Casimir-Polder intermolecular forces in minimal length theories

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Generalized uncertainty relations are known to provide a minimal length $\hbar\sqrt{\beta}$. The effect of such minimal length in the Casimir-Polder interactions between neutral atoms (molecules) is studied. The first order correction term in the minimal uncertainty parameter is derived and found to describe an attractive potential scaling as r^{-9}, as opposed to the well-known r^{-7} long range retarded potential.

I. INTRODUCTION

A complete understanding of the properties of the quantum vacuum is a point of central importance to both fundamental and applied physics [1]. A celebrated mechanical manifestation of the fluctuations of the electromagnetic field around its zero point (vacuum) configuration is the well-known Casimir effect [2] whereby an attractive force arises between two metallic neutral plates separated by a distance d.[1] The Casimir effect has recently received a lot of attention, mainly because of the following reasons: (i) advances in the experimental techniques leading to precision measurements [4,5] and to the possibility of measuring the dynamic Casimir effect [6]; (ii) increased importance of the Casimir forces in the field of microelectromechanical systems (MEMS) [7] and nanodevices [8]; (iii) relevance to physics beyond the standard model of particle physics, such as, for example, hypothetical extra-dimensional models [9]. The present work is a contribution to the last of the above points and tries to make a connection between the Casimir effect and models which are generally referred to as minimal length theories.

The concept of a minimal length arises naturally in quantum gravity. Indeed, when trying to resolve small distances, higher energies are needed, which eventually will affect the structure of space-time via their gravitational effects. Clearly this type of effect should occur at an energy scale of the order of the Planck mass. While it is generally stated that the incorporation of gravity within quantum field theory spoils the renormalizability of the latter, the fact that quantum gravity could play an important role in the suppression of infinities in quantum field theories was pointed out long ago [10].

Other models have been discussed in the literature which are also related to a minimal length such as, for example, extra-dimensional theories and/or noncommutative quantum field theories [11–13]. It is hoped that the energy (or equivalently length) scale associated to these models might turn out to be accessible experimentally in the next gen-

eration of high-energy accelerators such as the CERN LHC or the International Linear Collider.

Although not trivial to prove, it is natural to expect that a minimal distance should correspond at the quantum level to a minimal uncertainty. On the other hand, string theories have been shown to predict the existence of a *minimal length* in the form of an effective minimal uncertainty $(\Delta x)_{\min}$ [14]. This has triggered the study of quantum mechanical models based on generalized commutation relations such as

$$[\hat{x}, \hat{p}] = i\hbar(1 + \beta\hat{p}^2), \qquad (1)$$

which lead to a generalized uncertainty principle (GUP) which in turn provides a minimal uncertainty. In particular, in Ref. [15] a Hilbert space representation of the generalized uncertainty relations is constructed explicitly in a one-dimensional model, as well as in higher dimensions, identifying the states which realize the maximal localization (i.e. the minimal uncertainty) in position space. An important aspect of these theories is that the eigenstates of the position operator *are no longer physical states*, and so one is forced to introduce the so-called quasiposition representation, which consists in projecting the states onto the set of maximally localized states, instead of using the standard position representation obtained projecting the state vectors on the eigenstates of the position operator. Thus momentum states which are normally represented by plane waves $(2\pi)^{-3/2}\exp[(i/\hbar)\boldsymbol{p}\cdot\boldsymbol{x}] = \langle\boldsymbol{x}|\boldsymbol{p}\rangle$ are now replaced by the set of functions $\psi_{\boldsymbol{p}}^{ML}(\boldsymbol{x}) = \langle\psi_{\boldsymbol{x}}^{ML}|\boldsymbol{p}\rangle$, where $\psi_{\boldsymbol{x}}^{ML}$ are the maximally localized states (minimal uncertainty) around an average position $\boldsymbol{x}$. The particular form of these states depends on the number of dimensions and on the specific model considered. These type of models are also connected to noncommutative quantum field theories.

In Ref. [16] the author studies the standard Casimir effect of two perfectly conducting plates at a distance d computing the correction term arising within a model based on generalized uncertainty relations:

$$\mathcal{U}_{\text{Casimir}} = -\frac{\hbar c\pi^2}{720d^3}\left[1 - \pi^2\frac{99\,504}{56\,855}\left(\frac{\hbar\sqrt{\beta}}{d}\right)^2\right]. \qquad (2)$$

It is worth noting that the correction term due to the minimal length $(\Delta x)_{\min} \approx \hbar\sqrt{\beta}$ scales with a different

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[1]See, however, Ref. [3] for a different viewpoint which claims that there is no need to invoke vacuum fluctuations to explain the Casimir effect.

power law (d^{-6}), and it is repulsive. This result is used to obtain an upper bound on the minimal length which turns out to be $(\Delta x)_{\min} \lesssim 150$ nm.

In this work we take up the above scenario of generalized uncertainty relations and compute the correction term to the intermolecular Casimir-Polder interactions induced by a minimal length, expecting to find results similar to those of Ref. [16] [cf. Eq. (2)]. In particular, by performing a quantization procedure of the electromagnetic field as in Ref. [16], it will be shown that if a minimal length exists in nature the retarded Casimir-Polder interactions between neutral atoms and molecules will acquire, in addition to the standard r^{-7} interaction, a corrective term which scales as r^{-9}. However, as opposed to the plate-plate case, cf. Eq. (2), the new term has the same sign of the standard QED result, i.e. it describes an attractive interaction.

The remainder of the paper is organized as follows. In Sec. II we discuss the generalized uncertainty relations and define a set of maximally localized states; in Sec. III we discuss the quantization of the electromagnetic field in the presence of a minimal length; and in Sec. IV we derive the corrections to the Casimir-Polder intermolecular interactions due to a minimal length. Finally, in Sec. V we present the conclusions.

II. GENERALIZED UNCERTAINTY PRINCIPLE

Let us consider the generalized commutation relations of Eq. (1), for simplicity, in one dimension. From these modified commutation relations, one derives the generalized uncertainty principle (GUP) [15]

$$\triangle x \triangle p \geq \frac{\hbar}{2}[1 + \beta(\triangle p)^2 + \langle \hat{p} \rangle^2] \qquad \beta > 0, \quad (3)$$

which is found to be related to a minimal length [15,16]. Indeed, with a simple minimization procedure it is easily verified [15] that Eq. (3) implies an absolute minimal uncertainty $(\triangle x)_{\min} = \hbar\sqrt{\beta}$, which is obtained for those states such that $\langle \hat{p} \rangle = 0$ and $(\triangle p)^2 = \langle \hat{p}^2 \rangle = 1/\beta$. A Heisenberg algebra that satisfies the generalized commutation relations of Eq. (1) is represented on momentum space wave functions by

$$\hat{p}\psi(p) = p\psi(p), \qquad (4)$$

$$\hat{x}\psi(p) = i\hbar(1 + \beta p^2)\partial_p \psi(p). \qquad (5)$$

In Ref. [15] the eigenstates of the position operator have been shown to be nonphysical because their uncertainty in the position (which vanishes) is smaller than the absolute minimal uncertainty (or minimal length). One then can look for a set of maximally localized states for which the uncertainty in the position is minimal (i.e. equal to the minimal length) [17]. The procedure proposed by Kempf, Mangano, and Mann (KMM) [15] to find the maximally localized states consists in minimizing the value $(\Delta x)_\psi^2$ between those states which realize the generalized uncer-

tainty principle in Eq. (3) with the equality sign (*i.e. a squeezed state*):

$$(\Delta x)_\psi(\Delta p)_\psi = \tfrac{1}{2}|\langle[\hat{x}, \hat{p}]\rangle_\psi|. \qquad (6)$$

In the adopted momentum space representation Eq. (6) takes the form of a differential equation:

$$[i\hbar(1 + \beta p^2)\partial_p - \langle \hat{x} \rangle + i\hbar \frac{1 + \beta(\Delta p)^2 + \beta\langle p \rangle}{2(\Delta p)^2}$$
$$\times (\hat{p} - \langle \hat{p} \rangle)]\psi(p) = 0, \qquad (7)$$

which admits the normalized solution ($\langle \hat{x} \rangle = \xi, \langle \hat{p} \rangle = 0, \Delta p = 1/\sqrt{\beta}$):

$$\psi_\xi^{\mathrm{ML}}(p) = \sqrt{\frac{2\sqrt{\beta}}{\pi}}(1 + \beta p^2)^{-(1/2)}$$
$$\times \exp\left[-i\frac{\xi \arctan(\sqrt{\beta}p)}{\hbar\sqrt{\beta}}\right]. \qquad (8)$$

As opposed to ordinary quantum mechanics it turns out that such states, in addition to being normalizable, are of finite energy and no longer "orthogonal" i.e. their closure relation involves a finite function instead of a Dirac-δ distribution [$\delta(\xi - \xi')$]:

$$\langle \psi_{\xi'}^{ML}|\psi_\xi^{ML}\rangle = \frac{1}{\pi}\left[\frac{\xi - \xi'}{2\hbar\sqrt{\beta}} - \left(\frac{\xi - \xi'}{2\hbar\sqrt{\beta}}\right)^3\right]^{-1}\sin\left(\pi\frac{\xi - \xi'}{2\hbar\sqrt{\beta}}\right). \qquad (9)$$

Extension to higher dimensions.—In Ref. [17] the above ideas have been extended to a number n of arbitrary (spatial) dimensions. An important fact to recall is that there is no unique extension of Eq. (1) in more than one dimension. Indeed in order to preserve translational and rotational invariance, generalized commutation relations must take the form [15]

$$[\hat{x}_i, \hat{p}_j] = i\hbar[f(\hat{p}^2)\delta_{ij} + g(\hat{p}^2)\hat{p}_i\hat{p}_j], \qquad i, j = 1, \ldots, n. \qquad (10)$$

The functions $f(\hat{p}^2)$ and $g(\hat{p}^2)$ are not completely arbitrary. Relations between them can be found by imposing translational and rotational invariance. In the following we take up the standard choice of Ref. [17] which has become rather popular and has been the object of many phenomenological studies:

$$f(\hat{p}^2) = \frac{\beta\hat{p}^2}{\sqrt{1 + 2\beta\hat{p}^2} - 1}, \qquad g(\hat{p}^2) = \beta. \qquad (11)$$

A spectral representation can then be found such that

$$[\hat{x}_i, \hat{z}_j] = i\hbar\delta_{ij}, \qquad \hat{x}_i = i\hbar\partial_{z_i}. \qquad (12)$$

The KMM procedure consists in minimizing the position uncertainty within the set of squeezed states, see Ref. [15,17] for details, and yields the following *maximally localized states* around a mean position $\boldsymbol{\xi}$ (in the spectral

representation):

$$\psi_{\xi}^{ML}(z) = N\left(1 - \beta\frac{z^2}{2}\right)^{\alpha/2} \exp\left[-\frac{i}{\hbar}z\cdot\xi\right], \qquad (13)$$

where $\alpha = 1 + \sqrt{1 + n/2} \approx 2.58$ (in three spatial dimensions) is a numerical constant that characterizes the maximally localized states in the KMM approach. On the other hand, when the number of dimensions $n \geq 1$ the generalized uncertainty relations are not unique and different models (actually an infinite number of them) may be implemented [17] by choosing different functions $f(p^2)$ and/or $g(p^2)$ [cf. Eq. (11)] which will yield, in general, *different* maximally localized states which will not contain at all the numerical constant α. The *minimal* position uncertainty for the KMM maximally localized states is [18]

$$(\Delta x)_{\min} = \hbar\sqrt{\beta}\left[\frac{n}{8}\left(1 + \sqrt{1 + \frac{n}{2}}\right)\left(\frac{n + 2}{\sqrt{1 + n/2}} + 2\right)\right]^{1/2}$$

$$= \hbar\sqrt{\beta}\sqrt{\frac{n}{4}}\alpha, \qquad (14)$$

which in the case of three spatial dimensions gives $(\Delta x)_{\min} = 2.23533\,\hbar\sqrt{\beta}$.

The states given in Eq. (13) are normalizable states whose closure relation reads

$$\langle\psi_{(\xi)}^{ML}|\psi_{(\xi')}^{ML}\rangle = \tilde{\delta}_n(\xi - \xi')$$

$$= N^2 \int_{|z|\leq\sqrt{\frac{2}{\beta}}} d^n z\left(1 - \beta\frac{z^2}{2}\right)^{\alpha}$$

$$\times \exp\left[-\frac{i}{\hbar}z\cdot(\xi - \xi')\right], \qquad (15)$$

which turns out be a finite function for finite β. The normalization constant N can be chosen so that $\tilde{\delta}_n(\xi - \xi')$ reduces to the Dirac-δ distribution $\delta^n(\xi - \xi')$ in the limit of zero minimal length ($\beta \rightarrow 0$), i.e. $N = (2\pi\hbar)^{-n/2}$. Explicitly one has (for three spatial dimensions, $n = 3$)

$$\tilde{\delta}_3(\xi - \xi') = \Gamma(1 + \alpha)\frac{2^{\alpha}}{\pi^{3/2}(\hbar\sqrt{\beta})^3}\left(\frac{\sqrt{2}|\xi - \xi'|}{\hbar\sqrt{\beta}}\right)^{-(3/2+\alpha)}$$

$$\times J_{3/2+\alpha}\left(\frac{\sqrt{2}|\xi - \xi'|}{\hbar\sqrt{\beta}}\right). \qquad (16)$$

In Fig. 1 the closure function $\tilde{\delta}_3$ is plotted against its dimensionless argument. It can be easily verified that it satisfies the relation $\int d^3\xi\,\tilde{\delta}_3(\xi - \xi') = 1$.

For the purpose of deriving the first order correction in the minimal length of the Casimir-Polder forces, it will prove useful to expand the closure function, cf. Eq. (15), in powers of β and in terms of the Dirac-δ distribution. In order to properly do this, one must go to the momentum representation (see the Appendix for details), and one finds for the present model and within the KMM approach
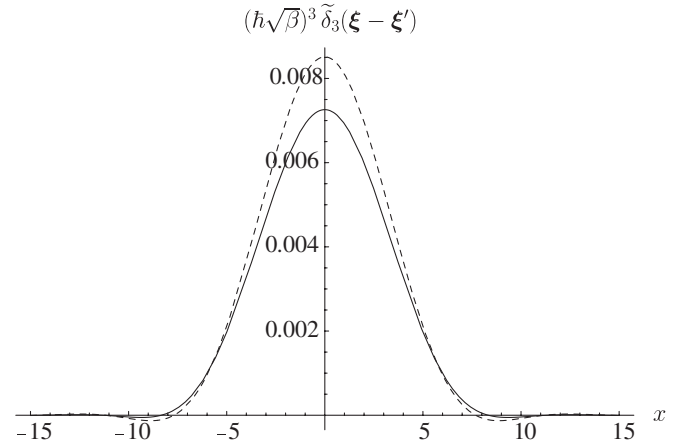

$(\hbar\sqrt{\beta})^3\,\tilde{\delta}_3(\xi - \xi')$

FIG. 1. Plot of the *regular* closure function $(\hbar\sqrt{\beta})^3\tilde{\delta}_3(\xi - \xi')$ along an arbitrary direction of the vector $\xi - \xi'$ (both positive and negative), versus the dimensionless quantity $x = \sqrt{2}|\xi - \xi'|/(\hbar\sqrt{\beta})$; (a) the dashed line is the result obtained [Eq. (16)] within the KMM procedure of Ref. [17], with $\alpha = 1 + \sqrt{1 + 3/2}$ corresponding to three spatial dimensions ($n = 3$); (b) the solid line is the result [Eq. (24) of this work] of the calculation performed with the (DGS) approach of Ref. [18]. This plot should not mislead the reader. Both functions satisfy the integral relation $\int d^3\xi'\tilde{\delta}_3(\xi - \xi') = 1$. The importance of numerical difference between the two functions, apparently inconsistent with this constraint, is reduced, when doing the integral, by the x^2 factor from the differential $d^3(\xi - \xi')$, and indeed both closure functions do integrate to 1, as has been checked both analytically and numerically.

$$\tilde{\delta}_n(r - r') \approx \left[1 + (\hbar\sqrt{\beta})^2\frac{\kappa}{2}\nabla_r^2 + \ldots\right]\delta^n(r - r'), \quad (17)$$

where $\kappa = 2 + \alpha$. We take the previous equation as the definition of the numerical constant κ whose value will, in general, depend on the model considered.[2] It should be noted that the KMM procedure to construct maximal localization states works properly only for the generalized commutation relations of Eq. (1). Detournay, Gabriel, and Spindel (DGS) [18] have proposed a better definition of maximally localized states, which is based on a minimization procedure on the subset of all physical states, and not just on the subset of squeezed states, and turns out to be suitable for more general commutation relations than those considered in Eq. (1). It should be realized that adopting this method (Ref. [18]) would only slightly affect our conclusions, namely, Eq. (17). The maximally localized states are in this approach given by Bessel functions. In the end result, cf. Eq. (17), we can anticipate that expanding the closure function in powers of the minimal length $(\hbar\sqrt{\beta})$

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[2]We emphasize that κ is a *numerical constant* and not a free parameter. Within the KMM procedure for obtaining the maximally localized states, its numerical value (within the model considered here) is fixed by the number of spatial dimensions only.

would merely amount to a change in the value of the numerical constant κ, which can, however, be expected to be always of order unity.

Let us discuss in further detail this point. In particular, we would like to show that adopting the DGS procedure, see Ref. [18], the resulting regular closure function $\tilde{\delta}_3(\boldsymbol{r} - \boldsymbol{r}')$, although represented by a different analytic expression, is numerically very close to that computed in the KMM procedure. Furthermore, its expansion in powers of β is as given in Eq. (17) but with a slightly lower value of the numerical constant κ. In Ref. [18] the authors show that the maximally localized states corresponding to the particular model being discussed here [cf. Eq. (11)] are given (in the spectral representation) in terms of Bessel functions:

$$\psi_{\boldsymbol{\xi}}^{ML}(z) = C_n \frac{J_\nu(\mu z)}{z^\nu} \exp\left[-\frac{i}{\hbar}\boldsymbol{\xi} \cdot z\right], \qquad (18)$$

where $z = |z|$; $\nu = \frac{n}{2} - 1$, $\mu = \pi\sqrt{\beta}/\sqrt{2}$ and the normalization constant C_n depends again on the number of dimensions that we are considering. These states are characterized by a *minimal* uncertainty which is given by

$$(\Delta x)_{\min} = \hbar\mu = \frac{j_{\nu,1}}{\sqrt{2}}\hbar\sqrt{\beta}, \qquad (19)$$

$j_{\nu,1}$ being the first zero of the Bessel function $J_\nu(x)$. For $n = 3$ we have $\nu = 1/2$ and $j_{\nu,1} = \pi$ so that the minimal length derived in the DGS procedure is $(\Delta x)_{\min} = 2.22144\,\hbar\sqrt{\beta}$ which is indeed smaller than the minimal length derived in the KMM approach.[3] With these states the closure function becomes

$$\langle\psi_{(\boldsymbol{\xi})}^{ML}|\psi_{(\boldsymbol{\xi}')}^{ML}\rangle = \tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}')$$

$$= C_n^2 \int_{|z|\leq\sqrt{2/\beta}} d^n z \frac{[J_\nu(\mu z)]^2}{z^{2\nu}}$$

$$\times \exp\left[-\frac{i}{\hbar}z \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}')\right]. \qquad (20)$$

Using the well-known expansion for the Bessel functions,

$$J_\nu(t) = \frac{1}{\Gamma(\nu+1)}\left(\frac{t}{2}\right)^\nu\left[1 - \frac{1}{\nu+1}\left(\frac{t}{2}\right)^2 + \dots\right], \quad (21)$$

the constant C_n can be chosen so that in the limit of a vanishing minimal length $\beta \to 0$ (or $\mu \to 0$) the regular closure function $\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}')$ reduces to the Dirac-δ distribution $\delta^n(\boldsymbol{\xi} - \boldsymbol{\xi}')$:

$$C_n = \frac{\Gamma(n/2)}{(2\pi\hbar)^{n/2}}\left(\frac{2}{\mu}\right)^{n/2-1}. \qquad (22)$$

In addition it is possible to derive an expansion of the closure function of the type in Eq. (17) also within the

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[3]Recall that the minimization in the DGS procedure is carried through within the set of all physical states, while in the KMM approach it is restricted to the squeezed states.

DGS procedure. One derives the following expansion in powers of β (see the Appendix for details):

$$\langle\psi_{(\boldsymbol{\xi})}^{ML}|\psi_{(\boldsymbol{\xi}')}^{ML}\rangle = \tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}')$$

$$\approx \left[1 + \frac{2 + (j_{\nu,1})^2/n}{2}(\hbar\sqrt{\beta})^2\nabla_{\boldsymbol{\xi}}^2 + \dots\right]$$

$$\times \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}'), \qquad (23)$$

thereby defining, also in the DGS procedure, a numerical constant κ, namely, $\kappa = 2 + (j_{\nu,1})^2/n$.

On the other hand, by analytically carrying through the integration in Eq. (20), one finds the closed expression (when $n = 3$)

$$\tilde{\delta}_3(\boldsymbol{\xi} - \boldsymbol{\xi}') = \frac{\sqrt{2}}{4\pi^4}\frac{1}{(\hbar\sqrt{\beta})^3}$$

$$\times \frac{2\mathrm{Si}(x) - \mathrm{Si}(x - 2\pi) - \mathrm{Si}(x + 2\pi)}{x}, \qquad (24)$$

where again $x = \sqrt{2}|\boldsymbol{\xi} - \boldsymbol{\xi}'|/(\hbar\sqrt{\beta})$ and $\mathrm{Si}(x) = \int_0^x dt \sin(t)/t$ is the sine-integral function. One can again verify that $\tilde{\delta}_3(\boldsymbol{\xi} - \boldsymbol{\xi}')$ satisfies the relation $\int d^3\boldsymbol{\xi}\,\tilde{\delta}_3(\boldsymbol{\xi} - \boldsymbol{\xi}') = 1$. Figure 1 shows a comparison of the closure functions derived in the KMM and DGS procedures. Table I summarizes some results within the two approaches for the model of the present work (model I).

The above discussion shows, in particular, that *even within the same model* adopting the KMM procedure or the more appropriate DGS method (where the minimization of the uncertainty is done over all physical states) will produce *slightly* different values of the numerical constant κ, thereby justifying the use of the widespread KMM approach at least within this particular model.

III. QED IN THE TEMPORAL GAUGE

In order to approach the quantization procedure in the presence of a minimal length, we review briefly the derivation of the temporal gauge $[A^0(\boldsymbol{r}, t) = c\varphi(\boldsymbol{r}, t) = 0]$ propagator in *standard* QED. The defining equation for the photon propagator in QED is

$$D^{ij}(\boldsymbol{r}, t; \boldsymbol{r}', t') = \frac{i}{\hbar}\langle 0|T[A^i(\boldsymbol{r}, t)A^j(\boldsymbol{r}', t')]|0\rangle, \qquad (25)$$

from which, applying through the classical equation of motion for the fields $A_i(\boldsymbol{r}, t)$ (Maxwell equations) in the temporal gauge, it is easily verified that

$$\left[\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta_m^\ell - \nabla^\ell\nabla_m\right]D^{mk}(\boldsymbol{r}, t; \boldsymbol{r}', t')$$

$$= -\frac{i}{\hbar c}\delta(t - t')\langle 0|[E^\ell(\boldsymbol{r}, t), A^k(\boldsymbol{r}', t')]|0\rangle, \qquad (26)$$

where the electric field operator has been introduced by $\boldsymbol{E}(\boldsymbol{r}, t) = -(1/c)\partial\boldsymbol{A}(\boldsymbol{r}, t)/\partial t$. Canonical quantization then

TABLE I.   Within the rotationally invariant model (model I) of Eq. (11) (see Refs. [15,17]), we compare the numerical constant κ and the minimal uncertainty as discussed in the text for both the KMM and DGS procedures of obtaining the maximally localized states. Recall that the constant $\alpha = 1 + \sqrt{1 + n/2}$ appears in the maximally localized states of the KMM procedure. We also give the same details for the direct product model (model II) used in Ref. [16], which is reviewed for the reader's benefit in the Appendix.

	κ	$\kappa(n = 3)$	$\frac{(\Delta x)_{\min}}{\hbar\sqrt{\beta}}(n)$	$\frac{(\Delta x)_{\min}}{\hbar\sqrt{\beta}}(n = 3)$
Model I (KMM)	$2 + \alpha$	$2 + 1 + \sqrt{5/2} \approx 4.581\,14$	$\sqrt{\frac{\pi}{4}}\alpha$	$\sqrt{\frac{3}{4}}\alpha \approx 2.2353$
Model I (DGS)	$2 + \frac{(j_{n/2-1,1})^2}{n}$	$2 + \frac{\pi^2}{3} \approx 5.289\,87$	$\frac{j_{n/2-1,1}}{\sqrt{2}}$	$\frac{\pi}{\sqrt{2}} \approx 2.221\,44$
Model II	$10/3$	$10/3$	$\sqrt{n}$	$\sqrt{3}$

prescribes the equal-time commutation relations [19]

$$[E^\ell(\boldsymbol{r}, t), A^k(\boldsymbol{r}', t)] = 4\pi i\hbar c \delta^{\ell k}\delta^3(\boldsymbol{r} - \boldsymbol{r}'), \quad (27)$$

and the differential equation for the QED photon propagator becomes

$$\left[\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta_m^\ell - \nabla^\ell\nabla_m\right]D^{mk}(\boldsymbol{r} - \boldsymbol{r}'; t - t')$$
$$= 4\pi\delta^{\ell k}\delta(t - t')\delta^3(\boldsymbol{r} - \boldsymbol{r}'). \quad (28)$$

The solution is found by going into a Fourier representation

$$D^{rj}(\boldsymbol{k}; \omega) = \int dt d^3\boldsymbol{r} e^{i\omega t}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}D^{rj}(\boldsymbol{r}, t), \quad (29)$$

and is given by the well-known result

$$D^{rj}(\boldsymbol{k}, \omega) = -\frac{4\pi}{(\omega/c)^2 - k^2 + i0}\left[\delta^{rj} - \frac{k^r k^j}{(\omega/c)^2}\right]. \quad (30)$$

As regards the calculation of the effect of a minimal length in the Casimir-Polder intermolecular interactions, following [20], it will be convenient to use the propagator in the mixed representation:

$$D^{ij}(\omega, \boldsymbol{r}) = \int_{-\infty}^{+\infty} dt e^{i\omega t}D^{ij}(t, \boldsymbol{r}), \quad (31)$$

which is found upon integrating in $\boldsymbol{k}$ space Eq. (30):

$$D^{ij}(\omega, \boldsymbol{r}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}D^{ij}(\boldsymbol{k}, \omega). \quad (32)$$

It is easily verified that $D^{ij}(\omega, \boldsymbol{r})$ satisfies the following differential equation:

$$\left[-\left(\frac{\omega^2}{c^2} + \nabla^2\right)\delta_m^\ell - \nabla^\ell\nabla_m\right]D^{mk}(\omega, \boldsymbol{r}) = 4\pi\delta^{\ell k}\delta^3(\boldsymbol{r}), \quad (33)$$

which is solved by [20]

$$D^{ij}(\omega, \boldsymbol{r}) = \left[\delta^{ij}\left(1 + \frac{ic}{|\omega|r} - \frac{c^2}{\omega^2 r^2}\right) + \frac{x^i x^k}{r^2}\left(\frac{3c^2}{\omega^2 r^2} - \frac{3ic}{|\omega|r} - 1\right)\right]\frac{e^{i|\omega|r/c}}{r}. \quad (34)$$

Quantization in minimal length theories.—Let us now discuss the quantization procedure of the electromagnetic field in a quantum world with a minimal length. We shall proceed following the scheme adopted in [16]. In this case the procedure of canonical quantization gets modified because it turns out that the equal-time commutation relations of the fields, see Eq. (27), are different. Indeed, now instead of expanding the field operators over plane waves (*position representation wave functions of momentum states*), one is forced to expand the fields over a set of maximally localized states $|\psi_{\boldsymbol{r}}^{ML}\rangle$ of average position $\boldsymbol{r}$ (*quasiposition representation of momentum states*):

$$A^i(\boldsymbol{r}, t) = \sum_\lambda \int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\left(\frac{2\pi\hbar c^2}{\omega_{\boldsymbol{p}}}\right)^{1/2}\left[a(\boldsymbol{p}, \lambda)\varepsilon^i(\boldsymbol{p}, \lambda)\right.$$
$$\times \langle\psi_{\boldsymbol{r}}^{ML}|\boldsymbol{p}\rangle e^{-i\omega_{\boldsymbol{p}}t} + a^\dagger(\boldsymbol{p}, \lambda)(\varepsilon^i(\boldsymbol{p}, \lambda))^*$$
$$\left.\times \langle\boldsymbol{p}|\psi_{\boldsymbol{r}}^{ML}\rangle e^{+i\omega_{\boldsymbol{p}}t}\right], \quad (35)$$

introducing creation and annihilation operators which do satisfy the usual commutation relations,

$$[a(\boldsymbol{p}, \lambda), a^\dagger(\boldsymbol{p}', \lambda')] = (2\pi)^3\delta^{\lambda,\lambda'}\delta^3(\boldsymbol{p} - \boldsymbol{p}'), \quad (36)$$

with all other commutators vanishing (*recall that momentum coordinates are commuting in the model which is being discussed here*). The equal-time commutation relations for the fields, cf. Eq. (27), are then easily found to be modified to

$$[E^\ell(\boldsymbol{r}, t), A^k(\boldsymbol{r}', t)] = 4\pi i\hbar c \delta^{\ell k}\langle\psi_{\boldsymbol{r}}^{ML}|\psi_{\boldsymbol{r}'}^{ML}\rangle$$
$$= 4\pi i\hbar c \delta^{\ell k}\tilde{\delta}_3(\boldsymbol{r} - \boldsymbol{r}'), \quad (37)$$

so that, comparing with Eq. (33), the photon propagator in the presence of a minimal length, $\tilde{D}^{ij}(\omega, \boldsymbol{r})$, is found from

$$\left[-\left(\frac{\omega^2}{c^2} + \nabla^2\right)\delta_m^\ell - \nabla^\ell\nabla_m\right]\tilde{D}^{mk}(\omega, \boldsymbol{r}) = 4\pi\delta^{\ell k}\tilde{\delta}_3(\boldsymbol{r}). \quad (38)$$

This approach to field quantization bears some similarity to that discussed in Ref. [21], where within a model of noncommutative space-time the real scalar field is defined as a mean value over coherent states. The noncommutativity is reflected by a modification of the Fourier transform replac-

ing ordinary plane waves by Gaussian wave packets. We remark that in Ref. [21] the authors derive for the Green function a differential equation which is similar to our Eq. (38), where the three-dimensional Dirac δ function is replaced by a Gaussian dumped by the noncommutative parameter which plays the role of the parameter β of this work. $\tilde{\delta}_3(\boldsymbol{r})$ being a regular function, it turns out that the standard QED photon propagator $D_{rj}(\omega, \boldsymbol{r})$ [from Eq. (33)] is the Green function of Eq. (38), the differential equation describing the photon propagator in *minimal length* QED, $\tilde{D}^{ij}(\omega, \boldsymbol{r})$. Therefore, it follows the central result of this work. *In minimal length QED the photon propagator in the mixed representation is given by a convolution of the regular closure function $\tilde{\delta}_3(\boldsymbol{r})$ with the standard QED photon propagator*:

$$\tilde{D}^{ik}(\omega, \boldsymbol{r}) = \int d^3 r' D^{ik}(\omega, \boldsymbol{r} - \boldsymbol{r}') \tilde{\delta}_3(\boldsymbol{r}'). \quad (39)$$

From Eq. (17) one can obtain, by means of partial integration, an expansion of the photon propagator in terms of the minimal length $\hbar\sqrt{\beta}$:

$$\tilde{D}^{ik}(\omega, \boldsymbol{r}) \approx D^{ik}(\omega, \boldsymbol{r}) + (\hbar\sqrt{\beta})^2 \frac{\kappa}{2} \nabla^2 D^{ik}(\omega, \boldsymbol{r}). \quad (40)$$

Using the explicit form of the QED temporal gauge propagator given in Eq. (34), it is easily verified that

$$\nabla^2 D^{ik}(\omega, \boldsymbol{r}) = -\frac{\omega^2}{c^2} D^{ik}(\omega, \boldsymbol{r}), \quad (41)$$

so that we finally get

$$\tilde{D}^{ik}(\omega, \boldsymbol{r}) \approx \left[1 - \frac{\kappa}{2}(\hbar\sqrt{\beta})^2 \frac{\omega^2}{c^2} \right] D^{ik}(\omega, \boldsymbol{r}). \quad (42)$$

A remark is in order at this point. It should be clear to the reader that the above approach to field quantization within a minimal length model breaks Lorentz invariance. This fact appears as well in other extensions of the standard model (SM) such as noncommutative quantum field theory, always being one of the major sources of debates between different authors. It should be noted that in Ref. [21] the question of Lorentz covariance is discussed and, within their noncommutative 2D model, successfully addressed. While completing this study the author became aware of a recent work [22] where a relativistic generalization of the Kempf algebra is proposed. Based on this new deformed algebra, it will probably be possible to define a Lorentz covariant field quantization which would account for a *fundamental minimal length*. Certainly this point deserves further study, but goes beyond the scope of the present work.

IV. CASIMIR-POLDER INTERMOLECULAR INTERACTIONS

The interaction of two neutral atoms (molecules) at rest in $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ and with electric dipole moments $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ is

described by the operator

$$V = -\boldsymbol{E}(\boldsymbol{r}_1, t) \cdot \boldsymbol{d}_1(t) - \boldsymbol{E}(\boldsymbol{r}_2, t) \cdot \boldsymbol{d}_2(t) \quad (43)$$

where $\boldsymbol{E}(\boldsymbol{r}, t)$ is the operator describing the electric field. It is well known that, employing standard perturbation methods of quantum field theory, the interaction of the neutral atoms (or molecules) at distances $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$, which are large compared to the atomic and/or molecular dimensions a: $r \gg a$,[4] is described in terms of (i) the dynamic polarizability tensor $\alpha^{ik}(\omega)$ of the atoms:

$$\alpha^{ik}(\omega) = \frac{i}{\hbar} \int_{-\infty}^{+\infty} d\tau \langle 0|T[d^i(\tau), d^k(0)]|0\rangle e^{+i\omega\tau}, \quad (44)$$

and (ii) the photon propagator in the mixed representation $D^{ik}(\omega, \boldsymbol{r})$ [20]. For the photon propagator we will use in the following the dyadic (tensor) notation $\mathsf{D}(\omega, \boldsymbol{r})$. Assuming that the polarizability tensors of the two atoms are isotropic, $\alpha_{(1,2)}^{ik}(\omega) = \delta^{ik}\alpha_{(1,2)}(\omega)$, one obtains

$$\alpha(\omega) = \frac{1}{3}\sum_n |d_{0n}|^2 \left(\frac{1}{\omega_{n0} - \omega - i0^+} + \frac{1}{\omega_{n0} + \omega - i0^+} \right), \quad (45)$$

and the final expression for the interaction potential can be cast as

$$\begin{aligned} U(r) = \frac{i\hbar}{2c^4} &\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \omega^4 \alpha_1(\omega)\alpha_2(\omega) \\ &\times \mathrm{Tr}[\mathsf{D}(\omega, \boldsymbol{r}) \cdot \mathsf{D}(\omega, \boldsymbol{r})]. \end{aligned} \quad (46)$$

When a minimal length is present in the above expression, one should replace the photon propagator $\mathsf{D}(\omega, \boldsymbol{r})$ with the modified propagator $\tilde{\mathsf{D}}(\omega, \boldsymbol{r})$. Using the approximate expression in Eq. (42) one finds

$$\begin{aligned} \mathrm{Tr}[\tilde{\mathsf{D}}(\omega, \boldsymbol{r}) \cdot \tilde{\mathsf{D}}(\omega, \boldsymbol{r})] \approx &\left[1 - \kappa(\hbar\sqrt{\beta})^2 \frac{\omega^2}{c^2} \right] \\ &\times \mathrm{Tr}[\mathsf{D}(\omega, \boldsymbol{r}) \cdot \mathsf{D}(\omega, \boldsymbol{r})], \end{aligned} \quad (47)$$

so that the interaction potential is given by the usual standard model result with an additional term describing the effects of the minimal length:

$$\begin{aligned} \delta U(r) = -\frac{i}{2}\frac{\hbar}{c^6}(\hbar\sqrt{\beta})^2\kappa &\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \\ &\times \omega^6 \alpha_1(\omega)\alpha_2(\omega) \mathrm{Tr}[\mathsf{D}(\omega, \boldsymbol{r}) \cdot \mathsf{D}(\omega, \boldsymbol{r})]. \end{aligned} \quad (48)$$

By using the explicit expression in Eq. (34) it is easily found that

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[4]In this field theoretic approach the distance r can otherwise be either much smaller (*short distances*) than the characteristic wavelength λ_0, of the spectra of the interacting atoms (or molecules), comparable to, or much larger than λ_0, (*long distances*).

$$\text{Tr}[\mathsf{D}(\omega, \boldsymbol{r}) \cdot \mathsf{D}(\omega, \boldsymbol{r})] = 2\left[1 + \frac{2ic}{|\omega|r} - \frac{5c^2}{\omega^2 r^2} - \frac{6ic^3}{|\omega|^3 r^3}\right.$$
$$\left. + \frac{3c^4}{\omega^4 r^4}\right]\frac{e^{2i|\omega|r/c}}{r^2}. \qquad (49)$$

The general expressions in Eqs. (46) and (48) can be simplified in the limiting cases of short distances ($a \ll r \ll \lambda_0$) and long distances ($r \gg \lambda_0$).

Short distances.—When $r \ll \lambda_0$, in the integral of Eq. (48) the values of $\omega \sim \omega_0 \sim c/\lambda_0$ are important so that $\omega r/c \ll 1$ and in Eq. (49) one is allowed to keep only the last term and to approximate the exponential with 1. Performing the straightforward computations one then finds an interaction:

$$U(r) = -\frac{2}{3r^6}\sum_{n,n'}\frac{|d_{n0}|^2|d_{n'0}|^2}{\hbar(\omega_{n0} + \omega_{n'0})}\left[1 - \kappa(\hbar\sqrt{\beta})^2\right.$$
$$\left. \times\left(\frac{\omega_{n0}^2 + \omega_{n'0}^2}{c^2}\right)\right], \qquad (50)$$

which scales as r^{-6} and is composed of two terms: the well-known London-type potential [20] and the correction term which is due to the minimal length. In this regime of short distances the minimal length correction affects only the strength of the interaction and does not change its power law.

Long distances.—In the limit of *long distances* $r \gg \lambda_0$ in the integral of Eq. (48) only the values of $\omega \lesssim c/r \ll \omega_0$ are important. When $\omega \gtrsim \omega_0$, the strongly oscillating complex exponential will suppress the integral. It is then possible to substitute the dynamic polarizabilities $\alpha_{1,2}(\omega)$ with their static values $\alpha_1(0)$ and $\alpha_2(0)$. We then find a correction of order $(\hbar\sqrt{\beta}/r)^2$ to the well-known r^{-7} Casimir-Polder interaction:

$$\delta U(r) = -\frac{129}{8\pi}\kappa\left(\frac{\hbar\sqrt{\beta}}{r}\right)^2\frac{\hbar c\alpha_1(0)\alpha_2(0)}{r^7}. \qquad (51)$$

Thus, in theories with a minimal length, a correction term arises in the Casimir-Polder atomic and molecular interactions at large distances. The correction has the same sign (*attractive potential*) but scales with a different power law (r^{-9}) relative to the QED result:

$$\mathcal{U}_{\text{Casimir-Polder}}(r) \approx -\frac{23}{4\pi}\frac{\hbar c\alpha_1(0)\alpha_2(0)}{r^7}$$
$$\times\left[1 + \frac{129}{56}\kappa\left(\frac{\hbar\sqrt{\beta}}{r}\right)^2\right]. \qquad (52)$$

This might be compared with the result of Ref. [16], cf. Eq. (2), which describes the Casimir energy of the plate-plate system. One can notice that, while in the plate-plate case the minimal length correction derived in Ref. [16] is *repulsive* and opposite to the standard result, in the case treated here of the interaction of two neutral atoms or molecules, the minimal length correction is *attractive*

i.e. of the same sign of the standard Casimir-Polder result. We should, however, be very careful in comparing the calculation of the Casimir effect of Ref. [16] with the present one. Indeed in Ref. [16] a different model of generalized uncertainty relations is explicitly taken up. As opposed to our Eq. (11) the model proposed in Ref. [16] given by $f(p^2) = 1 + \beta p^2$ and $g(p^2) = 0$ is the naive generalization of the one-dimensional model. Such a model is known [23] to be inconsistent with the KMM construction of maximally localized states. Because of the resulting noncommutativity of the coordinates, the resulting n differential equations [corresponding to Eq. (7)] of the squeezed states cannot be solved simultaneously [23]. A different way of extending the one-dimensional GUP of Eq. (1) is to take a direct product of it. This so-called direct product model (model II) clearly breaks rotational invariance as opposed to the model of this paper (model I). The maximally localized states are obtained by taking the product of the one-dimensional states, see Eq. (8), along the different dimensions. This model is therefore characterized by different maximally localized states, which will in turn produce different closure functions which, *when expanded in powers of β*, will provide different values for the numerical constant κ as defined by Eq. (17) (though we expect it to be always of order unity). In the Appendix we provide details of the closure function within this model and deduce the corresponding value of κ which turns out to be $\kappa = 10/3$. We may therefore conclude that the Casimir-Polder minimal length correction within the direct product model is also attractive. That the Casimir effect for parallel plates within the direct product model (model II) as calculated in Ref. [16] turns out to be repulsive may be due to the fact that the closure function there appears to have been computed, not with the maximally localized states, but with the formal position eigenstates which are not physical states. Setting this point definitively calls for a detailed analysis of the Casimir effect using the *proper* maximally localized states within either model I or II and goes beyond the scope of the present work.

The so-called Casimir-Polder force, for the atom-atom configuration, was first derived in 1948 [24], but it was not measured definitively until 1993 [25], by looking at the deflection of an atomic beam passing through two parallel plates. For earlier measurements related to the atom-plane configuration, see [26]. There has then been a consistent renewal of interest with an increase of the measurement's precision as in Ref. [27] where this tiny interaction between a neutral system and a surface is reported. The authors employed a cloud of ultracold atoms in a Bose-Einstein condensate (BEC) state. The range of distances explored was between 6 and 10 microns. See the recent review [28] for more details about the first generation of experiments, both in the Casimir effect and the related Casimir-Polder interactions, and for later developments and future directions.

It seems that a better direction to explore the Casimir-Polder interactions experimentally is to consider the plate-sphere (plate-atom) configuration. All the experimental advances have been achieved for this case [28]. This suggests the direction where the present theoretical work could develop: the calculation of the Casimir-Polder (plate-atom) interaction corrective term within a minimal length theory.

V. DISCUSSION AND CONCLUSIONS

We have studied the implications of models based on generalized commutation relations *i.e. with a minimal length* in the Casimir-Polder intermolecular interactions. The calculation is done following standard perturbation theoretical methods of quantum field theory and, in particular, the approach of I. E. Dzjalošinskij as illustrated in quantum field theory textbooks [20]. Here the interaction energy of two neutral atoms and/or molecules is related to the dynamic polarizability tensors of the two neutral bodies and to the photon Green function (propagator). The computation of the correction term due to the minimal length is thus carried out by discussing the QED photon propagator in the presence of a minimal length. Quantization of the electromagnetic field in the temporal gauge is performed in analogy to the canonical quantization procedure, the essential point being that the field operators $A^i(\boldsymbol{r}, t)$, instead of being decomposed over a complete set of plane waves (momentum eigenfunctions), are now decomposed over a complete set of maximally localized states. This approach has also been followed by the author of Ref. [16] in deriving the Casimir potential energy of the plate-plate system in the presence of a minimal length, cf. Eq. (2). The main point is that the equal-time commutation relations of the field operators, instead of being given in terms of a Dirac δ function, cf. Eq. (27), are now expressed by a finite regular function, cf. Eq. (37), and the photon propagator in minimal length QED is given by a convolution of the standard QED propagator with the regular function $\tilde{\delta}_3(\boldsymbol{r} - \boldsymbol{r}')$. In order to compute the lowest order correction term, the minimal length QED propagator is related to that of standard QED by performing an expansion in the minimal length parameter $\hbar\sqrt{\beta}$.

We have derived a corrective term to the (long distance) retarded Casimir-Polder interaction of two neutral atoms separated by a distance $r \gg \lambda_0$, finding, cf. Eq. (52), a new interaction term whose potential scales like r^{-9} as opposed to the standard r^{-7} result.

Clearly, should the minimal length $(\Delta x)_{\min} = \hbar\sqrt{\beta}$ be of the order of the Planck length L_P, the observability of this effect would be out of the question. In Ref. [29] the study of the harmonic oscillator and the hydrogen atom allowed one to derive an upper bound on the minimal length by comparing with precision measurements on hydrogenic atoms and for electrons trapped in strong magnetic fields. The upper bound obtained is

$$(\Delta x)_{\min} = \hbar\sqrt{\beta} < 10^{-1} \text{ fm.} \qquad (53)$$

Assuming that the constant β appearing in the deformed Heisenberg algebra is a universal constant, this upper bound would presumably preclude a possible observation with Casimir-Polder interaction measurements with accessible distances r which are typically in the range 80 nm $\leq r \leq 10 \ \mu$m [28]. Thus we conclude that, taking into account the upper bound of Ref. [29], the term derived in this work, cf. Eq. (52), would provide a correction whose relative strength we expect to be in the range

$$10^{-22} \lesssim \frac{\delta U}{U} = O\left[\left(\frac{\hbar\sqrt{\beta}}{r}\right)^2\right] \lesssim 10^{-18}, \qquad (54)$$

which is clearly beyond any foreseeable improvement in the precision of measurements of Casimir-Polder interactions. As discussed in Ref. [30] the rather strong upper bound in Eq. (53) could be avoided by assuming that the parameter β is not a universal constant and could vary from one system to another depending, for example, on the energy content of the system (the mass of the particle, for instance) or the strength of some interaction. Indeed, in Ref. [30], by making this hypothesis, the authors, through a comparison with the experimental results for ultracold neutron energy levels in a gravitational quantum well (GRANIT experiment) [31–33], derive a relaxed upper bound to the minimal length which turns out to be of the order of a few nanometers $[(\Delta x)_{\min} < 2.41 \text{ nm}]$. Even if this situation should apply, the relative strength of the corrective term will fall in the range

$$10^{-8} \lesssim \frac{\delta U}{U} = O\left[\left(\frac{\hbar\sqrt{\beta}}{r}\right)^2\right] \lesssim 10^{-4}, \qquad (55)$$

which is still out of the reach of current experiments in the Casimir-Polder interactions, whose precision is typically of 10^{-2} or 1% [28]. With an increased experimental sensitivity to smaller length scales, *of the order of a few nanometers*, and better experimental accuracy possible, perhaps, in the near future, the calculations described in the present work could find interesting applications and would provide a valuable tool to study the intimate structure of space-time.

ACKNOWLEDGMENTS

APPENDIX: DETAILS OF THE CLOSURE FUNCTIONS

In this appendix we give details of the series expansion in powers of β of the closure function, both in the model taken up in this work and in the one considered in Ref. [16]. In particular, we show that in all considered cases the following relation holds with different values of κ:

$$\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}') = \left[1 + \frac{\kappa}{2}(\hbar\sqrt{\beta})^2\nabla_{\boldsymbol{\xi}}^2 + O(\beta^2)\right]\delta^n(\boldsymbol{\xi} - \boldsymbol{\xi}'). \tag{A1}$$

It turns out that, in order to get the correct series expansion in powers of β of the closure function, one must resort to the unbounded momentum representation, while both Eq. (15) and Eq. (20), based on the spectral representation (compact space), can only reproduce correctly the first term of the expansion corresponding to $\beta = 0$. Let us consider this in detail within (i) the rotational invariant model object of this work, (model I) detailed both within the KMM and DGS procedure, and (ii) the direct product model of Ref. [16] (model II).

Rotational invariant model (model I).—In Eq. (15) we first perform a change of variables from the spectral representation to the momentum representation:

$$z_i = p_i\frac{\sqrt{1 + 2\beta p^2} - 1}{\beta p^2}, \quad \text{with} \quad p^2 = \boldsymbol{p} \cdot \boldsymbol{p} = \sum_i^n (p_i)^2. \tag{A2}$$

The Jacobian matrix is

$$\frac{\partial z_i}{\partial p_j} = \delta_{ij}a + 2p_ip_jb,$$

where

$$a = \frac{\sqrt{1 + 2\beta p^2} - 1}{\beta p^2},$$

$$b = \frac{-1 - \beta p^2 + \sqrt{1 + 2\beta p^2}}{\beta p^4\sqrt{1 + 2\beta p^2}}.$$

The Jacobian of the transformation J turns out to be given by

$$J = \det\left|\frac{\partial z_i}{\partial p_j}\right| = a^{n-1}(a + 2bp^2)$$

$$= \left(\frac{\sqrt{1 + 2\beta p^2} - 1}{\beta p^2}\right)^n \frac{1}{\sqrt{1 + 2\beta p^2}}.$$

(a) The KMM procedure for the maximally localized states.
Equation (15) becomes

$$\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}') = \int \frac{d^n\boldsymbol{p}}{(2\pi\hbar)^n}\left(\frac{\sqrt{1 + 2\beta p^2} - 1}{\beta p^2}\right)^{n+\alpha} \frac{1}{\sqrt{1 + 2\beta p^2}} \exp\left[-\frac{i}{\hbar}(\boldsymbol{\xi} - \boldsymbol{\xi}') \cdot \boldsymbol{p}\frac{\sqrt{1 + 2\beta p^2} - 1}{\beta p^2}\right]. \tag{A3}$$

We remark that the integration is unbounded. The integrand can be expanded in a power series around $\beta = 0$:

$$\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}') = \int \frac{d^n\boldsymbol{p}}{(2\pi\hbar)^n}\left[1 - \beta\left(\frac{n + 2 + \alpha}{2}p^2 - \frac{i}{2\hbar}(\boldsymbol{\xi} - \boldsymbol{\xi}') \cdot \boldsymbol{p}p^2\right) + O(\beta^2)\right]\exp\left[-\frac{i}{\hbar}(\boldsymbol{\xi} - \boldsymbol{\xi}') \cdot \boldsymbol{p}\right]. \tag{A4}$$

In the above expression one can make the substitution $\boldsymbol{p} \to i\hbar\nabla_{\boldsymbol{\xi}}$ and thereby perform the momentum integration obtaining

$$\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}') = \left[1 + \frac{2 + \alpha}{2}(\hbar\sqrt{\beta})^2\nabla_{\boldsymbol{\xi}}^2 + O(\beta^2)\right]\delta^n(\boldsymbol{\xi} - \boldsymbol{\xi}'), \tag{A5}$$

thus defining for this case the value of the numerical constant κ to be given by $\kappa = 2 + \alpha$. We note that the last expression has been obtained after using identities (due to partial integration) valid for distributions such as

$$[(\boldsymbol{\xi} - \boldsymbol{\xi}') \cdot \nabla_{\boldsymbol{\xi}}]\nabla_{\boldsymbol{\xi}}^2\delta^n(\boldsymbol{\xi} - \boldsymbol{\xi}') = -n\nabla_{\boldsymbol{\xi}}^2\delta^n(\boldsymbol{\xi} - \boldsymbol{\xi}'). \tag{A6}$$

(b) The DGS procedure for the maximally localized states.
Using the maximally localized states of the DGS procedure, one finds in the momentum representation

$$\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}') = C_n^2 \int d^n\boldsymbol{p}\left(\frac{\sqrt{1 + 2\beta p^2} - 1}{\beta p^2}\right)^n \frac{1}{\sqrt{1 + 2\beta p^2}} \frac{[J_\nu(\mu z)]^2}{z^{2\nu}}\bigg|_{z=\sqrt{z \cdot z}=(\sqrt{1+2\beta p^2}-1)/\beta\sqrt{p^2}}$$

$$\times \exp\left[-\frac{i}{\hbar}(\boldsymbol{\xi} - \boldsymbol{\xi}') \cdot \boldsymbol{p}\frac{\sqrt{1 + 2\beta p^2} - 1}{\beta p^2}\right], \tag{A7}$$

where again the momentum integration is unbounded and the integrand admits a well-defined series expansion in powers of β. Using Eq. (21) one finds, similarly to Eq. (A4),

$$\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}') = \int \frac{d^n\boldsymbol{p}}{(2\pi\hbar)^n}\left[1 - \beta\left(\frac{n + 2 + (j_{\nu,1})^2/n}{2}p^2 - \frac{i}{2\hbar}(\boldsymbol{\xi} - \boldsymbol{\xi}') \cdot \boldsymbol{p}p^2\right) + O(\beta^2)\right]\exp\left[-\frac{i}{\hbar}(\boldsymbol{\xi} - \boldsymbol{\xi}') \cdot \boldsymbol{p}\right], \tag{A8}$$

and performing the momentum integration as before we end up with

$$\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}') = \left[1 + \frac{2 + (j_{\nu,1})^2/n}{2} (\hbar\sqrt{\beta})^2 \nabla^2_{\boldsymbol{\xi}} \right.$$
$$\left. + O(\beta^2) \right] \delta^n(\boldsymbol{\xi} - \boldsymbol{\xi}'), \qquad (A9)$$

therefore defining for the DGS procedure a value of the numerical constant κ given by $\kappa = 2 + (j_{\nu,1})^2/n$.

Direct product model (model II).—We note that while in Ref. [16] the author announces to study the model

$$[\hat{x}_i, \hat{p}_j] = i\hbar(1 + \beta\hat{\boldsymbol{p}}^2)\delta_{ij}, \qquad i = 1, \ldots, n,$$
$$\hat{\boldsymbol{p}}^2 = \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}} = \sum_i^n (\hat{p}_i)^2, \qquad (A10)$$

then he uses, as maximally localized states, the direct product of the one-dimensional states.

More precisely, the proper definition of the direct product model (model II) is [23]

$$[\hat{x}_i, \hat{p}_j] = i\hbar(1 + \beta\hat{p}_i^2)\delta_{ij}, \qquad i = 1, \ldots, n. \quad (A11)$$

In this model the closure function is given by

$$\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}') = \int \frac{d^n\boldsymbol{p}}{(2\pi\hbar)^n} \prod_{j=1}^n \frac{1}{(1 + \beta p_j^2)^2}$$
$$\times \exp\left[+\frac{i}{\hbar}(\xi_j - \xi_j')\frac{\arctan(\sqrt{\beta}p_j)}{\sqrt{\beta}} \right], \quad (A12)$$

which factorizes in the product of n, identical, integrals

$$\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}') = \prod_{j=1}^n \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \frac{1}{(1 + \beta p^2)^2}$$
$$\times \exp\left[+\frac{i}{\hbar}(\xi_j - \xi_j')\frac{\arctan(\sqrt{\beta}p)}{\sqrt{\beta}} \right] \quad (A13)$$

$$= \prod_{j=1}^n \frac{1}{\hbar\sqrt{\beta}} \frac{1}{4} \frac{\sin[x_j]}{x_j - x_j^3/\pi^2}, \qquad (A14)$$

with $x_j = \pi(\xi_j - \xi_j')/(2\hbar\sqrt{\beta})$, while in Ref. [16] it is found that

$$\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}') = \prod_{i=1}^n \frac{\sin(\frac{\pi(\xi_j - \xi_j')}{2\hbar\sqrt{\beta}})}{\pi(\xi_j - \xi_j')} = \prod_{i=1}^n \frac{1}{\hbar\sqrt{\beta}} \frac{1}{2} \frac{\sin[x_j]}{x_j}.$$

We remark that in Ref. [16], in addition to some notational inconsistencies, the closure function appears to have been calculated improperly, as the $(1 + \beta p^2)$ factor appears in the denominator only with one power, while it should be squared, as one such factor is due to the change in the momentum measure and the other is from the square of the wave functions of the maximally localized states, see Ref. [15]. By performing a series expansion in powers of β of the integrand in Eq. (A12), we end up with the following:

$$\tilde{\delta}_n(\boldsymbol{\xi} - \boldsymbol{\xi}') = [1 + \tfrac{5}{3}(\hbar\sqrt{\beta})^2\nabla^2_{\boldsymbol{\xi}} + O(\beta^2)]\delta_n(\boldsymbol{\xi} - \boldsymbol{\xi}'),$$
$$(A15)$$

which defines for this model $\kappa = 10/3$, as reported in Table I.

Let us remark that within this model, given by the direct product of the one-dimensional algebra discussed in Sec. II, the KMM procedure gives the exact maximally localized states and so there is no need to distinguish between the KMM and DGS procedures [18].

It was, however, important to check that the sign of the constant κ is the same as in model I. Thus, even within the direct product model, the Casimir-Polder correction due to the minimal length turns out to be attractive.

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