Confining ensemble of dyons

Dmitri Diakonov* and Victor Petrov†

St. Petersburg NPI, Gatchina, 188 300, St. Petersburg, Russia
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We construct the integration measure over the moduli space of an arbitrary number of N kinds of dyons of the pure $SU(N)$ gauge theory at finite temperatures. The ensemble of dyons governed by the measure is mathematically described by a (supersymmetric) quantum field theory that is exactly solvable and is remarkable for a number of striking features: (i) The free energy has the minimum corresponding to the zero average Polyakov line, as expected in the confining phase; (ii) the correlation function of two Polyakov lines exhibits a linear potential between static quarks in any N-ality nonzero representation, with a calculable string tension roughly independent of temperature; (iii) the average spatial Wilson loop falls off exponentially with its area and the same string tension; (iv) at a critical temperature, the ensemble of dyons rearranges and deconfines; and (v) the estimated ratio of the critical temperature to the square root of the string tension is in excellent agreement with the lattice data.

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I. INTRODUCTION

Isolated dyons in the pure Yang-Mills theory are (anti)self-dual solutions of the equation of motion $D_\mu F_{\mu\nu} = 0$, which in an appropriate gauge are Abelian at large distances from the centers and carry unity electric and magnetic charges with respect to the Cartan generators C_m:

$$\pm \mathbf{E}_m = \mathbf{B}_m = \frac{1}{2}\frac{\mathbf{r}}{|\mathbf{r}|^3}C_m. \tag{1}$$

For the $SU(N)$ gauge group on which we focus in this paper, there are $N-1$ Cartan generators associated with the simple roots of the group $C_m = \mathrm{diag}(0, \ldots, 1, -1, 0, \ldots, 0)$, where the 1 is on the mth place, that are supplemented by the Nth generator $C_N = \mathrm{diag}(-1, 0, \ldots, 1)$ to make the set of N dyons electric- and magnetic-neutral. The first $N-1$ dyons are also called the Bogomolny-Prasad-Sommerfield (BPS) monopoles [1]. The last Nth dyon is sometimes called the Kaluza-Klein monopole: In the gauge where the first $N-1$ dyons are described by a static field, the last has time-dependent fields inside the core. However, by a periodic time-dependent gauge transformation, one can make the last one time-independent (at the cost of a time dependence inside other dyons); therefore, the distinction of the last dyon is illusory, and we shall treat all N of them on the same footing. The cyclic symmetry of N dyons is evident from the D-brane point of view [2,3].

In this paper, we explore the properties of a semiclassical vacuum built of a large number of dyons of N kinds.

To make the semiclassical calculation of the Yang-Mills partition function well defined, one needs (i) to expand about a true saddle point of the action and (ii) to be sure that the quantum fluctuation determinant is infrared-finite. For an isolated dyon, the first is true but the second is false. For an arbitrary superposition of N different-kind dyons, the second is true but the first is false. To satisfy both requirements, one can consider N dyons as constituents of the Kraan-van Baal-Lee-Lu (KvBLL) instantons with nontrivial holonomy [4,5], which are saddle points of the Yang-Mills partition function as they are exact self-dual solutions of the equations of motion. At the same time, the fluctuation determinant about the KvBLL instanton is infrared-finite (and actually exactly calculable [6]) since its total electric and magnetic charges are zero.

KvBLL instantons generalize the standard Belavin-Polyakov-Schwartz-Tuypkin (BPST) instantons [7] having trivial holonomy. The mere notion of dyons and of the KvBLL instantons (also called calorons) alike imply that the Yang-Mills field is periodic in the Euclidean time direction, as it is in the case of nonzero temperature. Therefore, we shall be considering the Yang-Mills partition function at finite temperature. However, the circumference of the compactified space can be gradually put to infinity, corresponding to the zero-temperature limit. In that limit, the temperature can be considered as an infrared regulator of the theory needed to distinguish between the trivial and the nontrivial holonomy. After all, the temperature of the Universe is 2.7 K $\neq 0$.

In this context, the holonomy is defined as the set $\{\mu_m\}$ of the gauge-invariant eigenvalues of the Polyakov loop L winding in the compactified time direction, at spatial infinity:

$$L = \mathrm{P}\exp\left(i\int_0^{1/T} dt A_4\right)_{|\vec{x}|\to\infty} = V\mathrm{diag}(e^{2\pi i\mu_1}, e^{2\pi i\mu_2}, \ldots, e^{2\pi i\mu_N})V^{-1}, \tag{2}$$

$$\sum_{m=1}^{N} \mu_m = 0.$$

By making a global gauge rotation, one can always order

*diakonov@thd.pnpi.spb.ru
†victorp@thd.pnpi.spb.ru

the eigenvalues such that

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N \leq \mu_{N+1} \equiv \mu_1 + 1, \tag{3}$$

which we shall assume done. If all eigenvalues are equal up to an integer, implying

$$\mu_m^{\rm triv} = \begin{cases} \frac{k}{N} - 1 & \text{when } m \leq k, \\ \frac{k}{N} & \text{when } m > k, \end{cases} \quad \text{where } k = 1, \ldots, N, \tag{4}$$

the Polyakov line belongs to the $SU(N)$ group center, and the holonomy is then said to be "trivial": $L_{\rm triv} = {\rm diag}(\exp(2\pi ik/N), \ldots, \exp(2\pi ik/N))$, $k = 1, \ldots, N$. Standard BPST instantons, as well as their genuine periodic generalization to nonzero temperatures by Harrington and Shepard [8], possess trivial holonomy, whereas for the KvBLL instantons the gauge-invariant eigenvalues of the Polyakov line assume, generally, nonequal values corresponding to a "nontrivial" holonomy. Among these, there is a special set of equidistant μ_m's that can be named a "maximally nontrivial" holonomy,

$$\mu_m^{\rm conf} = -\frac{1}{2} - \frac{1}{2N} + \frac{m}{N}, \tag{5}$$

having a distinguished property that it leads to ${\rm Tr}L = 0$. Since the average Polyakov line is zero in the confining phase, the set (5) can be also called the "confining" holonomy.

Whatever the set of μ's is equal to, it is a global characterization of the Yang-Mills system. Integrating over all possible μ's is equivalent to requesting that the total color charge of the system is zero [9]. Since μ's are constants, the ultimate partition function has to be extensive in these quantities, meaning $\mathcal{Z} = \exp[-F(\{\mu\}, T)V]$, where F is the free energy and V is the 3-volume. If $F(\{\mu\})$ has a minimum for a particular set $\{\mu\}$, integration over μ's is done by the saddle point method justified in the thermodynamic limit $V \to \infty$; hence, the Yang-Mills system settles at the minimum of the free energy as a function of μ's. The big question is whether the pure Yang-Mills theory has the minimum of the free energy at the confining holonomy (5) or elsewhere.

It has been argued long ago [9] that the pure Yang-Mills theory has N minima of the free energy at the trivial holonomy (4). To that end, one refers to the perturbative potential energy as a function of spatially constant A_4 [9,10]:

$$P^{\rm pert} = V \frac{(2\pi)^2 T^3}{3} \sum_{m>n}^{N} (\mu_m - \mu_n)^2 [1 - (\mu_m - \mu_n)]^2, \tag{6}$$

which, indeed, has N minima (all with zero energy) at the trivial holonomy (4), corresponding to N elements of the center of $SU(N)$. The confining holonomy (5) corresponds, on the contrary, to the nondegenerate maximum of (6), equal to

$$P^{\rm pert, max} = V \frac{(2\pi)^2 T^3}{180} \frac{N^4 - 1}{N^2}. \tag{7}$$

The large volume factor in Eq. (6) seemingly prohibits any configurations with nontrivial holonomy, dyons and KvBLL instantons included.

A loophole in this dyon-killing argument has been noticed in Ref. [11]: If one takes an *ensemble* of dyons, with their number proportional to the volume, the nonperturbative dyon-induced potential energy is also proportional to the volume and may hence override the perturbative one, possibly leading to another minimum of the full free energy. This scenario was made probable in Ref. [6], where it was shown that the nonperturbative potential energy induced by a dilute gas of the KvBLL instantons prevailed over the perturbative one at temperatures below some critical T_c estimated through Λ, the Yang-Mills scale parameter, and that the trivial holonomy was not the minimum of the full free energy anymore. Below that critical temperature, the KvBLL instantons dissociate into individual dyons. The problem, therefore, is to build the partition function for dissociated dyons and to check if the full free energy has a minimum at the confining holonomy (5). We get an affirmative answer to this question.

The moduli space of a single KvBLL instanton is characterized by $4N$ parameters (of which the classical action is independent); these can be conveniently chosen as $3d$ coordinates of N dyons constituting the instanton and their $U(1)$ phases, $3N + N = 4N$. In the part of the moduli space where all dyons are well separated, the KvBLL instanton becomes a sum of N types of BPS monopoles with a time-independent action density. At small separations between dyons, the action density of the KvBLL instanton is time-dependent and resembles that of the standard BPST instanton. The KvBLL instanton reduces to the standard BPST instanton in the two limiting cases: (i) trivial holonomy (all μ's are equal up to an integer) and any temperature and (ii) nontrivial holonomy but vanishing temperature, provided the separations between dyons shrink to zero as $\sim\rho^2 T$, where ρ is the standard instanton size.

The quantum weight of the KvBLL instanton is determined by a product of two factors: (i) the determinant of the moduli space metric and (ii) the small-oscillation determinant over nonzero modes in the KvBLL background. The latter has been computed exactly in Ref. [6] for the $SU(2)$ gauge group; recently, the result has been generalized to any $SU(N)$ [12]. The former is also known exactly (see the references and discussion in the next section). These achievements, however, are limited to the case of a single KvBLL instanton with unity topological charge. To build the dyon vacuum, one needs multi-instanton solutions, with the topological charge proportional to the volume, similar to the case of the instanton liquid model

[11,13]. Although there has been serious progress recently in constructing general multi-instanton solutions with nontrivial holonomy and their moduli space metric [14], a desirable explicit construction is still lacking.

Nevertheless, the moduli space metric can be constructed by combining the metric known for N different-kind dyons of the $SU(N)$ group with another known metric for same-kind dyons and by taking into due account the permutational symmetry between identical dyons. One of the two ingredients, the metric for different-kind dyons, is known exactly for all separations and involves only Coulomb-like interactions. The other ingredient related to the same-kind dyons is more complex. The metric for any separations between same-kind dyons allows for charge exchange between dyons and involves elliptic functions of separation. For two dyons of the same charge, the exact metric was found by Atiyah and Hitchin [15] from the requirement that the Riemann tensor constructed from the metric must be self-dual. When the separation between same-kind dyons exceeds their core sizes, charge exchange dies out exponentially with the separation, and the metric becomes simple and can be written for any number of same-kind dyons with the exponential precision [16,17]. It involves only the Coulomb-like interactions and is in fact very similar to that for different-kind dyons but with opposite signs in the Coulomb bonds. It is this marriage of the asymptotic form of the metric for same-kind dyons, valid with exponential precision, with the metric for different-kind dyons, valid for *any* separations, that we shall explore in this paper.

In fact, it may prove to be sufficient for an accurate description of the ensemble of dyons in the thermodynamic limit ($V \to \infty$), as dyons of the same kind repulse each other, whereas dyons of different kinds attract each other. Therefore, small separations between same-kind dyons, where our measure is only approximate, may be statistically unimportant. [Unfortunately, taking the large N limit does not help, since only nearest neighbors in color interact; hence, at any $N > 2$, there are only twice more different-kind bonds than same kind.] Indeed, we find that, despite an approximate integration measure, the Lorentz symmetry is, in a sense, restored at $T \to 0$: The "electric" string tension as determined from the correlation of Polyakov lines turns out to be independent of temperature and *equal* to the "magnetic" string tension determined from the area law for spatial Wilson loops, for all representations considered. The free energy itself also has a reasonable limit at $T \to 0$. However, to remain on the safe side, we claim the results only for sufficiently high temperatures (but below the deconfinement phase transition) where dyons do not overlap on the average, and the metric used is justified.

This study is exploratory as we ignore many essential ingredients of the full Yang-Mills theory. In particular, we consider only the ensemble of dyons of one duality, whereas CP invariance of the vacuum requires that there must be an equal number of self-dual and anti-self-dual configurations, up to the thermodynamic fluctuations $\sim\sqrt{V}$ [13]. We basically ignore the determinant over nonzero modes, taking from it only certain known salient features, such as the renormalization of the coupling constant and the perturbative potential energy (6). Our aim is to demonstrate that the integration measure over dyons has a drastic, probably decisive, effect on the ensemble of dyons, that the ensemble can be mathematically described by an exactly solvable field theory in three dimensions, and that the resulting vacuum built of dyons has certain features expected for the confining pure Yang-Mills theory.

A more phenomenological and lattice-oriented attempt to describe the ensemble of the KvBLL instantons has been proposed recently in Ref. [18].

II. INTEGRATION MEASURE OVER DYONS

A. Different-kind dyons

The metric of the moduli space of a single $SU(N)$ KvBLL instanton, written in terms of N different-kind dyons' coordinates and $U(1)$ phases, was first conjectured by Lee, Weinberg, and Yi [19], generalizing the previous work by Gibbons and Manton [17], and then confirmed by Kraan [20] by an explicit calculation of the zero-mode Jacobian from the Atiyah-Drinfeld-Hitchin-Manin-Nahm construction [21,22] for the $SU(N)$ caloron [23]. It was later checked independently in Ref. [24], also by an explicit calculation of the Jacobian.

There are several equivalent ways to present the metric of a single $SU(N)$ KvBLL instanton; we use here the form suggested originally by Gibbons and Manton [17] (although these authors considered another case—that of same-kind dyons; see the next subsection):

$$\begin{aligned} ds^2 &= G_{mn} d\mathbf{x}_m \cdot d\mathbf{x}_n + (d\psi_m + \mathbf{W}_{mm'} \cdot d\mathbf{x}_{m'}) \\ &\quad \times G^{-1}_{mn}(d\psi_n + \mathbf{W}_{nn'} \cdot d\mathbf{x}_{n'}), \\ m, n &= 1 \ldots N. \end{aligned} \tag{8}$$

Here $\mathbf{x}_m$, $m = 1 \ldots N$, are the $3d$ centers of dyons, and ψ_m are their $U(1)$ phases. G_{mn} is a symmetric matrix composed of the Coulomb interactions between dyons that are nearest "neighbors in color":

$$\begin{aligned} G_{mn} = {}& \delta_{mn}\left(4\pi\nu_m + \frac{1}{|\mathbf{x}_m - \mathbf{x}_{m-1}|} + \frac{1}{|\mathbf{x}_m - \mathbf{x}_{m+1}|}\right) \\ & - \frac{\delta_{m,n-1}}{|\mathbf{x}_m - \mathbf{x}_{m+1}|} - \frac{\delta_{m,n+1}}{|\mathbf{x}_m - \mathbf{x}_{m-1}|}, \end{aligned} \tag{9}$$

where $\nu_m = \mu_{m+1} - \mu_m$; $\nu_1 + \cdots + \nu_N = 1$ (see Sec. I). Periodicity in color indices is implied throughout the paper: $m = N + 1$ is equivalent to $m = 1$, and $m = 0$ is equivalent to $m = N$. $\mathbf{W}_{mn}$ are three $N \times N$ symmetric matrices composed of the electric charge-magnetic charge interaction potential $\mathbf{w}(\mathbf{x})$:

$$\mathbf{W}_{mn} = \delta_{mn}(\mathbf{w}(\mathbf{x}_m - \mathbf{x}_{m-1}) + \mathbf{w}(\mathbf{x}_m - \mathbf{x}_{m+1})) - \delta_{m,n-1}\mathbf{w}(\mathbf{x}_m - \mathbf{x}_{m+1}) - \delta_{m,n+1}\mathbf{w}(\mathbf{x}_m - \mathbf{x}_{m-1}), \tag{10}$$

where $\mathbf{w}(\mathbf{x})$ satisfies the equation $\epsilon^{abc}\partial_b \mathbf{w}_c = -\mathbf{x}_a/|\mathbf{x}|^3$; to solve it, one has to introduce a Dirac string singularity. Choosing the string, e.g., along the third axis and parametrizing the separation vector between two dyons in spherical coordinates $\mathbf{x} = |\mathbf{x}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, one finds $\mathbf{w}(\mathbf{x}) = (-\cot\theta\sin\phi, \cot\theta\cos\phi, 0)/|\mathbf{x}|$, such that the factor $(d\psi + \mathbf{w}\cdot d\mathbf{x})$ written for relative coordinates combines into

$$d\Sigma_3 = d\psi + \cos\theta d\phi, \tag{11}$$

which is a familiar 1-form encountered, e.g., in the theory of rigid-body rotations, being the third projection of the angular velocity. The fact that this quantity arises in the context of electric charge-magnetic charge interaction was known for quite a while [25]. A more general, basis-independent way to present the interaction is via the Wess-Zumino term [26].

The temperature factors T have been dropped in Eqs. (8)–(10) but can be restored any time from dimensions. We stress that the metric (8) is exact and valid for *any* 3d separations between dyons inside the KvBLL instanton, including the case when they strongly overlap. For more details on this metric, see Refs. [20,24].

The integration measure for the KvBLL instanton is

$$\prod_{m=1}^{N} d^3\mathbf{x}_m d\psi_m \sqrt{\det g}, \qquad \sqrt{\det g} = \det G, \tag{12}$$

where g is the full $4N \times 4N$ metric tensor given by Eq. (8). In computing $\det g$, one can shift $d\psi_m \to d\psi'_m = d\psi_m + \mathbf{W}_{mk}\cdot d\mathbf{x}_k$; therefore, $\det g = (\det G)^3 \det G^{-1} = (\det G)^2$, and hence $\sqrt{\det g} = \det G$, where G is given by Eqs. (9) [24]. In that reference, it was also shown that, in the limit of trivial holonomy or small temperatures, the integration measure (12) reduces to the well-known one for the standard BPST instanton [27].

B. Same-kind dyons

For multi-KvBLL instantons, a new element appears; namely, two or more *same-kind* dyons are present. For two dyons of the same kind, the metric splits into a flat 4d space for center-of-mass coordinates and a nonflat 4d space M_2 for relative coordinates $\mathbf{r}$ and ψ. Self-duality implies that M_2 is a self-dual Einstein manifold. Gibbons and Pope [28] proposed the following form for the metric:

$$ds^2 = f^2dr^2 + a^2d\Sigma_1^2 + b^2d\Sigma_2^2 + c^2d\Sigma_3^2, \qquad \sqrt{g} = |fabc|\sin\theta, \tag{13}$$

where

$$d\Sigma_1 = -\sin\psi d\theta + \cos\psi\sin\theta d\phi,$$
$$d\Sigma_2 = \cos\psi d\theta + \sin\psi\sin\theta d\phi,$$

and $d\Sigma_3$ is the third "angular velocity" (11); a, b, c, and f are functions of the dyon separation r. Self-duality requires that the Riemann tensor built from the metric (13) satisfies $R_{\alpha\beta\gamma\delta} = \frac{1}{2}\sqrt{g}\epsilon_{\gamma\delta\kappa\lambda}R^{\kappa\lambda}_{\alpha\beta}$, which leads to the system of first-order equations

$$\frac{1}{f}\frac{da}{dr} = \frac{b^2 + c^2 - a^2}{2bc} - \lambda \quad \text{and cyclic permutations of } a, b, c, \tag{14}$$

where $\lambda = 0$ or 1. The value $\lambda = 1$ is chosen from symmetry considerations [29]. Equations (14) have a simple solution [16]

$$f = -\beta\sqrt{1 + \frac{2\alpha}{r}}, \qquad a = b = \beta r\sqrt{1 + \frac{2\alpha}{r}}, \qquad c = \frac{2\alpha\beta}{\sqrt{1 + \frac{2\alpha}{r}}} \tag{15}$$

with any α, β. We fix α, β from the asymptotics of the metric of two dyons of the same kind m at $r \to \infty$: $\beta = \sqrt{2\pi\nu_m}$, $\alpha = -1/(2\pi\nu_m)$ [30]. To get the full 8×8 metric tensor in terms of the two dyons' coordinates

$$\mathbf{x}_1 = \mathbf{X} + \tfrac{1}{2}\mathbf{r}, \qquad d\psi_1 = d\Psi + \tfrac{1}{2}d\psi,$$
$$\mathbf{x}_2 = \mathbf{X} - \tfrac{1}{2}\mathbf{r}, \qquad d\psi_2 = d\Psi - \tfrac{1}{2}d\psi,$$

we add to Eq. (13) the flat metric for center-of-mass coordinates $8\pi\nu(d\mathbf{X}\cdot d\mathbf{X} + d\Psi d\Psi)$ and obtain the metric which is very similar to Eq. (8):

$$ds^2 = \tilde{G}_{ij}d\mathbf{x}_i\cdot d\mathbf{x}_j + (d\psi_i + \tilde{\mathbf{W}}_{ii'}\cdot d\mathbf{x}_{i'}) \times \tilde{G}^{-1}_{ij}(d\psi_j + \tilde{\mathbf{W}}_{jj'}\cdot d\mathbf{x}_{j'}), \qquad i, j = 1, 2, \tag{16}$$

$$\tilde{G}_{ij} = \begin{pmatrix} 4\pi\nu_m - \frac{2}{|\mathbf{x}_1 - \mathbf{x}_2|} & \frac{2}{|\mathbf{x}_1 - \mathbf{x}_2|} \\ \frac{2}{|\mathbf{x}_1 - \mathbf{x}_2|} & 4\pi\nu_m - \frac{2}{|\mathbf{x}_1 - \mathbf{x}_2|} \end{pmatrix}, \tag{17}$$

$$\tilde{W}_{ij} = \begin{pmatrix} -\mathbf{w}(\mathbf{x}_1 - \mathbf{x}_2) & \mathbf{w}(\mathbf{x}_1 - \mathbf{x}_2) \\ \mathbf{w}(\mathbf{x}_1 - \mathbf{x}_2) & -\mathbf{w}(\mathbf{x}_1 - \mathbf{x}_2) \end{pmatrix}, \qquad \mathbf{w}(\mathbf{x}) = \frac{1}{|\mathbf{x}|}(-\cot\theta\sin\phi, \cot\theta\cos\phi, 0). \tag{18}$$

Note the opposite sign of Coulomb interactions in Eq. (17) as compared to Eq. (9).

It is easy to generalize Eqs. (16)–(18) to *any* number of dyons, all of the mth kind. One extends the summation in Eq. (16) from $i, j \leq 2$ to $i, j \leq K$, where K is the number of same-kind dyons, and modifies Eqs. (17) and (18) as

$$\tilde{G}_{ij} = \begin{cases} 4\pi\nu_m - \sum_{k\neq i} \frac{2}{|\mathbf{x}_i - \mathbf{x}_k|}, & i = j, \\ \frac{2}{|\mathbf{x}_i - \mathbf{x}_j|}, & i \neq j, \end{cases}$$
$$\tilde{W}_{ij} = \begin{cases} -2\sum_{k\neq i} \mathbf{w}(\mathbf{x}_i - \mathbf{x}_k), & i = j, \\ 2\mathbf{w}(\mathbf{x}_i - \mathbf{x}_j), & i \neq j. \end{cases} \tag{19}$$

Equations (16) and (19) were derived by Gibbons and Manton [17] (with other coefficients related to another scale convention) from considering the classical equations of motion for K identical monopoles at large separations.

Although Eq. (15), from where the metric (16) stems, is an exact solution of the Einstein self-duality equation, it is believed that Eq. (16) is valid only for large separations $r > 1/2\pi\nu_m T$ (we restore the explicit temperature factor here). Note that a very similar metric (8) for different-kind dyons is proven to be valid at any separations. A somewhat superficial reason for the difference between same- and different-kind dyons was noted in Ref. [19]: While the metric (8) is positive-definite, the metric (16) goes to zero at $r = 1/2\pi\nu_m T$. A deeper reason is that, while the metric (8) describes a system with total electric and magnetic charges zero (the KvBLL instanton), the metric (16) is applied to a system where neither is zero.

A nontrivial solution of the Einstein self-duality equation (14) was found by Atiyah and Hitchin (AH) [15] (for more details, see [29]). In the AH solution, the a, b, c functions are given by elliptic integrals, and $a(r) \neq b(r)$. It follows then from Eq. (13) that the relative $U(1)$ orientation angle ψ enters the metric in an essential way; in particular, a shift of ψ is not an isometry anymore. The functions a, b, c of the AH solution differ from those of the solution (15) by terms of the order of $\sim\exp(-4\pi\nu_m T r)$ which die out exponentially at large separations [16]. Therefore, at large r the AH solution takes the form of Eq. (15) such that ψ enters the metric in a trivial way, as in Eq. (16).

For the AH solution, the metric determinant goes to zero at even larger $r = 1/(4\nu_m T)$ [15,16]. Physically, this point corresponds to an axially symmetric two-monopole solution where two monopoles coincide. When dyons overlap, what should be called "separation" becomes ambiguous; r is defined only in the context of a concrete parametrization of the field. In the ensemble, the zero of the metric determinant means a vanishing contribution to the partition function, actually imposing a very strong repulsion between same-kind dyons. The same is true for the approximate metric (16), which we shall use below. One can think that the approximate metric (16), because of the strong repulsion it imposes, strongly suppresses in the statistical mechanics sense configurations with close same-kind dyons, where the metric becomes inaccurate. In other words, the approximate metric (16) may be accurate for statistically important configurations. This hypothesis needs a detailed study, of course. Its consequences, however, turn out to be reasonable.

C. Combining the metric for same-kind and different-kind dyons

The explicit form of the metric tensor for K KvBLL instantons made of N kinds of dyons is not known (for the latest development, see Ref. [14]). Below, we suggest an ansatz for this metric, satisfying the known requirements. One has to combine the metric (8) for N different-kind dyons with that for K same-kind ones. The solution of the problem is almost obvious if one takes the approximate metric (16) for same-kind dyons, as it has exactly the same form as the metric (8) for different-kind dyons. Since the metric cannot "know" to which instanton a particular dyon belongs, it must be symmetric under permutations of any pair of dyons of the same kind. Importantly, the metric of the moduli space of self-dual solutions must be hyper-Kähler [29].

Let indices $m, n = 1 \ldots N$ refer to the dyon kind (or "color") and indices $i, j, k = 1 \ldots K$ number the dyons of the same kind. The coordinates of the ith dyon of the mth kinds are $(\mathbf{x}_{mi}, \psi_{mi}) \equiv y^\alpha_{mi} \equiv y^\alpha_A$, $\alpha = 1, 2, 3, 4$. To shorten notations, we introduce instead of the multi-index (mi) a single index $A = (mi)$ running from 1 to KN.

We write the full metric tensor as

$$\begin{aligned} ds^2 &= g_{A\alpha,B\beta} dy^\alpha_A dy^\beta_B \\ &= G_{AB} d\mathbf{x}_A \cdot d\mathbf{x}_B + (d\psi_A + \mathbf{W}_{AA'} \cdot d\mathbf{x}_{A'}) \\ &\quad \times G^{-1}_{AB}(d\psi_B + \mathbf{W}_{B,B'} \cdot d\mathbf{x}_{B'}), \end{aligned} \tag{20}$$

where, explicitly,

$$\begin{aligned} G_{AB} &= G_{mi,nj} \\ &= \delta_{mn}\delta_{ij}\bigg(4\pi\nu_m + \sum_k \frac{1}{|\mathbf{x}_{mi} - \mathbf{x}_{m-1,k}|} \\ &\quad + \sum_k \frac{1}{|\mathbf{x}_{mi} - \mathbf{x}_{m+1,k}|} - 2\sum_{k\neq i} \frac{1}{|\mathbf{x}_{mi} - \mathbf{x}_{mk}|}\bigg) \\ &\quad - \frac{\delta_{m,n-1}}{|\mathbf{x}_{mi} - \mathbf{x}_{m+1,j}|} - \frac{\delta_{m,n+1}}{|\mathbf{x}_{mi} - \mathbf{x}_{m-1,j}|} \\ &\quad + 2\frac{\delta_{mn}}{|\mathbf{x}_{mi} - \mathbf{x}_{mj}|}\bigg|_{i\neq j}, \end{aligned} \tag{21}$$

$$\begin{aligned} \mathbf{W}_{AB} &= \mathbf{W}_{mi,nj} \\ &= \delta_{mn}\delta_{ij}\bigg(\sum_k \mathbf{w}(\mathbf{x}_{mi} - \mathbf{x}_{m-1,k}) + \sum_k \mathbf{w}(\mathbf{x}_{mi} - \mathbf{x}_{m+1,k}) \\ &\quad - 2\sum_{k\neq i} \mathbf{w}(\mathbf{x}_{mi} - \mathbf{x}_{mk})\bigg) - \delta_{m,n-1}\mathbf{w}(\mathbf{x}_{mi} - \mathbf{x}_{m+1,j}) \\ &\quad - \delta_{m,n+1}\mathbf{w}(\mathbf{x}_{mi} - \mathbf{x}_{m-1,j}) + 2\delta_{mn}\mathbf{w}(\mathbf{x}_{mi} - \mathbf{x}_{mj})|_{i\neq j}. \end{aligned} \tag{22}$$

The inverse matrix G^{-1}_{AB} in (20) is understood according to the relation $G^{-1}_{AC}G_{CB} = \delta_{AB} = \delta_{mn}\delta_{ij}$.

Note that the Coulomb bonds in Eq. (21) for the same-kind dyons have an opposite sign from those for the neighbor kind and have a twice larger coefficient. The coefficients -1, 2, -1 are actually the scalar products of simple roots of the $SU(N)$ group, supplemented by an additional nonsimple root to make the matrix cyclic-symmetric. This remark allows the generalization of Eqs. (21) and (22) to any Lie group.

The constructed $4KN \times 4KN$ metric tensor $g_{A\alpha,B\beta}$ is hyper-Kähler. It means that there exist three "complex structures" $I(a)$, $a = 1, 2, 3$ (all three are $4KN \times 4KN$ matrices) such that

$$I(a)g = gI(a)^T \quad (\text{"T" means transposed}) \tag{23}$$

and which satisfy the Pauli algebra

$$I(a)I(b) = \epsilon^{abc}I(c) - \delta^{ab}\mathbf{1}. \tag{24}$$

Related to $I(a)$, there are 3 Kähler symplectic 2-forms

$$\omega(a) = \Omega(a)_{B\beta,C\gamma}dy_B^\beta \wedge dy_C^\gamma, \qquad \Omega(a) = -\Omega(a)^T, \tag{25}$$

where

$$\Omega(a) = I(a)g. \tag{26}$$

The 2-forms $\omega(a)$ are closed:

$$d\omega(a) = 0 \quad \text{or} \quad \frac{\partial}{\partial y_A^\alpha}\Omega(a)_{B\beta,C\gamma}dy_A^\alpha \wedge dy_B^\beta \wedge dy_C^\gamma = 0. \tag{27}$$

Explicitly, the 3 Kähler forms $\omega(a)$ have the same form as in Ref. [17] for same-kind dyons, only G_{AB} and $\mathbf{W}_{AB}$ should be now taken from Eqs. (21) and (22):

$$\omega(a) = 2(d\psi_A + \mathbf{W}_{AA'} \cdot d\mathbf{x}_{A'}) \wedge dx_A^a - G_{BC}\epsilon^{abc}dx_B^b \wedge dx_C^c. \tag{28}$$

With G_{AB} and $\mathbf{W}_{AB}$ given by Eqs. (21) and (22), the 3 Kähler forms $\omega(a)$ [or $\Omega(a)_{B\beta,C\gamma}$] are fixed from Eq. (28), and the complex structures $I(a)$ are found from inverting Eq. (26). We have checked that the algebra (23) and (24) is then satisfied for any choice of $\mathbf{w}(\mathbf{x})$ in Eq. (22). It is the closure of the 2-forms, Eq. (27), that requests that $\mathbf{w}(\mathbf{x})$ is the electric charge-magnetic charge interaction potential satisfying the equation $\epsilon^{abc}\partial_b \mathbf{w}_c = -\mathbf{x}_a/|\mathbf{x}|^3$.

We note further properties of the constructed G_{AB} and $\mathbf{W}_{AB}$:

(i) Symmetry: $G_{AB} = G_{BA}$, $\mathbf{W}_{AB} = \mathbf{W}_{BA}$, meaning, of course, $G_{mi,nj} = G_{nj,mi}$, $\mathbf{W}_{mi,nj} = \mathbf{W}_{nj,mi}$;

(ii) Overall "neutrality": $\sum_{nj} G_{mi,nj} = 4\pi\nu_m$, $\sum_{mi} G_{mi,nj} = 4\pi\nu_n$, $\sum_{nj}\mathbf{W}_{mi,nj} = 0$, $\sum_{mi}\mathbf{W}_{mi,nj} = 0$;

(iii) Identity loss: Dyons of the same kind are indistinguishable, meaning mathematically that $\det G$ is symmetric under permutation of any pair of dyons ($i \leftrightarrow j$) of the same kind m;

(iv) Factorization: In the geometry when dyons fall into K well separated clusters of N dyons of all kinds in each, $\det G$ factorizes into a product of exact integration measures for K KvBLL instantons, $\det G = (\det G_1)^K$, where G_1 is given by Eq. (9).

The integration measure over the moduli space of K KvBLL instantons of the $SU(N)$ gauge group is

$$\prod_{i=1}^{K}\prod_{m=1}^{N}\int d\mathbf{x}_{mi}d\psi_{mi}\sqrt{\det g}, \qquad \sqrt{\det g} = \det G. \tag{29}$$

In deriving the last relation, we notice that, in the determinant, $d\psi_{mi}$ can be shifted by $\mathbf{W}_{mi,m'i'} \cdot d\mathbf{x}_{m'i'}$; hence, $\det g = (\det G)^3 \det G^{-1} = (\det G)^2$, and therefore $\sqrt{\det g} = \det G$, where the $KN \times KN$ matrix G is given by Eq. (21). We have also checked this result by an explicit calculation of the determinant of the full $4KN \times 4KN$ metric tensor g. Since G is independent of the $U(1)$ angles ψ_{mi}, integration over ψ can be omitted.

D. Dyons' fugacity

Fugacity is a term from statistical mechanics of grand canonical ensembles (where the number of particles is not fixed) denoting the weight with which a particle contributes to the grand partition function. Let there be K_m dyons of the mth kind, $m = 1 \ldots N$. For a neutral system of K KvBLL instantons, the number of dyons of every kind is equal, $K_1 = \cdots = K_N = K$; however, we shall consider the general case of nonequal K's for the time being: One can always project to the neutrality condition. For an arbitrary set of K's, G is a $(K_1 + \cdots + K_N) \times (K_1 + \cdots + K_N)$ matrix given by Eq. (21).

We write the partition function of the grand canonical ensemble as a sum over all numbers of dyons of each kind:

$$\mathcal{Z} = \sum_{K_1\ldots K_N}\frac{1}{K_1!\ldots K_N!}\prod_{m=1}^{N}\prod_{i=1}^{K_m}\int(d^3\mathbf{x}_{mi}f)\det G(\mathbf{x}), \tag{30}$$

where f is the x-independent factor—the fugacity—accompanying every integral over $\mathbf{x}$. Since $\det G(\mathbf{x})$ is symmetric under permutation of same-kind dyons, the identity factorials are needed to avoid counting the same configuration more than once. If one likes to impose the overall neutrality condition, viz. that only configurations with the equal number of dyons of different kinds contribute to the partition function ($K_1 = \cdots = K_N = K$), one integrates Eq. (30) over auxiliary angles:

$$\begin{aligned}\mathcal{Z}^{\text{neutr}} &= \sum_{K_1\ldots K_N}\int_0^{2\pi}\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_N}{2\pi}\frac{e^{i\theta_1(K_2-K_1)}}{K_1!}\cdots\frac{e^{i\theta_N(K_1-K_N)}}{K_N!} \\ &\quad\times\prod_{m=1}^{N}\prod_{i=1}^{K_m}\int(d^3\mathbf{x}_{mi}f)\det G(\mathbf{x}) \\ &= \sum_K\frac{1}{(K!)^N}\prod_{m=1}^{N}\prod_{i=1}^{K}\int(d^3\mathbf{x}_{mi}f)\det G(\mathbf{x}).\end{aligned} \tag{31}$$

We shall see below, however, that the neutrality condition will be taken care of dynamically; therefore, the additional integration (31) is, in fact, unnecessary.

As fugacity is x-independent, it can be established from the limit when all dyons are grouped into N-plets of different-kind dyons, forming infinitely dilute neutral KvBLL instantons, such that the measure factorizes into a product of individual instanton measures. The latter is known to be [24]

$$\prod_{m=1}^{N}\int d^3\mathbf{x}_m \,\det G_1(\mathbf{x})2^{2N}\pi^{3N}, \tag{32}$$

where $G_1(\mathbf{x})$ is the $N\times N$ matrix (9) for *one* KvBLL instanton; see Eq. (12). This must be multiplied by the factor $[(\mu^4/T)/\sqrt[4]{2\pi g^2}]^N$ coming from $4N$ zero modes of the instanton. Here μ is the ultraviolet cutoff, and g is the bare coupling constant given at that cutoff [6,24]. Multiplication by this factor makes Eq. (32) dimensionless, as it should be. In addition, Eq. (32) is multiplied by the exponent of minus the classical action of the instanton, equal to $(\Lambda/\mu)^{(11/3)N}$, where Λ is the Yang-Mills scale parametrizing the coupling constant in the Pauli-Villars regularization scheme, and by the dimensionless factor $[\det(-D^2)]^{-1}$, where D^2 is the Laplace operator in the instanton background. The last factor arises from integration over nonzero modes; it is understood that the small-oscillation determinant is normalized to the free (zero field) determinant and UV regularized by the Pauli-Villars method. It is known that the normalized and regularized $[\det(-D^2)]^{-1}$ is proportional to $\mu^{-(N/3)}$, times the exponent of minus the perturbative potential energy (6), times a slowly varying function of dyon separations [6,31].

Combining all factors, we observe that the Pauli-Villars mass μ cancels out (as it should in a renormalizable theory), and we obtain the dyon fugacity

$$f=\frac{\Lambda^4}{T}\frac{4\pi}{g^4}c, \tag{33}$$

where c is proportional to $\Lambda^{-(1/3)}$; it is made dimensionless by a combination of temperature and dyon separations. The relative (un)importance of c in the dynamics of the ensemble, as compared to the measure factor (30), is illustrated by the powers of Λ: Their ratio is $(-1/3){:}4=-1/12$. For the time being, we shall set $c=1$ and recall it in the discussion in Sec. IV. The coupling g^2 in (33) starts to "run" at the two-loop level not included here. Ultimately, its precise argument is determined self-consistently from the action density of the ensemble [32]. In the study of the large-N behavior, it will be important that $c=\mathcal{O}(1)$ whereas $1/g^4=\mathcal{O}(N^2)$; hence, the fugacity $f=\mathcal{O}(N^2)$.

III. DYON PARTITION FUNCTION AS A QUANTUM FIELD THEORY

We now face an interesting problem of finding the correlation functions in the ensemble of dyons whose grand partition function is given by Eq. (30). The renormalized Yang-Mills scale parameter Λ creeps in via the fugacity (33); therefore, all physical quantities will be henceforth expressed through Λ. The temperature also enters explicitly via Eq. (33); the temperature factors are understood in all Coulomb bonds in the matrix G (21) as well, to make them dimensionless. Thus, the partition function and the ensuing correlation functions depend, generally, on both Λ and T.

The ensemble defined by the partition function (30) is a very unusual one, as it is governed by the determinant of a matrix G whose dimension is equal to the number of particles and not by the exponent of the interaction energy, as is common in statistical mechanics. Of course, one can always write $\det G=\exp\mathrm{Tr}\log G\equiv\exp(U_{\rm int})$; however, then the interaction potential $U_{\rm int}$ will contain not only 2-body, but also 3-, 4-, 5-, ...-body forces that are increasingly important. At the same time, the statistical mechanics of an ensemble governed by the determinant-induced interactions can be transformed into an equivalent quantum field theory which considerably simplifies its handling.

To that end, we first notice that a matrix determinant can be presented as a result of the integration over a finite number of anticommuting Grassmann variables [33]

$$\det G=\int\prod_A d\psi_A^\dagger d\psi_A\exp(\psi_A^\dagger G_{AB}\psi_B), \tag{34}$$

where the usual convention [33] for anticommuting integration variables is understood:

$$\begin{gathered}\psi_A\psi_B+\psi_B\psi_A=0,\qquad \psi_A^\dagger\psi_B^\dagger+\psi_B^\dagger\psi_A^\dagger=0,\\ \psi_A^\dagger\psi_B+\psi_B\psi_A^\dagger=0\quad\text{for any}A,B,\\ \int d\psi_A^\dagger d\psi_A=0,\qquad \int d\psi_A^\dagger d\psi_A\psi_A^\dagger\psi_A=1.\end{gathered} \tag{35}$$

In our case, $A=(mi)$ is a multi-index, where $m=1\ldots N$ is the dyon kind, and $i=1\ldots K_m$ is the number of a dyon of the mth kind. We rewrite identically the partition function (30) as

$$\begin{aligned}Z=&\sum_{K_1\ldots K_N}\frac{f^{K_1}\ldots f^{K_N}}{K_1!\ldots K_N!}\prod_{m=1}^{N}\prod_{i=1}^{K_m}\int d^3\mathbf{x}_{mi}\\ &\times\int d\psi_{mi}^\dagger d\psi_{mi}\exp(\psi_{mi}^\dagger G_{mi,nj}\psi_{nj}),\end{aligned} \tag{36}$$

where $G_{mi,nj}$ is a matrix made of Coulomb interactions [Eq. (21)], and f is the fugacity (33) where we put $c=1$. Having obtained G in the exponent, it is now possible to express its Coulomb matrix elements from path integrals, extending the Polyakov trick [34] to anticommuting variables.

A. Off-diagonal elements: Ghost fields

We first present the off-diagonal ($i \neq j$) elements of $\psi^\dagger G\psi$ by means of a functional integration over anticommuting (or ghost) fields. In the next subsection, we present the diagonal ($i = j$) elements with the help of a functional integration over commuting (boson) fields.

Let us consider the Gaussian path integral over N anticommuting fields $\chi_m(\mathbf{x})$ coupled to the anticommuting source $\sum_i[\psi_{mi}\delta(\mathbf{x}-\mathbf{x}_{mi}) - \psi_{m-1,i}\delta(\mathbf{x}-\mathbf{x}_{m-1,i})]$:

$$\mathcal{Y} = \prod_{m=1}^{N}\int D\chi_m^\dagger D\chi_m \exp\int d\mathbf{x}\sum_m\Big\{\frac{T}{4\pi}\partial\chi_m^\dagger\cdot\partial\chi_m + i\sum_i[(\psi_{mi}^\dagger\chi_m(\mathbf{x}) + \chi_m^\dagger(\mathbf{x})\psi_{mi})\delta(\mathbf{x}-\mathbf{x}_{mi}) - (\psi_{m-1,i}^\dagger\chi_m(\mathbf{x}) + \chi_m^\dagger(\mathbf{x})\psi_{m-1,i})\delta(\mathbf{x}-\mathbf{x}_{m-1,i})]\Big\}. \tag{37}$$

Although we do not write it explicitly to save space, we assume that $\mathcal{Y}$ is normalized to the same path integral with the kinetic term but without the source term. The subscript m is periodic: $m = N+1$ is equivalent to $m = 1$, and $m = 0$ means $m = N$.

The path integration of an action that is quadratic in anticommuting variables is performed in the same way as the Gaussian path integral over bosonic variables, with the result

$$\mathcal{Y} = \exp\Big[\frac{1}{T}\sum_{m,i,j}\Big(\frac{\psi_{mi}^\dagger\psi_{mj}}{|\mathbf{x}_{mi}-\mathbf{x}_{mj}|} - \frac{\psi_{mi}^\dagger\psi_{m-1,j}}{|\mathbf{x}_{mi}-\mathbf{x}_{m-1,j}|} - \frac{\psi_{m-1,i}^\dagger\psi_{mj}}{|\mathbf{x}_{m-1,i}-\mathbf{x}_{mj}|} + \frac{\psi_{m-1,i}^\dagger\psi_{m-1,j}}{|\mathbf{x}_{m-1,i}-\mathbf{x}_{m-1,j}|}\Big)\Big]. \tag{38}$$

Owing to the cyclic summation over m, the last term doubles the first one, and we correctly reproduce the off-diagonal ($i \neq j$) part of $\psi_{mi}^\dagger G_{mi,nj}\psi_{nj}$ in Eq. (36) [cf. the last two lines in Eq. (21)]. However, the sum in Eq. (38) contains an extra diagonal divergent term $2\psi_{mi}^\dagger\psi_{mi}/|\mathbf{x}_{mi}-\mathbf{x}_{mi}|$ which is absent in Eq. (36) and, hence, should be canceled.

B. Diagonal elements: Boson fields

Next, we present the diagonal ($i = j$) part of $\psi^\dagger G\psi$ by means of a Gaussian integration over bosonic fields v_m, w_m.

Let us consider

$$\mathcal{X} = \prod_{m=1}^{N}\int Dv_m Dw_m \exp\int d\mathbf{x}\sum_m\Big\{\frac{T}{4\pi}\partial v_m\cdot\partial w_m + \sum_i[(\psi_{mi}^\dagger\psi_{mi}\delta(\mathbf{x}-\mathbf{x}_{mi}) - \psi_{m-1,i}^\dagger\psi_{m-1,i} \times \delta(\mathbf{x}-\mathbf{x}_{m-1,i}))v_m(\mathbf{x}) + (\delta(\mathbf{x}-\mathbf{x}_{mi}) - \delta(\mathbf{x}-\mathbf{x}_{m-1,i}))w_m(\mathbf{x})]\Big\}. \tag{39}$$

To make this path integral formally convergent, one assumes that the integration over either v_m or w_m goes along the imaginary axis. As in the case of ghost fields in the previous subsection, we do not write it explicitly but assume that $\mathcal{X}$ is normalized to the same path integral with the kinetic term but without the source term.

Integrating (39) over w_m, we obtain a functional δ function:

$$\delta\Big(-\frac{T}{4\pi}\partial^2 v_m + \sum_i[\delta(\mathbf{x}-\mathbf{x}_{mi}) - \delta(\mathbf{x}-\mathbf{x}_{m-1,i})]\Big),$$

whose solution is

$$v_m(\mathbf{x}) = -\frac{1}{T}\sum_i\Big(\frac{1}{|\mathbf{x}-\mathbf{x}_{mi}|} - \frac{1}{|\mathbf{x}-\mathbf{x}_{m-1,i}|}\Big), \qquad \sum_m v_m(\mathbf{x}) = 0. \tag{40}$$

The Jacobian following from the δ function $\det(-(T/4\pi)\partial^2)$ cancels with the same Jacobian from the normalization integral. Substituting $v_m(x)$ from Eq. (40) back into Eq. (39) and using the cyclic symmetry of the summation over m, we obtain

$$\mathcal{X} = \exp\Big[-\frac{1}{T}\sum_{m,i,j}\Big(2\frac{\psi_{mi}^\dagger\psi_{mj}}{|\mathbf{x}_{mi}-\mathbf{x}_{mj}|} - \frac{\psi_{mi}^\dagger\psi_{mi}}{|\mathbf{x}_{mi}-\mathbf{x}_{m-1,j}|} - \frac{\psi_{mi}^\dagger\psi_{mi}}{|\mathbf{x}_{mi}-\mathbf{x}_{m+1,j}|}\Big)\Big]. \tag{41}$$

The divergent term at $i = j$, namely, $-2\psi_{mi}^\dagger\psi_{mi}/|\mathbf{x}_{mi}-\mathbf{x}_{mi}|$, cancels exactly the unwanted extra term in Eq. (38), and we reproduce precisely the diagonal term in $\psi_{mi}^\dagger G_{mi,nj}\psi_{nj}$ of Eq. (36) [cf. the first two lines in Eq. (21)].

Thus, we have fully reproduced the factor $\exp(\psi^\dagger G\psi)$ in the partition function (36) with the help of the integration over anticommuting ghost ($\chi_m^\dagger$, χ_m) and ordinary boson (v_m, w_m) variables. The Coulomb interactions have been traded for kinetic energy terms of those fields. Apparently, v_m are $U(1)^{N-1}$ Abelian electric potentials, and w_m are their dual fields.

C. Synthesis: The equivalent quantum field theory

We now use Eqs. (37) and (39) to rewrite identically the partition function (36). We have

$$\mathcal{Z} = \prod_{m=1}^{N} \int D\chi_m^\dagger D\chi_m Dv_m Dw_m \exp \int d\mathbf{x} \sum_m \left[\frac{T}{4\pi} (\partial \chi_m^\dagger \cdot \partial \chi_m + \partial v_m \cdot \partial w_m) \right]$$
$$\times \sum_{K_1=0}^{\infty} \frac{f^{K_1}}{K_1!} \left(\int d\mathbf{x}_1 \int d\psi_1^\dagger d\psi_1 \exp[4\pi\nu_1 \psi_1^\dagger \psi_1 + i\psi_1^\dagger(\chi_1(x_1) - \chi_2(x_1)) + i(\chi_1^\dagger(x_1) - \chi_2^\dagger(x_1))\psi_1 + (v_1(x_1)\right.$$
$$- v_2(x_1))\psi_1^\dagger\psi_1 + (w_1(x_1) - w_2(x_1))]\Big)^{K_1} \times \ldots \times \sum_{K_N=0}^{\infty} \frac{f^{K_N}}{K_N!} \left(\int d\mathbf{x}_N \int d\psi_N^\dagger d\psi_N \exp[4\pi\nu_N \psi_N^\dagger \psi_N + i\psi_N^\dagger(\chi_N(x_N)\right.$$
$$\left. - \chi_1(x_N)) + i(\chi_N^\dagger(x_N) - \chi_1^\dagger(x_N))\psi_N + (v_N(x_N) - v_1(x_N))\psi_N^\dagger\psi_N + (w_N(x_N) - w_1(x_N))] \right)^{K_N}. \qquad (42)$$

In writing (42), we have used the fact that K_m identical integrals over $d\mathbf{x}_{mi} d\psi_{mi}^\dagger d\psi_{mi}$ appear in the partition function, where $i = 1 \ldots K_m$ is a "dumb" index labeling integration variables. Therefore, one representative of such an integral for every dyon kind m is taken to the power K_m.

In each line in Eq. (42), integration over $d\psi_m^\dagger d\psi_m$ can be trivially performed, given the rules (35): It reduces to expanding the exponents in Eq. (42) to the terms bilinear in $\psi_m^\dagger$, ψ_m. For example, for $m = 1$, we get

$$\sum_{K_1=0}^{\infty} \frac{1}{K_1!} \left(f \int d\mathbf{x} [4\pi\nu_1 + (\chi_1^\dagger - \chi_2^\dagger)(\chi_1 - \chi_2)(\mathbf{x}) + (v_1 - v_2)(\mathbf{x})] e^{(w_1 - w_2)(\mathbf{x})} \right)^{K_1}$$
$$= \exp\left(f \int d\mathbf{x} [4\pi\nu_1 + (\chi_1^\dagger - \chi_2^\dagger)(\chi_1 - \chi_2) + (v_1 - v_2)] e^{w_1 - w_2} \right) \qquad (43)$$

and similarly for other values of m. We obtain

$$\mathcal{Z} = \prod_{m=1}^{N} \int D\chi_m^\dagger D\chi_m Dv_m Dw_m \exp \int d\mathbf{x} \sum_m \left\{ \frac{T}{4\pi} (\partial \chi_m^\dagger \cdot \partial \chi_m + \partial v_m \cdot \partial w_m) + f[4\pi\nu_m + (v_m - v_{m+1}) + (\chi_m^\dagger - \chi_{m+1}^\dagger)(\chi_m - \chi_{m+1})] e^{w_m - w_{m+1}} \right\}. \qquad (44)$$

Given the cyclic symmetry in the summation over m, the terms without derivatives can be rewritten in a nicer way. We introduce the function

$$\mathcal{F}(w) \equiv \sum_{m=1}^{N} e^{w_m - w_{m+1}} \qquad (45)$$

and recall that $\nu_m = \mu_{m+1} - \mu_m$, where μ_m are the eigenvalues of the Polyakov line; see the introduction. The terms without derivatives in Eq. (44) can be written as

$$f\left[(-4\pi\mu_m + v_m) \frac{\partial \mathcal{F}}{\partial w_m} + \chi_m^\dagger \frac{\partial^2 \mathcal{F}}{\partial w_m \partial w_n} \chi_n \right]$$

(summation over repeated indices is understood), where

$$\frac{\partial \mathcal{F}}{\partial w_m} = e^{w_m - w_{m+1}} - e^{w_{m-1} - w_m},$$
$$\frac{\partial^2 \mathcal{F}}{\partial w_m \partial w_n} = \delta_{mn}(e^{w_m - w_{m+1}} + e^{w_{m-1} - w_m}) - \delta_{m,n-1} e^{w_m - w_{m+1}} - \delta_{m,n+1} e^{w_{m-1} - w_m}.$$

The final result for the dyon partition function is

$$\mathcal{Z} = \int D\chi^\dagger D\chi Dv Dw \exp \int d^3x \left\{ \frac{T}{4\pi} (\partial_i \chi_m^\dagger \partial_i \chi_m + \partial_i v_m \partial_i w_m) + f\left[(-4\pi\mu_m + v_m) \frac{\partial \mathcal{F}}{\partial w_m} + \chi_m^\dagger \frac{\partial^2 \mathcal{F}}{\partial w_m \partial w_n} \chi_n \right] \right\}. \qquad (46)$$

Equation (46) should be divided by the normalization integral being the same expression but with zero fugacity f. In fact, the normalization integral is unity and can be omitted. Indeed, integrating over v_m gives $\delta(-(T/4\pi)\partial^2 w_m)$, whose only solution is $w_m = \text{const}$, whereas the Jacobian is $\det^{-1}(-(T/4\pi)\partial^2)$. This Jacobian, however, is immediately canceled by the integral over the ghost fields χ_m. Therefore, the quantum field theory defined by Eq. (46) is the full result for the dyon partition function.

IV. GROUND STATE: "CONFINING" HOLONOMY PREFERRED

The fields v_m enter the partition function (46) only linearly. Therefore, they can be integrated out right away, giving rise to a δ function

$$\int Dv_m \to \delta\left(-\frac{T}{4\pi} \partial^2 w_m + f \frac{\partial \mathcal{F}}{\partial w_m} \right). \qquad (47)$$

This δ function restricts possible fields w_m over which one still has to integrate in Eq. (46). Let $\bar{w}_m$ be a solution to the argument of the δ function. Integrating over small fluctuations about $\bar{w}$ gives the Jacobian

$$\mathrm{Jac} = \det{}^{-1}\left(-\frac{T}{4\pi}\partial^2\delta_{mn} + f\frac{\partial^2\mathcal{F}}{\partial w_m \partial w_n}\Big|_{w=\bar{w}}\right). \quad (48)$$

Remarkably, exactly the same functional determinant (but in the numerator) arises from integrating over the ghost fields, in the same background $\bar{w}$:

$$\int D\chi^\dagger D\chi \exp\int d^3x\left[\frac{T}{4\pi}\partial_i\chi_m^\dagger\partial_i\chi_m + f\chi_m^\dagger\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\chi_n\right] = \det\left(-\frac{T}{4\pi}\partial^2\delta_{mn} + f\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\right). \quad (49)$$

Therefore, all quantum corrections cancel *exactly* between the boson and ghost fields (a characteristic feature of supersymmetry), and the ensemble of dyons is basically governed by a classical-field theory [35].

To find the ground state, we examine the fields' potential energy being $-4\pi f\mu_m\partial\mathcal{F}/\partial w_m$, which we prefer to write restoring $\nu_m = \mu_{m+1} - \mu_m$ and $\mathcal{F}$ as

$$\mathcal{P} = 4\pi f\sum_m \nu_m e^{w_m - w_{m+1}}. \quad (50)$$

For constant fields w_m, this is multiplied by the volume; therefore, one has to find the stationary point for any given set of ν_m's. It leads to the equations

$$\frac{\partial\mathcal{P}}{\partial w_1} = 4\pi f(\nu_1 e^{w_1-w_2} - \nu_N e^{w_N - w_1}) = 0,$$
$$\frac{\partial\mathcal{P}}{\partial w_2} = 4\pi f(\nu_2 e^{w_2-w_3} - \nu_1 e^{w_1 - w_2}) = 0, \ldots \quad (51)$$

whose solution is

$$e^{w_1-w_2} = \frac{(\nu_1\nu_2\nu_3\ldots\nu_N)^{(1/N)}}{\nu_1},$$
$$e^{w_2-w_3} = \frac{(\nu_1\nu_2\nu_3\ldots\nu_N)^{(1/N)}}{\nu_2}, \text{ etc.} \quad (52)$$

The solution corresponds to all terms in Eq. (50) being equal, despite *a priori* nonequal ν_m's. Putting it back into Eq. (50), we obtain

$$\mathcal{P} = 4\pi f N(\nu_1\nu_2\ldots\nu_N)^{(1/N)}, \qquad \nu_1 + \nu_2 + \cdots + \nu_N = 1. \quad (53)$$

The maximum is achieved when all ν's are equal:

$$\nu_1 = \nu_2 = \cdots = \nu_N = \frac{1}{N}, \qquad \mathcal{P}^{\max} = 4\pi f. \quad (54)$$

Equal ν's correspond to the "maximal nontrivial" or confining holonomy; see Eq. (5). Since there are no quantum corrections, the free energy of the dyons ensemble is simply proportional to the classical potential energy $F = -\mathcal{P}V$. Therefore, the maximum of $\mathcal{P}$ corresponds to the minimum of the free energy. Thus, the free energy of the grand canonical ensemble has the minimum at the confining values of the holonomy (see the introduction). In the minimum, the free energy is

$$F^{\min} = -4\pi f V = -\frac{16\pi^2}{g^4}\Lambda^4\frac{V}{T} = -\frac{N^2}{4\pi^2}\frac{\Lambda^4}{\lambda^2}\frac{V}{T},$$
$$\lambda \equiv \frac{\alpha_s N}{2\pi} = \frac{g^2 N}{8\pi^2}, \quad (55)$$

and there are no corrections to this result. In the last equation, we have introduced the N-independent 't Hooft coupling λ.

Let us make a few comments. First, the free energy (55) has the correct behavior at large N. Second, $V/T = V^{(4)}$ is in fact the $4d$ volume of the $R^3 \times S^1$ space. Although we do not expect our theory to be valid at small temperatures (where the measure we use for same-kind dyons is probably incomplete), Eq. (55) can be formally extended to the zero-temperature limit, as it correctly reproduces the extensive dependence on the $4d$ volume. Third, Eq. (55) gives, in fact, the density of dyons. One can introduce separate fugacities f_m for dyons of the mth kind into the partition function (36); then the average number of dyons is found from the obvious relation

$$\langle K_m\rangle = \frac{\partial \log\mathcal{Z}}{\partial\log f_m}\Big|_{f_m=f}.$$

With separate fugacities, the result (55) is modified by replacing $f \to (f_1 f_2\ldots f_N)^{1/N}$; hence,

$$\langle K_m\rangle = -f_m\frac{\partial F^{\min}}{\partial f_m}\Big|_{f_m=f} = \frac{1}{N}4\pi f V = \frac{N}{4\pi^2}\frac{\Lambda^4}{\lambda^2}V^{(4)}, \quad (56)$$

i.e., a finite (and equal) density of each kind of dyons in the 4-volume, meaning also the finite density of the KvBLL instantons. From the three-dimensional point of view, the $3d$ density of dyons (and KvBLL instantons) is increasing as the temperature goes down: There are more and more instantons sitting on top of each other in $3d$ but spread over the compactified time direction.

Let us add a few comments of speculative nature as they extend what is actually done here. We attempt to make contact with the phenomenology of the pure glue $SU(N)$ Yang-Mills theory. In the real world, there must be as many anti-self-dual dyons in the vacuum as there are dual ones, up to thermodynamic fluctuations $\sim\sqrt{V}$. For a crude estimate, we make the simplest assumption that adding anti-self-dual dyons just doubles the free energy. If the topological angle θ is introduced, one has to change dyon fugacities $f \to f e^{i\theta/N}$ and antidyon fugacities $f \to f e^{-i\theta/N}$, such that the KvBLL instanton whose fugacity is f^N acquires a phase $e^{i\theta}$ and the anti-instanton acquires a phase $e^{-i\theta}$ [37]. After minimization in w_m and ν_m which goes as before, the free energy (55) becomes

$$F = -4\pi f 2\cos\frac{\theta}{N}V = -\frac{16\pi^2}{g^4}\Lambda^4 2\cos\frac{\theta}{N}\frac{V}{T}, \quad (57)$$

leading to the topological susceptibility

$$\langle Q_T^2\rangle = \int d^4x \left\langle \frac{\mathrm{Tr}F\tilde{F}(x)}{16\pi^2}\frac{\mathrm{Tr}F\tilde{F}(0)}{16\pi^2}\right\rangle = \frac{1}{V^{(4)}}\frac{\partial^2 F}{\partial\theta^2}\bigg|_{\theta=0} = \frac{32\pi^2}{N^2 g^4}\Lambda^4 = \frac{1}{2\pi^2}\frac{\Lambda^4}{\lambda^2}. \tag{58}$$

We see that the topological susceptibility is stable at large N as it is expected from the N-counting rules.

The free energy is related, via the trace anomaly, to the so-called gluon condensate [11,32]

$$F \simeq -\frac{11N}{12}\frac{\langle \mathrm{Tr}F_{\mu\nu}^2\rangle}{16\pi^2}V^{(4)},$$

from which we find

$$\frac{\langle \mathrm{Tr}F_{\mu\nu}^2\rangle}{16\pi^2} \simeq N\frac{12}{11}\frac{1}{2\pi^2}\frac{\Lambda^4}{\lambda^2} = N\frac{12}{11}\langle Q_T^2\rangle. \tag{59}$$

It is the expected N dependence of the condensate.

As the temperature increases, the perturbative potential energy (6) becomes increasingly important since its contribution grows as T^4 with respect to the nonperturbative one. The perturbative energy arises from the small-oscillation determinant $[\det(-D^2)]^{-1}$ denoted as c in Eq. (33). If we naively add up the dyon-induced free energy (55) and the perturbative energy (7) both computed at the maximally nontrivial holonomy (54), we obtain the full free energy

$$\left(-\frac{32\pi^2}{g^4}\frac{\Lambda^4}{T} + T^3\frac{(2\pi)^2}{180}\frac{N^4-1}{N^2}\right)V.$$

It becomes positive and hence less favorable than the zero energy of the trivial holonomy at the temperature

$$T_c^4 = \frac{45}{2\pi^4}\frac{N^4}{N^4-1}\frac{\Lambda^4}{\lambda^2}. \tag{60}$$

At this temperature, the deconfinement phase transition is expected. We see that T_c is stable in N as it should be on general grounds. For a numerical estimate at $N=3$, we take $\lambda = 1/4$ compatible with the commonly assumed freezing of α_s at the value of 0.5 and $\Lambda = 200$ MeV in the Pauli-Villars scheme. We then obtain from Eqs. (58)–(60) the topological susceptibility, the gluon condensate, and the critical temperature $(189\ \mathrm{MeV})^4$, $(255\ \mathrm{MeV})^4$, and 278 MeV, respectively, being in reasonable agreement with the phenomenological and lattice values. More robust quantities (from both the theoretical and the lattice viewpoints) are those measured in units of the string tension; such a comparison will be made in the next section.

From now on, we shall assume we are far enough below the critical temperature that the minimum of the free energy implies the confining holonomy [Eq. (54)]. From Eq. (52), we learn that at the minimum all constant parts of w_m's are equal (up to a possible difference in $2\pi i k$, with integer k, which does not change the exponents of w). Let us note that had we imposed the overall neutrality condition of the dyon ensemble by an additional integration over the θ angles [see Eq. (31)], it would be equivalent to shifting $w_m \to w_m + i\theta_m$. Since in Eq. (46) one integrates over all functions w_m including their constant parts, an additional integration over θ's is unnecessary, and the neutrality condition is imposed automatically.

The triviality of the free energy (55) (which is due to the cancellation between boson and ghost quantum determinants) does not mean the triviality of the ensemble: Dyons are, in fact, strongly correlated, as we shall see in the last sections. To study correlations, one has to insert source terms into the partition function (46). With the sources switched on, the fields w_m are allowed to be x-dependent. Therefore, one has to retain the term $-4\pi f\mu_m\partial\mathcal{F}/\partial w_m$, which we rewrite using Eqs. (50) and (54) as

$$\text{action} = \int d^3x \frac{4\pi f}{N}\mathcal{F}(w), \tag{61}$$

where $\mathcal{F}(w)$ is defined in Eq. (45).

Finally, we note that the equation of motion for the fields w_m, following from the δ function (47), is known as the periodic Toda lattice [38], which has plenty of soliton solutions. In particular, there are many one-dimensional domain-wall solutions interpolating between $w_m - w_n = 2\pi i k_{mn}$ and $2\pi i k'_{mn}$, where k, k' are integers. Why do they not contribute to the partition function? The answer is that any soliton is x-dependent, and an overall shift of the soliton is a zero mode of the operator (48) resulting in an integration over the soliton position in space. However, it is also a zero mode of the identical operator for ghosts (49), leading to a vanishing ghost determinant. Therefore, any soliton gives a zero contribution to the partition function. However, solitons may and will generally contribute to the correlation functions.

V. CORRELATION FUNCTION OF POLYAKOV LINES

In the gauge where $A_4(\mathbf{x})$ is chosen to be time-independent, the Polyakov line is $\mathrm{Tr}L(\mathbf{z}) = \mathrm{Tr}\exp(iA_4(\mathbf{z})/T)$. The A_4 field of K KvBLL instantons away from their cores is Abelian [23] and can be gauge chosen to be diagonal:

$$A_4(\mathbf{z})/T = \delta_{mn}\left[2\pi\mu_m + \frac{1}{2T}\sum_i\left(\frac{1}{|\mathbf{z}-\mathbf{x}_{mi}|} - \frac{1}{|\mathbf{z}-\mathbf{x}_{m-1,i}|}\right)\right].$$

Comparing it with Eq. (40), we observe that A_4 can be written as

$$A_4(\mathbf{z})/T = \mathrm{diag}\left(2\pi\mu_m - \frac{1}{2}v_m(\mathbf{z})\right),$$
$$\mathrm{Tr}L(\mathbf{z}) = \sum_m \exp\left(2\pi i\mu_m - \frac{i}{2}v_m(\mathbf{z})\right). \tag{62}$$

Therefore, to compute the vacuum average of any number of Polyakov lines, one has to add a source term to the partition function (46):

$$\sum_{m_1,m_2\ldots}\exp\bigg[\epsilon_{m_1}\bigg(2\pi i\mu_{m_1}-\frac{i}{2}\int d\mathbf{x}v_m(\mathbf{x})\delta_{mm_1}\delta(\mathbf{x}-\mathbf{z_1})\bigg)+\epsilon_{m_2}\bigg(2\pi i\mu_{m_2}-\frac{i}{2}\int d\mathbf{x}v_m(\mathbf{x})\delta_{mm_2}\delta(\mathbf{x}-\mathbf{z_2})\bigg)+\cdots\bigg],\tag{63}$$

where $\mathbf{z}_{1,2\ldots}$ are the points in space where Polyakov lines are placed and $\epsilon_{m_1,m_2\ldots}=\pm1$ depending on whether one takes $L=\exp(iA_4/T)$ or $L^\dagger=\exp(-iA_4/T)$.

The source term is linear in v_m, which means that integration over v_m in the partition function with a source produces a δ function (47) as before whose argument is now shifted by the source:

$$\int Dv_m\to\delta\bigg(-\frac{T}{4\pi}\partial^2w_m+f\frac{\partial\mathcal{F}}{\partial w_m}-\epsilon_{m_1}\frac{i}{2}\delta(\mathbf{x}-\mathbf{z_1})\delta_{mm_1}-\epsilon_{m_2}\frac{i}{2}\delta(\mathbf{x}-\mathbf{z_2})\delta_{mm_2}-\cdots\bigg).\tag{64}$$

The correlation function of any number of widely separated Polyakov lines in the fundamental representation is given by the path integral with δ functions:

$$\begin{aligned}\langle\mathrm{Tr}L(\mathbf{z}_1)\mathrm{Tr}L(\mathbf{z}_2)\ldots\rangle=&\sum_{m_1,m_2,\ldots}e^{2\pi i(\mu_{m_1}+\mu_{m_2}+\cdots)}\int Dw_m\\&\times\exp\bigg(\int d\mathbf{x}\frac{4\pi f}{N}\mathcal{F}(w)\bigg)\\&\cdot\prod_m\delta\bigg(-\frac{T}{4\pi}\partial^2w_m+f\frac{\partial\mathcal{F}}{\partial w_m}\\&-\frac{i}{2}\delta(\mathbf{x}-\mathbf{z_1})\delta_{mm_1}\\&-\frac{i}{2}\delta(\mathbf{x}-\mathbf{z_2})\delta_{mm_2}-\cdots\bigg)\\&\times\det\bigg(-\frac{T}{4\pi}\partial^2\delta_{mn}+f\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\bigg).\end{aligned}\tag{65}$$

It is understood that Eq. (65) is divided by the same expression but without the source. The last factor comes from integrating over the ghost fields.

The strategy is to find all possible solutions of the δ functions, substitute them into the action (61), and sum over $m_{1,2\ldots}$. Note that, whatever functions w_m solve the δ functions, the Jacobian arising from those δ functions is again canceled exactly by the ghost determinant. Therefore, there will be no corrections to a classical calculation.

A. Average of a single line

The average $\langle\mathrm{Tr}L\rangle$ is expected to be zero for the confining holonomy, but let us check how it follows from the general equation (65). In this case, there is only one δ-function source in Eq. (65). One has to solve the equation

$$-\frac{T}{4\pi}\partial^2w_m+f\frac{\partial\mathcal{F}}{\partial w_m}=\frac{i}{2}\delta(\mathbf{x}-\mathbf{z_1})\delta_{mm_1}$$

and plug the solution into the action (61). The solution is $w_m(\mathbf{x})\approx\delta_{mm_1}(i/2T)/|\mathbf{x}-\mathbf{z}_1|$ near the source, where the Laplacian is the leading term and $\partial\mathcal{F}/\partial w_m$ can be neglected. At large distances from the source, w_m decays; therefore, $\partial\mathcal{F}/\partial w_m$ can be expanded to the linear order in w_m. The solution decreases exponentially with the distance. At intermediate distances, the nonlinearity is essential. However, whatever is the precise form of the solution of this nonlinear equation, the action is finite and independent of m_1, as there is a perfect cyclic symmetry in m_1. Therefore, the action factors out from the summation over m_1, and we obtain

$$\langle\mathrm{Tr}L\rangle=\mathrm{const}\sum_{m_1}\exp(2\pi i\mu_{m_1})=0,\tag{66}$$

as expected in the confining phase. We use here the "maximally nontrivial" holonomy (5), which has been shown in Sec. IV to bring the free energy to the minimum.

B. Heavy-quark potential

The correlation function of two Polyakov lines in the fundamental representation at spatial points $\mathbf{z}_1$ and $\mathbf{z}_2$ is

$$\begin{aligned}\langle\mathrm{Tr}L(\mathbf{z}_1)\mathrm{Tr}L^\dagger(\mathbf{z}_2)\rangle=&\sum_{m_1,n_1}e^{2\pi i(\mu_{m_1}-\mu_{n_1})}\int Dw_m\\&\times\exp\bigg(\int d\mathbf{x}\frac{4\pi f}{N}\mathcal{F}(w)\bigg)\\&\cdot\prod_m\delta\bigg(-\frac{T}{4\pi}\partial^2w_m+f\frac{\partial\mathcal{F}}{\partial w_m}\\&-\frac{i}{2}\delta(\mathbf{x}-\mathbf{z_1})\delta_{mm_1}+\frac{i}{2}\delta(\mathbf{x}-\mathbf{z_2})\delta_{mn_1}\bigg)\\&\times\det\bigg(-\frac{T}{4\pi}\partial^2\delta_{mn}+f\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\bigg).\end{aligned}\tag{67}$$

We are interested in the asymptotics of the correlator (67) at large source separations $|\mathbf{z}_1-\mathbf{z}_2|\to\infty$. We shall see in a moment that w_m's solving the δ functions fall off exponentially from the sources $w\sim\exp(-M|\mathbf{z}-\mathbf{z}_{1,2}|)/|\mathbf{z}-\mathbf{z}_{1,2}|$; therefore, the generally nonlinear equations on w_m can be linearized far from the sources. The same Yukawa (or, more precisely, Coulomb) functions are the solutions close to the sources, as the leading term there is the Laplacian, and the $\partial\mathcal{F}/\partial w_m$ term can be neglected. In the intermediate range, the nonlinearity is essential, but it has no influence on the asymptotics of the potential between two infinitely heavy quarks—only on the residue of the correlator. The action acquires the $|\mathbf{z}_1-\mathbf{z}_2|$-dependent contribution from the range of integration far away from both sources where $w_m(x)$ is small. Therefore, to find the asymptotics of the heavy-quark potential, one can take $\partial\mathcal{F}/\partial w_m$ to the linear order in w_m and $\mathcal{F}(w)$ to the quadratic order. We have for small w_m

$$\mathcal{F}(w) = \sum_m e^{w_m - w_{m+1}} \approx N + \frac{1}{2} w_m \mathcal{M}_{mn} w_n, \qquad \frac{\partial \mathcal{F}}{\partial w_m} \approx \mathcal{M}_{mn} w_n, \tag{68}$$

where $\mathcal{M}$ is the matrix made of scalar products of the simple roots of the gauge group, supplemented by a non-simple root to make it periodic: $\mathcal{M}_{mn} = \mathrm{Tr} C_m C_n$; see Eq. (1). In our case of $SU(N)$,

$$\mathcal{M} = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & -1 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}. \tag{69}$$

The $SU(2)$ group is a special case where this matrix is

$$\mathcal{M}^{(2)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{70}$$

The orthonormalized eigenvectors are the pairs

$$V_n^{(k,1)} = \sqrt{\frac{2}{N}} \cos\left(\frac{2\pi k}{N} n\right) \quad \text{and} \quad V_n^{(k,2)} = \sqrt{\frac{2}{N}} \sin\left(\frac{2\pi k}{N} n\right) \tag{71}$$

corresponding to the twice-degenerate eigenvalues

$$\mathcal{M}^{(k)} = \left(2 \sin\frac{\pi k}{N}\right)^2, \qquad k = 1 \ldots \left[\frac{N-1}{2}\right]. \tag{72}$$

There is also an eigenvector $V_n^{(0)} = \cos(2\pi \cdot 0/N \cdot n)/\sqrt{N} = (1, 1, \ldots, 1)/\sqrt{N}$ with a nondegenerate zero eigenvalue, and, in the case of even N, there is an additional eigenvector $V_n^{(N/2)} = \cos(2\pi \cdot N/2/N \cdot n)/\sqrt{N} = (1, -1, 1, \ldots, -1)/\sqrt{N}$ with a nondegenerate eigenvalue equal to 2^2. In other words, the eigenvalues are

$$\mathcal{M}^{(k)} = \left(2 \sin\frac{\pi k}{N}\right)^2, \qquad k = 0, \ldots, N-1, \tag{73}$$

where the pairs of eigenvalues corresponding to k and $N - k$ are apparently degenerate.

In the linearized form, the δ functions in Eq. (67) impose the equations

$$-\partial^2 w_m + M^2 \mathcal{M}_{mn} w_n = \frac{2\pi i}{T}(\delta_{mm_1}\delta(\mathbf{x} - \mathbf{z}_1) - \delta_{mn_1}\delta(\mathbf{x} - \mathbf{z}_2)), \tag{74}$$

where we have introduced the "dual photon" mass

$$M^2 = \frac{4\pi f}{T} = \frac{16\pi^2\Lambda^4}{g^4 T^2} = \mathcal{O}(N^2). \tag{75}$$

Equations (74) are best solved in the momentum space:

$$w_m(\mathbf{p}) = \frac{2\pi i}{T}\left(\frac{1}{\mathbf{p}^2 + M^2\mathcal{M}}\right)_{mn} E_n(\mathbf{p}), \quad \text{where } E_n = \delta_{nm_1} e^{i\mathbf{p}\cdot\mathbf{z}_1} - \delta_{nn_1} e^{i\mathbf{p}\cdot\mathbf{z}_2}. \tag{76}$$

This must be put into the action (61) where $\mathcal{F}(w)$ is to be expanded to the quadratic order. We have

$$\begin{aligned} &\int d^3\mathbf{x} \frac{4\pi f}{N} \frac{1}{2} w_m(\mathbf{x}) \mathcal{M}_{mn} w_n(\mathbf{x}) \\ &= \frac{2\pi f}{N} \int \frac{d^3\mathbf{p}}{(2\pi)^3} w_m(\mathbf{p}) \mathcal{M}_{mn} w_n(-\mathbf{p}) \\ &= -\frac{2\pi f}{N} \frac{(2\pi)^2}{T^2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} E_m(\mathbf{p}) \\ &\quad \times \left(\frac{1}{\mathbf{p}^2 + M^2\mathcal{M}}\right)_{mp} \mathcal{M}_{pq} \left(\frac{1}{\mathbf{p}^2 + M^2\mathcal{M}}\right)_{qn} E_n(-\mathbf{p}) \\ &= -\frac{(2\pi)^3 f}{NT^2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} E_m(\mathbf{p}) \\ &\quad \times \sum_{l=1}^{N} V_m^{(l)} \frac{1}{\mathbf{p}^2 + M^2\mathcal{M}^{(l)}} \mathcal{M}^{(l)} \frac{1}{\mathbf{p}^2 + M^2\mathcal{M}^{(l)}} V_n^{(l)} E_n(-\mathbf{p}), \end{aligned} \tag{77}$$

where we have diagonalized the matrices by the orthogonal transformation built of the eigenvectors $V_m^{(l)}$ corresponding to the eigenvalues $\mathcal{M}^{(l)}$. We now pick from $E_m(\mathbf{p})E_n(-\mathbf{p})$ the cross terms depending on $\mathbf{z}_1 - \mathbf{z}_2$ as only they are relevant for the interaction. The inverse Fourier transform is

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot(\mathbf{z}_1 - \mathbf{z}_2)}}{(\mathbf{p}^2 + M^2\mathcal{M}^{(l)})^2} = \frac{1}{8\pi} \frac{e^{-|\mathbf{z}_1 - \mathbf{z}_2| M\sqrt{\mathcal{M}^{(l)}}}}{M\sqrt{\mathcal{M}^{(l)}}}.$$

Therefore, we continue the chain of Eqs. (77) and write

$$(77) = \frac{2\pi^2 f}{NT^2 M} V_{m_1}^{(l)} \sqrt{\mathcal{M}^{(l)}} e^{-|\mathbf{z}_1 - \mathbf{z}_2| M\sqrt{\mathcal{M}^{(l)}}} V_{n_1}^{(l)}, \tag{78}$$

where summation over all eigenvalues labeled by l is implied. The coefficient -2 arises because cross terms in $E_m E_n$ have a negative relative sign, and there are two such terms. We obtain from Eq. (67)

$$\langle \mathrm{Tr} L(\mathbf{z}_1) \mathrm{Tr} L^\dagger(\mathbf{z}_2)\rangle = \sum_{m_1, n_1} \exp\left(2\pi i(\mu_{m_1} - \mu_{n_1}) + \frac{2\pi^2 f}{NT^2 M} \times V_{m_1}^{(l)} \sqrt{\mathcal{M}^{(l)}} e^{-|\mathbf{z}_1 - \mathbf{z}_2| M\sqrt{\mathcal{M}^{(l)}}} V_{n_1}^{(l)}\right). \tag{79}$$

At large separations $|\mathbf{z}_1 - \mathbf{z}_2|$ between the point sources, the second term in Eq. (79) is exponentially small, and one can Taylor expand it. The zero-order term is zero as it is the product of two independent sums over m_1 and n_1; i.e., it is the product of unconnected $\langle \mathrm{Tr} L(\mathbf{z}_1)\rangle \langle \mathrm{Tr} L^\dagger(\mathbf{z}_2)\rangle = 0$, as explained in the previous subsection. In the first nonzero order, we get

$$\langle \mathrm{Tr}L(\mathbf{z}_1)\mathrm{Tr}L^\dagger(\mathbf{z}_2)\rangle = \frac{2\pi^2 f}{NT^2M}\sum_{l=1}^{[N/2]}\sqrt{\mathcal{M}^{(l)}}\,e^{-|\mathbf{z}_1-\mathbf{z}_2|M\sqrt{\mathcal{M}^{(l)}}} \times \sum_{m_1,n_1=1}^{N}\exp(2\pi i(\mu_{m_1}-\mu_{n_1})) \times V^{(l)}_{m_1}V^{(l)}_{n_1}. \tag{80}$$

It is a sum of exponentially decaying contributions with the exponents determined by the eigenvalues $\mathcal{M}^{(l)}$; see Eq. (73). The weight of the lth contribution is determined by the summation over m_1, n_1. For $l = 1, \ldots, [\frac{N-1}{2}]$, eigenvalues are twice-degenerate, and we use the eigenvectors (71). At even N, the highest eigenvalue is nondegenerate, the corresponding eigenvector being $V_n^{(N/2)} = \cos\pi n/\sqrt{N}$. Summation over m_1, n_1 in Eq. (80) gives

$$\sum_{m_1,n_1=1}^{N}\exp\left(2\pi i\frac{m_1-n_1}{N}\right)\frac{2}{N}\left(\cos\frac{2\pi l m_1}{N}\cos\frac{2\pi l n_1}{N}+\sin\frac{2\pi l m_1}{N}\sin\frac{2\pi \ln_1}{N}\right) = N\delta_{l,1} \quad \text{for all } l = 1\ldots\left[\frac{N-1}{2}\right], \text{ any } N;$$
$$\sum_{m_1,n_1=1}^{N}\exp\left(2\pi i\frac{m_1-n_1}{N}\right)\frac{1}{N}\cos\pi m_1\cos\pi n_1 = N\delta_{N,2} \quad \text{for } l=\frac{N}{2}, \text{ even } N. \tag{81}$$

We see that only the exponent with the *lowest* eigenvalue $\sqrt{\mathcal{M}^{(1)}} = 2\sin\frac{\pi}{N}$ contributes in Eq. (79) to the correlator of Polyakov lines in the fundamental representation; higher eigenvalues decouple through orthogonality. We thus obtain

$$\langle \mathrm{Tr}L(\mathbf{z}_1)\,\mathrm{Tr}L^\dagger(\mathbf{z}_2)\rangle = \frac{2\pi^2 f}{NT^2M}2\sin\frac{\pi}{N}N \times \exp\left(-|\mathbf{z}_1-\mathbf{z}_2|M2\sin\frac{\pi}{N}\right) \tag{82}$$

plus exponentially small corrections from the expansion of Eq. (79) to higher orders. This should be compared with the standard definition of the heavy-quark potential

$$\langle \mathrm{Tr}L(\mathbf{z}_1)\,\mathrm{Tr}L^\dagger(\mathbf{z}_2)\rangle = C\exp\left(-\frac{V(\mathbf{z}_1-\mathbf{z}_2)}{T}\right),$$

from which we deduce the linear heavy-quark potential at large separations:

$$V(\mathbf{z}_1-\mathbf{z}_2) = |\mathbf{z}_1-\mathbf{z}_2|MT2\sin\frac{\pi}{N} = \sigma|\mathbf{z}_1-\mathbf{z}_2|, \qquad C = \mathcal{O}(N^0), \tag{83}$$

with the "string tension"

$$\sigma = MT2\sin\frac{\pi}{N} = T\sqrt{\frac{4\pi f}{T}}2\sin\frac{\pi}{N} = 8\pi\frac{\Lambda^2}{g^2}\sin\frac{\pi}{N} = \frac{\Lambda^2}{\lambda}\frac{N}{\pi}\sin\frac{\pi}{N}. \tag{84}$$

In the last equation in the chain, the N-independent 't Hooft coupling λ has been used. We see that the string tension turns out to be (i) independent of the temperature [39] and (ii) independent of N at large N, as expected. In reality, we expect that anti-self-dual dyons not accounted for here double M^2, and, hence, the dyon-induced string tension is actually $\sqrt{2}$ times bigger. A more robust quantity (from both the theoretical and the lattice viewpoints) is the ratio $T_c\sqrt{\sigma}$ since in this ratio the poorly known parameters Λ and λ cancel out; see Eq. (60):

$$\frac{T_c}{\sqrt{\sigma}} = \left(\frac{45}{4\pi^4}\frac{\pi^2N^2}{(N^4-1)\sin^2\frac{\pi}{N}}\right)^{(1/4)} \xrightarrow{N\to\infty} \frac{1}{\pi}\left(\frac{45}{4}\right)^{(1/4)} + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{85}$$

The values are compared to those measured in lattice simulations of the pure $SU(N)$ gauge theories [40] in Table I demonstrating good agreement. The relatively large 4% deviation for the $SU(2)$ group may be related to the fact that we have determined T_c in Sec. IV by comparing the free energy for confining and trivial holonomy, that is, assuming a first-order transition, whereas for $N = 2$ it is actually a second-order one.

In Table II, we add the comparison of the topological susceptibility (58) measured in units of the string tension, with the lattice data [41]. The agreement is also remarkably good, given the approximate nature of the model.

C. N-ality and k strings

All irreducible representations of the $SU(N)$ group fall into N classes: those that appear in the direct product of any number of adjoint representations and those that appear in the direct product of any number of adjoint representations with the irreducible representation being the rank-k antisymmetric tensor, $k = 1, \ldots, N-1$. "N-ality" is said to be zero in the first case and equal to k in the second. N-ality-zero representations transform trivially under the center of the group Z_N; the rest acquire a phase $2\pi k/N$.

One expects that there is no asymptotic linear potential between static color sources in the adjoint representation as such sources are screened by gluons. If a representation is found in a direct product of some number of adjoint

TABLE I. Deconfinement temperature $T_c/\sqrt{\sigma}$ from Eq. (85) (upper row) and from lattice simulations [40] (lower row).

$N=2$	3	4	6	8	∞
(0.7425)	0.6430	0.6150	0.5967	0.5906	$0.5830 + \frac{0.4795}{N^2} + \frac{0.5006}{N^4} + \ldots$
0.7091(36)	0.6462(30)	0.6344(81)	0.6101(51)	0.5928(107)	$0.5970(38) + \frac{0.449(29)}{N^2}$ (fit)

representations and a rank-k antisymmetric representation, the adjoint ones "cancel out" as they can be all screened by an appropriate number of gluons. Therefore, from the confinement viewpoint, all N-ality $= k$ representations are equivalent, and there are only $N-1$ string tensions, $\sigma(k,N)$ being the coefficients in the asymptotic linear potential for sources in the antisymmetric rank-k representation. Its dimension is $d(k,N) = \frac{N!}{k!(N-k)!}$, and the eigenvalue of the quadratic Casimir operator is $C(k,N) = \frac{N+1}{2N}k(N-k)$.

The value $k=1$ corresponds to the fundamental representation, whereas $k=N-1$ corresponds to the representation conjugate to the fundamental [quarks and antiquarks]. In general, the rank-$(N-k)$ antisymmetric representation is conjugate to the rank-k one; it has the same dimension and the same string tension $\sigma(k,N) = \sigma(N-k,N)$. Therefore, for odd N, all string tensions appear in equal pairs; for even N, apart from pairs, there is one privileged representation with $k=\frac{N}{2}$ which has no pair and is real. The total number of different string tensions is thus $[\frac{N}{2}]$.

The behavior of $\sigma(k,N)$ as function of k and N is an important issue as it discriminates between various confinement mechanisms. On general N-counting grounds, one can only infer that at large N and $k \ll N$, $\sigma(k,N)/\sigma(1,N) = (k/N)(1+\mathcal{O}(1/N^2))$ [42]. In this subsection, we show that the dyon ensemble leads to the sine law for the k strings

$$\sigma(k,N) = \text{const}\sin\frac{\pi k}{N} \quad \text{(plus temperature-dependent corrections)}, \tag{86}$$

satisfying the above requirement on the asymptotics. The sine behavior has been found in certain supersymmetric theories [43]. Here it follows from a direct calculation of the correlator of Polyakov lines in the rank-k antisymmetric representation.

We first show that there is no asymptotic linear potential between adjoint sources. If A_4 is diagonal and given by Eq. (62), the eigenvalues of the Polyakov loop in the adjoint representation are $\exp(\pm i(A_{4m} - A_{4n})/T)$, and there are $N-1$ unity eigenvalues. Therefore, the average of the adjoint line is nonzero, and the correlator of two such lines tends asymptotically to a nonzero constant.

Let the Polyakov line in the fundamental representation be $L(\mathbf{z}) = \exp(iA_4(\mathbf{z})/T) = \text{diag}(z_1, z_2, \ldots, z_N)$, where $z_m = \exp(2\pi i\mu_m - \frac{i}{2}v_m(\mathbf{z}))$; see Eq. (62). The Polyakov lines in the antisymmetric rank-k representation are then

$$\begin{aligned} L(1,N) &= \text{Tr}L = \sum_{m=1}^{N} z_m, \qquad k=1,\\ L(2,N) &= \frac{1}{2}((\text{Tr}L)^2 - \text{Tr}L^2) = \sum_{m<n}^{N} z_m z_n, \qquad k=2,\\ L(3,N) &= \frac{1}{6}((\text{Tr}L)^3 - 3\text{Tr}L^2\text{Tr}L + 2\text{Tr}L^3) = \sum_{m<n<p}^{N} z_m z_n z_p,\\ &\qquad k=3,\\ L(k,N) &= \sum_{m_1<m_2<\cdots<m_k}^{N} z_{m_1} z_{m_2}\ldots z_{m_k}. \end{aligned} \tag{87}$$

Therefore, any general $L(k,N)(\mathbf{z})$ placed at the $3d$ point $\mathbf{z}$ serves as a source

$$\sum_{m_1<m_2<\cdots<m_k}^{N} \exp\Big[2\pi i(\mu_{m_1} + \cdots + \mu_{m_k}) - \frac{i}{2}(v_{m_1}(\mathbf{z}) + \cdots + v_{m_k}(\mathbf{z}))\Big]$$

for the v_m field, which should be put into the partition function (46).

To get the correlation function of two lines in k representation, we proceed as in Sec. V B and arrive at the generalization of Eq. (80):

TABLE II. Topological susceptibility $\langle Q_T^2\rangle^{1/4}/\sqrt{\sigma}$ from Eq. (58) (upper row) and from lattice simulations [41] (lower row).

$N=2$	3	4	5	∞
0.5	0.439	0.420	0.412	$0.399 + \frac{0.328}{N^2} + \frac{0.243}{N^4} + \ldots$
0.4831(56)	0.434(10)	0.387(17)	0.387(21)	$0.376(20) + \frac{0.43(10)}{N^2}$ (fit)

$$\langle L(k,N)(\mathbf{z}_1)L^\dagger(k,N)(\mathbf{z}_2)\rangle = \frac{2\pi^2 f}{NT^2M}\sum_{l=1}^{[(N/2)]}\sqrt{\mathcal{M}^{(l)}}\,e^{-|\mathbf{z}_1-\mathbf{z}_2|M\sqrt{\mathcal{M}^{(l)}}}\cdot\sum_{m_1<m_2<\cdots<m_k}^{N}\sum_{n_1<n_2<\cdots<n_k}^{N}\exp 2\pi i(\mu_{m_1}+\cdots+\mu_{m_k} - \mu_{n_1}-\cdots-\mu_{n_k})[V^{(l)}_{m_1}+\cdots+V^{(l)}_{m_k}][V^{(l)}_{n_1}+\cdots+V^{(l)}_{n_k}]. \tag{88}$$

In deriving Eq. (88), it is important that the maximally nontrivial holonomy (5) is used, leading to $\langle L(k,N)\rangle = 0$, $k = 1\dots N-1$. Higher powers of $\exp(-|\mathbf{z}_1-\mathbf{z}_2|M\sqrt{\mathcal{M}^{(l)}})$ have been neglected.

Again, the correlation function of Polyakov lines is a sum of exponentially decaying contributions with the exponents determined by the eigenvalues $\mathcal{M}^{(l)}$; see Eq. (73). The weight of the lth contribution is given by the sum over $m_{1,2\dots k}$ and $n_{1,2\dots k}$. We recall the eigenvectors $V^{(l)}$ (71) and observe the following important orthogonality relation:

$$\sum_{m_1<m_2<\cdots<m_k}^{N}\sum_{n_1<n_2<\cdots<n_k}^{N}\exp\left[2\pi i\frac{m_1+\cdots+m_k-n_1-\cdots-n_k}{N}\right][V^{(l)}_{m_1}+\cdots+V^{(l)}_{m_k}][V^{(l)}_{n_1}+\cdots+V^{(l)}_{n_k}] = \begin{cases} N\delta_{lk} & \text{for all twice-degenerate eigenvalues } l = 1\dots\left[\frac{N-1}{2}\right],\ \text{any } N\geq 2k, \\ N\delta_{N,2k} & \text{for the highest, nondegenerate eigenvalue } l=\frac{N}{2},\ \text{even } N. \end{cases} \tag{89}$$

[The orthogonality relation (81) is a particular case of this general one, corresponding to $k = 1$. The derivation of these relations is elementary when one presents the eigenvectors in the exponential form.]

The above orthogonality relations imply that the correlator of the lines in rank-k antisymmetric tensor representation *couples only to the single exponent* determined by the kth eigenvalue $\sqrt{\mathcal{M}^{(k)}} = 2\sin\frac{\pi k}{N}$, $N\geq 2k$; all of the rest of the eigenvalues decouple [44]. Therefore, the correlation function (88) is

$$\langle L(k,N)(\mathbf{z}_1)L^\dagger(k,N)(\mathbf{z}_2)\rangle = \frac{2\pi^2 f}{NT^2M}2\sin\frac{\pi k}{N}N \times\exp(-|\mathbf{z}_1-\mathbf{z}_2|M\sqrt{\mathcal{M}^{(k)}}), \tag{90}$$

and, hence, the general-k string tension is

$$\sigma(k,N) = MT\sqrt{\mathcal{M}^{(k)}} = MT2\sin\frac{\pi k}{N} = \frac{\Lambda^2}{\lambda}\frac{N}{\pi}\sin\frac{\pi k}{N} \tag{91}$$

as announced. Lattice simulations [45] support this regime, whereas another lattice study [46] gives somewhat smaller values but within 2 standard deviations from the values following from Eq. (91). For a general discussion of the sine regime for k strings, which is favored from many viewpoints, see [42].

VI. AREA LAW FOR SPATIAL WILSON LOOPS

The area behavior of the *spatial* Wilson loops is not directly related to the linear confining potential; however, it is believed that in a confining theory the spatial Wilson loop must exhibit the area law. The reason is that (i) at $T\to 0$, Lorentz symmetry is restored, and, therefore, the spatial loop must behave in the same way as the timelike one whose area law is related to the linear confining potential, and (ii) at high T, the spatial loop eventually becomes a timelike loop from the 2 + 1 dimensions' point of view, which has to obey the area law to fulfill confinement in $3d$. Therefore, it is very plausible that the spatial Wilson loop has the area behavior at any temperatures. It is expected that the spatial string tension is roughly constant below the deconfinement transition and eventually grows as $\sim T^2$ at very high temperatures where the theory is basically three-dimensional.

In this section, we demonstrate that the dyon ensemble induces the area law for spatial Wilson loops and that the string tension coincides with that found in the previous section from the correlators of the Polyakov lines. We think that it is an interesting result since (a) the way we derive the string tension for spatial loops is very different from that for Polyakov lines and (b) in a sense, it demonstrates that our ensemble restores Lorentz symmetry at low temperatures, despite its three-dimensional formulation.

The condition that A_4 is time-independent only partially fixes the gauge: One can still perform time-independent gauge transformations. This freedom can be used to make A_4 diagonal (i.e., Abelian). This necessarily implies Dirac string singularities, which are pure gauge artifacts as they do not carry any energy. Moreover, the Dirac strings' directions are also subject to the freedom of the gauge choice. In Refs. [4,23], the gauge choice in the explicit construction of the KvBLL instanton was such that the Dirac strings were connecting individual dyon constituents of the instanton. This choice is, however, not convenient in the ensemble of dyons as dyons have to lose their "memory" to which particular instanton they belong. The natural gauge is where all Dirac strings of all dyons are directed to infinity along some axis, e.g., along the z axis. The dyons' field in this gauge is given explicitly in Ref. [47] [for the $SU(2)$ group].

In this gauge, the magnetic field of dyons beyond their cores is also Abelian and is a superposition of the Abelian fields of individual dyons. For large Wilson loops in which we are interested, the field of a large number of dyons contribute as they have a slowly decreasing $1/|\mathbf{x}-\mathbf{x}_i|$ asymptotics; hence, the use of the field outside the cores is justified. Owing to self-duality,

$$[B_i(\mathbf{x})]_{mn} = [\partial_i A_4(\mathbf{x})]_{mn} = -\frac{T}{2}\delta_{mn}\partial_i v_m(\mathbf{x}); \qquad (92)$$

cf. Eq. (40). Since A_i is Abelian beyond the cores, one can use the Stokes theorem for the spatial Wilson loop:

$$W \equiv \mathrm{Tr}\mathrm{P}\exp i\oint A_i dx^i = \mathrm{Tr}\exp i\int B_i d^2\sigma^i = \sum_m \exp\left(-i\frac{T}{2}\int d^2\sigma^i\partial_i v_m\right). \qquad (93)$$

Equation (93) may look contradictory as we first use $B_i = \mathrm{curl}A_i$ and then $B_i = \partial_i A_4$. Actually, there is no contradiction as the last equation is true up to Dirac string singularities which carry away the magnetic flux. If the Dirac string pierces the surface spanning the loop, it gives a quantized contribution $\exp(2\pi i\cdot\mathrm{integer}) = 1$; one can also use the gauge freedom to direct Dirac strings parallel to the loop surface, in which case there is no contribution from the Dirac strings at all.

Let us take a flat Wilson loop lying in the (xy) plane at $z=0$. Then Eq. (93) is continued as

$$W = \sum_m \exp\left(-i\frac{T}{2}\int_{x,y\in\mathrm{area}} d^3x\,\partial_z v_m\delta(z)\right) = \sum_m \exp\left(i\frac{T}{2}\int_{x,y\in\mathrm{area}} d^3x\, v_m\partial_z\delta(z)\right). \qquad (94)$$

It means that the average of the Wilson loop in the dyons ensemble is given by the partition function (46) with the source

$$\sum_m \exp\left(i\frac{T}{2}\int d^3x\, v_m\frac{d\delta(z)}{dz}\theta(x,y\in\mathrm{area})\right),$$

where $\theta(x,y\in\mathrm{area})$ is a step function equal to unity if x, y belong to the area inside the loop and equal to zero otherwise. Again, the source shifts the argument of the δ function arising from the integration over the v_m variables, and the average Wilson loop in the fundamental representation is given by the equation

$$\langle W\rangle = \sum_{m_1}\int Dw_m \exp\left(\int d\mathbf{x}\frac{4\pi f}{N}\mathcal{F}(w)\right)\cdot\prod_m\delta\left(-\frac{T}{4\pi}\partial^2 w_m + f\frac{\partial\mathcal{F}}{\partial w_m} + \frac{iT}{2}\delta_{mm_1}\frac{d\delta(z)}{dz}\theta(x,y\ \mathrm{area})\right)\times\det\left(-\frac{T}{4\pi}\partial^2\delta_{mn} + f\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\right). \qquad (95)$$

Therefore, one has to solve the nonlinear equations on w_m's with a source along the surface of the loop

$$-\partial^2 w_m + M^2(e^{w_m - w_{m+1}} - e^{w_{m-1}-w_m}) = -2\pi i\delta_{mm_1}\frac{d\delta(z)}{dz}\theta(x,y\in\mathrm{area}), \qquad M^2 = \frac{4\pi f}{T}, \qquad (96)$$

for all m_1, plug it into the action $(4\pi f/N)\mathcal{F}(w)$, and sum over m_1. In order to evaluate the average of the Wilson loop in a general antisymmetric rank-k representation, one has to take the source in Eq. (96) as $-2\pi i\delta'(z)(\delta_{mm_1}+\cdots+\delta_{mm_k})$ and sum over $m_1<\cdots<m_k$ from 1 to N; see Eq. (87). Again, the ghost determinant cancels exactly the Jacobian from the fluctuations of w_m about the solution; therefore, the classical-field calculation is exact.

Contrary to the case of the Polyakov lines, one cannot, generally speaking, linearize Eq. (96) in w_m but has to solve the nonlinear equations as they are. With no source in the right-hand side, Eq. (96) is known as the periodic Toda lattice, and it is integrable for any N. It has an hierarchy of soliton solutions constructed in Refs. [38,48]. Below, we modify those solutions in such a way that they satisfy Eq. (96) with a source in the right-hand side. We call them “pinned solitons”; their action determines the string tensions. We obtain below for the “magnetic” k-string tension

$$\sigma(k,N) = MT2\sin\frac{\pi k}{N} = \frac{\Lambda^2}{\lambda}\frac{N}{\pi}\sin\frac{\pi k}{N}, \qquad (97)$$

which coincides exactly with the “electric” string tension (91) found from the correlators of the Polyakov lines.

A. Construction of string solitons

Let us find the pinned solitons corresponding to Wilson loops in a general antisymmetric rank-k representation of the $SU(N)$ gauge group. First of all, we rewrite Eq. (96) for the difference fields $w_{m,m+1} = w_m - w_{m+1}$ as only the differences enter the action:

$$
\begin{aligned}
-w''_{12} + M^2(2e^{w_{12}} - e^{w_{N1}} - e^{w_{23}}) &= -2\pi i\delta'(z)[(\delta_{1,m_1} + \cdots + \delta_{1,m_k}) - (\delta_{2,m_1} + \cdots + \delta_{2,m_k})],\\
-w''_{23} + M^2(2e^{w_{23}} - e^{w_{12}} - e^{w_{34}}) &= -2\pi i\delta'(z)[(\delta_{2,m_1} + \cdots + \delta_{2,m_k}) - (\delta_{3,m_1} + \cdots + \delta_{3,m_k})], \quad \cdots\\
-w''_{N-1,N} + M^2(2e^{w_{N-1,N}} - e^{w_{N-2,N-1}} - e^{w_{N,1}}) &= -2\pi i\delta'(z)[(\delta_{N-1,m_1} + \cdots + \delta_{N-1,m_k}) - (\delta_{N,m_1} + \cdots + \delta_{N,m_k})],\\
-w''_{N,1} + M^2(2e^{w_{N,1}} - e^{w_{N-1,N}} - e^{w_{12}}) &= -2\pi i\delta'(z)[(\delta_{N,m_1} + \cdots + \delta_{N,m_k}) - (\delta_{1,m_1} + \cdots + \delta_{1,m_k})].
\end{aligned}
\tag{98}
$$

At all z except one point $z = 0$, where there is a source, $w_{m,m+1}$ satisfy free (zero source) equations. Solutions for very similar equations have been constructed in Refs. [38,48]. Adjusting them to our case, we write the general solutions of Eq. (98) with the zero right-hand side:

$$
\begin{aligned}
w^{(k)}_{m,m+1}(z) &= \ln\frac{[1+\gamma\varkappa^{k(m-1)}E^{(k)}(z)][1+\gamma\varkappa^{k(m+1)}E^{(k)}(z)]}{[1+\gamma\varkappa^{km}E^{(k)}(z)]^2}, \qquad k = 1,\ldots,N-1,\\
E^{(k)}(z) &= \exp(-M\sqrt{\mathcal{M}^{(k)}}z), \qquad \sqrt{\mathcal{M}^{(k)}} = 2\sin\frac{\pi k}{N}, \qquad \varkappa = \exp\left(\pm\frac{2\pi i}{N}\right).
\end{aligned}
\tag{99}
$$

The complex parameter γ is arbitrary; there is also freedom in choosing the sign of the phase of $\varkappa$. Equations (99) describe free "domain-wall" solitons on the axis $-\infty < z < +\infty$; it is easy to check that their actions do not depend on γ and are given by Eq. (97). Were γ real, it could be understood as an overall shift of the domain wall by z_0 where $E^{(k)}(z_0) = \gamma$. However, we need pinned soliton solutions with a nonzero $\delta'(z)$ source in the right-hand side. To find these, we have to take one free solution at $z > 0$ and another free solution at $z < 0$, where the two solutions may differ only by the value of the so far arbitrary complex parameter γ and by the sign of the phase of $\varkappa$.

We fix $\varkappa = \exp(+\frac{2\pi i}{N})$ at $z > 0$ and $\varkappa = \exp(-\frac{2\pi i}{N})$ at $z < 0$. The condition that only the imaginary parts of $w_{m,m+1}$ have jumps at $z = 0$ requires that γ is a pure phase factor, $\gamma = e^{i\alpha}$ at $z > 0$, and $\gamma = e^{-i\alpha}$ at $z < 0$. Indeed, with these definitions, $w^{(k)}_{m,m+1}$ at $z < 0$ are complex conjugates of the same functions at $z > 0$; the real parts of $w^{(k)}_{m,m+1}$ are continuous functions at $z = 0$, whereas the imaginary parts may have jumps. We define the logarithms in Eq. (99) such that they have cuts along the negative axis. Thus, the general form of a "pinned" soliton solving Eq. (98) with a nonzero source is

$$
w^{(k)}_{m,m+1}(z) = \begin{cases} \ln\frac{[1+\gamma\varkappa^{k(m-1)}E^{(k)}(z)][1+\gamma\varkappa^{k(m+1)}E^{(k)}(z)]}{[1+\gamma\varkappa^{km}E^{(k)}(z)]^2}, & z > 0,\\ \ln\frac{[1+\gamma^*\varkappa^{*k(m-1)}E^{(k)}(-z)][1+\gamma^*\varkappa^{*k(m+1)}E^{(k)}(-z)]}{[1+\gamma^*\varkappa^{*km}E^{(k)}(-z)]^2}, & z < 0, \end{cases} \qquad m = 1,\ldots,N, \qquad k = 1,\ldots,N-1. \tag{100}
$$

We have now to choose the phase factor γ such that the functions $w^{(k)}_{m,m+1}(z)$ have $\pm 2\pi i$ jumps at $z = 0$ in accordance with the source in the right-hand side of Eq. (98). We note that at $z \to \pm\infty$ the arguments of the logarithms tend to $e^{i\cdot 0}$; hence, all functions tend to zero at $z \to \pm\infty$. As one varies $|z|$ from ∞ to 0, the arguments of the logarithms travel in the complex plane, ending up at the *real axis* at $z \to 0$. The trajectories for $z > 0$ and for $z < 0$ are mirror images of one another since $w^{(k)}_{m,m+1}(-z) = (w^{(k)}_{m,m+1}(z))^*$. If at $z \to 0$ the trajectories end up at the positive semiaxis, the function has no jump since the logarithm is uniquely defined there. If at $z \to 0$ the trajectories end up at the negative semiaxis, the function has a $\pm 2\pi i$ jump owing to the cut of the logarithm along the negative semiaxis. The sign of the jump depends on whether the trajectory approaches the cut from above or from below. For given N and k, the only handle ruling the behavior of the trajectories in the complex plane is the phase factor γ. We shall show below that one can find γ such that a given function $w_{m,m+1}$ has a needed jump. But before presenting explicit solutions for $k = 1, 2, \ldots$, let us show that the string tension for a general k representation is given by Eq. (97).

To find the string tension, one needs to compute the action on the solution (100):

$$
\begin{aligned}
\text{action}\,(k,N) &= \frac{4\pi f}{N}\int d^3x \sum_{m=1}^{N}[\exp(w^{(k)}_{m,m+1}) - 1]\\
&\quad\times \theta(x, y \in \text{area}) = -\sigma(k,N)\,\text{area},
\end{aligned}
$$

where we have subtracted the constant related to the vacuum. It is understood that the solution (100) is valid for x, y inside the loop because of the θ function in (96) that we have omitted for brevity; outside the loop, there is no source, and $w_{m,m+1}(z) = 0$ is compatible with the equation. At the loop boundary, $w_{m,m+1}(z)$ interpolates between (100) and zero. Substituting the solution (100), we obtain for the string tension

$$\sigma(k,N) = -\frac{4\pi f}{N}\int_{-\infty}^{\infty} dz \sum_{m=1}^{N}[\exp(w^{(k)}_{m,m+1}(z)) - 1]$$
$$= \frac{4\pi f}{N}\sum_{m=1}^{N}\int_{-\infty}^{\infty} dz(\varkappa^k - 2 + \varkappa^{-k}) \times \frac{\gamma\varkappa^{km}E^{(k)}(z)}{(1+\gamma\varkappa^{km}E^{(k)}(z))^2},$$

where for $z < 0$ one has to change $E^{(k)}(z) \to E^{(k)}(-z) = 1/E^{(k)}(z)$, $\varkappa \to \varkappa^* = 1/\varkappa$, $\gamma \to \gamma^* = 1/\gamma$. In fact, the integrand is invariant under such change; therefore, one can proceed with the above expression integrating from $-\infty$ to $+\infty$: The integral equals $1/(M\sqrt{\mathcal{M}^{(k)}})$ and does not depend either on m or γ. Therefore, there are N equal terms in the sum, and we obtain finally the string tension

$$\sigma(k,N) = 4\pi f\frac{\varkappa^k - 2 + \varkappa^{-k}}{2M\sin\frac{\pi k}{N}} = MT2\sin\frac{\pi k}{N}$$

as announced.

B. Wilson loop in the fundamental representation, $k = 1$

It is easy to verify that, if one takes $\arg(\gamma)$ inside one of the N equal-length intervals covering the whole 2π range,

$$\arg(\gamma) \in \left(\pi, \pi - \frac{2\pi}{N}\right), \quad \left(\pi - \frac{2\pi}{N}, \pi - \frac{4\pi}{N}\right), \quad \left(\pi - \frac{4\pi}{N}, \pi - \frac{6\pi}{N}\right), \ldots, \left(-\pi + \frac{2\pi}{N}, -\pi\right), \tag{101}$$

Eq. (100) for $k = 1$ gives the solutions of Eq. (98) corresponding to a single source at $m_1 = 1, 2, 3, \ldots, N$, respectively. For example, taking $\arg(\gamma)$ inside the first interval makes the functions w_{12} and $w_{N,1}$ discontinuous at $z = 0$, where their imaginary parts have a 2π jump in accordance with the source term $2\pi i\delta'(z)\delta_{m1}$. All of the rest functions are continuous. When one moves $\arg(\gamma)$ to the second interval in (101), the functions w_{12} and w_{23} have jumps in accordance with the source term $2\pi i\delta'(z)\delta_{m2}$, while all other functions are continuous, and so on. An example of the solutions for $N = 3$ is shown in Fig. 1, where $\arg(\gamma)$ is taken from the first interval, in this case $(\pi, \frac{\pi}{3})$. When $\arg(\gamma)$ is taken from the second interval $(\frac{\pi}{3}, -\frac{\pi}{3})$ or from the third interval $(-\frac{\pi}{3}, -\pi)$, the functions change cyclically $w_{12} \to w_{23} \to w_{31} \to w_{12}$.

The action density also varies as a function of $\arg(\gamma)$ but is periodic with a period of $\frac{2\pi}{N}$. At N points in the middle of the intervals (101), namely, at $\arg(\gamma) = \pi - [(2m_1 - 1)\pi/N]$, the action density is real; otherwise, it is generally complex. It is remarkable that the action itself, or the string tension, is real and does not depend on γ. It means that $\arg(\gamma)$ is a new string Goldstone mode, if one allows γ to be a function of $2d$ string coordinates—in addition to the usual Goldstone modes associated with long-wave deformations of the string surface.

C. Strings for higher representations, $k \geq 2$

The Wilson loop in the antisymmetric rank-k tensor representation is a source for k functions $w_{m_1}, \ldots, w_{m_k}$, where the numbers $m_1 < \cdots < m_k$ can lie anywhere on the circle $(1, 2, \ldots, N)$. However, Toda equations do not have solutions for all configurations of $m_1 \ldots m_k$. Configurations with no classical solutions presumably give much smaller contributions to the Wilson loop at large areas than configurations that do generate solitons as they are stationary points.

The strategy for finding pinned solitons corresponding to Wilson loops in higher representations is simple: One takes the general solution (100) at a certain value of k and varies the phase of γ from π to $-\pi$. For any k, there will be *continuous* intervals of $\arg(\gamma)$ for which the functions $w^{(k)}_{m,m+1}(z)$ satisfy Toda equations (98) with a $\delta'(z)$ source in the right-hand side corresponding to certain sets of numbers $m_1 < m_2 < \cdots < m_k$. For all intervals of $\arg(\gamma)$, the pinned soliton action and hence the k-string tension are given by Eq. (97) and are thus degenerate in $\arg(\gamma)$.

We did not attempt to enumerate systematically the rapidly growing variety of solitons at arbitrary N and k. We find it more instructive to describe all pinned solitons of the $SU(6)$ group, which is sufficiently "rich" as it possesses nontrivial strings with $k = 1$, 2, and 3.

For $k = 1$, the solutions have been in fact given in Sec. VI B: Six equal-length intervals of $\arg(\gamma) \in (\pi, \frac{2\pi}{3})$, $(\frac{2\pi}{3}, \frac{\pi}{3})$, $(\frac{\pi}{3}, 0)$, $(0, -\frac{\pi}{3})$, $(-\frac{\pi}{3}, -\frac{2\pi}{3})$, $(-\frac{2\pi}{3}, -\pi)$ corre-

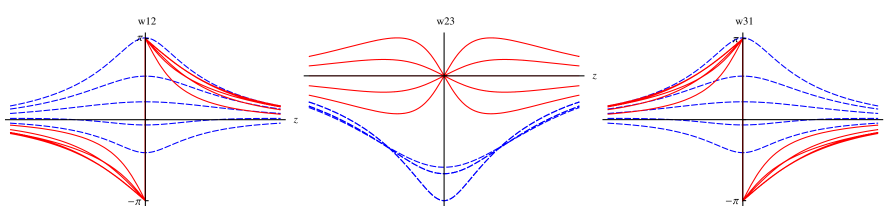

FIG. 1 (color online). A bunch of profile functions w_{12} (left), w_{23} (middle), and w_{31} (right) inside the $SU(3)$ string for five values of the parameter γ: $\arg(\gamma) = (4, 5, 6, 7, 8) \times (\pi/9)$. The solid red curves display imaginary parts, and the dashed blue curves display real parts of w_{12}, w_{23}, and w_{31}, respectively, as functions of the distance z from the Wilson loop plane. The string tension (the action) is identical for all five curves.

spond to solutions with a single source placed at $m_1 = 1, 2, 3, 4, 5$, and 6, respectively. In all cases, the string tension is $\sigma(1, 6) = 2MT\sin\frac{\pi}{6}$.

For $k = 2$, three equal-length intervals $\arg(\gamma) \in (\pi, \frac{\pi}{3})$, $(\frac{\pi}{3}, -\frac{\pi}{3})$, $(-\frac{\pi}{3}, -\pi)$ correspond to the double $k = 2$ sources at $m_1 = 1$, $m_2 = 4$; $m_1 = 2$, $m_2 = 5$; $m_1 = 3$, $m_2 = 6$, respectively. In all cases, the string tension is $\sigma(2, 6) = 2MT\sin\frac{2\pi}{6}$.

For $k = 3$, two equal-length intervals $\arg(\gamma) \in (\pi, 0), (0, -\pi)$ correspond to the triple $k = 3$ sources at $m_1 = 1$, $m_2 = 3$, $m_3 = 5$ and $m_1 = 2$, $m_2 = 4$, $m_3 = 6$, respectively. In all cases, the string tension is $\sigma(3, 6) = 2MT\sin\frac{3\pi}{6}$.

As a matter of fact, $k = 3$, $N = 6$ is a particular case of the general rank-k representation of the $SU(2k)$ group. [Another example is the $k = 1$ representation of the $SU(2)$ group, which simultaneously is a particular case of a fundamental representation considered in the previous subsection.] For all k and $N = 2k$, there are pinned solitons generated by k sources placed at $m_1 = 1$, $m_2 = 3$, $m_3 = 5, \ldots,$ if $\arg(\gamma) \in (\pi, 0)$, and placed at $m_1 = 2$, $m_2 = 4$, $m_3 = 6, \ldots,$ if $\arg(\gamma) \in (0, -\pi)$. The string tension is given by Eq. (97), where one puts $N = 2k$, and is degenerate in $\arg(\gamma)$.

To summarize this section, we have shown that, to find the spatial Wilson loop averaged over the ensemble of dyons, one needs to solve a chain of Toda equations with a $\delta'(z)$ source in the right-hand side. We have solved those equations for any N and Wilson loop representation k, finding pinned solitons in the transverse direction to the loop surface. The solutions generalize the famous double-layer solutions for the string in the $3d$ Georgi-Glashow model by Polyakov [34]. The resulting ''magnetic'' string tension is proportional to $\sin\frac{\pi k}{N}$ and coincides exactly with the ''electric'' string tension found in Sec. V from the correlators of the Polyakov lines. We have observed that the Toda equations with a source allow a continuous set of solutions for the string profile, characterized by a phase $\arg(\gamma) \in (-\pi, \pi)$, all with the same string tension. It means that, in addition to the usual Goldstone modes related to deformations of the string surface, there must be an extra Goldstone mode related to the string profile. Therefore, the string theory is more complicated than given by the standard Nambu-Goto action, which may have important implications for both theory and phenomenology.

VII. SUMMARY

Generalizing previous work on the subject, we have written down the metric of the moduli space for an arbitrary number of N kinds of dyons in the pure $SU(N)$ gauge theory. Assuming that it is mainly the metric and not the fluctuation determinant about dyons that defines the ensemble of interacting dyons, we have presented the grand partition function of the ensemble (where the number of particles is not fixed beforehand but found from the minimum of the free energy at given temperature) as a path integral over $N - 1$ Abelian electric potentials v_m and their duals w_m, as well as over $N - 1$ ghost fields $\chi_m^\dagger, \chi_m$. The resulting quantum field theory of those fields turns out to be exactly solvable owing to the cancelation between boson and ghost loops—a feature similar to that observed in supersymmetric theories. It enables one to make exact statements about the dyon ensemble: to find its free energy and correlation functions.

The free energy appears to have the minimum at the ''maximal nontrivial'' holonomy corresponding to the confining zero value of the average Polyakov line. Calculating the correlation functions of Polyakov lines in various N-ality $= k$ representations (where $k = 1, \ldots N - 1$), we find the asymptotic linear confining potential with the k-string tension proportional to $\sin\frac{\pi k}{N}$, the coefficient being calculated through the Yang-Mills scale parameter Λ and the 't Hooft coupling λ. The actual value of λ has to be determined self-consistently at the 2-loop level not considered here. Taking $\lambda = \frac{1}{4}$ compatible with phenomenology, we observe a reasonable agreement of the estimated deconfinement temperature T_c, the string tension σ, the gluon condensate, and the topological susceptibility with what is known from lattice simulations and phenomenology. A more robust ratio $T_c/\sqrt{\sigma}$ independent of Λ and λ is in surprisingly good agreement with the lattice data taken at $N = 3$, 4, 6, and 8, given the approximate nature of the model.

We have also shown that the dyon ensemble leads to the area law for the average of *spatial* Wilson loops for any nonzero N-ality. The calculated spatial (''magnetic'') string tension coincides with the ''electric'' string tension found from Polyakov lines for all N and k. We find this coincidence interesting as it indicates the restoration of Lorentz symmetry at low temperatures. Since the formalism used is three-dimensional at finite temperatures, the restoration of Lorentz symmetry at $T \to 0$ is by no means trivial.

We do not pretend to have answered all of the questions and obtained a realistic confining theory, as we have ignored essential ingredients of the full Yang-Mills theory, enumerated in the introduction. Our aim was to demonstrate that the integration measure over dyons has a drastic, probably decisive, effect on the ensemble of dyons that the ensemble can be mathematically described by an exactly solvable field theory in three dimensions and that the resulting semiclassical vacuum built of dyons has many features expected for the confining pure Yang-Mills theory.

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