength of the diquark coupling constant G_d.

In Fig. 4 we show the critical temperature T_c as a function of Δ_0 in the G_0G theory and in the BCS mean field theory. While the critical temperature is not strongly modified by the diquark fluctuations in a wide range of Δ_0, the difference between the two is up to 20% in the strong coupling region with $\Delta_0 \simeq 200$ MeV. In Fig. 5, we show the pseudogap $\Delta_{\rm pg}$ at the critical temperature T_c. In a wide range of the coupling, the pseudogap is of the order of 100 MeV, which is as large as the diquark condensate $\Delta_{\rm sc}$ at zero temperature. Such a behavior means that the two-flavor color superconductivity at moderate density is in the BCS-BEC crossover region and quite like the high temperature superconductivity in cuprates [6,7]. Since $\Delta_{\rm sc}$ vanishes at $T = T_c$, the large pseudogap will bring a significant effect at and above T_c, such as the non-Fermi liquid behavior. In Fig. 6, we show the temperature dependence of the diquark condensate $\Delta_{\rm sc}$ and pseudogap $\Delta_{\rm pg}$ at two values of Δ_0. With increasing temperature, while the diquark condensate decreases, the pseudogap increases from zero. At low temperature, especially at zero temperature, we can safely neglect the pseudogap.

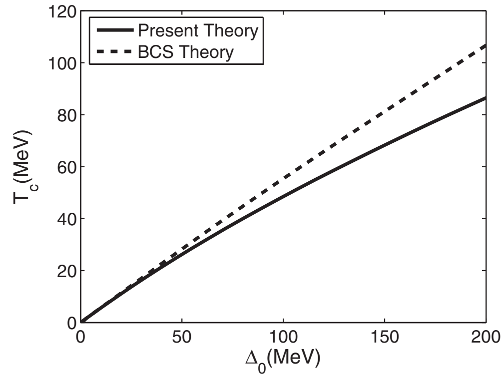

FIG. 4. The phase transition temperature T_c for two-flavor color superconductivity as a function of the diquark condensate Δ_0 at zero temperature in the BCS mean field theory (dashed line) and in the G_0G theory with diquark fluctuations (solid line).

While the pseudogap is small at low temperature and dominates the system only near and above T_c, the diquark fluctuations bring a significant contribution to thermodynamics at any temperature. In the low temperature region, the temperature behavior of the pseudogap is significantly important, since it can tell us whether the coefficient Z_1 or Z_2 dominates the pair fluctuations. In Fig. 7 we show the pseudogap at low temperature. In the region of $T/T_c \leq 0.1$, it obeys a perfect power law $\Delta_{\rm pg} \propto T^{3/4}$, which means that Z_1 is the dominant one for the pair susceptibility.

Considering the uncondensed diquarks, the total thermodynamic potential Ω can be expressed as

$$\Omega = \Omega_{\rm cond} + \Omega_{\rm quark} + \Omega_{\rm diquark}, \tag{72}$$

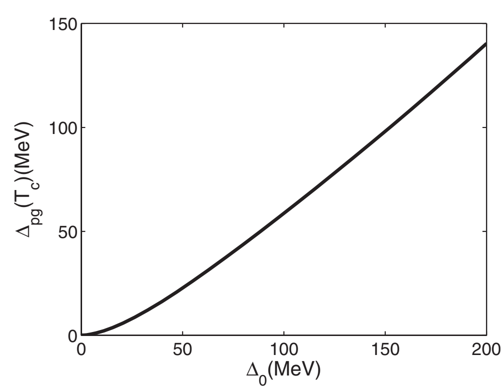

FIG. 5. The pseudogap $\Delta_{\rm pg}$ in two-flavor color superconductivity at the critical temperature T_c as a function of Δ_0.

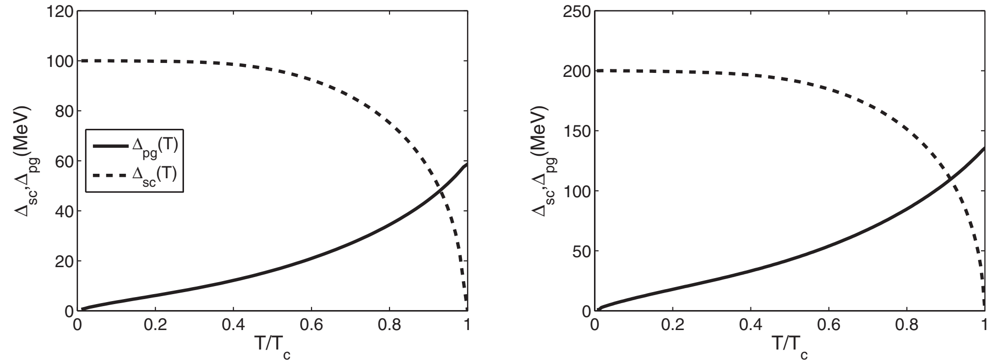

FIG. 6. The diquark condensate Δ_{sc} (dashed lines) and pseudogap Δ_{pg} (solid lines) in two-flavor color superconductivity as functions of temperature scaled by T_c for $\Delta_0 = 100$ MeV (left panel) and 200 MeV (right panel).

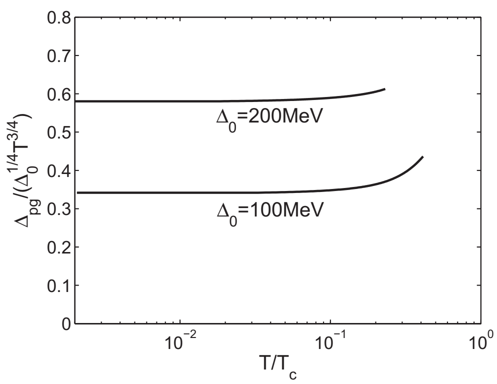

FIG. 7. The temperature dependence of the pseudogap Δ_{pg} scaled by $\Delta_0^{1/4}T^{3/4}$ for $\Delta_0 = 100$ MeV and 200 MeV.

where the condensate and quark contributions take the same form as in the BCS theory, and the diquark contribution can be written as

$$\Omega_{\text{diquark}} = \sum_q \ln[1 - 4G_d\chi(q)]. \tag{73}$$

Since the coefficient Z_1 controls the pair fluctuations at low temperature, the specific heat satisfies the power law $C \propto T^{3/2}$. As we mentioned above, the diquark contribution can be neglected only at sufficiently weak coupling. While the color superconductor at moderate baryon density may not reach the BEC condition, the effect of diquark fluctuations on the thermodynamics may be remarkable, and it may bring significant astrophysical consequences, such as the cooling process in compact stars.

VI. SUMMARY

We have generalized the nonrelativistic G_0G formalism of BCS-BEC crossover to relativistic fermion systems. The theory can describe the superfluidity/superconductivity with arbitrary strength of attractive interaction, both in the symmetric phase and the symmetry breaking phase. The beyond-BCS effect at strong coupling brings in thermally excited bosons and contributes a pseudogap to fermion excitations. In such a formalism, we confirmed that there exists a BCS-NBEC-RBEC crossover in relativistic fermion systems.

For color superconductivity at moderate baryon density, while the BEC state cannot be reached, the effect of diquark fluctuations is still remarkable and the naive BCS mean field theory breaks down when the temperature is close to the critical value. We investigate the two-flavor color superconductivity at a quark chemical potential $\mu =$ 350–500 MeV where the gap at zero temperature is of the order of 100 MeV. We found that the beyond-BCS effect strongly suppresses the transition temperature, and the pseudogap is very large near the critical temperature. This may strongly modify the thermodynamics of quark matter and bring significant astrophysical consequences in the study of compact stars.

Such a theory can be applied to not only the diquark condensate ($\langle qq\rangle$) at finite baryon density but also the chiral condensate ($\langle q\bar{q}\rangle$) at finite temperature and pion superfluidity at finite isospin density. The observation of $q\bar{q}$ bound states in strongly coupled quark-gluon plasma [26] and a large thermal quark mass above the chiral phase transition temperature in lattice QCD [27] indicates strongly the significance of the $q\bar{q}$ Bose-Einstein condensation and the quark pseudogap effect [12,13]. The study in this direction is in progress [28].

ACKNOWLEDGMENTS

We thank Dr. Mei Huang for drawing our attention to [6,7] and Gaofeng Sun for a careful check on the calculations in the Appendix. This work is supported by Grant No. NSFC10428510, No. 10575058, and the major state basic research developing program 2007CB815000.

APPENDIX: PAIR SUSCEPTIBILITY AND ITS EXPANSION COEFFICIENTS

In this appendix, we evaluate the explicit expression of the pair susceptibility and its momentum expansion.

Completing the trace in Dirac space and the Matsubara summation over the fermion frequencies, we obtain from Eqs. (11), (16), and (30)

$$\chi(q) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\bigg[\bigg(\frac{1 - f(E_\mathbf{k}^-) - f(\xi_{\mathbf{q}-\mathbf{k}}^-)}{E_\mathbf{k}^- + \xi_{\mathbf{q}-\mathbf{k}}^- - q_0}\frac{E_\mathbf{k}^- + \xi_\mathbf{k}^-}{2E_\mathbf{k}^-} - \frac{f(E_\mathbf{k}^-) - f(\xi_{\mathbf{q}-\mathbf{k}}^-)}{E_\mathbf{k}^- - \xi_{\mathbf{q}-\mathbf{k}}^- + q_0}\frac{E_\mathbf{k}^- - \xi_\mathbf{k}^-}{2E_\mathbf{k}^-}\bigg)\bigg(\frac{1}{2} - \frac{\mathbf{k}\cdot(\mathbf{q}-\mathbf{k}) - m^2}{2\epsilon_\mathbf{k}\epsilon_{\mathbf{q}-\mathbf{k}}}\bigg)$$
$$+ \bigg(\frac{1 - f(E_\mathbf{k}^-) - f(\xi_{\mathbf{q}-\mathbf{k}}^+)}{E_\mathbf{k}^- + \xi_{\mathbf{q}-\mathbf{k}}^+ + q_0}\frac{E_\mathbf{k}^- - \xi_\mathbf{k}^-}{2E_\mathbf{k}^-} - \frac{f(E_\mathbf{k}^-) - f(\xi_{\mathbf{q}-\mathbf{k}}^+)}{E_\mathbf{k}^- - \xi_{\mathbf{q}-\mathbf{k}}^+ - q_0}\frac{E_\mathbf{k}^- + \xi_\mathbf{k}^-}{2E_\mathbf{k}^-}\bigg)\bigg(\frac{1}{2} + \frac{\mathbf{k}\cdot(\mathbf{q}-\mathbf{k}) - m^2}{2\epsilon_\mathbf{k}\epsilon_{\mathbf{q}-\mathbf{k}}}\bigg)\bigg]$$
$$+ (E_\mathbf{k}^\pm, \xi_\mathbf{k}^\pm, q_0 \to E_\mathbf{k}^\mp, \xi_\mathbf{k}^\mp, -q_0). \tag{A1}$$

Taking its first and second order derivatives with respect to q_0, we have

$$Z_1 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E_\mathbf{k}^-}\bigg[\frac{1 - f(E_\mathbf{k}^-) - f(\xi_\mathbf{k}^-)}{E_\mathbf{k}^- + \xi_\mathbf{k}^-} + \frac{f(E_\mathbf{k}^-) - f(\xi_\mathbf{k}^-)}{E_\mathbf{k}^- - \xi_\mathbf{k}^-}\bigg] - (E_\mathbf{k}^\pm, \xi_\mathbf{k}^\pm \to E_\mathbf{k}^\mp, \xi_\mathbf{k}^\mp),$$
$$Z_2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E_\mathbf{k}^-}\bigg[\frac{1 - f(E_\mathbf{k}^-) - f(\xi_\mathbf{k}^-)}{(E_\mathbf{k}^- + \xi_\mathbf{k}^-)^2} - \frac{f(E_\mathbf{k}^-) - f(\xi_\mathbf{k}^-)}{(E_\mathbf{k}^- - \xi_\mathbf{k}^-)^2}\bigg] + (E_\mathbf{k}^\pm, \xi_\mathbf{k}^\pm \to E_\mathbf{k}^\mp, \xi_\mathbf{k}^\mp). \tag{A2}$$

Using the relation $(E_\mathbf{k}^\pm)^2 - (\xi_\mathbf{k}^\pm)^2 = \Delta^2$, the coefficients can be rewritten as

$$Z_1 = \frac{1}{2\Delta^2}\bigg[n - 2\int \frac{d^3\mathbf{k}}{(2\pi)^3}(f(\xi_\mathbf{k}^-) - f(\xi_\mathbf{k}^+))\bigg],$$
$$Z_2 = \frac{1}{2\Delta^4}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\bigg[\frac{(E_\mathbf{k}^-)^2 + (\xi_\mathbf{k}^-)^2}{E_\mathbf{k}^-}(1 - 2f(E_\mathbf{k}^-)) - 2\xi_\mathbf{k}^-(1 - 2f(\xi_\mathbf{k}^-))\bigg] + (E_\mathbf{k}^\pm, \xi_\mathbf{k}^\pm \to E_\mathbf{k}^\mp, \xi_\mathbf{k}^\mp). \tag{A3}$$

Taking the second order derivative of the susceptibility χ with respect to $\mathbf{q}$, we obtain another coefficient,

$$\xi^2 = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\bigg[\frac{1}{2E_\mathbf{k}^-}\bigg(\frac{1 - f(E_\mathbf{k}^-) - f(\xi_\mathbf{k}^-)}{E_\mathbf{k}^- + \xi_\mathbf{k}^-} + \frac{f(E_\mathbf{k}^-) - f(\xi_\mathbf{k}^-)}{E_\mathbf{k}^- - \xi_\mathbf{k}^-}\bigg)\frac{\epsilon_\mathbf{k}^2 - \mathbf{k}^2x^2}{\epsilon_\mathbf{k}^3} - \bigg(\frac{1}{E_\mathbf{k}^-}\bigg(\frac{1 - f(E_\mathbf{k}^-) - f(\xi_\mathbf{k}^-)}{(E_\mathbf{k}^- + \xi_\mathbf{k}^-)^2}$$
$$- \frac{f(E_\mathbf{k}^-) - f(\xi_\mathbf{k}^-)}{(E_\mathbf{k}^- - \xi_\mathbf{k}^-)^2}\bigg) + \frac{2f'(\xi_\mathbf{k}^-)}{\Delta^2}\bigg)\frac{\mathbf{k}^2x^2}{\epsilon_\mathbf{k}^2} - \bigg(\frac{1 - f(E_\mathbf{k}^-) - f(\xi_\mathbf{k}^+)}{E_\mathbf{k}^- + \xi_\mathbf{k}^+}\frac{E_\mathbf{k}^- - \xi_\mathbf{k}^-}{2E_\mathbf{k}^-} - \frac{f(E_\mathbf{k}^-) - f(\xi_\mathbf{k}^+)}{E_\mathbf{k}^- - \xi_\mathbf{k}^+}\frac{E_\mathbf{k}^- + \xi_\mathbf{k}^-}{2E_\mathbf{k}^-}$$
$$- \frac{1 - 2f(E_\mathbf{k}^-)}{2E_\mathbf{k}^-}\bigg)\frac{\epsilon_\mathbf{k}^2 - \mathbf{k}^2x^2}{2\epsilon_\mathbf{k}^4}\bigg] + (E_\mathbf{k}^\pm, \xi_\mathbf{k}^\pm \to E_\mathbf{k}^\mp, \xi_\mathbf{k}^\mp) \tag{A4}$$

with $x = \cos\theta$ and $f'(x)$ being the first order derivative of the Fermi-Dirac distribution.

In the nonrelativistic limit with $k_f \ll m$, $|\mu - m|$, $\Delta \ll m$, all the terms including antifermion dispersions can be safely neglected, and the relativistic dispersions are reduced to $\xi_\mathbf{k} = \mathbf{k}^2/(2m) - (\mu - m)$ and $E_\mathbf{k} = \sqrt{\xi_\mathbf{k}^2 + \Delta^2}$. Taking into account $|\mathbf{q}| \ll m$, we have

$$\chi_{\rm NR}(q) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\bigg[\frac{1 - f(E_\mathbf{k}) - f(\xi_{\mathbf{q}-\mathbf{k}})}{E_\mathbf{k} + \xi_{\mathbf{q}-\mathbf{k}} - q_0}\frac{E_\mathbf{k} + \xi_\mathbf{k}}{2E_\mathbf{k}} - \frac{f(E_\mathbf{k}) - f(\xi_{\mathbf{q}-\mathbf{k}})}{E_\mathbf{k} - \xi_{\mathbf{q}-\mathbf{k}} + q_0}\frac{E_\mathbf{k} - \xi_\mathbf{k}}{2E_\mathbf{k}}\bigg], \tag{A5}$$

which is just the same as the one given in [6,7], and the expansion coefficients Z_1, Z_2, and ξ^2 are reduced to [6,7]

$$Z_1 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E_\mathbf{k}}\bigg[\frac{1 - f(E_\mathbf{k}) - f(\xi_\mathbf{k})}{E_\mathbf{k} + \xi_\mathbf{k}} + \frac{f(E_\mathbf{k}) - f(\xi_\mathbf{k})}{E_\mathbf{k} - \xi_\mathbf{k}}\bigg] = \frac{1}{2\Delta^2}\bigg[n - 2\int \frac{d^3\mathbf{k}}{(2\pi)^3}f(\xi_\mathbf{k})\bigg],$$
$$Z_2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E_\mathbf{k}}\bigg[\frac{1 - f(E_\mathbf{k}) - f(\xi_\mathbf{k})}{(E_\mathbf{k} + \xi_\mathbf{k})^2} - \frac{f(E_\mathbf{k}) - f(\xi_\mathbf{k})}{(E_\mathbf{k} - \xi_\mathbf{k})^2}\bigg]$$
$$= \frac{1}{2\Delta^4}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\bigg[\frac{E_\mathbf{k}^2 + \xi_\mathbf{k}^2}{E_\mathbf{k}}(1 - 2f(E_\mathbf{k})) - 2\xi_\mathbf{k}(1 - 2f(\xi_\mathbf{k}))\bigg],$$
$$\xi^2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\bigg[\frac{1}{4mE_\mathbf{k}}\bigg(\frac{1 - f(E_\mathbf{k}) - f(\xi_\mathbf{k})}{E_\mathbf{k} + \xi_\mathbf{k}} + \frac{f(E_\mathbf{k}) - f(\xi_\mathbf{k})}{E_\mathbf{k} - \xi_\mathbf{k}}\bigg) - \frac{\mathbf{k}^2}{6m^2}\bigg(\frac{1}{E_\mathbf{k}}\bigg(\frac{1 - f(E_\mathbf{k}) - f(\xi_\mathbf{k})}{(E_\mathbf{k} + \xi_\mathbf{k})^2} - \frac{f(E_\mathbf{k}) - f(\xi_\mathbf{k})}{(E_\mathbf{k} - \xi_\mathbf{k})^2}\bigg)$$
$$+ \frac{2f'(\xi_\mathbf{k})}{\Delta^2}\bigg)\bigg]. \tag{A6}$$

In the RBEC limit with $\mu \to 0$, we can expand Z_1 in powers of μ, $Z_1 \simeq R\mu + O(\mu^3)$, with the expansion coefficient R given by

$$R = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{1-2f(E_\mathbf{k})}{E_\mathbf{k}^3} - 2\frac{\epsilon_\mathbf{k}^2}{E_\mathbf{k}^2}\frac{f'(E_\mathbf{k})}{\Delta^2} + 2\frac{f'(\epsilon_\mathbf{k})}{\Delta^2}\right], \tag{A7}$$

where $E_\mathbf{k} = \sqrt{\epsilon_\mathbf{k}^2 + \Delta^2}$ is the dispersion at $\mu = 0$, and Z_2 and ξ^2 can be simplified as

$$\begin{aligned} Z_2 &= \frac{1}{\Delta^4}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{E_\mathbf{k}^2+\epsilon_\mathbf{k}^2}{E_\mathbf{k}}(1-2f(E_\mathbf{k})) - 2\epsilon_\mathbf{k}(1-2f(\epsilon_\mathbf{k}))\right], \\ \xi^2 &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{1}{2\Delta^2}\left((1-2f(\epsilon_\mathbf{k})) - \frac{\epsilon_\mathbf{k}}{E_\mathbf{k}}(1-2f(E_\mathbf{k}))\right)\frac{\epsilon_\mathbf{k}^2 - \mathbf{k}^2x^2}{\epsilon_\mathbf{k}^3} - \left(\frac{1}{\Delta^4}\left(\frac{E_\mathbf{k}^2+\epsilon_\mathbf{k}^2}{E_\mathbf{k}}(1-2f(E_\mathbf{k})) - 2\epsilon_\mathbf{k}(1-2f(\epsilon_\mathbf{k}))\right)\right.\right. \\ &\quad \left.\left. + \frac{2f'(\epsilon_\mathbf{k})}{\Delta^2}\right)\frac{\mathbf{k}^2x^2}{\epsilon_\mathbf{k}^2} - \left(\frac{1}{2\Delta^2}\left(\frac{E_\mathbf{k}^2+\epsilon_\mathbf{k}^2}{E_\mathbf{k}}(1-2f(E_\mathbf{k})) - 2\epsilon_\mathbf{k}(1-2f(\epsilon_\mathbf{k}))\right) - \frac{1-2f(E_\mathbf{k})}{2E_\mathbf{k}}\right)\frac{\epsilon_\mathbf{k}^2 - \mathbf{k}^2x^2}{2\epsilon_\mathbf{k}^4}\right]. \end{aligned} \tag{A8}$$

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