All-order consistency of 5D supergravity vacua

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We show that the maximally supersymmetric vacua of minimal $d = 5$ $N = 1$ sugra remain maximally supersymmetric solutions when taking into account higher order corrections.

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The question of whether a given supergravity solution is a consistent background for string propagation to all orders in perturbation theory is an interesting though hard one. It has of course long been known that pp-waves with flat transverse space provide such backgrounds since all the scalar invariants vanish identically—recently the class of exact sugra solutions with vanishing scalar invariants was investigated in [1]—but they are exceptional.

A class of sugra solutions for which a proof of all-order consistency is highly desirable are the maximally supersymmetric solutions, and for most of them an answer is known: in Ref. [2], Kallosh and Rajaraman, by making use of superspace methods, showed the all-order consistency of $aDS_2 \times S^2$ in minimal $N = 2$ $d = 4$ sugra, of $aDS_5 \times S^5$ in type IIB, and also of $aDS_4 \times S^7$ and $aDS_7 \times S^4$ in M-theory. Since the associated Minkowski and Kowalski-Glikmann solutions [3–5] are pp-waves, we must conclude that all the maximally supersymmetric solutions in minimal $N = 2$ $d = 4$, type IIB-sugra or M-sugra are all-order consistent.

The way Kallosh and Rajaraman attacked the problem leans heavily on the fact that the Riemann tensor and the field strengths are covariantly constant with respect to (w.r.t.) the Levi-Cività connection. This covariantly constancy, however, is due to the fact that the solutions describe symmetric spacetimes, G/H, with G-invariant field strengths. Interestingly, the vast majority of maximally supersymmetric solutions are described by symmetric spaces, and one can envisage similar arguments to apply for their consistency.

The strange ducks in the pond are the maximally supersymmetric solutions in $d = 5$ sugra [6–10]. Solutions such as the near-horizon-BMPV solution or the Gödel space, are *not* symmetric [11]: rather, they describe homogeneous, naturally reductive spacetimes with compatible fluxes. As such, there is a metric compatible connection that parallelizes the Riemann tensor and the field strengths, but it is not the Levi-Cività one. We then should ask ourselves the question whether the maximally supersymmetric solutions of $d = 5$ $N = 1$ sugra are all-order exact or not, and how to attack the problem. The answer to the last question lies in the way one would construct, and indeed constructs, higher order sugra actions in lower dimensions, be they string inspired or not.

Partial results have recently been obtained in Refs. [12–14] and these works instigated the current investigations: the symmetric solution $aDS_2 \times S^3$ was shown to be maximally supersymmetric in Ref. [14] and $aDS_3 \times S^2$ in Ref. [13] in a theory with $\mathrm{Tr}(A \wedge^2 R)$ corrections. This theory was constructed in [12] and they also derived the conditions for the existence of a maximally supersymmetric aDS_5.

Dealing with on-shell sypersymmetry in systems with higher orders is quite cumbersome: since it is on-shell, the supersymmetry transformations ‘‘need to know about’’ the equations of motion, so that *à priori* we would have to face the possibility of a lengthy Noether procedure in order to construct the theory.[1] A possible off-shell formulation of supersymmetry evades this problem by fixing the supersymmetry transformations once and for all, so that the brunt of the effort goes to the construction of the action. A similar problem occurs with the dependent fields such as the spin connection. Indeed, trying to construct a higher order theory in the first order prescription is cumbersome, as it calls for the elimination of the dependent fields through the use of its equation of motion. In conclusion, we are looking for a supergravity description in which the dependent fields are already fixed and is as off-shell as possible.

Having such a prescription, then, we can discuss the solutions to the equations defining unbroken supersymmetry, the so-called Killing spinor equation: as was to be expected, the solutions preserving all supersymmetries are the ones from ordinary sugra. The more daunting task lies in showing that maximally supersymmetric solutions always solve the equations of motion that can be derived from an arbitrary sugralike action, in particular, from an action that describes stringy corrections. This can however be accomplished by making use of the Killing spinor identities [16,17], which we will abbreviate as KSIs.

In Sec. II we will apply the program we sketched above to sugralike actions constructed using the superconformal approach. In this approach, one starts out with locally superconformal invariance, for which an off-shell formulation exists, to build superconformally invariant actions.

[1] See e.g. [15] and references therein, for the latest advances in the determination of the supersymmetric extention of the R^4 corrections in M-sugra.

Once such an action is known, one imposes certain gauge choices in order to break the superconformal invariance and to obtain (Poincaré) supergravities. Since the discussion of the superconformal approach in Sec. II will be brief, but hopefully concise, we refer the reader to Ref. [18] as a possible starting point to the extensive literature on the subject.

In order to declutter the technicalities, however, we will start by discussing and applying our strategy in the less involved case of minimal $N = 1$ $d = 5$ sugra.

I. OFF-SHELL SUPERSYMMETRY IN MINIMAL $N = 1$ $d = 5$ SUGRA

In Ref. [19], Zucker derived an off-shell formulation of minimal $d = 5$ $N = 1$ sugra based on an off-shell multiplet of dimension (48|48):[2] it consists of the fields from the on-shell multiplet, namely, the Fünfbein $e_\mu{}^a$, the vector field A_μ, and the gravitino ψ_μ, and a bunch of auxiliary fields: the fermions χ and λ and the bosons φ, V_a, v^{ab}, and the $\mathfrak{su}(2)$-triplet fields $\vec{t}$ and $\vec{V}_a$. The supersymmetry variations for purely bosonic configurations are

$$\delta\Psi_a = \nabla_a\epsilon + \frac{1}{2\sqrt{3}}(3\not{F}\gamma_a - \gamma_a\not{F})\epsilon + \frac{1}{2}\gamma_{abc}\epsilon v^{bc} - \frac{i}{2}\vec{\sigma}\epsilon\vec{V}_a + 2i\gamma_a\vec{\sigma}\epsilon\vec{t}, \tag{1}$$

$$-2\delta\chi = \epsilon\varphi - 6i\vec{\sigma}\epsilon\vec{t} - \gamma^a\epsilon V_a + \frac{i}{2}\gamma^a\vec{\sigma}\epsilon\vec{V}_a - 2\not{v}\epsilon, \tag{2}$$

$$\begin{aligned} 4\delta\lambda = {}& \epsilon\bigg[\nabla_a V^a + \frac{1}{4}\vec{V}_a\cdot\vec{V}^a - V_aV^a + \frac{5!}{2}\vec{t}\cdot\vec{t} + \varphi^2 \\ &+ 2v_{ab}v^{ab} + \frac{4}{\sqrt{3}}v^{ab}F_{ab}\bigg] - 2\gamma_a\epsilon\bigg(\nabla_b v^{ab} \\ &- \frac{1}{2}\varepsilon^{abcde}v_{bc}vde - \frac{2}{\sqrt{3}}\varepsilon^{abcde}F_{bc}v_{de}\bigg) \\ &+ \gamma^{abc}\epsilon\nabla_a v_{bc} + \gamma^{ab}\epsilon F_a{}^c v_{bc}. \end{aligned} \tag{3}$$

The analysis for the existence of maximally supersymmetric solutions of the above off-shell Killing spinor equation is straightforward: from Eq. (2), we see that $\varphi = \vec{t} = V_a = \vec{V}_a = v_{ab} = 0$ which automatically trivializes Eq. (3). Equation (1) then reduces to

$$0 = \nabla_a\epsilon + \frac{1}{2\sqrt{3}}(3\not{F}\gamma_a - \gamma_a\not{F})\epsilon, \tag{4}$$

which is nothing but the Killing spinor equation for minimal on-shell $N = 1$ $d = 5$ sugra. Clearly, the configurations solving the above equations are the ones enumerated by Gauntlett *et al.* in Ref. [10], but we must ask ourselves whether they automatically solve the equations of motion that can be derived from an off-shell action based on Zucker's formulation.

The way we want to show this to be the case was pioneered by Kallosh and Ortín [16], and first used to show that some equations of motion (e.o.m.s) are implied by supersymmetry (susy) in Ref. [17]. This approach goes by the name of *Killing spinor identities* and exploits the fact that the invariance of an action under a (super)symmetry implies the identity (introducing a superset of fields $\Phi^A = \{B^a, F^\alpha\}$)

$$\delta\mathcal{S} = \int_5 \delta\Phi^A\frac{\delta(\sqrt{g}\mathcal{S})}{\sqrt{g}\delta\Phi^A} = \int_5 \delta\Phi^A\mathcal{E}_A \to 0 = \delta\Phi^A\mathcal{E}_A(\Phi), \tag{5}$$

where we introduced the notation in which the equation of motion for a field Φ^A is written as $\mathcal{E}_A(\Phi) = 0$. If we then consider the functional derivative of the last equation in (5) w.r.t. some fermion field and evaluate the resulting identity for purely bosonic configurations that solve the Killing spinor equations, i.e. $F^\alpha = \delta_\epsilon F^\alpha|_{F=0} = 0$, we end up with

$$0 = \frac{\delta}{\delta F^\beta}[\delta_\epsilon B^a]\bigg|_{F=0}\mathcal{E}_a. \tag{6}$$

This equation is the Killing spinor identity and must hold for any supersymmetric system.

Let us start analyzing the implications of the KSIs by calculating the one w.r.t. the auxiliar field λ. A short calculation results in

$$0 = \mathcal{E}_i(\vec{t})\bar{\epsilon}\sigma^i - 2i\mathcal{E}(\varphi)\bar{\epsilon} - 2i\mathcal{E}^a(V)\bar{\epsilon}\gamma_a + 2i\mathcal{E}_{ab}(v)\bar{\epsilon}\gamma^{ab} - 4\mathcal{E}^a_i(\vec{V})\bar{\epsilon}\gamma_a\sigma^i. \tag{7}$$

Since we are interested in maximally supersymmetric solutions, the above identity holds for all ϵ which together with the properties of the γ- and σ-matrices implies that

$$0 = \mathcal{E}_i(\vec{t}) = \mathcal{E}(\varphi) = \mathcal{E}^a(V) = \mathcal{E}_{ab}(v) = \mathcal{E}^a_i(\vec{V}). \tag{8}$$

In ordinary language, this means that maximally supersymmetric solutions of the off-shell Killing spinor equations automatically solve the e.o.m. of the auxiliar fields. The only nontrivial e.o.m.s that remain are the ones for the bosonic on-shell fields, and they can be derived from the gravitino KSI

$$0 = 4\mathcal{E}^\mu_a(e)\bar{\epsilon}\gamma^a + \sqrt{3}\mathcal{E}^\mu(A)\bar{\epsilon}, \tag{9}$$

where we already used the results in (8). Applying the same reasoning as before, we reach the conclusion that we also identically solve the "Einstein" and the "Maxwell" equations.

The fact that the KSIs oblige the e.o.m.s to be identically satisfied for maximally supersymmetric configurations

[2] The minimal off-shell gravity multiplet has dimension (40, 40), but the quadratic sugra action based on this multiplet, leads to a constraint imposing the metric to be noninvertible. Further arguments for extension of the minimal multiplet are given in [20].

should have been expected: indeed, since for maximally supersymmetric configurations we can factor out the explicit appearance of the Killing spinor, ϵ, in the KSIs and decompose the latter into independent tensor-structure blocks. The crux of the matter is that, since the fields are distinguishable due to their symmetry properties, such as R-symmetry, there can only be one e.o.m. per block, whence a maximally supersymmetric configuration always solves the e.o.m.s.

The conclusion thus far is that, if we use Zucker's off-shell multiplet to construct effective actions, then the maximally supersymmetric solutions are the ones from $d = 5$ $N = 1$ sugra and whatever action one writes down, they always solve the corresponding equations of motion. How does coupling minimal sugra to matter multiplets change this picture?

II. VACUA AND COUPLING TO VECTOR MULTIPLETS

The field content of on-shell $d = 5$ $N = 1$ sugra coupled to n vector multiplets[3] is a Fünfbein $e_\mu{}^a$, a (symplectic-Majorana) gravitino ψ^i_μ ($i = 1, 2$), $n + 1$ vector fields A^I_μ ($I = 1, \ldots, n + 1$), and n scalars ϕ: the scalars parametrize a *very special manifold* through the $n + 1$ sections $h^I(\phi)$ that are constrained to satisfy

$$\mathcal{C}_{IJK} h^I h^J h^K = 1, \tag{10}$$

where $\mathcal{C}$ is a constant, completely symmetric 3-tensor.

In order to arrive at this on-shell sugra by means of the superconformal approach, one starts by introducing one Weyl multiplet, $n + 1$ vector multiplets, and one hypermultiplet. The Weyl multiplet is the superconformal analogue of the graviton-multiplet and consists of the Fünfbein e^a_μ, the gravitino ψ^i_μ, the vectors b_μ and $\mathrm{V}^{(ij)}_\mu$, a symplectic-Majorana spinor ξ, and the scalars T_{ab} and D. The $n + 1$ superconformal vector multiplets consist of vectors A^I_μ, gaugini λ^{Ii}, the *unconstrained* scalars h^I, and the auxiliar fields $\mathrm{Y}^{I(ij)}$. The hypermultiplet, then, consists of scalars A^i_j and spinors ζ_i constrained by suitable reality conditions.

The superconformal Killing spinor equations, i.e. the variation of the fermionic fields under supersymmetry variations with parameter ϵ and conformal supersymmetries with parameter η evaluated for vanishing fermions, then read

$$\delta\psi_a = \mathcal{D}_a\epsilon + {}_a\not{\mathrm{T}}\epsilon - \gamma_a\eta. \tag{11}$$

$$\delta\chi = \mathrm{D}\epsilon - 2\gamma^c\gamma^{ab}\epsilon\mathcal{D}_a\mathrm{T}_{bc} - 2\gamma^a\epsilon\varepsilon_{abcde}\mathrm{T}^{bc}\mathrm{T}^{de} + 8\not{\mathrm{T}}\eta + 2\not{R}(\mathrm{V})\epsilon, \tag{12}$$

$$\delta\Omega = \mathrm{Y}^I\epsilon - h^I\eta - \tfrac{1}{2}\not{\mathcal{D}}h^I\epsilon - \tfrac{1}{2}\not{F}^I\epsilon, \tag{13}$$

$$\delta\zeta^i = \not{\mathcal{D}}\mathsf{A}^i_j\epsilon^j - 2\not{\mathrm{T}}\epsilon^j\mathsf{A}^i_j + 3\eta^j\mathsf{A}^i_j. \tag{14}$$

where the $\mathfrak{su}(2)$ indices of ϵ, η, $R(\mathrm{V})$, and Y^I are implicit but present. In the above formulas, we used

$$\mathcal{D}\epsilon = \nabla\epsilon + \tfrac{1}{2}\gamma\not{b}\epsilon + \mathrm{V}\epsilon, \qquad \mathcal{D}\mathrm{T}_{ab} = \nabla\mathrm{T}_{ab} - b\mathrm{T}_{ab}, \tag{15}$$

At this point we must discuss the gauge fixings in order to get rid of the fields that are not part of the on-shell sugra. This process actually consists of two parts as we must not only break the superconformal symmetry down to Poincaré symmetry, but also get rid of the auxiliar fields. A clear exposition of the traditional gauge fixing program in sugra is given in Ref. [18], which has the advantage of using physically sound criteria to select gauge fixings. One of these concerns the normalization of the Einstein-Hilbert term: the gauge fixing they impose is that the normalization of the Einstein-Hilbert term is canonical, and together with the D-e.o.m. this implies Eq. (10). In the generic case, however, this criterion is rather cumbersome: instead we will follow Ref. [14], which has the advantage that the gauge-fixed Killing spinor equations have a simple form.[4]

The role of the gauge fixings is to break the superconformal symmetry down to Poincaré symmetry, which in the case at hand means breaking dilatations (D), conformal translations (K_a), the R-symmetry [$\mathfrak{su}(2)$], as well as the special supersymmetries (S). The conformal translations are broken by the K-gauge $b_\mu = 0$, and the rest of the symmetries are broken by imposing conditions on the compensating hypermultiplet: R-symmetry is broken by the condition $\mathcal{D}\mathsf{A} = 0$, which is consistent with the condition $\mathsf{A}^2 = -2$ for breaking the dilatational symmetry. S-symmetry, then, is broken by the condition $\zeta^i = 0$.

At this point the fields that do not match up with the on-shell sugra are T, V, D, and χ in the Weyl multiplet, the Y^I and one of the h^I and λ^I from the vector multiplets. In sugra, most of them are auxiliar fields and are therefore to be eliminated by the use of their equation of motion. In fact we should be a bit more specific: in sugra the equation of motion for D, abbreviated as $\mathcal{E}(\mathrm{D}) = 0$, imposes the constraint in Eq. (10), since D appears linearly in the action and hence acts as a Lagrange multiplier. Similarly, $\mathcal{E}^{ab}(\mathrm{T}) = 0$ results in $-2\mathrm{T} = C_{IJK}h^I h^J F^K$ and integration over χ rids us of the unwanted gaugino. Lastly, the e.o.m. for V would, ignoring fermionic contributions, identify V with the pullback of the $\mathfrak{su}(2)$ connection characterizing the quaternionic-Kähler manifold spanned by the *noncompensating* hypermultiplets.

One of the most important implications of the gauge fixings is that, in order for them to not break the ordinary

[3]Hypermultiplets can be introduced as well, with the same result, but will not be treated in order to make the discussion a tad lighter. Also, in this section we shall follow the conventions of [12].

[4]The spirit of this program is however the same as the canonical one as displayed in Fig. 1 of Ref. [18].

supersymmetries, any supersymmetry transformation must be accompanied by a compensating S-symmetry transformation. Indeed, the supersymmetry condition $\delta\zeta^i = 0$ means that

$$\eta = \tfrac{2}{3}\not{\mathrm{T}}\epsilon, \tag{16}$$

which as promised leads to a simple form of the resulting Killing spinor equations.

The analysis of the off-shell Killing spinor equations is then straightforward and results in

$$\begin{aligned} dh^I &= 0, & 0 &= \nabla_\mu\epsilon + {}_\mu\not{\mathrm{T}}\epsilon - \tfrac{2}{3}\gamma_\mu\not{\mathrm{T}}\epsilon, \\ F^I &= -\tfrac{4}{3}h^I\mathrm{T}, & d\mathrm{T} &= 0, \\ \mathrm{Y}^I &= 0, & \nabla_b\mathrm{T}^{ba} &= \tfrac{1}{3}\varepsilon^{abcde}\mathrm{T}_{bc}\mathrm{T}_{de}, \\ R(\mathrm{V}) &= 0, & D &= \tfrac{8}{3}\mathrm{T}_{ab}\mathrm{T}^{ab}. \end{aligned} \tag{17}$$

The fact that $R(\mathrm{V}) = 0$ means that V is pure gauge, whence we can eliminate it. It also means that the hyperscalars in the compensating hypermultiplet are constant. The first three equations in the right column then mean that the maximally supersymmetric solutions are once again given by the ones from minimal on-shell sugra with T playing the role of the field strength of the graviphoton.

Let us start the discussion of the KSIs by calculating the one for the auxiliar spinor χ, namely

$$0 = \mathcal{E}_{ab}(\mathrm{T}) + 4\mathrm{T}_{ab}\mathcal{E}(\mathrm{D}), \tag{18}$$

$$0 = \nabla_\mu\mathcal{E}(\mathrm{D}), \tag{19}$$

$$0 = h^I\mathcal{E}_{Iij}(\mathrm{Y}) = \mathcal{E}^\mu_{ij}(\mathrm{V}). \tag{20}$$

This result may, seeing the similar result in the foregoing section, seem strange, yet it makes perfect sense: remember that in ordinary sugra $\mathcal{E}(\mathrm{D}) = 0$ imposes the constraint (10). The analysis of the BPS equation, however, says nothing, and in fact cannot say anything, about the normalization of the h^I. Rather, in order to embed the maximally supersymmetric vacua of minimal sugra consistently, one has to solve $\mathcal{E}(\mathrm{D}) = 0$, after which $\mathcal{E}(\mathrm{T})$ vanishes identically.

The KSI w.r.t. the gaugini λ^I then implies

$$0 = \nabla_\mu\mathcal{E}_{Iij}(\mathrm{Y}) = \mathcal{E}_I(h) = \mathcal{E}^\mu_I(A), \tag{21}$$

meaning that the e.o.m. for the gauge fields A^I and the scalars h^I are automatically satisfied. The above KSI does, however, mean that we should check explicitly whether $\mathcal{E}_{Iij}(\mathrm{Y})$ really vanishes for our solutions. But its index-structure implies that in order to construct it we must always use Y or V and since they vanish for the vacua, we must conclude that $\mathcal{E}(\mathrm{Y}) = 0$ for the vacua.

The one equation left to check is the Einstein equation, which can only reside in the gravitino KSI. In fact if we impose that we already solved all the other equations of motion, we automatically find that maximal supersymmetry implies that $\mathcal{E}^\mu_a(e) = 0$.

In conclusion, we see that the question about maximally supersymmetric solutions and their all-order consistency reduces to an embedding problem that determines the constant values of the scalars h^I in terms of the parameters determining the maximally supersymmetric solutions. As advertised in Ref. [21], the embedding formula $\mathcal{E}(\mathrm{D}) = 0$ defines a deformation of very special geometry which should have a profound influence on, for example, the attractor mechanism.

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