PHYSICAL REVIEW D **76,** 047502 (2007)

Strains and axial outflows in the field of a rotating black hole

D. Bini*

Istituto per le Applicazioni del Calcolo "M. Picone," CNR I-00161 Rome, Italy and ICRA, University of Rome "La Sapienza," I-00185 Rome, Italy

F. de Felice†

Dipartimento di Fisica, Università di Padova, and INFN, Sezione di Padova, Via Marzolo 8, I-35131 Padova, Italy

A. Geralico‡

Physics Department and ICRA, University of Rome "La Sapienza," I-00185 Rome, Italy

(Received 20 April 2007; published 31 August 2007)

We study the behavior of an initially spherical bunch of particles accelerated along trajectories parallel to the symmetry axis of a rotating black hole. We find that, under suitable conditions, curvature and inertial strains compete to generate jetlike structures. This is a purely kinematical effect which does not account by itself for physical processes underlying the formation of jets. In our analysis a crucial role is played by the property of the electric and magnetic part of the Weyl tensor to be Lorentz invariant along the axis of symmetry in Kerr spacetime.

DOI: 10.1103/PhysRevD.76.047502 PACS numbers: 04.20.Cv

In the present work we study the relative behavior of a bunch of particles constrained to move parallel to the axis of symmetry of a Kerr black hole under the combined effects of a given acceleration and of the background curvature. Ultrarelativistic particles are expected to be produced near highly magnetized rapidly rotating neutron stars or as a consequence of the complicated accretion phenomena in the vicinity of an active black hole (see e.g. [1–3]). For our purposes, we imagine an abundance of such particles near the poles of the collapsed system. In the case of axial acceleration we find that curvature and inertial strains compete to the formation of jetlike structures from an initial spherical configuration. We want to stress that we neither propose a specific model of jets nor make any assumption about the physical processes that are responsible for the initial acceleration, but only discuss in detail a general relativistic kinematical effect that should be considered in any jet model which assumes as a driving machine a rotating black hole.

In this paper Greek indices run from 0 to 3, Latin indices run from 1 to 3, and hatted indices indicate tetrad components. Furthermore units are chosen so that $c = 1 = G$.

Consider Kerr spacetime, whose line element in Kerr-Schild coordinates $(t, x^1 = x, x^2 = y, x^3 = z)$ is given by

$$\begin{aligned} \mathrm{d}s^2 &= g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \equiv (\eta_{\alpha\beta} + 2H k_\alpha k_\beta)\mathrm{d}x^\alpha \mathrm{d}x^\beta, \\ H &= \frac{\mathcal{M} r^3}{r^4 + a^2 z^2}, \end{aligned} \tag{1}$$

where $\eta_{\alpha\beta}$ is the flat spacetime metric and

$$k_\alpha \mathrm{d}x^\alpha = -\mathrm{d}t - \frac{(rx + ay)\mathrm{d}x + (ry - ax)\mathrm{d}y}{r^2 + a^2} - \frac{z}{r}\mathrm{d}z, \tag{2}$$

with r implicitly defined by $(x^2 + y^2)/(r^2 + a^2) + z^2/r^2 = 1$. Here $\mathcal{M}$ and a are the total mass and specific angular momentum characterizing the spacetime; in geometrized units they have both the dimension of a length.

Consider then the family of static observers on and nearby the symmetry axis. It is understood that these observers only exist outside the ergosphere. Their four velocity is aligned with the Killing temporal direction

$$m = \frac{1}{M}\partial_t, \qquad M = \sqrt{-g_{tt}}, \tag{3}$$

with dual $m^\flat = -M(\mathrm{d}t - M_a \mathrm{d}x^a)$, where M and $M_a = -g_{ta}/g_{tt}$ $(a = 1, 2, 3)$ are the lapse and shift functions, respectively. The spacetime metric (1) can then also be written as [4]

$$\begin{aligned} \mathrm{d}s^2 &= -M^2(\mathrm{d}t - M_a \mathrm{d}x^a)^2 + \gamma_{ab}\mathrm{d}x^a \mathrm{d}x^b, \\ \gamma_{ab} &= g_{ab} + M^2 M_a M_b. \end{aligned} \tag{4}$$

We now construct an orthonormal frame adapted to the static observers. First consider the set of three unitary vector fields $\epsilon(m)_{\hat a} = (1/\sqrt{\gamma_{aa}})(\partial_a + M_a \partial_t)$; then fix one of these, say $\epsilon(m)_{\hat 3}$, and define the following orthonormal frame adapted to the family of observers m:

$$\begin{aligned} E(m)_{\hat 1} &= f(m)_{\hat 1}, \\ E(m)_{\hat 2} &= \frac{1}{\sqrt{1 - w_{12}^2}}[f(m)_{\hat 2} - w_{12} f(m)_{\hat 1}], \\ E(m)_{\hat 3} &= \epsilon(m)_{\hat 3}, \end{aligned} \tag{5}$$

where

*binid@icra.it
†fernando.defelice@pd.infn.it
‡geralico@icra.it

$$f(m)_{\hat 1} = \frac{1}{\sqrt{1-v_{13}^2}}[\epsilon(m)_{\hat 1} - v_{13}\epsilon(m)_{\hat 3}],$$
$$f(m)_{\hat 2} = \frac{1}{\sqrt{1-v_{23}^2}}[\epsilon(m)_{\hat 2} - v_{23}\epsilon(m)_{\hat 3}], \tag{6}$$

$v_{ab} = \epsilon(m)_{\hat a} \cdot \epsilon(m)_{\hat b} = \gamma_{ab}/\sqrt{\gamma_{aa}\gamma_{bb}}$ (no sum over the indices a and b is meant here) and $w_{12} = f(m)_{\hat 1} \cdot f(m)_{\hat 2}$, with the dot product denoting scalar multiplication.

Note that the frame (5) is well behaved on the rotation axis; in fact setting $x = 0 = y$ we have

$$m = \sqrt{\frac{z^2+a^2}{\Delta_z}}\partial_t, \qquad E(m)_{\hat 1} = \partial_x, \qquad E(m)_{\hat 2} = \partial_y,$$
$$E(m)_{\hat 3} = \sqrt{\frac{\Delta_z}{z^2+a^2}}\left[\partial_z + \frac{2\mathcal{M}z}{\Delta_z}\partial_t\right], \tag{7}$$

where $\Delta_z = z^2 - 2\mathcal{M}z + a^2 = (z-z_+)(z-z_-)$, with $z_\pm = \mathcal{M} \pm \sqrt{\mathcal{M}^2 - a^2}$.

Consider now a set of particles moving along the z-direction, on and nearby the axis of symmetry, with four velocity

$$U = \gamma[m + \nu E(m)_{\hat 3}], \qquad \gamma = (1-\nu^2)^{-1/2}, \tag{8}$$

where the instantaneous linear velocity $\nu = \nu(z)$, relative to the local static observers, is in general a function of the coordinate z; here and throughout the paper the physical velocity is in units of light velocity. These particles are accelerated except perhaps those which move strictly on the axis of symmetry. Their history forms a congruence $\mathcal{C}_U$ of ∞^2 world lines and each of them can be parametrized by the pair (x, y) of the spatial coordinates. A frame adapted to this kind of orbits can be fixed with the triad

$$E(U)_{\hat 1} = E(m)_{\hat 1}, \qquad E(U)_{\hat 2} = E(m)_{\hat 2},$$
$$E(U)_{\hat 3} = \gamma[\nu m + E(m)_{\hat 3}], \tag{9}$$

obtained by boosting along z the corresponding triad (5) adapted to the static observers.

The congruence is accelerated with nonvanishing components along the three spatial directions $E(U)_{\hat a}$. It is now possible to study the components of the vector Y connecting the specific curve of the congruence $\mathcal{C}_U$, which we fix along the rotation axis with $x = 0 = y$ with tangent vector $\tilde U$ and nearby world lines of the same congruence. The reference world line is accelerated along the direction $E(\tilde U)_{\hat 3}$, i.e. $a(\tilde U) = a(\tilde U)^{\hat 3} E(\tilde U)_{\hat 3}$, with

$$a(\tilde U)^{\hat 3} = \frac{\gamma}{\sqrt{\Delta_z(z^2+a^2)}}\left[\gamma^2\nu\frac{d\nu}{dz}\Delta_z + \mathcal{M}\frac{z^2-a^2}{z^2+a^2}\right]$$
$$= \frac{d}{dz}\left(\frac{\gamma}{\sqrt{\tilde\gamma_{33}}}\right), \tag{10}$$

where $\tilde\gamma_{33} = (z^2+a^2)/\Delta_z$ denotes the restriction on the axis of that metric coefficient.

The connecting vector Y is defined to undergo Lie transport along U, i.e. $\mathcal{L}_U Y = 0$ and with respect to the spatial frame (9)

$$\dot Y^{\hat 0} = -Y^{\hat a} a(U)_{\hat a}, \tag{11}$$

$$\dot Y^{\hat a} + [\omega_{(\mathrm{fw},U,E)} \times \vec Y]^{\hat a} + K(U)^{\hat a}{}_{\hat b} Y^{\hat b} = 0, \tag{12}$$

with the operation "$\times$" denoting exterior product in the observer local rest space. Differentiating Eq. (12) with respect to the proper time of U (hereafter denoted by a dot) we obtain the "deviation equation"

$$\ddot Y^{\hat a} + \mathcal{K}_{(U,E)}{}^{\hat a}{}_{\hat b} Y^{\hat b} = 0,$$
$$\mathcal{K}_{(U,E)}{}^{\hat a}{}_{\hat b} = [T_{(\mathrm{fw},U,E)} - S(U) + \mathcal{E}(U)]^{\hat a}{}_{\hat b}, \tag{13}$$

where $\mathcal{K}_{(U,E)}{}^{\hat a}{}_{\hat b}$ are the components of the "deviation" matrix.

The kinematical tensor $K(U)$ in Eq. (12) is defined as $P(U)^\mu_\alpha P(U)^\beta_\nu \nabla_\mu U^\nu = -K(U)^\beta{}_\alpha$, where $P(U)^\alpha_\beta = \delta^\alpha_\beta + U^\alpha U_\beta$ projects orthogonally to U. Moreover, it is standard to write $K(U) = \omega(U) - \theta(U)$, where $\omega(U)$ is an antisymmetric tensor representing the vorticity of the congruence $\mathcal{C}_U$ and $\theta(U)$ is a symmetric tensor representing the expansion. $\omega_{(\mathrm{fw},U,E)}$ is a vector representing the angular velocity of the spatial triad $E_{\hat a}$ with respect to a Fermi-Walker transported triad along U

$$P(U)\nabla_U E_{\hat a} = \omega_{(\mathrm{fw},U,E)} \times E_{\hat a}. \tag{14}$$

The quantity $\mathcal{E}(U)^\alpha{}_\gamma = R^\alpha{}_{\beta\gamma\delta}U^\beta U^\delta$ appearing in Eq. (13) is the electric part of the Riemann tensor as measured by the observer U. $S(U)$ is the strain tensor defined by

$$S(U)_{\hat a\hat b} = \nabla(U)_{\hat b} a(U)_{\hat a} + a(U)_{\hat a} a(U)_{\hat b}, \tag{15}$$

while the tensor $T_{(\mathrm{fw},U,E)}$ is given by

$$T_{(\mathrm{fw},U,E)}{}^{\hat a}{}_{\hat b} = \delta^{\hat a}_{\hat b}\omega^2_{(\mathrm{fw},U,E)} - \omega^{\hat a}_{(\mathrm{fw},U,E)}\omega_{(\mathrm{fw},U,E)\hat b} - \epsilon^{\hat a}{}_{\hat b\hat f}\dot\omega^{\hat f}_{(\mathrm{fw},U,E)} - 2\epsilon^{\hat a}{}_{\hat f\hat c}\omega^{\hat f}_{(\mathrm{fw},U,E)}K(U)^{\hat c}{}_{\hat b}, \tag{16}$$

with $\epsilon_{\hat a\hat b\hat c}$ being the Levi-Civita alternating symbol. We refer to [5] for a detailed derivation of these equations as well as the discussion about the corresponding observer-dependent analysis. The deviation Eq. (13) describes the interplay between tidal forces, relative strains, and properties of the reference triad.

The reference world line plays the role of a "fiducial observer" and the most relevant quantities associated with it are easily obtained by setting $x = 0 = y$ in their expressions for the general world line of the congruence. In order to determine the deviations measured by the "fiducial observer'' with respect to the chosen frame we need to evaluate all the kinematical fields of the congruence (the acceleration $a(U)$, the vorticity $\omega(U)$, and the expansion $\theta(U)$), the electric part of the Weyl tensor $\mathcal{E}(U)$, the strain tensor $S(U)$, and the characterization of the spatial triad (9) with respect to a Fermi-Walker frame as given by the tensor $T_{(\mathrm{fw},U,E)}$. To simplify the notation we shall hereafter omit the tilde $\tilde{()}$ being understood that all the quantities we shall now consider are defined on the symmetry axis.

Let us proceed evaluating the quantities which enter the relative deviation Eq. (13).

(i) The only nonvanishing components of the kinematical tensors $\omega(U)$ and $\theta(U)$ are given by

$$\theta(U)_{\hat{3}\hat{3}} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\gamma\nu}{\sqrt{\gamma_{33}}}\right),$$
$$\omega(U)_{\hat{1}\hat{2}} = \gamma(1+\nu)\frac{2a\mathcal{M}z}{\sqrt{\Delta_z}(z^2+a^2)^{3/2}}. \tag{17}$$

(ii) The vector $\omega_{(\mathrm{fw},U,E)}$ is given by

$$\omega_{(\mathrm{fw},U,E)} = \omega_{(\mathrm{fw},U,E)\hat{3}}E(U)_{\hat{3}},$$
$$\omega_{(\mathrm{fw},U,E)\hat{3}} \equiv \omega(U)_{\hat{1}\hat{2}}. \tag{18}$$

(iii) The electric part of the Weyl tensor is a diagonal matrix with respect to the adapted frame (9), namely

$$\mathcal{E}(U) = \mathcal{M}z\frac{z^2-3a^2}{(z^2+a^2)^3}\mathrm{diag}[1,1,-2]. \tag{19}$$

Form (19) of the electric part of the Weyl tensor coincides with that calculated with respect to the static frame (7), namely $\mathcal{E}(U)_{\hat{a}\hat{b}} = \mathcal{E}(m)_{\hat{a}\hat{b}}$. In fact, since the fiducial observer moves on the axis of rotation of the Kerr metric and this axis is also a special tidal axis [6–8] then the curvature is unaffected by the Lorentz boost (8) and (9). It is not so, however, for the inertial terms.

(iv) The only nonvanishing components of the tensor $T_{(\mathrm{fw},U,E)}$ turn out to be

$$T_{(\mathrm{fw},U,E)\hat{1}\hat{1}} = T_{(\mathrm{fw},U,E)\hat{2}\hat{2}} = -\omega_{(\mathrm{fw},U,E)\hat{3}}{}^2,$$
$$T_{(\mathrm{fw},U,E)\hat{1}\hat{2}} = -T_{(\mathrm{fw},U,E)\hat{2}\hat{1}} = -\dot{\omega}_{(\mathrm{fw},U,E)\hat{3}}$$
$$= -\frac{\mathcal{M}a\nu}{(z^2+a^2)^2(1-\nu)}\left[2\gamma^2 z\frac{\mathrm{d}\nu}{\mathrm{d}z} - \frac{(5z^2-a^2)\Delta_z+z^4-a^4}{(z^2+a^2)\Delta_z}\right]. \tag{20}$$

(v) The nonzero components of the strain tensor $S(U)^{\hat{a}}{}_{\hat{b}}$ can be written as

$$S(U)_{\hat{1}\hat{1}} = S(U)_{\hat{2}\hat{2}} = -\omega_{(\mathrm{fw},U,E)\hat{3}}{}^2 + \mathcal{M}z\frac{z^2-3a^2}{(z^2+a^2)^3},$$
$$S(U)_{\hat{1}\hat{2}} = -S(U)_{\hat{2}\hat{1}} = -\dot{\omega}_{(\mathrm{fw},U,E)\hat{3}},\ S(U)_{\hat{3}\hat{3}}$$
$$= \frac{\gamma}{\sqrt{\gamma_{33}}}\frac{\mathrm{d}}{\mathrm{d}z}[a(U)^{\hat{3}}] + [a(U)^{\hat{3}}]^2. \tag{21}$$

From Eqs. (19)–(21) it follows that the deviation matrix $\mathcal{K}_{(U,E)}$ has only the nonzero component $\mathcal{K}_{(U,E)\hat{3}\hat{3}} = -S(U)_{\hat{3}\hat{3}} + \mathcal{E}(U)_{\hat{3}\hat{3}}$; hence, as expected, the particles emitted nearby the axis in the z-direction will be relatively accelerated in the z-direction only.

The spatial components of the connecting vector Y are then obtained by integrating the deviation equation (13) which now reads

$$\ddot{Y}^{\hat{1}} = 0, \qquad \ddot{Y}^{\hat{2}} = 0, \qquad \ddot{Y}^{\hat{3}} = -\mathcal{K}_{(U,E)\hat{3}\hat{3}}Y^{\hat{3}}. \tag{22}$$

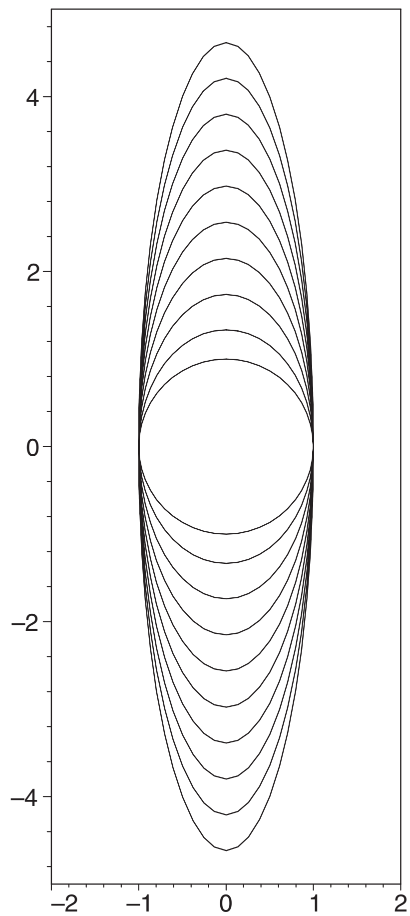

FIG. 1. The spreading of an initially circular bunch of particles on the $y^{\hat{1}}$-$y^{\hat{3}}$ plane emitted along the z-axis at $z_0/\mathcal{M} = 2$ is shown for the choice of parameters $a/\mathcal{M} = 0.5$, $\bar{\kappa} = 1.5$, and $\mathcal{M}A = 0.3$. The behavior is similar for different values of $\bar{\kappa}$ and $\mathcal{M}A$. The curves correspond to increasing values of the coordinate $z/\mathcal{M} = [2, 4, 6, 8, 10, 12, 14, 16, 18, 20]$. It can be shown that the spreading along the z-axis becomes faster and faster for high values of $\mathcal{M}A$.

Turning the dependence on the proper time τ_U into a dependence on z according to the relation $dz/d\tau_U = \gamma\nu/\sqrt{\gamma_{33}}$ the solution for the spatial components of Y reads

$$Y^{\hat{1}} = Y_0^{\hat{1}}, \qquad Y^{\hat{2}} = Y_0^{\hat{2}}, \qquad Y^{\hat{3}} = C\frac{\gamma\nu}{\sqrt{\gamma_{33}}}, \tag{23}$$

where C is a constant. This result shows that the conditions imposed on the particles of the bunch to move parallel to the axis of rotation is assured by a suitable balancing among the gravitoelectric (curvature) tensor (19), the inertial tensor (20), and the strain tensor (21). Were the curvature not invariant under Lorentz boosts, the balance in the transverse direction with respect to the axis of rotation would not be assured.

For any given acceleration $a(U) = A(z)E(U)_{\hat{3}}$ of the reference world line the corresponding instantaneous linear spatial velocity $\bar{\nu}$, say, can be obtained by integrating Eq. (10):

$$\frac{\bar{\gamma}}{\sqrt{\gamma_{33}}} = \int_{z_0}^{z} A(z)dz + \bar{\kappa} \equiv F(z), \tag{24}$$

where $\bar{\kappa}$ is a constant. In the case of geodesic motion $A(z) = 0$ and $\bar{\kappa}$ is just the conserved particle's energy per unit mass.

Let us then consider the case of the reference world line $\tilde{U}$ constantly accelerated, namely, with $a(\tilde{U})^{\hat{3}} = A =$ const. The instantaneous linear velocity relative to a local static observer is given by

$$\nu_A = \left[1 - \frac{1}{\gamma_{33}[\bar{\kappa} + A(z - z_0)]^2}\right]^{1/2}, \tag{25}$$

having set $F(z) = \bar{\kappa} + A(z - z_0)$; the positive value has been selected for ν_A in order to consider outflows. The solution (23) for the components of the connecting vector turns out to be

$$Y^{\hat{1}} = Y_0^{\hat{1}}, \qquad Y^{\hat{2}} = Y_0^{\hat{2}}, \qquad Y^{\hat{3}} = Y_0^{\hat{3}}\frac{\gamma_A\nu_A}{\sqrt{\gamma_{33}}\bar{\kappa}\nu_A^0}, \tag{26}$$

where the constant C of Eq. (23) has been fixed as $C = Y_0^{\hat{3}}/(\bar{k}\nu_A^0)$ with $\nu_A^0 = \nu_A(z_0)$ and $Y_0^{\hat{a}} = Y^{\hat{a}}(z_0)$, z_0 being the starting point on the axis.

Figure 1 shows the behavior of an initially spherical bunch of particles in the $Y^{\hat{1}}$-$Y^{\hat{3}}$ plane for increasing values of the coordinate z. We clearly see a stretching along the z-axis leading to a collimated axial outflow of matter, clearly suggestive of an "astrophysical jet." Note that in this case the acceleration acts contrarily to the curvature tidal effect, namely, as we have already seen, $S(U)$ and $\mathcal{E}(U)$ act in competition leading to a quite unexpected result.

The stretching shown in Fig. 1 for uniformly accelerated outgoing particles persists also with a general acceleration. This behavior appears to be independent of the acceleration mechanism itself. This is interesting because accelerations are naturally expected in astrophysical processes. Observed jets, for instance, are always associated with magnetohydrodynamical properties which are expected to provide a suitable acceleration mechanism. Whatever the mechanism is, this analysis is consistent with their evolution.

[1] B. Mashhoon and J. C. McClune, Mon. Not. R. Astron. Soc. **262**, 881 (1993).

[2] B. Punsly, *Black Hole Gravitohydromagnetics* (Springer-Verlag, Berlin, 2001).

[3] *Relativistic Flows in Astrophysics*, edited by A. W. Guthmann, M. Georganopoulos, A. Marcowith, and K. Manolaku, Lecture Notes in Physics, Vol. 589 (Springer, New York, 2002).

[4] R. T. Jantzen, P. Carini, and D. Bini, Ann. Phys. (N.Y.) **215**, 1 (1992).

[5] D. Bini, F. de Felice, and A. Geralico, Classical Quantum Gravity **23**, 7603 (2006).

[6] B. Mashhoon, Phys. Lett. A **122**, 299 (1987); **163**, 7 (1992).

[7] J. K. Beem and P. E. Parker, J. Math. Phys. (N.Y.) **31**, 819 (1990); G. S. Hall and A. D. Hossack, *ibid.* **34**, 5897 (1993).

[8] J. K. Beem and S. G. Harris, Gen. Relativ. Gravit. **25**, 939 (1993); **25**, 963 (1993).