Effect of virtual pairs for relativistic bound-state problems

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We use the variational method within the Hamiltonian formalism of quantum field theory to derive relativistic two-, (four)-, and (six)-body wave equations for scalar particles interacting via a massive or massless mediating scalar field (the scalar Yukawa model). Fock-space variational trial states [2 + (4) + (6)] are used to derive the relativistic two-body system. The equations are shown to have the Schrödinger nonrelativistic limit, with Coulombic interparticle potentials in the case of a massless mediating field and Yukawa interparticle potentials in the case of a massive mediating field. The results show that the inclusion of virtual pairs has a large effect for the binding energy of the system at strong coupling. In the case of the discovery of a *scalar* Higgs particle in upcoming experiments, we may apply the present results to real, physical systems.

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I. INTRODUCTION

The study of few-body systems in relativistic quantum field theory (QFT) has played an important role in the development of modern physics. It provides a stringent test of our idea about the nature of particle interactions. For example, the spectrum of the hydrogen atom was an early test of quantum electrodynamics. The problem of describing relativistic bound states in QFT was solved in 1951 by Bethe and Salpeter [1], at least in principle. Its practical implementation, particularly for strongly coupled systems, is another matter.

In earlier works by Darewych [2], a modified variational method in the Hamiltonian formalism of QFT was proposed for deriving relativistic few-body wave equations. The variational method in Hamiltonian QFT is close to the traditional Schrödinger type of description of few-body systems and is straightforwardly generalizable to systems of more than two particles [3]. In principle, the variational approach does not depend on the magnitude of the coupling strength, in contrast to perturbation theory, which generally relies on an expansion in a small parameter. In practice, the variational approach is only as good as the trial states that are used in its implementation. The variational method has been used sparingly in QFT, perhaps due to the difficulty of constructing realistic, yet tractable, trial states. The construction of realistic yet tractable trial states, suitable at strong coupling, is a key task in the variational approach.

In a paper by Ding and Darewych [4], the variational method was used to derive relativistic two-body (particle-antiparticle) wave equations for the scalar Yukawa theory with a simple trial state $|N\bar{N}\rangle$, in which scalar particles, which we shall term "nucleons," interact through a mediating scalar field ("pions"), which may be massive or massless (as in the original Wick-Cutkosky model [5]). The results obtained in [4] were based on the simplest possible nucleon-antinucleon Fock-space trial state that can be written. Such an approximation is acceptable at weak coupling, but it is increasingly less reliable as the coupling increases.

In a recent paper by Emami-Razavi and Darewych [6], the efficiency of the variational method was demonstrated (especially at strong coupling) using a trial state comprised of two and four Fock-space states ($|N\bar{N}\rangle + |N\bar{N}N\bar{N}\rangle$). The main purpose of that paper was to study the importance of the effect of virtual pairs on the two-body energy, particularly for strong coupling. Approximate variational two-body ground-state solutions of the relativistic equations were obtained for various strengths of coupling, for both massive and massless mediating fields. A comparison of the two-body binding energies with other calculations (for example, [7] or [8]) was presented.

In this paper, we study *the effect of additional virtual pairs* on the two-body ground-state energy using a trial state comprised of two, four, and six Fock-space states ($|N\bar{N}\rangle + |N\bar{N}N\bar{N}\rangle + |N\bar{N}N\bar{N}N\bar{N}\rangle$). The novelty of the present work is to use the additional six-body $|N\bar{N}N\bar{N}N\bar{N}\rangle$ Fock-space states in the calculation of the two-body ground-state energy. We will see that it is a very challenging problem, since not only is it extremely difficult to derive the relativistic 2 + (4) + (6)-body wave equations, but also it is computationally demanding to find the two-body binding energies for different coupling constants.

The presentation of this paper is as following. We recall the model, its reformulation, the quantized theory, and the variational method in Sec. II and summarize previous results obtained using the simplest $|N\bar{N}\rangle$ trial state (and a short reminder of the $|N\bar{N}\rangle + |N\bar{N}N\bar{N}\rangle$ case [6]) in Sec. III. Section IV contains the derivation of the coupled, relativistic, momentum-space equations, using an improved trial state of the type $|N\bar{N}\rangle + |N\bar{N}N\bar{N}\rangle + |N\bar{N}N\bar{N}N\bar{N}\rangle$. These equations, along with their approximate solutions, are our principal results. The nonrelativistic limit of the equations is also considered in that section.

Approximate, variational solutions for the two-body ground-state energies and wave functions at various strengths of the coupling are presented in Sec. V, and a comparison with other calculations is presented. Discussions and concluding remarks are given in Sec. VI.

II. SCALAR YUKAWA MODEL

The Hamiltonian density in the reformulated model [2,6] is given by

$$\mathcal{H}(x) = \mathcal{H}_\phi(x) + \mathcal{H}_\chi(x) + \mathcal{H}_{I_1}(x) + \mathcal{H}_{I_2}(x), \quad (2.1)$$

where

$$\mathcal{H}_\phi(x) = p_{\phi^*}(x)p_\phi(x) + \nabla\phi^*(x)\cdot\nabla\phi(x) + m^2\phi^*(x)\phi(x), \quad (2.2)$$

$$\mathcal{H}_\chi(x) = \tfrac{1}{2}p_{\chi_0}^2 + \tfrac{1}{2}(\nabla\chi_0)^2 + \tfrac{1}{2}\mu^2\chi_0^2, \quad (2.3)$$

$p_{\phi^*} = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}^*$, $p_\phi = \partial\mathcal{L}/\partial\dot{\phi}^* = \dot{\phi}$, and $p_{\chi_0} = \partial\mathcal{L}/\partial\dot{\chi}_0 = \dot{\chi}_0$ are conjugate momenta,

$$\mathcal{H}_{I_1}(x) = g\phi^*(x)\phi(x)\chi_0(x), \quad (2.4)$$

$$\mathcal{H}_{I_2}(x) = -\frac{g^2}{2}\int dx'\phi^*(x)\phi(x)D(x-x')\phi^*(x')\phi(x'), \quad (2.5)$$

where $dx = d^N x dt$ in N spatial plus time dimensions, and

$$D(x-x') = \int\frac{dk}{(2\pi)^{N+1}}e^{-ik\cdot(x-x')}\frac{1}{\mu^2-k^2+i\epsilon}, \quad (2.6)$$

where $dk = d^{N+1}k$ and $k^2 = k^\nu k_\nu$.

Scalar mass m nucleons are interacting via a mediating real scalar (pion) field. $\phi(x)$ is the field corresponding to the nucleon (or antinucleon). The mediating field $\chi(x)$ can be massless (i.e., $\mu = 0$) or massive ($\mu \neq 0$). $\chi_0(x)$ is a solution of the free field equation, while $D(x-x')$ is a covariant Green function. g is the coupling constant, and we use the unit $\hbar = c = 1$.

We construct a quantum field theory, based on the reformulated Hamiltonian (2.1). Our notation is

$$\phi(x) = \int d^N q\frac{1}{[(2\pi)^N 2\omega_q]^{1/2}}[A(\mathbf{q})e^{-iq\cdot x} + B^\dagger(\mathbf{q})e^{iq\cdot x}], \quad (2.7)$$

$$\chi_0(x) = \int d^N p\frac{1}{[(2\pi)^N 2\Omega_p]^{1/2}}[d(\mathbf{p})e^{-ip\cdot x} + d^\dagger(\mathbf{p})e^{ip\cdot x}], \quad (2.8)$$

where $\omega_q = (\mathbf{q}^2+m^2)^{1/2}$, $\Omega_p = (\mathbf{p}^2+\mu^2)^{1/2}$, $q\cdot x = q^\nu x_\nu$, and $q^\nu = (q^0 = \omega_q, \mathbf{q})$.

The momentum-space operators $A^\dagger$, A, $B^\dagger$, B describe the creation ($A^\dagger$, $B^\dagger$) and annihilation (A, B) of free nucleons and antinucleons, respectively, while d, $d^\dagger$ describe the annihilation and creation of the free mediating-field pions. They satisfy the usual commutation relations; the nonvanishing ones are

$$[A(\mathbf{p}), A^\dagger(\mathbf{q})] = [B(\mathbf{p}), B^\dagger(\mathbf{q})] = \delta^N(\mathbf{p}-\mathbf{q}), \quad (2.9)$$

$$[d(\mathbf{p}), d^\dagger(\mathbf{q})] = \delta^N(\mathbf{p}-\mathbf{q}). \quad (2.10)$$

We normal order the Hamiltonian (now denoted by $:\hat{H}:$), since we are not concerned with vacuum-energy questions in this work. The vacuum state $|0\rangle$ is defined by $A_\mathbf{p}|0\rangle = B_\mathbf{p}|0\rangle = d_\mathbf{k}|0\rangle = 0$.

In the Hamiltonian formalism of QFT, we seek solutions of the equation

$$\hat{P}^\beta|\Psi\rangle = Q^\beta|\Psi\rangle, \quad (2.11)$$

where $\hat{P}^\beta = (\hat{H}, \hat{\mathbf{P}})$ is the energy-momentum operator of the QFT, and $Q^\beta = (E, \mathbf{Q})$ is the energy-momentum eigenvalue. Since the $\beta = 0$ component of Eq. (2.11) is generally impossible to solve, one needs to resort to approximation schemes. The variational approximation corresponds to finding approximate solutions to (2.11) by using the variational principle

$$\delta\langle\Psi_{\rm trial}|\hat{H} - E|\Psi_{\rm trial}\rangle_{t=0} = 0, \quad (2.12)$$

where $|\Psi_{\rm trial}\rangle$ is a suitably chosen trial state containing adjustable features (parameters, functions).

A slightly simpler model, in which ϕ is real, is often considered. In that case, there are only particles and no antiparticles. These models (ϕ complex, i.e., two-component field, or ϕ real) are closely related, since the forces among particles (and/or antiparticles) are only attractive (i.e., in the scalar Yukawa theory, they are gravity-like rather than electromagneticlike). The virtual-annihilation interaction (which arises in the particle-antiparticle case) is a contactlike interaction that has only a small effect on the total energy, as is pointed out in [4].

III. SIMPLE VARIATIONAL TWO-PARTICLE TRIAL STATE AND 2 + (4) WAVE EQUATIONS

We shall quote some results from the earlier work of Ding and Darewych [4] to make it easier to follow the present paper. The simplest nucleon-antinucleon trial state that can be chosen is

$$|\psi_2\rangle = \int d^N p_1 d^N p_2 F(\mathbf{p}_1, \mathbf{p}_2)A^\dagger(\mathbf{p}_1)B^\dagger(\mathbf{p}_2)|0\rangle, \quad (3.1)$$

where F is a normalizable, adjustable function, i.e., such that $\langle\psi_2|\psi_2\rangle = \int d^N p_1 d^N p_2|F(\mathbf{p}_1, \mathbf{p}_2)|^2$ is finite (it will be taken to be unity). The matrix elements needed to implement the variational principle (2.12) are

$$\langle\psi_2 | :\hat{H}_\phi - E: | \psi_2\rangle = \int d^N p_1 d^N p_2 F^*(\mathbf{p}_1, \mathbf{p}_2) F(\mathbf{p}_1, \mathbf{p}_2)[\omega_{p_1} + \omega_{p_2} - E], \tag{3.2}$$

and

$$\begin{aligned}\langle\psi_2 | :\hat{H}_I: | \psi_2\rangle_{t=0} &= \langle\psi_2 | :\hat{H}_{I_2}: | \psi_2\rangle_{t=0} = -\int d^N p_1 d^N p_2 d^N p_1' d^N p_2' F^*(\mathbf{p}_1', \mathbf{p}_2') F(\mathbf{p}_1, \mathbf{p}_2)\mathcal{K}_{2,2} \\ &= -\frac{g^2}{8(2\pi)^N}\int d^N p_1 d^N p_2 d^N p_1' d^N p_2' F^*(\mathbf{p}_1', \mathbf{p}_2') F(\mathbf{p}_1, \mathbf{p}_2)\delta^N(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2')\frac{1}{\sqrt{\omega_{p_1}\omega_{p_2}\omega_{p_1'}\omega_{p_2'}}} \\ &\quad\times\left[\frac{1}{\mu^2 - (p_1' - p_1)^2} + \frac{1}{\mu^2 - (p_2' - p_2)^2} + \frac{1}{\mu^2 - (p_1 + p_2)^2} + \frac{1}{\mu^2 - (p_1' + p_2')^2}\right],\end{aligned} \tag{3.3}$$

where we call $\mathcal{K}_{2,2}(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_1, \mathbf{p}_2)$ the kernel in Eq. (3.3), and we will use it in Sec. IV. $\mathcal{K}_{2,2}$ is the following expression:

$$\begin{aligned}\mathcal{K}_{2,2} &= \frac{g^2}{8(2\pi)^N}\delta^N(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2) \\ &\quad\times\frac{1}{\sqrt{\omega_{p_1}\omega_{p_2}\omega_{p_1'}\omega_{p_2'}}}\left[\frac{1}{\mu^2 - (p_1' - p_1)^2}\right. \\ &\quad+ \frac{1}{\mu^2 - (p_2' - p_2)^2} + \frac{1}{\mu^2 - (p_1 + p_2)^2} \\ &\quad\left.+ \frac{1}{\mu^2 - (p_1' + p_2')^2}\right].\end{aligned} \tag{3.4}$$

We see that in Eq. (3.3) the kernel (momentum-space potential) contains all tree-level Feynman diagrams, namely, one-pion exchange [the first two terms inside the brackets under the integral of (3.3)] and virtual annihilation (the last two terms inside these brackets).

In the rest frame $\mathbf{Q} = 0$ and $F(\mathbf{p}_1, \mathbf{p}_2) = f(\mathbf{p}_1)\delta^N(\mathbf{p}_1 + \mathbf{p}_2)$, the relativistic momentum-space particle-antiparticle wave equation is

$$\begin{aligned}[2\omega_p - E]f(\mathbf{p}) &= \frac{g^2}{8(2\pi)^N}\int d^N p' f(\mathbf{p}')\frac{1}{\omega_p\omega_{p'}} \\ &\quad\times\left[\frac{2}{\mu^2 + (\mathbf{p}' - \mathbf{p})^2 - (\omega_p - \omega_{p'})^2}\right. \\ &\quad\left.- \frac{1}{4\omega_p^2 - \mu^2} - \frac{1}{4\omega_{p'}^2 - \mu^2}\right].\end{aligned} \tag{3.5}$$

In the nonrelativistic limit $|\mathbf{p}|^2 \ll m^2$, Eq. (3.5) becomes

$$\begin{aligned}\left[\frac{\mathbf{p}^2}{m} - \epsilon\right]f(\mathbf{p}) &= \frac{g^2}{4(2\pi)^N m^2}\int d^N p' f(\mathbf{p}') \\ &\quad\times\left[\frac{1}{\mu^2 + (\mathbf{p}' - \mathbf{p})^2} - \frac{1}{4m^2 - \mu^2}\right],\end{aligned} \tag{3.6}$$

where $\epsilon = E - 2m$. In coordinate space, the Fourier transform of Eq. (3.6) is just the usual time-independent Schrödinger equation for the relative motion of the particle-antiparticle system:

$$-\frac{\hbar^2}{m}\nabla^2\psi(\mathbf{r}) + V(r)\psi(\mathbf{r}) = \epsilon\psi(\mathbf{r}), \tag{3.7}$$

where the potential is

$$V(r) = -\alpha\frac{e^{-\mu r}}{r} + \frac{4\pi\alpha}{4m^2 - \mu^2}\delta^3(\mathbf{r}), \tag{3.8}$$

and $\alpha = g^2/16\pi m^2$ is the effective dimensionless coupling constant. This potential is a sum of an attractive Yukawa potential, due to one-pion exchange, and a repulsive (if $\mu < 2m$) contact potential, due to virtual annihilation.

The simplest two-body trial state (3.1) considered above is a reasonable approximation for the case of weak coupling ($\alpha \ll 1$), but it becomes increasingly inadequate as α increases. More flexible and elaborate trial states would include terms that account for two (and more) pion exchange effects and additional (virtual) nucleon-antinucleon pair effects.

Hwang and Karmanov [9] have recently calculated the two-body rest mass in the Wick-Cutkosky model using light-front dynamics, in which they study the contributions of Fock sectors with increasing numbers of exchanged quanta. Their results show that the two-body ground-state binding energy increases with the inclusion of multipion exchanges but not dramatically.

In a recent paper [6], the effects on the binding energy (particularly at strong coupling) of using trial states that allow for additional (virtual) particle-antiparticle pairs were examined by Emami-Razavi and Darewych, namely,

$$|\psi_t\rangle = |\psi_2\rangle + |\psi_4\rangle, \tag{3.9}$$

where $|\psi_2\rangle$ is the trial state (3.1), while

$$\begin{aligned}|\psi_4\rangle &= |A^\dagger B^\dagger A^\dagger B^\dagger\rangle \\ &= \int d^N p_1 d^N p_2 d^N p_3 d^N p_4 G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) \\ &\quad\times A^\dagger(\mathbf{p}_1)B^\dagger(\mathbf{p}_2)A^\dagger(\mathbf{p}_3)B^\dagger(\mathbf{p}_4)|0\rangle,\end{aligned} \tag{3.10}$$

and $G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ is another adjustable function.

For $F = 0$, we see that $|\psi_t\rangle = |\psi_4\rangle$ is the simplest possible trial state suitable for describing a four-body bound system of two particles and two antiparticles ("quadronium") of rest energy (mass) $E \leq 4m$, as was considered in Ref. [3]. However, for F, G both nonzero, the trial state (3.9) is suitable for describing two-body bound states of rest energy (mass) $E \leq 2m$, with the accommodation of a virtual particle-antiparticle Fock state. [For $E \leq 4m$, the trial state (3.9) is suitable for describing the bound states of quadronium with an allowance for virtual pair annihilation.] Approximate numerical solutions obtained with the 2 + (4) trial states (3.9) are presented in Ref. [6]. The results show that the two-body ground-state binding energy increases dramatically with the inclusion of virtual pairs [especially at strong coupling compared to the case of simple trial state (3.1)].

IV. 2 + (4) + (6) WAVE EQUATIONS

In this paper, our focus shall be to examine the effects on the two-body binding energy (particularly at strong coupling) of using trial states that allow for additional (virtual) particle-antiparticle pairs. Therefore, we shall use a trial state that contains two-pair (i.e., four-body) and three-pair (i.e., six-body) Fock-state components, namely,

$$|\psi_t\rangle = |\psi_2\rangle + |\psi_4\rangle + |\psi_6\rangle, \tag{4.1}$$

where $|\psi_2\rangle$ is the simple trial state (3.1), while $|\psi_4\rangle$ is that given in Eq. (3.10) and

$$|\psi_6\rangle = |A^\dagger B^\dagger A^\dagger B^\dagger A^\dagger B^\dagger\rangle = \int d^N p_1 d^N p_2 \ldots d^N p_6 S(\mathbf{p}_1, \ldots, \mathbf{p}_6) A^\dagger(\mathbf{p}_1) B^\dagger(\mathbf{p}_2) \times A^\dagger(\mathbf{p}_3) B^\dagger(\mathbf{p}_4) A^\dagger(\mathbf{p}_5) B^\dagger(\mathbf{p}_6)|0\rangle, \tag{4.2}$$

where $S(\mathbf{p}_1, \ldots, \mathbf{p}_6)$ is another adjustable function.

For $F = 0$ and $S = 0$, we see that $|\psi_t\rangle = |\psi_4\rangle$ is the simplest possible trial state suitable for describing a four-body bound system of two particles and two antiparticles (quadronium) of rest energy (mass) $E \leq 4m$, as was considered by Emami-Razavi and Darewych [3]. Moreover, for $G = 0$ and $S = 0$, we see that $|\psi_t\rangle = |\psi_2\rangle$ is the simplest possible trial state suitable for describing a two-body bound system as was considered in ref. [4]. However, for F, G, and S all nonzero, the trial state (4.1) is suitable for describing two-body bound states of rest energy (mass) $E \leq 2m$, with the accommodation of two and three virtual particle-antiparticle pairs in the trial state.

The matrix elements

$$\langle\psi_i|{:}\hat{H}{:}|\psi_j\rangle, \qquad i, j = 2, 4, 6, \tag{4.3}$$

needed to implement the variational principle (2.12) are given in the appendix. Note that for the present choice of our trial state we have

$$\langle\psi_i|{:}\hat{H}_{I_1}{:}|\psi_j\rangle = 0 \quad \text{and} \quad \langle\psi_i|{:}\hat{H}_\chi{:}|\psi_j\rangle = 0. \tag{4.4}$$

This means that the trial state (4.1) is incapable of describing a process involving the emission or absorption of physical (as opposed to virtual) pions.

For arbitrary variations of the functions F, G, and S, the variational principle (2.12), taking time $t = 0$, leads to the following coupled integral equations for the wave-function coefficients F, G, and S:

$$F(\mathbf{p}_1, \mathbf{p}_2)[\omega_{p_1} + \omega_{p_2} - E] = \int d^N p_1' d^N p_2' F(\mathbf{p}_1', \mathbf{p}_2') \mathcal{K}_{2,2}(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_1, \mathbf{p}_2) + \int d^N p_1' d^N p_2' d^N p_3' d^N p_4' G(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_3', \mathbf{p}_4') \mathcal{K}_{2,4}(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_3', \mathbf{p}_4', \mathbf{p}_1, \mathbf{p}_2) + \int d^N p_1' d^N p_2' \ldots d^N p_6' S(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_3', \mathbf{p}_4', \mathbf{p}_5', \mathbf{p}_6') \mathcal{K}_{2,6}(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_3', \mathbf{p}_4', \mathbf{p}_5', \mathbf{p}_6', \mathbf{p}_1, \mathbf{p}_2), \tag{4.5}$$

$$G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)[\omega_{p_1} + \omega_{p_2} + \omega_{p_3} + \omega_{p_4} - E] = \int d^N p_1' d^N p_2' F(\mathbf{p}_1', \mathbf{p}_2') \mathcal{K}_{4,2}(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) + \int d^N p_1' d^N p_2' d^N p_3' d^N p_4' G(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_3', \mathbf{p}_4') \times \mathcal{K}_{4,4}(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_3', \mathbf{p}_4', \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) + \int d^N p_1' d^N p_2' \ldots d^N p_6' S(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_3', \mathbf{p}_4', \mathbf{p}_5', \mathbf{p}_6') \times \mathcal{K}_{4,6}(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_3', \mathbf{p}_4', \mathbf{p}_5', \mathbf{p}_6', \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4), \tag{4.6}$$

and

$$S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6)[\omega_{p_1}+\omega_{p_2}+\omega_{p_3}+\omega_{p_4}+\omega_{p_5}+\omega_{p_6}-E]$$
$$=\int d^Np'_1d^Np'_2F(\mathbf{p}'_1,\mathbf{p}'_2)\mathcal{K}_{6,2}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6)$$
$$+\int d^Np'_1d^Np'_2d^Np'_3d^Np'_4G(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4)\mathcal{K}_{6,4}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6)$$
$$+\int d^Np'_1d^Np'_2\ldots d^Np'_6S(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}'_5,\mathbf{p}'_6)\mathcal{K}_{6,6}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}'_5,\mathbf{p}'_6,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6), \quad (4.7)$$

respectively, where the kernels $\mathcal{K}_{2,2}$, $\mathcal{K}_{2,4}$, $\mathcal{K}_{2,6}$, $\mathcal{K}_{4,4}$, $\mathcal{K}_{4,2}$, $\mathcal{K}_{6,4}$, $\mathcal{K}_{6,2}$, $\mathcal{K}_{4,6}$, and $\mathcal{K}_{6,6}$ are given in the appendix.

It is instructive to consider the nonrelativistic limit ($\mathbf{p}^2/m^2 \ll 1$) of Eqs. (4.5), (4.6), and (4.7). These are, respectively,

$$F(\mathbf{p}_1,\mathbf{p}_2)\left[\frac{\mathbf{p}_1^2+\mathbf{p}_2^2}{2m}-\epsilon_2\right]=\int d^Np'_1d^Np'_2F(\mathbf{p}'_1,\mathbf{p}'_2)\tilde{\mathcal{K}}_{2,2}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}_1,\mathbf{p}_2)$$
$$+\int d^Np'_1d^Np'_2d^Np'_3d^Np'_4G(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4)\tilde{\mathcal{K}}_{2,4}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}_1,\mathbf{p}_2)$$
$$+\int d^Np'_1d^Np'_2\ldots d^Np'_6S(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}'_5,\mathbf{p}'_6)\tilde{\mathcal{K}}_{2,6}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}'_5,\mathbf{p}'_6,\mathbf{p}_1,\mathbf{p}_2), \quad (4.8)$$

$$G(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\left[\frac{\mathbf{p}_1^2+\mathbf{p}_2^2+\mathbf{p}_3^2+\mathbf{p}_4^2}{2m}-\epsilon_4\right]$$
$$=\int d^Np'_1d^Np'_2F(\mathbf{p}'_1,\mathbf{p}'_2)\tilde{\mathcal{K}}_{4,2}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)$$
$$+\int d^Np'_1d^Np'_2d^Np'_3d^Np'_4G(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4)\tilde{\mathcal{K}}_{4,4}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)$$
$$+\int d^Np'_1d^Np'_2\ldots d^Np'_6S(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}'_5,\mathbf{p}'_6)\tilde{\mathcal{K}}_{4,6}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}'_5,\mathbf{p}'_6,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4), \quad (4.9)$$

and

$$S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6)\left[\frac{\mathbf{p}_1^2+\mathbf{p}_2^2+\mathbf{p}_3^2+\mathbf{p}_4^2+\mathbf{p}_5^2+\mathbf{p}_6^2}{2m}-\epsilon_6\right]$$
$$=\int d^Np'_1d^Np'_2F(\mathbf{p}'_1,\mathbf{p}'_2)\tilde{\mathcal{K}}_{6,2}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6)$$
$$+\int d^Np'_1d^Np'_2d^Np'_3d^Np'_4G(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4)\tilde{\mathcal{K}}_{6,4}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6)$$
$$+\int d^Np'_1d^Np'_2\ldots d^Np'_6S(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}'_5,\mathbf{p}'_6)\tilde{\mathcal{K}}_{6,6}(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3,\mathbf{p}'_4,\mathbf{p}'_5,\mathbf{p}'_6,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6), \quad (4.10)$$

where $\epsilon_2 = E - 2m$, $\epsilon_4 = E - 4m$, and $\epsilon_6 = E - 6m$. The nonrelativistic kernels are denoted by $\tilde{\mathcal{K}}_{2,2}$, $\tilde{\mathcal{K}}_{2,4}$, $\tilde{\mathcal{K}}_{2,6}$, $\tilde{\mathcal{K}}_{4,4}$, $\tilde{\mathcal{K}}_{4,2}$, $\tilde{\mathcal{K}}_{6,4}$, $\tilde{\mathcal{K}}_{6,2}$, $\tilde{\mathcal{K}}_{4,6}$, and $\tilde{\mathcal{K}}_{6,6}$. The (more familiar) coordinate-space representation of (4.8), (4.9), and (4.10) in $N = 3$ spatial dimensions is

$$-\frac{1}{2m}\left(\sum_{i=1}^{2}\nabla_i^2\right)\Psi(\mathbf{r}_1,\mathbf{r}_2)+(V_{2,2}(\mathbf{r}_1,\mathbf{r}_2)-\epsilon_2)\Psi(\mathbf{r}_1,\mathbf{r}_2)=-\int d^3r_3d^3r_4\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)V_{2,4}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$$
$$-\int d^3r_3d^3r_4d^3r_5d^3r_6\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\mathbf{r}_5,\mathbf{r}_6)V_{2,6}(\mathbf{r}_3,\mathbf{r}_4,\mathbf{r}_5,\mathbf{r}_6), \quad (4.11)$$

$$-\frac{1}{2m}\left(\sum_{i=1}^{4}\nabla_i^2\right)\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)+(V_{4,4}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)-\epsilon_4)\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$$
$$=-\Psi(\mathbf{r}_1,\mathbf{r}_2)V_{4,2}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)-\int d^3r_5d^3r_6\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\mathbf{r}_5,\mathbf{r}_6)V_{4,6}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\mathbf{r}_5,\mathbf{r}_6), \quad (4.12)$$

and

$$-\frac{1}{2m}\left(\sum_{i=1}^{6}\nabla_i^2\right)\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\mathbf{r}_5,\mathbf{r}_6)+(V_{6,6}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\mathbf{r}_5,\mathbf{r}_6)-\epsilon_6)\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\mathbf{r}_5,\mathbf{r}_6)$$
$$=-\Psi(\mathbf{r}_1,\mathbf{r}_2)V_{6,2}(\mathbf{r}_3,\mathbf{r}_4,\mathbf{r}_5,\mathbf{r}_6)-\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)V_{6,4}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\mathbf{r}_5,\mathbf{r}_6), \tag{4.13}$$

where $\Psi(\mathbf{r}_1,\mathbf{r}_2)$, $\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$, and $\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\mathbf{r}_5,\mathbf{r}_6)$ are Fourier transforms of $F(\mathbf{p}_1,\mathbf{p}_2)$, $G(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)$, and $S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6)$, respectively. We denote the diagonal and off-diagonal potentials by $V_{i,j}$ ($i,j=2,4,6$). Explicitly, in $N=3$ spatial dimensions, they are the following:

$$V_{2,2}=-\alpha\frac{e^{-\mu|r_1-r_2|}}{|\mathbf{r}_1-\mathbf{r}_2|}+\alpha\frac{4\pi}{4m^2-\mu^2}\delta^3(\mathbf{r}_1-\mathbf{r}_2), \tag{4.14}$$

$$\begin{aligned}V_{2,4}=V_{4,2}&=-\alpha\frac{e^{-\mu|r_1-r_3|}}{|\mathbf{r}_1-\mathbf{r}_3|}\delta^3(\mathbf{r}_3-\mathbf{r}_4)-\alpha\frac{e^{-\mu|r_2-r_3|}}{|\mathbf{r}_2-\mathbf{r}_3|}\delta^3(\mathbf{r}_3-\mathbf{r}_4)\\&\quad+\alpha\frac{4\pi}{4m^2-\mu^2}\delta^3(\mathbf{r}_1-\mathbf{r}_3)\delta^3(\mathbf{r}_1-\mathbf{r}_4)\\&\quad+\alpha\frac{4\pi}{4m^2-\mu^2}\delta^3(\mathbf{r}_2-\mathbf{r}_3)\delta^3(\mathbf{r}_2-\mathbf{r}_4),\end{aligned} \tag{4.15}$$

$$\begin{aligned}V_{2,6}=V_{6,2}&=3\times\alpha\frac{4\pi}{4m^2-\mu^2}\delta^3(\mathbf{r}_3-\mathbf{r}_4)\delta^3(\mathbf{r}_3-\mathbf{r}_5)\\&\quad\times\delta^3(\mathbf{r}_3-\mathbf{r}_6),\end{aligned} \tag{4.16}$$

$$\begin{aligned}V_{4,4}&=-\alpha\sum_{j=1}^{3}\sum_{k=j+1}^{4}\frac{e^{-\mu|r_j-r_k|}}{|\mathbf{r}_j-\mathbf{r}_k|}\\&\quad+\sum_{j=1}^{3}\sum_{k=j+1}^{4}{}'\,\alpha\frac{4\pi}{4m^2-\mu^2}\delta(\mathbf{r}_j-\mathbf{r}_k)\quad\text{for } j<k,\end{aligned} \tag{4.17}$$

where $\sum'_{k=a}u_k$ means $u_a+u_{a+2}+u_{a+4}+\cdots$.

This means that $V_{4,4}$ is composed of two types of terms. The first ones are corresponding to pairwise Yukawa interactions among particles and antiparticles, and the second terms correspond to repulsive virtual-annihilation interactions among odd-even or even-odd indices of j and k among particles and antiparticles.

$$\begin{aligned}V_{4,6}=V_{6,4}&=3\alpha\frac{4\pi}{4m^2-\mu^2}\delta^3(\mathbf{r}_3-\mathbf{r}_5)\delta^3(\mathbf{r}_3-\mathbf{r}_6)\\&\quad+3\alpha\frac{4\pi}{4m^2-\mu^2}\delta^3(\mathbf{r}_4-\mathbf{r}_5)\delta^3(\mathbf{r}_4-\mathbf{r}_6)\\&\quad-3\alpha\frac{e^{-\mu|r_3-r_5|}}{|\mathbf{r}_3-\mathbf{r}_5|}\delta^3(\mathbf{r}_5-\mathbf{r}_6)\\&\quad-3\alpha\frac{e^{-\mu|r_4-r_5|}}{|\mathbf{r}_4-\mathbf{r}_5|}\delta^3(\mathbf{r}_5-\mathbf{r}_6),\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}V_{6,6}&=-\alpha\sum_{j=1}^{5}\sum_{k=j+1}^{6}\frac{e^{-\mu|r_j-r_k|}}{|\mathbf{r}_j-\mathbf{r}_k|}\\&\quad+\sum_{j=1}^{5}\sum_{k=j+1}^{6}{}'\,\alpha\frac{4\pi}{4m^2-\mu^2}\delta(\mathbf{r}_j-\mathbf{r}_k)\quad\text{for } j<k,\end{aligned} \tag{4.19}$$

where $\sum'_{k=a}u_k$ means $u_a+u_{a+2}+u_{a+4}+\cdots$.

Once again, this means that $V_{6,6}$ is composed of two types of terms. Fifteen terms are corresponding to pairwise Yukawa interactions among particles and antiparticles, and nine terms correspond to repulsive virtual-annihilation interactions among odd-even or even-odd indices of j and k among particles and antiparticles.

We recall that $\alpha=g^2/16\pi m^2$ is the effective dimensionless coupling constant.

Note that all of the potentials $V_{i,j}$ are superpositions of Yukawa potentials, corresponding to one-meson exchange, and contact (delta-function) potentials, corresponding to virtual nucleon-antinucleon annihilation. Of course, if $V_{2,4}=V_{4,2}=V_{2,6}=V_{6,2}=V_{4,6}=V_{6,4}=0$, Eqs. (4.11), (4.12), and (4.13) reduce to uncoupled two-, four-, and six-particle Schrödinger equations, respectively.

We note that Eq. (4.11) can be rewritten as

$$\begin{aligned}-\frac{1}{2m}\left(\sum_{i=1}^{2}\nabla_i^2\right)\Psi(\mathbf{r}_1,\mathbf{r}_2)+(V_{2,2}(\mathbf{r}_1,\mathbf{r}_2)-\epsilon_2)\Psi(\mathbf{r}_1,\mathbf{r}_2)&=\alpha\int d^3r_3\frac{e^{-\mu|r_1-r_3|}}{|\mathbf{r}_1-\mathbf{r}_3|}\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_3)\\&\quad+\alpha\int d^3r_3\frac{e^{-\mu|r_2-r_3|}}{|\mathbf{r}_2-\mathbf{r}_3|}\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_3)\\&\quad-\alpha\frac{4\pi}{4m^2-\mu^2}\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_1,\mathbf{r}_1)-\alpha\frac{4\pi}{4m^2-\mu^2}\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_2,\mathbf{r}_2)\\&\quad-3\alpha\frac{4\pi}{4m^2-\mu^2}\int d^3r_3\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_3,\mathbf{r}_3,\mathbf{r}_3).\end{aligned} \tag{4.20}$$

We see that it has the form of an inhomogeneous two-body Schrödinger equation, where the inhomogeneous term on the right-hand side (which would ordinarily be zero in a two-body Schrödinger equation) is provided by the four-body and six-body wave functions and the $V_{2,4}$ and $V_{2,6}$ potentials.

It is not possible to obtain exact, analytic solutions of the coupled wave equations, even in the nonrelativistic limit. Therefore, we shall consider some approximate, variational solutions.

V. APPROXIMATE, VARIATIONAL SOLUTIONS FOR THE RELATIVISTIC PARTICLE-ANTIPARTICLE GROUND STATE

Approximate, variational solutions of the coupled relativistic equations (4.5), (4.6), and (4.7) can be obtained by choosing explicit analytic forms with adjustable parameters for the functions F, G, and S, that is, taking the trial state to be of the form

$$|\psi_t\rangle = a_2|\psi_{t_2}\rangle + a_4|\psi_{t_4}\rangle + a_6|\psi_{t_6}\rangle, \tag{5.1}$$

where a_2, a_4, and a_6 are linear parameters, and $|\psi_{t_2}\rangle$, $|\psi_{t_4}\rangle$, $|\psi_{t_6}\rangle$ are ψ_2 [Eq. (3.1)], ψ_4 [Eq. (3.10)], and ψ_6 [Eq. (4.2)], respectively, but with F, G, and S being explicit analytic forms containing additional adjustable parameters.

Optimization of the linear parameters a_2, a_4, and a_6 corresponds to diagonalizing the matrix

$$H_t = \begin{bmatrix} \langle\psi_{t_2}|{:}\hat{H}{:}|\psi_{t_2}\rangle & \langle\psi_{t_2}|{:}\hat{H}{:}|\psi_{t_4}\rangle & \langle\psi_{t_2}|{:}\hat{H}{:}|\psi_{t_6}\rangle \\ \langle\psi_{t_4}|{:}\hat{H}{:}|\psi_{t_2}\rangle & \langle\psi_{t_4}|{:}\hat{H}{:}|\psi_{t_4}\rangle & \langle\psi_{t_4}|{:}\hat{H}{:}|\psi_{t_6}\rangle \\ \langle\psi_{t_6}|{:}\hat{H}{:}|\psi_{t_2}\rangle & \langle\psi_{t_6}|{:}\hat{H}{:}|\psi_{t_4}\rangle & \langle\psi_{t_6}|{:}\hat{H}{:}|\psi_{t_6}\rangle \end{bmatrix}, \tag{5.2}$$

where $\langle\psi_{ti}|\psi_{ti}\rangle = 1$ $(i = 2, 4, 6)$ are taken, and the expressions for the matrix elements are those given in the appendix, evaluated at time $t = 0$.

For the ground state of the two(plus four and plus six)-body system, in the rest frame, we choose the simple forms

$$F(\mathbf{p}_1, \mathbf{p}_2) = f(\mathbf{p}_1)\delta^N(\mathbf{p}_1 + \mathbf{p}_2), \tag{5.3}$$

$$G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = f(\mathbf{p}_1)f(\mathbf{p}_2)f(\mathbf{p}_3)\delta^N(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4), \tag{5.4}$$

and

$$S(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6) = f(\mathbf{p}_1)f(\mathbf{p}_2)\ldots f(\mathbf{p}_5)\delta^N(\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_6), \tag{5.5}$$

where

$$f(\mathbf{p}_i) = \frac{1}{(\mathbf{p}_i^2 + p_0^2)^\nu}. \tag{5.6}$$

We note that there are two adjustable parameters p_0 and ν in our choice of F, G, and S; hence, the three eigenvalues E_{t_1}, E_{t_2}, and E_{t_3} (and corresponding eigenvectors) are functions of these parameters. We keep ν fixed at the hydrogenic ground-state value ($\nu = 2$). The occurrence of minimum of $\langle H\rangle$ at values of ν larger than 1.5 avoids the domain $(3/4, 1)$ in which the momentum-space integral equations become singular, so fixing ν at 2 also avoids this problem. For a given value of the coupling constant, the best approximation to the two-body ground-state energy (rest mass) for the chosen forms of F, G, and S corresponds to those values of the parameters for which E_{t_1} [the smallest of the three eigenvalues of (5.2)] is a minimum.

The computation of $E = E_{t_1}$ requires the evaluation of troublesome multidimensional integrals. We worked them out by using the "Monte Carlo" method [10]. The technique involves a combination of importance sampling and the adaptive Monte Carlo code. We present the nucleon-antinucleon ground-state energy E (rest mass of the bound system) for various values of the coupling constant α, for both the massless- ($\mu = 0$) and the massive- ($\mu/m = 0.15$) exchange cases.

Numerical results are given in Tables I and II. Energies E_n and p_0 are given in units of m (i.e., $m = 1$). Note that the error estimates in the Monte Carlo results grow significantly with increasing n (number of particles) and α (the coupling constant) of the scalar field theory under study. We quote the following examples. For the massless-exchange case ($\mu/m = 0$), the error increases from about 4% for $\alpha \sim 0.5$ to about 10% at $\alpha = 0.9$. For

TABLE I. Nucleon-antinucleon ground-state energy and wave-function parameters for various α, massless-exchange ($\mu = 0$) case.

α	E, $\lvert\psi_2\rangle$ (p_0) (Ref. [4])	Eq. (5.7)	E, $\lvert\psi_2\rangle + \lvert\psi_4\rangle$ (p_0) (Ref. [6])	a_4/a_2 (Ref. [6])	E, $\lvert\psi_2\rangle + \lvert\psi_4\rangle + \lvert\psi_6\rangle$ (p_0) (present results)	a_4/a_2 (present results)	a_6/a_2 (present results)
0.1	1.997 ± 0.002 (0.049)	1.997	1.995 ± 0.005 (0.070)	10^{-8}	1.993 ± 0.007 (0.085)	$\sim 10^{-8}$	$\sim 10^{-11}$
0.2	1.990 ± 0.003 (0.094)	1.989	1.981 ± 0.015 (0.150)	10^{-6}	1.973 ± 0.025 (0.155)	$\sim 10^{-6}$	$\sim 10^{-9}$
0.3	1.979 ± 0.003 (0.133)	1.977	1.962 ± 0.030 (0.215)	7×10^{-5}	1.947 ± 0.040 (0.220)	$\sim 7 \times 10^{-5}$	$\sim 10^{-7}$
0.4	1.964 ± 0.003 (0.166)	1.958	1.910 ± 0.035 (0.255)	0.000 15	1.865 ± 0.050 (0.260)	0.000 15 ± (0.000 02)	$\sim 10^{-6}$
0.5	1.947 ± 0.003 (0.196)	1.932	1.864 ± 0.035 (0.325)	0.001 05	1.790 ± 0.070 (0.335)	0.001 10 ± (0.000 15)	$\sim 10^{-5}$
0.6	1.927 ± 0.003 (0.222)	1.897	1.773 ± 0.035 (0.465)	0.008 50	1.695 ± 0.080 (0.475)	0.008 80 ± (0.000 70)	0.000 25 ± (0.000 02)
0.7	1.904 ± 0.004 (0.246)	1.852	1.704 ± 0.040 (0.535)	0.021 45	1.595 ± 0.100 (0.535)	0.021 60 ± (0.001 75)	0.002 05 ± (0.000 18)
0.8	1.880 ± 0.004 (0.267)	1.789	1.620 ± 0.038 (0.605)	0.037 50	1.480 ± 0.120 (0.610)	0.037 70 ± (0.002 90)	0.011 25 ± (0.001 10)
0.9	1.854 ± 0.005 (0.287)	1.695	1.524 ± 0.040 (0.640)	0.055 80	1.360 ± 0.135 (0.650)	0.056 10 ± (0.004 25)	0.022 30 ± (0.002 20)
1	1.827 ± 0.005 (0.305)	1.414	1.381 ± 0.040 (0.680)	0.072 65			

TABLE II. Nucleon-antinucleon ground-state energy and wave-function parameters for various α, massive-exchange ($\mu = 0.15$ m) case.

α	E, $\vert\psi_2\rangle$ (p_0) (Ref. [4])	E, $\vert\psi_2\rangle + \vert\psi_4\rangle$ (p_0) (Ref. [6])	a_4/a_2 (Ref. [6])	E, $\vert\psi_2\rangle + \vert\psi_4\rangle + \vert\psi_6\rangle$ (p_0) (present results)	a_4/a_2 (present results)	a_6/a_2 (present results)
0.3	1.999 ± 0.001 (0.010)	1.997 ± 0.003 (0.070)	10^{-7}	1.995 ± 0.005 (0.080)	$\sim 10^{-7}$	$\sim 10^{-10}$
0.4	1.997 ± 0.001 (0.120)	1.990 ± 0.005 (0.220)	4×10^{-5}	1.985 ± 0.015 (0.225)	4×10^{-5} $\pm(5 \times 10^{-6})$	$\sim 10^{-7}$
0.5	1.991 ± 0.002 (0.159)	1.967 ± 0.035 (0.315)	5.4×10^{-4}	1.947 ± 0.045 (0.320)	5.5×10^{-4} $\pm(5 \times 10^{-5})$	$\sim 10^{-6}$
0.6	1.983 ± 0.002 (0.192)	1.915 ± 0.035 (0.440)	0.003 95	1.865 ± 0.070 (0.445)	$0.004\,00 \pm (0.000\,25)$	3.5×10^{-5} $\pm(0.3) \times 10^{-5}$
0.7	1.973 ± 0.002 (0.220)	1.848 ± 0.035 (0.525)	0.011 50	1.755 ± 0.095 (0.530)	$0.012\,00 \pm (0.001\,05)$	$0.000\,17 \pm (0.15) \times 10^{-4}$
0.8	1.960 ± 0.003 (0.245)	1.746 ± 0.035 (0.585)	0.025 07	1.625 ± 0.110 (0.590)	$0.025\,00 \pm (0.002\,10)$	$0.001\,55 \pm (0.000\,15)$
0.9	1.946 ± 0.003 (0.267)	1.627 ± 0.035 (0.620)	0.041 60	1.485 ± 0.125 (0.635)	$0.041\,75 \pm (0.003\,60)$	$0.013\,75 \pm (0.001\,35)$
1	1.930 ± 0.004 (0.287)	1.487 ± 0.035 (0.645)	0.058 05			

the massive-exchange case ($\mu/m = 0.15$), the error increases from about 2% for $\alpha \sim 0.5$ to about 8% at $\alpha = 0.9$.

Figure 1 is a plot of E/m versus α for the massless-exchange case $\mu/m = 0$ (the Wick-Cutkosky model). This figure also contains results obtained using the simplest form of trial state (i.e., $G = 0$ and $S = 0$), by Ding and Darewych [4], and the trial state (i.e., $G \neq 0$ and $S = 0$) that had been calculated by Emami-Razavi and Darewych [6].

We also include calculations of the two-body bound states for the present model in the Feshbach-Villars formalism, using an "empty" vacuum state $|\tilde{0}\rangle$ annihilated by the full field operator, i.e., $\phi|\tilde{0}\rangle = 0$, which yield two-body Klein-Gordon-like equations that have analytic solutions

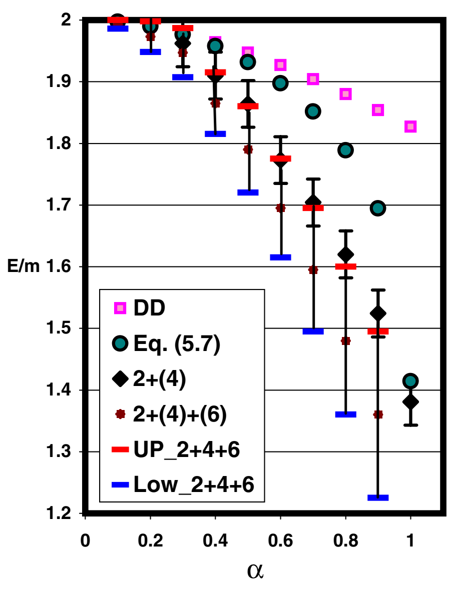

FIG. 1 (color online). Two-body ground-state energy (rest mass) E/m as a function of the dimensionless coupling constant $\alpha = g^2/16\pi m^2$ for the massless-exchange case ($\mu = 0$). Curves from top to bottom: Variational with $\psi_t = \psi_2$ [Ding and Darewych (DD) [4], Eq. (3.5)]; "empty vacuum" Feshbach-Villars formulation [Eq. (5.7) with $n = 1$]; variational results with $\psi_t = \psi_2 + \psi_4$ [Eqs. (5.2), (5.3), (5.4), (5.5), and (5.6) [6]]; present variational results with $\psi_t = \psi_2 + \psi_4 + \psi_6$ [Eqs. (5.2), (5.3), (5.4), (5.5), and (5.6)].

with the energy spectrum [11]

$$E_n = m\sqrt{2\left(1 + \sqrt{1 - \left(\frac{\alpha}{n}\right)^2}\right)}, \qquad (5.7)$$

where $n = 1, 2, 3, \ldots$ is the principal quantum number. For the ground state, this predicts a critical value $\alpha_c = 1$, at which $E_1/m = \sqrt{2}$.

There are a number various other calculations of the two-body binding energy for this model, beginning with the original ladder Bethe-Salpeter calculations of Wick and Cutkosky [5] and corresponding light-front-dynamics ladder results, for example, Ji and Furnstahl [12], Mangin-Brinet and Carbonell [13], and Bakker, van Iersel, and Piljman [14]. We do not plot them in Fig. 1, as they lie above our simplest ($G = 0$ and $S = 0$) results. We believe that the relatively low binding energies predicted in all of these cases are a reflection of the inadequacy of the ladder approximation (except at very low values $\alpha \ll 1$ of the coupling constant). There are also other approaches for the relativistic bound-state problems. For example, one can see the work of Todorov and his quasipotential approach in Ref. [15].

Numerical values corresponding to Fig. 1 are listed in Table I, along with the associated optimal values of the parameters a_2, a_4, a_6, and p_0, which specify the approximate ground-state wave functions (we keep ν fixed at the hydrogenic ground-state value $\nu = 2$). Note that the $G = 0$ and $S = 0$ results and $S = 0$ only ($G \neq 0$) in Table I are taken from Refs. [4,6]. They are included here for comparison purposes.

The present results indicate that the simplest ($G = 0$ and $S = 0$, Ref. [4]) approximation and the improved ones ($G \neq 0$ and $S = 0$, Ref. [6], or $G \neq 0$ and $S \neq 0$, present results) are similar "to within numerical uncertainties" in the domain of weak coupling ($0 \leq \alpha \lesssim 0.2$), where relativistic effects are small. However, the simplest ($G = 0$ and $S = 0$) approximation underestimates the binding energy considerably for larger values of α, showing that the simplest approximation is not reliable except at low α. We note that the linear parameters a_4 and a_6 of the trial state [cf. Eq. (5.1)] are small (relative to a_2) at low α and increase steadily as α increases. This indicates that the four-body and six-body Fock-space components ψ_4 and ψ_6 of the trial state (5.1) become increasingly important with increasing α.

We could not obtain meaningful numerical results for $\alpha \gtrsim 1$. This likely indicates that the Wick-Cutkosky model (massless case, $\mu = 0$) has no two-body bound-state solutions for α larger than some critical value α_c, as happens in the one-body Klein-Gordon case (the one-body Klein-Gordon equation with a scalar Coulombic potential $-\frac{\alpha}{r}$ has the bound-state energy spectrum $\frac{E}{m} = \pm\sqrt{1-(\alpha^2/4n^2)}$, which has no real $1s$ eigenvalues for $\alpha > 2$ [16]).

Our present variational solutions with F, G, and S all $\neq 0$ [i.e., 2 + (4) + (6)] predict stronger binding than any of the other calculations. However, we cannot predict a precise value of α_c since our numerical Monte Carlo results become unreliable for $\alpha \gtrsim 0.8$.

Table II contains variational results analogous to those of Table I for the $\mu/m = 0.15$ case in numerical form. We also list in Table II the corresponding optimal values of the adjustable parameters p_0, a_4/a_2, and a_6/a_2 (with $\nu = 2$), at which the expectation value of the Hamiltonian is a minimum. These parameters specify the ground-state wave-function components ψ_2, ψ_4, and ψ_6. Note that the variation of these parameters with α is qualitatively similar to the $\mu = 0$ case.

We have also plotted the ground-state "nucleon-antinucleon" energy as a function of α for the massive "pion" exchange case (with $\mu/m = 0.15$) in Fig. 2, along with results of some other workers, obtained by various approaches. There are many such results. It would not be possible to plot all of them in one figure, since it would become too cluttered and illegible. We therefore include a selection of results in the figure. Most of the results in Fig. 2 have been taken from the paper of Nieuwenhuis and Tjon [7] (corresponding numerical values are given in [4]). Note that, for this massive-exchange case, the interaction is short range, and binding sets in for α at about 0.3.

The top curve, indicating weakest binding, is the ladder Bethe-Salpeter approximation [5]. The one below it corresponds to the simplest variational results ($G = 0$ and $S = 0$, Ref. [4]) in the present formalism. The next curve below that is a one-time reduction of the Bethe-Salpeter equation due to Gross [17]. It contains one-quantum exchange and relativistic retardation effects.

Results obtained using the "empty vacuum" Feshbach-Villars formalism [11] lie very close to the Gross results; hence, they are also not plotted in Fig. 2. Both predict a critical value α_c beyond which there are no two-body bound states. Indeed, the Gross equation with retardation gives $\alpha_c \simeq 1.3$, where $E/m \simeq 1.4$, whereas the Feshbach-Villars result is $\alpha_c = 1.2087$ and $E/m = 1.483\,86$ [11].

The curve below the Gross results in Fig. 2 is from the numerical calculation by Nieuwenhuis and Tjon [7] using the Feynman-Schwinger representation. It does not include loop effects.

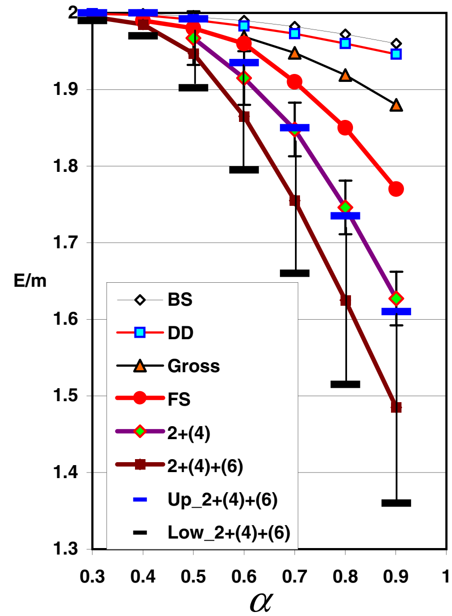

FIG. 2 (color online). Two-body ground-state energy (rest mass) E/m as a function of the dimensionless coupling constant $\alpha = g^2/16\pi m^2$ for the massive-exchange case ($\mu/m = 0.15$). Curves from top to bottom: Bethe-Salpeter (BS) in ladder approximation; variational with $\psi_t = \psi_2$ [Ding and Darewych (DD) [4], Eq. (3.5)]; Gross equation with retardation [17]; Feynman-Schwinger (FS) formulation (Nieuwenhuis and Tjon [7]); variational results with $\psi_t = \psi_2 + \psi_4$ [Eqs. (5.2), (5.3), (5.4), (5.5), and (5.6)] [6]; present variational results with $\psi_t = \psi_2 + \psi_4 + \psi_6$ [Eqs. (5.2), (5.3), (5.4), (5.5), and (5.6)].

The curve below Nieuwenhuis and Tjon [7] corresponds to the improved ($G \neq 0$) relativistic variational approximation [6], which contains tree-level effects and retardation in the two-body sector, as well as virtual nucleon-antinucleon pairs from the four-body sector (and coupling between these sectors). As in the massless-exchange case, no meaningful result is presented in Ref. [6] for $\alpha_c > 1$, which may be suggestive of the existence of a critical value of the coupling constant with $\alpha_c \simeq 1$.

The lowest curve corresponds to the present results ($G \neq 0$ and $S \neq 0$) relativistic variational approximation, which contains tree-level effects and retardation in the two-body sector, as well as virtual nucleon-antinucleon pairs from the four-body and six-body sectors (and coupling between these sectors).

VI. DISCUSSIONS AND CONCLUDING REMARKS

We have used the variational method in a reformulated version of the Hamiltonian formalism of quantum field theory to derive relativistic wave equations for two-body bound states in the scalar Yukawa model, in which massive, scalar nucleons N and "antinucleons" $\bar{N}$ of mass m interact via scalar meson exchange, which may be massive ($\mu \neq 0$) or massless ($\mu = 0$). The novel feature of the present work is the use of a trial state of the form

$|\psi_t\rangle = |\psi_2\rangle + |\psi_4\rangle + |\psi_6\rangle$, where $|\psi_2\rangle$, $|\psi_4\rangle$, and $|\psi_6\rangle$ represent, respectively, two-, four-, and six-body Fock-state components of the form $|\psi_2\rangle = |F(N\bar{N})\rangle$, $|\psi_4\rangle = |G(N\bar{N}N\bar{N})\rangle$, and $|\psi_6\rangle = |S(N\bar{N}N\bar{N}N\bar{N})\rangle$, respectively, where F, G, and S represent, respectively, adjustable two-, four-, and six-body wave-function coefficients. For purposes of describing two-body bound states of rest energy $E < 2m$, the four- and six-body components allow for virtual nucleon-antinucleon effects.

The variational principle leads to three coupled, multidimensional momentum-space relativistic integral wave equations for the two-, four-, and six-body coefficient functions F, G, and S. The interaction kernels (momentum-space potentials) in these equations contain one-meson exchange and virtual pair annihilation diagrams, along with retardation effects. These equations are Salpeter-like (rather than Klein-Gordon-like); that is, they admit only positive-energy solutions. We have also presented the nonrelativistic limit of these equations. In the coordinate-space representation, they are coupled two-, four-, and six-body Schrödinger equations, with attractive Yukawa interparticle potentials (or Coulombic, if $\mu = 0$) corresponding to one-pion exchange and contact (delta-function) potentials corresponding to virtual-annihilation effects.

We determine the two-body ground-state energy (rest mass) $E(\alpha)$ and wave-function parameters for both the massless- ($\mu = 0$) and the massive- ($\mu/m = 0.15$) exchange cases, in the domains $0 < \alpha < 1$ and $0.3 \leq \alpha < 1$, respectively. The ground-state energy is found to decrease steadily from $2m$ with increasing strength of the coupling. It was shown previously by Emami-Razavi and Darewych [6] that the ψ_4 component of the trial state $\psi_t = \psi_2 + \psi_4$ becomes increasingly important as α increases, and the binding energy at high α is much larger than that obtained with $\psi_t = \psi_2$ only. Our present results show that the ψ_6 component of the trial state $\psi_t = \psi_2 + \psi_4 + \psi_6$ also becomes increasingly important as α increases, and the binding energy at high α is much larger than that obtained with $\psi_t = \psi_2 + \psi_4$ (especially after the value of $\alpha \gtrsim 0.7$). No meaningful solutions were found for $\alpha \gtrsim 1$.

In order to explain the present numerical results adequately with respect to our Monte Carlo methods, we divide the scale of different α on three different categories. The first one corresponds to $0.1 \leq \alpha \leq 0.3$ ($\mu = 0$, massless-exchange case) and $0.3 \leq \alpha \leq 0.4$ ($\mu = 0.15$, massive-exchange case). In the scale of $0.1 \leq \alpha \leq 0.9$, we call the first case "low" coupling. The second case corresponds to $0.4 \leq \alpha \leq 0.6$ ($\mu = 0$, massless-exchange case) and $0.5 \leq \alpha \leq 0.6$ ($\mu = 0.15$, massive-exchange case). We call the second case "high" coupling. Finally, the last case (the third one) corresponds to $0.7 \leq \alpha \leq 0.9$ ($\mu = 0$, massless-exchange case) and $0.7 \leq \alpha \leq 0.9$ ($\mu = 0.15$, massive-exchange case). We call the third case "very high" coupling.

In all cases, the binding energy increases with the addition of consecutive virtual pairs. The numerical results of E/m versus α for all the cases are as the following. The results of $\psi_t = \psi_2 + \psi_4 + \psi_6$ are below $\psi_t = \psi_2 + \psi_4$ and the results of $\psi_t = \psi_2 + \psi_4$ are below $\psi_t = \psi_2$ only. Also, the binding energy of the massless-exchange case ($\mu = 0$) is bigger than the massive-exchange case ($\mu = 0.15$).

Within numerical uncertainties, the numerical results for the first case (low coupling) are close to each other for the trial states $\psi_t = \psi_2 + \psi_4 + \psi_6$, $\psi_t = \psi_2 + \psi_4$, and $\psi_t = \psi_2$. For the case of high coupling, we see that in both cases ($\psi_t = \psi_2 + \psi_4$ or the present results $\psi_t = \psi_2 + \psi_4 + \psi_6$) the inclusion of virtual pairs has a large effect for the binding energy of the system. One should note that, due to the extremely difficult calculations for the additional six-body case in this work, we have a relatively large error estimate for the ground-state energy solutions. Nevertheless, the present results confirm that the addition of a six-body Fock state increases again the binding energy of the system at high coupling. For the last case, namely, very high coupling, once again, our numerical results confirm that the additional virtual pairs increase the binding energy of the system notably. Our numerical Monte Carlo results become unreliable for $\alpha \gtrsim 0.8$ as mentioned before.

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APPENDIX

We recall that the total Hamiltonian of this scalar QFT is $\hat{H}(t) = \int d^N x \hat{\mathcal{H}}(x)$, where the Hamiltonian density $\hat{\mathcal{H}}(x)$ is given in Eqs. (2.1), (2.2), (2.3), (2.4), (2.5), and (2.6).

The Hamiltonian matrix corresponding to the generalized particle-antiparticle trial state (5.1) can be written as

$$H_t = \begin{bmatrix} \langle\psi_{t_2}|{:}\hat{H}{:}|\psi_{t_2}\rangle & \langle\psi_{t_2}|{:}\hat{H}{:}|\psi_{t_4}\rangle & \langle\psi_{t_2}|{:}\hat{H}{:}|\psi_{t_6}\rangle \\ \langle\psi_{t_4}|{:}\hat{H}{:}|\psi_{t_2}\rangle & \langle\psi_{t_4}|{:}\hat{H}{:}|\psi_{t_4}\rangle & \langle\psi_{t_4}|{:}\hat{H}{:}|\psi_{t_6}\rangle \\ \langle\psi_{t_6}|{:}\hat{H}{:}|\psi_{t_2}\rangle & \langle\psi_{t_6}|{:}\hat{H}{:}|\psi_{t_4}\rangle & \langle\psi_{t_6}|{:}\hat{H}{:}|\psi_{t_6}\rangle \end{bmatrix}. \tag{A1}$$

For the two-particle sector, the matrix elements and the kernel $\mathcal{K}_{2,2}$ are given in Eqs. (3.2), (3.3), and (3.4).

For the four-particle system, the matrix element corresponding to the rest-plus-kinetic energy is

$$\langle\psi_4|{:}\hat{H}_\phi - E{:}|\psi_4\rangle = \int d^N p_1 \ldots d^N p_4 G^*(\mathbf{p}_1, \ldots, \mathbf{p}_4) \times G(\mathbf{p}_1, \ldots, \mathbf{p}_4) \times [\omega_{p_1} + \ldots + \omega_{p_4} - E]. \tag{A2}$$

The matrix element corresponding to the interactions has the structure

$$\langle\psi_4|{:}\hat{H}_I{:}|\psi_4\rangle = [G1] + [G2] + [G3] + [G4] + [G5] + [G6] + [A1] + [A2] + [A3] + [A4],$$

which means that there are six terms corresponding to the one-quantum exchange interaction (all of them attractive, i.e., gravitylike) and four terms corresponding to virtual-annihilation interaction among the particle and antiparticle pairs.

$$\langle\psi_4|\hat{H}_I{:}|\psi_4\rangle = -\int d^N p_1 \ldots d^N p_4 d^N p'_1 \ldots d^N p'_4 \times G^*(\mathbf{p}'_1, \ldots, \mathbf{p}'_4) G(\mathbf{p}_1, \ldots, \mathbf{p}_4) \times \mathcal{K}_{4,4}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3, \mathbf{p}'_4, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4), \quad \text{(A3)}$$

where $\mathcal{K}_{4,4}$ is the following expression:

$$\begin{aligned}
\mathcal{K}_{4,4} = \frac{g^2}{4(2\pi)^N}\Bigg\{&\Bigg[\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_1 + \mathbf{p}'_3 - \mathbf{p}_1 - \mathbf{p}_3)}{\sqrt{\omega_{p'_1}\omega_{p'_3}\omega_{p_1}\omega_{p_3}}}\frac{1}{\mu^2 - (p_1 - p'_1)^2}\Bigg]_{G1} \\
&+ \Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_2 + \mathbf{p}'_3 - \mathbf{p}_2 - \mathbf{p}_3)}{\sqrt{\omega_{p'_2}\omega_{p'_3}\omega_{p_2}\omega_{p_3}}}\frac{1}{\mu^2 - (p_2 - p'_2)^2}\Bigg]_{G2} \\
&+ \Bigg[\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_1 + \mathbf{p}'_2 - \mathbf{p}_1 - \mathbf{p}_2)}{\sqrt{\omega_{p'_1}\omega_{p'_2}\omega_{p_1}\omega_{p_2}}}\frac{1}{\mu^2 - (p_2 - p'_2)^2}\Bigg]_{G3} \\
&+ \Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_4 + \mathbf{p}'_3 - \mathbf{p}_3 - \mathbf{p}_4)}{\sqrt{\omega_{p'_4}\omega_{p'_3}\omega_{p_3}\omega_{p_4}}}\frac{1}{\mu^2 - (p_3 - p'_3)^2}\Bigg]_{G4} \\
&+ \Bigg[\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_4 + \mathbf{p}'_1 - \mathbf{p}_4 - \mathbf{p}_1)}{\sqrt{\omega_{p'_4}\omega_{p'_1}\omega_{p_4}\omega_{p_1}}}\frac{1}{\mu^2 - (p_1 - p'_1)^2}\Bigg]_{G5} \\
&+ \Bigg[\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\frac{\delta^N(\mathbf{p}'_4 + \mathbf{p}'_2 - \mathbf{p}_4 - \mathbf{p}_2)}{\sqrt{\omega_{p'_4}\omega_{p'_2}\omega_{p_4}\omega_{p_2}}}\frac{1}{\mu^2 - (p_4 - p'_4)^2}\Bigg]_{G6} \\
&+ \Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_4 + \mathbf{p}'_3 - \mathbf{p}_3 - \mathbf{p}_4)}{\sqrt{\omega_{p'_4}\omega_{p'_3}\omega_{p_3}\omega_{p_4}}}\frac{1}{\mu^2 - (p_4 + p_3)^2}\Bigg]_{A1} \\
&+ \Bigg[\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_4 + \mathbf{p}'_1 - \mathbf{p}_4 - \mathbf{p}_1)}{\sqrt{\omega_{p'_4}\omega_{p'_1}\omega_{p_4}\omega_{p_1}}}\frac{1}{\mu^2 - (p_4 + p_1)^2}\Bigg]_{A2} \\
&+ \Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_2 + \mathbf{p}'_3 - \mathbf{p}_2 - \mathbf{p}_3)}{\sqrt{\omega_{p'_2}\omega_{p'_3}\omega_{p_2}\omega_{p_3}}}\frac{1}{\mu^2 - (p'_3 + p'_2)^2}\Bigg]_{A3} \\
&+ \Bigg[\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_1 + \mathbf{p}'_2 - \mathbf{p}_1 - \mathbf{p}_2)}{\sqrt{\omega_{p'_1}\omega_{p'_2}\omega_{p_1}\omega_{p_2}}}\frac{1}{\mu^2 - (p'_1 + p'_2)^2}\Bigg]_{A4}\Bigg\}. \quad \text{(A4)}
\end{aligned}$$

The matrices $\langle\psi_4|{:}\hat{H}{:}|\psi_2\rangle$, $\langle\psi_6|{:}\hat{H}{:}|\psi_2\rangle$, and $\langle\psi_6|{:}\hat{H}{:}|\psi_4\rangle$ are complex conjugates of $\langle\psi_2|{:}\hat{H}{:}|\psi_4\rangle$, $\langle\psi_2|{:}\hat{H}{:}|\psi_6\rangle$, and $\langle\psi_4|{:}\hat{H}{:}|\psi_6\rangle$, respectively, and we do not write them again. Note that the expressions of $\mathcal{K}_{2,4}$, $\mathcal{K}_{2,6}$, and $\mathcal{K}_{4,6}$ can be obtained from $\mathcal{K}_{4,2}$, $\mathcal{K}_{6,2}$, and $\mathcal{K}_{6,4}$, respectively. One has to change the variables $\mathbf{p}'_1$ to $\mathbf{p}_1$, $\mathbf{p}'_2$ to $\mathbf{p}_2$, etc., and vice versa $\mathbf{p}_1$ to $\mathbf{p}'_1$, $\mathbf{p}_2$ to $\mathbf{p}'_2$, etc., inside the expressions $\mathcal{K}_{4,2}$, $\mathcal{K}_{6,2}$, and $\mathcal{K}_{6,4}$ in order to obtain $\mathcal{K}_{2,4}$, $\mathcal{K}_{2,6}$, and $\mathcal{K}_{4,6}$, respectively. The matrix elements $\langle\psi_2|{:}\hat{H}{:}|\psi_4\rangle$, $\langle\psi_2|{:}\hat{H}{:}|\psi_6\rangle$, and $\langle\psi_4|{:}\hat{H}{:}|\psi_6\rangle$, which couple the two-, four-, and six-body sectors, respectively, have the following expressions:

$$\langle\psi_2|{:}\hat{H}{:}|\psi_4\rangle = \langle\psi_2|{:}\hat{H}_I{:}|\psi_4\rangle = -\int d^N p_1 \ldots d^N p_4 d^N p'_1 d^N p'_2 \times F^*(\mathbf{p}'_1, \mathbf{p}'_2) G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) \times \mathcal{K}_{4,2}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4), \quad \text{(A5)}$$

where $\mathcal{K}_{4,2}$ is

$$\begin{aligned}
\mathcal{K}_{4,2} = \frac{g^2}{4(2\pi)^N}\Bigg[&\frac{\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)}{\sqrt{\omega_{p'_1}\omega_{p_1}\omega_{p_3}\omega_{p_4}}}\frac{\delta^N(\mathbf{p}'_1 - \mathbf{p}_1 - \mathbf{p}_3 - \mathbf{p}_4)}{\mu^2 - (p_1 - p'_1)^2} \\
&+ \frac{\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)}{\sqrt{\omega_{p'_2}\omega_{p_2}\omega_{p_3}\omega_{p_4}}}\frac{\delta^N(\mathbf{p}'_2 - \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)}{\mu^2 - (p_2 - p'_2)^2} \\
&+ \frac{\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)}{\sqrt{\omega_{p'_1}\omega_{p_1}\omega_{p_3}\omega_{p_4}}}\frac{\delta^N(\mathbf{p}'_1 - \mathbf{p}_1 - \mathbf{p}_3 - \mathbf{p}_4)}{\mu^2 - (p_4 + p_1)^2} \\
&+ \frac{\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)}{\sqrt{\omega_{p'_2}\omega_{p_2}\omega_{p_3}\omega_{p_4}}}\frac{\delta^N(\mathbf{p}'_2 - \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)}{\mu^2 - (p_3 + p_2)^2}\Bigg]; \quad \text{(A6)}
\end{aligned}$$

$$\langle\psi_2|{:}\hat{H}{:}|\psi_6\rangle = \langle\psi_2|{:}\hat{H}_I{:}|\psi_6\rangle = -\int d^N p'_1 d^N p'_2 d^N p_1 \ldots d^N p_6 \times F^*(\mathbf{p}'_1, \mathbf{p}'_2) S(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6) \times \mathcal{K}_{6,2}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6), \quad \text{(A7)}$$

where $\mathcal{K}_{6,2}$ is

$$\mathcal{K}_{6,2} = \frac{g^2}{4(2\pi)^N}\bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\frac{\delta^N(\mathbf{p}_3 + \mathbf{p}_4 + \mathbf{p}_5 + \mathbf{p}_6)}{\sqrt{\omega_{p_3}\omega_{p_4}\omega_{p_5}\omega_{p_6}}}\frac{1}{\mu^2 - (p_5 + p_6)^2} + \delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2) \times \frac{\delta^N(\mathbf{p}_3 + \mathbf{p}_4 + \mathbf{p}_5 + \mathbf{p}_6)}{\sqrt{\omega_{p_3}\omega_{p_4}\omega_{p_5}\omega_{p_6}}}\frac{1}{\mu^2 - (p_4 + p_5)^2} + \delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2) \times \frac{\delta^N(\mathbf{p}_3 + \mathbf{p}_4 + \mathbf{p}_5 + \mathbf{p}_6)}{\sqrt{\omega_{p_3}\omega_{p_4}\omega_{p_5}\omega_{p_6}}}\frac{1}{\mu^2 - (p_3 + p_6)^2}\bigg]; \tag{A8}$$

and

$$\langle\psi_4|{:}\hat{H}{:}|\psi_6\rangle = \langle\psi_4|{:}\hat{H}_I{:}|\psi_6\rangle = -\int d^N p'_1 \ldots d^N p'_4 d^N p_1 \ldots d^N p_6 G^*(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3, \mathbf{p}'_4)S(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6) \times \mathcal{K}_{6,4}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3, \mathbf{p}'_4, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6), \tag{A9}$$

where $\mathcal{K}_{6,4}$ is

$$\begin{aligned}\mathcal{K}_{6,4} = \frac{g^2}{4(2\pi)^N}\bigg\{&\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_3 - \mathbf{p}_3 - \mathbf{p}_5 - \mathbf{p}_6)}{\sqrt{\omega_{p'_3}\omega_{p_3}\omega_{p_5}\omega_{p_6}}}\frac{1}{\mu^2 - (p_5 + p_6)^2}\\
&+ \delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_1 - \mathbf{p}_1 - \mathbf{p}_5 - \mathbf{p}_6)}{\sqrt{\omega_{p'_1}\omega_{p_1}\omega_{p_5}\omega_{p_6}}}\frac{1}{\mu^2 - (p_6 + p_1)^2}\\
&+ \delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_4 - \mathbf{p}_6)\frac{\delta^N(\mathbf{p}'_3 - \mathbf{p}_5 - \mathbf{p}_4 - \mathbf{p}_3)}{\sqrt{\omega_{p'_3}\omega_{p_5}\omega_{p_4}\omega_{p_3}}}\frac{1}{\mu^2 - (p_4 + p_3)^2}\\
&+ \delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_2 - \mathbf{p}_2 - \mathbf{p}_5 - \mathbf{p}_6)}{\sqrt{\omega_{p'_2}\omega_{p_2}\omega_{p_5}\omega_{p_6}}}\frac{1}{\mu^2 - (p_2 + p_5)^2}\\
&+ \delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_1 - \mathbf{p}_5)\frac{\delta^N(\mathbf{p}'_4 - \mathbf{p}_6 - \mathbf{p}_4 - \mathbf{p}_1)}{\sqrt{\omega_{p'_4}\omega_{p_6}\omega_{p_4}\omega_{p_1}}}\frac{1}{\mu^2 - (p_4 + p_1)^2}\\
&+ \delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_4 - \mathbf{p}_4 - \mathbf{p}_5 - \mathbf{p}_6)}{\sqrt{\omega_{p'_4}\omega_{p_4}\omega_{p_5}\omega_{p_6}}}\frac{1}{\mu^2 - (p_4 + p_5)^2}\\
&+ \delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_3 - \mathbf{p}_3 - \mathbf{p}_5 - \mathbf{p}_6)}{\sqrt{\omega_{p'_3}\omega_{p_3}\omega_{p_5}\omega_{p_6}}}\frac{1}{\mu^2 - (p'_3 - p_3)^2}\\
&+ \delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\delta^N(\mathbf{p}'_1 - \mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_3 - \mathbf{p}_1 - \mathbf{p}_5 - \mathbf{p}_6)}{\sqrt{\omega_{p'_3}\omega_{p_1}\omega_{p_5}\omega_{p_6}}}\frac{1}{\mu^2 - (p'_3 - p_5)^2}\\
&+ \delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_1 - \mathbf{p}_1 - \mathbf{p}_6 - \mathbf{p}_5)}{\sqrt{\omega_{p'_1}\omega_{p_1}\omega_{p_6}\omega_{p_5}}}\frac{1}{\mu^2 - (p'_1 - p_1)^2}\\
&+ \delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_2 - \mathbf{p}_2 - \mathbf{p}_6 - \mathbf{p}_5)}{\sqrt{\omega_{p'_2}\omega_{p_2}\omega_{p_6}\omega_{p_5}}}\frac{1}{\mu^2 - (p'_2 - p_2)^2}\\
&+ \delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_4 - \mathbf{p}_4 - \mathbf{p}_5 - \mathbf{p}_6)}{\sqrt{\omega_{p'_4}\omega_{p_4}\omega_{p_5}\omega_{p_6}}}\frac{1}{\mu^2 - (p'_4 - p_4)^2}\\
&+ \delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_2 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_4 - \mathbf{p}_2 - \mathbf{p}_5 - \mathbf{p}_6)}{\sqrt{\omega_{p'_4}\omega_{p_2}\omega_{p_5}\omega_{p_6}}}\frac{1}{\mu^2 - (p'_4 - p_6)^2}\bigg\}.\end{aligned} \tag{A10}$$

For the six-particle system ($n = 6$), the matrix element corresponding to the rest-plus-kinetic energy is

$$\langle\psi_6|{:}\hat{H}_\phi - E{:}|\psi_6\rangle = \int d^N p_1 d^N p_2 \ldots d^N p_6 S^*(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_6)S(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_6) \times [\omega_{p_1} + \omega_{p_2} + \omega_{p_3} + \omega_{p_4} + \omega_{p_5} + \omega_{p_6} - E]. \tag{A11}$$

The matrix element corresponding to the interactions has the structure

$$\langle\psi_6|:\hat{H}_I:|\psi_6\rangle = [G1] + [G2] + \cdots + [G14] + [G15] + [A1] + [A2] + \cdots + [A8] + [A9], \tag{A12}$$

which means that there are 15 terms $[n(n-1)/2 = 15]$ corresponding to the one-quantum exchange interaction (all of them attractive, i.e., gravitylike) and 9 terms $(n^2/4 = 9)$ corresponding to virtual-annihilation interaction among the particle and antiparticle pairs.

$$\langle\psi_6|:\hat{H}_I:|\psi_6\rangle = -\int d^N p_1 \ldots d^N p_6 d^N p'_1 \ldots d^N p'_6 S^*(\mathbf{p}'_1, \ldots, \mathbf{p}'_6) S(\mathbf{p}_1, \ldots, \mathbf{p}_6) \mathcal{K}_{6,6}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3, \mathbf{p}'_4, \mathbf{p}'_5, \mathbf{p}'_6, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6). \tag{A13}$$

The expression of $\mathcal{K}_{6,6}$ is a long expression. We divide it in two parts. The first one is the corresponding Yukawa interactions (all of them attractive, i.e., gravitylike), and the second part is the corresponding virtual-annihilation interactions. Hence, we can write $\mathcal{K}_{6,6} = \mathcal{K}^G_{6,6} + \mathcal{K}^A_{6,6}$; $\mathcal{K}^G_{6,6}$ and $\mathcal{K}^A_{6,6}$ are the following expressions, respectively:

$$\begin{aligned}
\mathcal{K}^G_{6,6} = \frac{g^2}{4(2\pi)^N}\Bigg\{&\Bigg[\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_1 + \mathbf{p}'_2 - \mathbf{p}_1 - \mathbf{p}_2)}{\sqrt{\omega_{p'_1}\omega_{p'_2}\omega_{p_1}\omega_{p_2}}}\frac{\delta^N(\mathbf{p}'_5 - \mathbf{p}_5)\delta^N(\mathbf{p}'_6 - \mathbf{p}_6)}{\mu^2 - (p_2 - p'_2)^2}\Bigg]_{G_1}\\
&+\Bigg[\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_1 + \mathbf{p}'_3 - \mathbf{p}_1 - \mathbf{p}_3)}{\sqrt{\omega_{p'_1}\omega_{p'_3}\omega_{p_1}\omega_{p_3}}}\frac{\delta^N(\mathbf{p}'_5 - \mathbf{p}_5)\delta^N(\mathbf{p}'_6 - \mathbf{p}_6)}{\mu^2 - (p_1 - p'_1)^2}\Bigg]_{G_2}\\
&+\Bigg[\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_1 + \mathbf{p}'_4 - \mathbf{p}_1 - \mathbf{p}_4)}{\sqrt{\omega_{p'_1}\omega_{p'_4}\omega_{p_1}\omega_{p_4}}}\frac{\delta^N(\mathbf{p}'_5 - \mathbf{p}_5)\delta^N(\mathbf{p}'_6 - \mathbf{p}_6)}{\mu^2 - (p_1 - p'_1)^2}\Bigg]_{G_3}\\
&+\Bigg[\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_1 + \mathbf{p}'_5 - \mathbf{p}_1 - \mathbf{p}_5)}{\sqrt{\omega_{p'_1}\omega_{p'_5}\omega_{p_1}\omega_{p_5}}}\frac{\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\delta^N(\mathbf{p}'_6 - \mathbf{p}_6)}{\mu^2 - (p_5 - p'_5)^2}\Bigg]_{G_4}\\
&+\Bigg[\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_1 + \mathbf{p}'_6 - \mathbf{p}_1 - \mathbf{p}_6)}{\sqrt{\omega_{p'_1}\omega_{p'_6}\omega_{p_1}\omega_{p_6}}}\frac{\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\delta^N(\mathbf{p}'_5 - \mathbf{p}_5)}{\mu^2 - (p_6 - p'_6)^2}\Bigg]_{G_5}\\
&+\Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_2 + \mathbf{p}'_3 - \mathbf{p}_2 - \mathbf{p}_3)}{\sqrt{\omega_{p'_2}\omega_{p'_3}\omega_{p_2}\omega_{p_3}}}\frac{\delta^N(\mathbf{p}'_5 - \mathbf{p}_5)\delta^N(\mathbf{p}'_6 - \mathbf{p}_6)}{\mu^2 - (p_2 - p'_2)^2}\Bigg]_{G_6}\\
&+\Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_2 + \mathbf{p}'_4 - \mathbf{p}_2 - \mathbf{p}_4)}{\sqrt{\omega_{p'_2}\omega_{p'_4}\omega_{p_2}\omega_{p_4}}}\frac{\delta^N(\mathbf{p}'_5 - \mathbf{p}_5)\delta^N(\mathbf{p}'_6 - \mathbf{p}_6)}{\mu^2 - (p_4 - p'_4)^2}\Bigg]_{G_7}\\
&+\Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_2 + \mathbf{p}'_5 - \mathbf{p}_2 - \mathbf{p}_5)}{\sqrt{\omega_{p'_2}\omega_{p'_5}\omega_{p_2}\omega_{p_5}}}\frac{\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\delta^N(\mathbf{p}'_6 - \mathbf{p}_6)}{\mu^2 - (p_5 - p'_5)^2}\Bigg]_{G_8}\\
&+\Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_2 + \mathbf{p}'_6 - \mathbf{p}_2 - \mathbf{p}_6)}{\sqrt{\omega_{p'_2}\omega_{p'_6}\omega_{p_2}\omega_{p_6}}}\frac{\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\delta^N(\mathbf{p}'_5 - \mathbf{p}_5)}{\mu^2 - (p_6 - p'_6)^2}\Bigg]_{G_9}\\
&+\Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_3 + \mathbf{p}'_4 - \mathbf{p}_3 - \mathbf{p}_4)}{\sqrt{\omega_{p'_3}\omega_{p'_4}\omega_{p_3}\omega_{p_4}}}\frac{\delta^N(\mathbf{p}'_5 - \mathbf{p}_5)\delta^N(\mathbf{p}'_6 - \mathbf{p}_6)}{\mu^2 - (p_3 - p'_3)^2}\Bigg]_{G_{10}}\\
&+\Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_3 + \mathbf{p}'_5 - \mathbf{p}_3 - \mathbf{p}_5)}{\sqrt{\omega_{p'_3}\omega_{p'_5}\omega_{p_3}\omega_{p_5}}}\frac{\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\delta^N(\mathbf{p}'_6 - \mathbf{p}_6)}{\mu^2 - (p_3 - p'_3)^2}\Bigg]_{G_{11}}\\
&+\Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_3 + \mathbf{p}'_6 - \mathbf{p}_3 - \mathbf{p}_6)}{\sqrt{\omega_{p'_3}\omega_{p'_6}\omega_{p_3}\omega_{p_6}}}\frac{\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)\delta^N(\mathbf{p}'_5 - \mathbf{p}_5)}{\mu^2 - (p_6 - p'_6)^2}\Bigg]_{G_{12}}\\
&+\Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_4 + \mathbf{p}'_5 - \mathbf{p}_4 - \mathbf{p}_5)}{\sqrt{\omega_{p'_4}\omega_{p'_5}\omega_{p_4}\omega_{p_5}}}\frac{\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_6 - \mathbf{p}_6)}{\mu^2 - (p_4 - p'_4)^2}\Bigg]_{G_{13}}\\
&+\Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_4 + \mathbf{p}'_6 - \mathbf{p}_4 - \mathbf{p}_6)}{\sqrt{\omega_{p'_4}\omega_{p'_6}\omega_{p_4}\omega_{p_6}}}\frac{\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_5 - \mathbf{p}_5)}{\mu^2 - (p_4 - p'_4)^2}\Bigg]_{G_{14}}\\
&+\Bigg[\delta^N(\mathbf{p}'_1 - \mathbf{p}_1)\delta^N(\mathbf{p}'_2 - \mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_5 + \mathbf{p}'_6 - \mathbf{p}_5 - \mathbf{p}_6)}{\sqrt{\omega_{p'_5}\omega_{p'_6}\omega_{p_5}\omega_{p_6}}}\frac{\delta^N(\mathbf{p}'_3 - \mathbf{p}_3)\delta^N(\mathbf{p}'_4 - \mathbf{p}_4)}{\mu^2 - (p_5 - p'_5)^2}\Bigg]_{G_{15}}\Bigg\},
\end{aligned} \tag{A14}$$

and

$$\begin{aligned}
\mathcal{K}^{A}_{6,6} = \frac{g^2}{4(2\pi)^N}\Bigg\{&\left[\delta^N(\mathbf{p}'_3-\mathbf{p}_3)\delta^N(\mathbf{p}'_4-\mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_1+\mathbf{p}'_2-\mathbf{p}_1-\mathbf{p}_2)}{\sqrt{\omega_{p'_1}\omega_{p'_2}\omega_{p_1}\omega_{p_2}}}\frac{\delta^N(\mathbf{p}'_5-\mathbf{p}_5)\delta^N(\mathbf{p}'_6-\mathbf{p}_6)}{\mu^2-(p_1+p_2)^2}\right]_{A_1}\\
&+\left[\delta^N(\mathbf{p}'_2-\mathbf{p}_2)\delta^N(\mathbf{p}'_3-\mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_1+\mathbf{p}'_4-\mathbf{p}_1-\mathbf{p}_4)}{\sqrt{\omega_{p'_1}\omega_{p'_4}\omega_{p_1}\omega_{p_4}}}\frac{\delta^N(\mathbf{p}'_5-\mathbf{p}_5)\delta^N(\mathbf{p}'_6-\mathbf{p}_6)}{\mu^2-(p_1+p_4)^2}\right]_{A_2}\\
&+\left[\delta^N(\mathbf{p}'_2-\mathbf{p}_2)\delta^N(\mathbf{p}'_3-\mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_1+\mathbf{p}'_6-\mathbf{p}_1-\mathbf{p}_6)}{\sqrt{\omega_{p'_1}\omega_{p'_6}\omega_{p_1}\omega_{p_6}}}\frac{\delta^N(\mathbf{p}'_4-\mathbf{p}_4)\delta^N(\mathbf{p}'_5-\mathbf{p}_5)}{\mu^2-(p'_1+p'_6)^2}\right]_{A_3}\\
&+\left[\delta^N(\mathbf{p}'_1-\mathbf{p}_1)\delta^N(\mathbf{p}'_4-\mathbf{p}_4)\frac{\delta^N(\mathbf{p}'_2+\mathbf{p}'_3-\mathbf{p}_2-\mathbf{p}_3)}{\sqrt{\omega_{p'_2}\omega_{p'_3}\omega_{p_2}\omega_{p_3}}}\frac{\delta^N(\mathbf{p}'_5-\mathbf{p}_5)\delta^N(\mathbf{p}'_6-\mathbf{p}_6)}{\mu^2-(p'_2+p'_3)^2}\right]_{A_4}\\
&+\left[\delta^N(\mathbf{p}'_1-\mathbf{p}_1)\delta^N(\mathbf{p}'_3-\mathbf{p}_3)\frac{\delta^N(\mathbf{p}'_2+\mathbf{p}'_5-\mathbf{p}_2-\mathbf{p}_5)}{\sqrt{\omega_{p'_2}\omega_{p'_5}\omega_{p_2}\omega_{p_5}}}\frac{\delta^N(\mathbf{p}'_4-\mathbf{p}_4)\delta^N(\mathbf{p}'_6-\mathbf{p}_6)}{\mu^2-(p_2+p_5)^2}\right]_{A_5}\\
&+\left[\delta^N(\mathbf{p}'_1-\mathbf{p}_1)\delta^N(\mathbf{p}'_2-\mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_3+\mathbf{p}'_4-\mathbf{p}_3-\mathbf{p}_4)}{\sqrt{\omega_{p'_3}\omega_{p'_4}\omega_{p_3}\omega_{p_4}}}\frac{\delta^N(\mathbf{p}'_5-\mathbf{p}_5)\delta^N(\mathbf{p}'_6-\mathbf{p}_6)}{\mu^2-(p'_4+p'_3)^2}\right]_{A_6}\\
&+\left[\delta^N(\mathbf{p}'_1-\mathbf{p}_1)\delta^N(\mathbf{p}'_2-\mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_3+\mathbf{p}'_6-\mathbf{p}_3-\mathbf{p}_6)}{\sqrt{\omega_{p'_3}\omega_{p'_6}\omega_{p_3}\omega_{p_6}}}\frac{\delta^N(\mathbf{p}'_4-\mathbf{p}_4)\delta^N(\mathbf{p}'_5-\mathbf{p}_5)}{\mu^2-(p_3+p_6)^2}\right]_{A_7}\\
&+\left[\delta^N(\mathbf{p}'_1-\mathbf{p}_1)\delta^N(\mathbf{p}'_2-\mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_4+\mathbf{p}'_5-\mathbf{p}_4-\mathbf{p}_5)}{\sqrt{\omega_{p'_4}\omega_{p'_5}\omega_{p_4}\omega_{p_5}}}\frac{\delta^N(\mathbf{p}'_3-\mathbf{p}_3)\delta^N(\mathbf{p}'_6-\mathbf{p}_6)}{\mu^2-(p'_4+p'_5)^2}\right]_{A_8}\\
&+\left[\delta^N(\mathbf{p}'_1-\mathbf{p}_1)\delta^N(\mathbf{p}'_2-\mathbf{p}_2)\frac{\delta^N(\mathbf{p}'_5+\mathbf{p}'_6-\mathbf{p}_5-\mathbf{p}_6)}{\sqrt{\omega_{p'_5}\omega_{p'_6}\omega_{p_5}\omega_{p_6}}}\frac{\delta^N(\mathbf{p}'_3-\mathbf{p}_3)\delta^N(\mathbf{p}'_4-\mathbf{p}_4)}{\mu^2-(p'_5+p'_6)^2}\right]_{A_9}\Bigg\}. \qquad \text{(A15)}
\end{aligned}$$

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