Confinement in $\mathcal{N} = 1$ supersymmetric QCD: One step beyond Seiberg's duality

M. Shifman[1] and A. Yung[1,2,3]

[1]*William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA*
[2]*Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia*
[3]*Institute of Theoretical and Experimental Physics, Moscow 117259, Russia*

We consider $\mathcal{N} = 1$ supersymmetric quantum chromodynamics (SQCD) with the gauge group U(N_c) and $N_c + N$ quark flavors. N_c flavors are massless; the corresponding squark fields develop (small) vacuum expectation values (VEVs) on the Higgs branch. Extra N flavors are endowed with small (and equal) mass terms. We study this theory through its Seiberg's dual: U(N) gauge theory with $N_c + N$ flavors of "dual quark" fields plus a gauge-singlet mesonic field M. The original theory is referred to as "quark theory" while the dual one is termed "monopole theory." The suggested mild deformation of Seiberg's procedure changes the dynamical regime of the monopole theory from infrared free to asymptotically free at large distances. We show that, upon condensation of the dual quarks, the dual theory supports non-Abelian flux tubes (strings). Seiberg's duality is extended beyond purely massless states to include light states on both sides. Being interpreted in terms of the quark theory, the monopole-theory flux tubes are supposed to carry chromoelectric fields. The string junctions—confined monopole-theory monopoles—can be viewed as "constituent quarks" of the original quark theory. We interpret closed strings as glueballs of the original quark theory. Moreover, there are string configurations formed by two junctions connected by a pair of different non-Abelian strings. These can be considered as constituent quark mesons of the quark theory.

I. INTRODUCTION

In the mid-1990s Seiberg argued [1,2] that two distinct $\mathcal{N} = 1$ Yang-Mills theories with appropriately chosen matter (usually referred to as electric/magnetic theories) can be equivalent in the infrared (IR) limit. Seiberg's duality was tested in a number of nontrivial ways (it passed all tests with flying colors), was understood in the framework of string/D-brane theory, and became an important tool in the realm of strongly coupled gauge theories. Seiberg's duals have different gauge groups, say, SU(N_c) in the electric theory versus SU($N_f - N_c$) in the magnetic one. The matter sectors of the electric and magnetic theories from the dual pair are related to each other in a well-defined manner.

Seiberg's duality was most extensively used in the so-called conformal window [3], in which both the original theory and its dual are conformally invariant. Typically, while one theory is strongly coupled the other one is at weak coupling which allows one to perform a fully quantitative analysis of the weakly coupled theory, with predictions for scaling dimensions applicable to the strongly coupled dual theory in the conformal regime.

Seiberg's duality holds outside the conformal window too. It relates to each other massless states of the dual theories, one of which is infrared free. Moreover, by itself it does not provide us with a confinement mechanism based on formation of confining strings (flux tubes). The questions we address are: can Seiberg's duality be generalized to include theories which at large distances exhibit non-Abelian string-based confinement? Can this duality con-

nect with each other not only massless states but also massive ones?

Below we will argue that the answers to these two questions are positive. To construct dual pairs with the above properties we will need a modest extension of ideas developed in connection with Seiberg's duality previously [3–5]. As is well known [1–4], large vacuum expectation values (VEVs) of the squark fields in the electric theory translate into large mass terms of the dual quarks in the magnetic theory and *vice versa*. With large VEVs we Higgs a part of the gauge group. The corresponding gauge bosons become heavy and can be integrated out. On the other side of duality the corresponding dual quarks become heavy and can be integrated out. This procedure changes the IR behavior of each theory from the given dual pair, but their IR equivalence remains intact.

We want to change the conformal (or IR free regime) in the infrared limit into a confining regime. To this end the above procedure must be modified. The modification we suggest is as follows. We start from an $\mathcal{N} = 1$ SQCD with a certain number of the quark fields. The dynamical scale of this theory is Λ_Q. Quark fields from a judiciously chosen subset are assumed to develop VEVs which are small on the scale of Λ_Q. The remaining quark fields are endowed with a common mass term m_q which is also small compared to Λ_Q, so that all "hadrons" are dynamical; none can be integrated out. We argue that within the framework of this deformation of Seiberg's procedure, on the other side of duality, the IR free regime is deformed to give rise to a theory which supports flux tubes (strings) at weak coupling and confines non-Abelian (dual) monopoles. A number of

states in this theory are light in the sense that their mass tends to zero in the limit $m_q \rightarrow 0$. We argue that via duality these light states are in one-to-one correspondence with the light states of the original theory. Thus, duality gets extended to include (in addition to massless moduli) a part of the spectrum which is light compared to the natural dynamical scale Λ_Q *but not massless.* We refer to such dual pairs as quark/monopole theories to distinguish them from Seiberg's electric/magnetic theories.

Extended duality allows us to analyze the monopole theory at weak coupling and make a number of highly nontrivial predictions for the quark-theory light sector which is at strong coupling. In fact, the monopole theory we get is close to the so-called *M* model developed previously [6] with the purpose of studying non-Abelian strings and confined non-Abelian monopoles in $\mathcal{N} = 1$ super-Yang-Mills theories. Some distinctions of the monopole theory from the *M* model (e.g. non-Bogomol'nyi-Prasad-Sommerfield (BPS) nature of the flux tubes) do not produce drastic changes of the dynamical pattern we arrive at compared to the dynamical pattern of the *M* model.

As was mentioned, the non-Abelian monopoles must be important in Seiberg's duality being related to dual quarks. We take one step further suggesting that the non-Abelian monopoles *are* the dual quarks. The dual quark fields condense providing (small) masses to all gauge bosons of the monopole theory. The way the monopole theory is Higgsed is very peculiar—it corresponds to baryon-operator dominated vacuum in the quark theory. Confined monopoles of the monopole theory are to be interpreted as certain "constituent quarks" of the quark theory. Both form N-plets of the global unbroken SU(N) theory which is present in the quark and monopole theories, on both sides of the extended duality.

To explain how this works we have to be more specific. What are the non-Abelian monopoles?

Although the full answer is not yet known there are certain results which were obtained recently and which have a natural interpretation in the framework of "the non-Abelian monopole" hypothesis.

Let us start from the Abelian 't Hooft-Polyakov monopole [7] of which we know everything. Suppose we have a model with the SU(2) gauge group broken down to U(1) by condensation of adjoint scalars a^a, $a = 1, 2, 3$. One can always align the adjoint condensate along the third axis in the SU(2) space,

$$\langle a^a \rangle = \delta^{a3} \langle a^3 \rangle.$$

The size of the 't Hooft-Polyakov monopole is of the order of $\langle a^3 \rangle^{-1}$ while its mass is of the order of $\langle a^3 \rangle / g^2$, where g^2 is the gauge coupling constant. In the non-Abelian limit $\langle a^3 \rangle \rightarrow 0$ the size of this state formally tends to infinity and its mass tends to zero, see [8] for a thorough discussion. The monopole seems to disappear, at least classically, from the physical spectrum.

However, from recent supersymmetric studies we learned that in some cases the monopoles become stabilized in the non-Abelian limit by quantum effects. In particular, this happens to confined monopoles in $\mathcal{N} = 2$ SQCD [9,10] with the gauge group U(N), the Fayet-Iliopoulos (FI) term for the U(1) factor, and $N_f = N$ quark flavors. This theory supports non-Abelian flux tubes— strings [9–12]. They are formed upon quark condensation triggered by the FI term. The string orientational zero modes are associated with the rotation of their color flux inside the non-Abelian gauge subgroup SU(N). Non-Abelian strings originate from the Abelian Z_N strings in the special regime which ensures vanishing of VEVs of the adjoint scalars, i.e. exactly in the non-Abelian limit we are interested in. The internal dynamics of the orientational modes of the non-Abelian strings are described by the two-dimensional (2, 2) supersymmetric $CP(N - 1)$ model on the string world sheet [9–12].

The monopoles are confined by these strings. In fact, elementary monopoles are nothing but the junctions of elementary Z_N strings [9,13]. It is possible to trace the fate of the confined monopoles [9]—from the Abelian regime (with nonvanishing VEVs of the adjoint fields) where they are just the 't Hooft-Polyakov monopoles slightly deformed by confinement effects, deep into the non-Abelian limit of vanishing VEVs of the adjoint fields. In this limit confined monopoles are seen as kinks of the $CP(N - 1)$ model on the string world sheet. They are stabilized by quantum effects. Their size and mass are determined by the dynamical scale of the $CP(N - 1)$ model. Thus, in quantum theory confined monopoles do not disappear from the physical spectrum in the non-Abelian limit.[1]

As was recently shown [6], breaking $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$ we do not necessarily destroy the above confined non-Abelian monopoles. In particular, they still exist in the limit when the adjoint fields decouple altogether and no traces of "Abelization" of the theory remain.

These results lead us to the following conjecture. Since the monopoles (stabilized by quantum effects) survive in the non-Abelian limit in the confining phase it is plausible to suggest that they can exist also in other phases matching Seiberg's notion of dual quarks. It is natural then that the theory dual to $\mathcal{N} = 1$ SQCD is formulated as a gauge theory of non-Abelian monopoles. In particular, in our construction the monopoles (a.k.a the quark fields of the monopole theory) are in the Higgs phase. We will show that this leads to formation of the non-Abelian strings in the monopole theory whose tension is proportional to a (fractional) power of m_q, which in the original (quark)

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[1]A somewhat different approach to non-Abelian monopoles was developed in [14].

theory must be interpreted as flux tubes of a "chromo-electric" field.

To make our statement as clear as possible it is worth comparing it with what was achieved in the Seiberg-Witten solution [15]. The low-energy limit of the $\mathcal{N} = 2$ theory is a U(1) theory. Upon dualization it becomes $\mathcal{N} = 2$ SQED with matter fields representing massless monopoles. This theory is IR free. Then the original theory is slightly deformed by a mass term of the adjoint chiral superfield. To the leading order this deformation does not break $\mathcal{N} = 2$ supersymmetry of low-energy SQED. However, it changes the IR free regime into the Higgs regime. The monopoles condense, and BPS-saturated light strings of the Abrikosov-Nielsen-Olesen type [16] ensue. These are interpreted as the confining strings of the underlying electric theory.

Our goal is a similar scenario but in non-Abelian version in $\mathcal{N} = 1$ theories. The string we get is non-Abelian and non-BPS. The world sheet theory (besides translational and supertranslational moduli fields) is a nonsupersymmetric $CP(N-1)$ model, which has its own dynamics in the infrared. In particular, kinks of the $CP(N-1)$ model are N-plets of a global symmetry. In this sense our construction is closer to the QCD string, whatever it might be.

The paper is organized as follows. In Secs. II and III we introduce the quark and the monopole theories related by extended Seiberg's duality, and identify unbroken global symmetries. In Sec. IV we study the pattern of physical scales relevant to both theories, imposing the requirement of weak coupling in the monopole theory. We show that this regime is self-consistent and then study the spectrum of elementary excitations in the monopole theory. In Secs. V and VI we thoroughly discuss the emergence of the non-Abelian strings and confined monopoles in the monopole theory. The tension of the string is shown to be small (vanishing in the limit $m_q \to 0$). All hadrons related to these strings are light in the scale of Λ_Q. Arguably, they represent a variety of light dual counterparts in the quark theory. The latter, being strongly coupled, tells us nothing about this rich set of low-lying states. In Sec. VII which can be viewed as a conceptual culmination we continue the discussion of the light sectors and suggest a quark-theory interpretation of the results obtained in the monopole theory. Section VIII presents brief conclusions. Appendix A summarizes our notation. In Appendix B we discuss peculiarities of the 't Hooft matching conditions in our construction.

II. QUARK THEORY

Our quark theory[2] is $\mathcal{N} = 1$ SQCD with the gauge group $U(N_c) = SU(N_c) \times U(1)$ and $N_c + N$ flavors of fundamental matter—let us call it quarks. As usual each

quark flavor is described by two chiral superfields, Q and $\tilde{Q}$, one in the fundamental another in the antifundamental representation of $U(N_c)$.

We will endow N quark flavors (out of $N_c + N$) with equal mass terms m_q, while the remaining N_c quark fields have vanishing mass terms. Our color/flavor notation is as follows: the quark supermultiplets are the chiral superfields Q^{kA} and $\tilde{Q}_{Ak}$ where

$$k = 1, \ldots, N_c$$

and

$$A = 1, \ldots, (N_c + N)$$

are the color and flavor indices, respectively. The coupling constants of $SU(N_c)$ and $U(1)$ gauge factors are denoted by $(g_{Q2})^2$ and $(g_{Q1})^2$, respectively. The subscript Q will remind us that we deal with the quark theory.

In the supersymmetric vacuum the scalar components of the quark multiplets q and $\tilde{q}$ are subject to N_c^2 real D-term conditions

$$q^A T^a \bar{q}_A - \bar{\tilde{q}}^A T^a \tilde{q}_A = 0, \qquad (a = 1, 2, \ldots, N_c^2 - 1);$$
$$q^A T \bar{q}_A - \bar{\tilde{q}}^A T \tilde{q}_A = 0, \qquad (2.1)$$

where T^a are generators of the $SU(N_c)$ normalized by the condition

$$\mathrm{Tr}\, T^a T^b = \tfrac{1}{2}\delta^{ab},$$

while T is the U(1) generator which we choose to be $T = 1/2$.

The massless quark flavors—there are N_c such flavors—develop VEVs breaking both $SU(N_c)$ and $U(1)$ gauge groups. Then the theory is fully Higgsed. It has a vacuum manifold, the Higgs branch whose dimension is

$$\dim \mathcal{H}_Q = 4N_c^2 - N_c^2 - N_c^2 = 2N_c^2. \qquad (2.2)$$

Here we take into account the fact that we have $4N_c^2$ real variables q^{kP} and $\tilde{q}_{Pk}$ with the flavor index P,

$$P = 1, \ldots, N_c,$$

which describe massless squarks; we subtract N_c^2 real D-term conditions and N_c^2 gauge phases (for the $U(N_c)$ gauge group) eaten by the Higgs mechanism. In other words, $2N_c^2 - 2$ real degrees of freedom (out of $4N_c^2$) enter the non-Abelian gauge supermultiplets and acquire "masses" $\sim \Lambda_Q$ where Λ_Q is the dynamical scale parameter of the quark theory. Two real degrees of freedom enter the U(1) gauge supermultiplet and acquire masses equal to that of the (Higgsed) photon.

The Higgs branch can be described in a gauge invariant way by the meson and baryon chiral moduli [1,2,4][3]

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[2]A summary of our notation and conventions is presented in Appendix A.

[3]Note that unlike [2] only the product of the baryon operators $\tilde{B}B$ is gauge invariant in the case of the $U(N_c)$ gauge group.

$$\langle \tilde{q}_S q^P \rangle, \qquad \tilde{B}B, \qquad (2.3)$$

subject to the condition

$$\det \langle \tilde{q}_S q^P \rangle - \tilde{B}B = \Lambda_Q^{2N_c - N} m_q^N, \qquad (2.4)$$

where

$$B = \frac{1}{N_c!} \varepsilon_{k_1 \ldots k_{N_c}} q^{k_1 1} \ldots q^{k_{N_c} N_c},$$

$$\tilde{B} = \frac{1}{N_c!} \varepsilon^{k^1 \ldots k^{N_c}} \tilde{q}_{1 k_1} \ldots \tilde{q}_{N_c k_{N_c}}. \qquad (2.5)$$

One can view

$$\Lambda_{Q,le} = (\Lambda_Q^{2N_c - N} m_q^N)^{1/(2N_c)} \qquad (2.6)$$

as a scale parameter of the effective low-energy theory emerging at momenta below m_q. However, this parameter will play no role in what follows.

Using $\mathrm{SU}(N_c) \times \mathrm{SU}(N_c)$ flavor rotations we can always transform the matrix of the vacuum expectation values $\langle \tilde{q}_S q^P \rangle$ to a diagonal form,

$$\langle \tilde{q}_S q^P \rangle \to \delta_S^P \mathcal{Q}_P, \qquad (2.7)$$

where N_c parameters $\mathcal{Q}_P$ determine the position of the vacuum on the vacuum manifold. Generically $\mathcal{Q}_1 \neq \mathcal{Q}_2 \neq \ldots \neq \mathcal{Q}_{N_c}$. We will assume that all $\mathcal{Q}$'s are non-degenerate and of the same order of magnitude,

$$\mathcal{Q}_1 \sim \mathcal{Q}_2 \sim \ldots \sim \mathcal{Q}_{N_c} \sim \mathcal{Q}. \qquad (2.8)$$

Finally, we will assume that

$$\mathcal{Q} \ll \Lambda_Q^2, \qquad (2.9)$$

see below for a more detailed discussion. Equation (2.9) implies the strong coupling regime.

Now, let us make a step back and set $m_q = 0$. Then the quark theory at the Lagrangian level has

$$\mathrm{SU}(N_c + N)_L \times \mathrm{SU}(N_c + N)_R \times \mathrm{U}(1)_R \times \mathrm{U}(1)_A \qquad (2.10)$$

global symmetry. The axial $\mathrm{U}(1)_A$ symmetry is anomalous. The $\mathrm{U}(1)_R$ symmetry is chosen to be nonanomalous with respect to non-Abelian gauge bosons. However, it appears to be anomalous with respect to the $\mathrm{U}(1)$ gauge bosons, see Appendix B. Note the absence of the global baryon $\mathrm{U}(1)$ symmetry. The baryon charge is gauged in our theory.

If $m_q = 0$ this global group is broken by quarks VEVs down to

$$\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R. \qquad (2.11)$$

The number of broken generators is $2N_c^2 + 4N N_c$.

The global symmetry (2.11) is broken by the quark mass $m_q \neq 0$ down to a diagonal

$$\mathrm{SU}(N). \qquad (2.12)$$

When a small mass m_q is switched on, some of the spontaneously broken generators are broken explicitly too and, thus, interpolate pseudo-Goldstone (pG) rather than the Goldstone states. The number of true massless states is given by the dimension of the Higgs branch (2.2) while the number of the pseudo-Goldstone states is

$$N_{\mathrm{pG}} = 4N N_c. \qquad (2.13)$$

These pG states (interpolated by $\tilde{Q}_S Q^{\dot{K}}$ where $\dot{K} = N_c + 1, \ldots, N_c + N$ while $S = 1, \ldots, N_c$, see Appendix A) have masses

$$m_{\mathrm{pG}} \sim m_q. \qquad (2.14)$$

The pseudo-Goldstones $\tilde{Q}_S Q^{\dot{K}}$ are in the fundamental representation of the global unbroken $\mathrm{SU}(N)$.

Other light states are not easily seen on this side of duality but can be inferred from its dual description (see Sec. IV), in particular, a vector multiplet and "pions" $\tilde{q}_{\dot{L}} q^{\dot{K}}$ in the adjoint representation of the global $\mathrm{SU}(N)$.

The first coefficient of the β function of the quark theory is

$$b_Q = 2N_c - N. \qquad (2.15)$$

To make it positive (so the theory is asymptotically free) it is sufficient to have $N < 2N_c$. However, we will limit ourselves to even smaller values of N. We will consider the quark theory well below the left edge of the conformal window [3],

$$N < \frac{N_c}{2}. \qquad (2.16)$$

Moreover, we will assume the quark mass terms to be the smallest dimensional parameter in the quark theory,

$$m_q \ll \sqrt{\mathcal{Q}} \ll \Lambda_Q. \qquad (2.17)$$

This condition means that N quark flavors with vanishing VEVs are dynamical and do not decouple, while the condition (2.9) ensures that the quark theory is in the strong coupling regime. If the quark VEVs were much larger than Λ_Q^2 the quark theory would be in the Higgs phase at weak coupling. Instead, at small quark VEVs the theory is in the strong coupling regime and, although the Higgs and confinement phases are analytically connected in theories with fundamental matter [17], it is more convenient to speak of the quark theory in terms of confinement.

III. MONOPOLE THEORY

To begin with, let us put $m_q = \mathcal{Q} = 0$ in the quark theory. In this case we can just follow Seiberg [1] and accept that the dual description of the original quark theory (Seiberg's "magnetic dual") is given by an $\mathcal{N} = 1$ supersymmetric gauge theory with certain matter fields. The gauge group is $\mathrm{U}(N)$. The $\mathcal{N} = 1$ vector multiplets consist of the $\mathrm{U}(1)$ gauge fields A_μ and $\mathrm{SU}(N)$ gauge field A_μ^a,

(here $a = 1, \ldots, N^2 - 1$) and their Weyl fermion superpartners λ_α, and λ_α^a. The spinorial index of λ's runs over $\alpha = 1, 2$.

There are $N + N_c$ flavors in the monopole theory. From the standpoint of the monopole theory *per se* each flavor is represented by a pair of superfields h^{kA} and $\tilde{h}_{Ak}$ with the lowest (scalar) components h^{kA} and $\tilde{h}_{Ak}$ and the Weyl fermions ψ^{kA} and $\tilde{\psi}_{Ak}$, all in the fundamental representation of the SU(N) gauge group. Here $k = 1, \ldots, N$ is the color index while A is the flavor index, $A = 1, \ldots, (N + N_c)$. Seiberg termed these h fields dual quarks. From the standpoint of the original quark theory the dual quarks are to be interpreted as monopole fields.

In addition, the monopole theory has a gauge neutral "mesonic" field M_A^B which is locally related to the $\tilde{Q}_A Q^B$ composite chiral operator of the original quark theory [1,4],

$$\tilde{Q}_A Q^B = \kappa M_A^B, \tag{3.1}$$

where κ is an energy scale to be determined below.

The mesonic field is coupled to the dual quark fields via the superpotential [1,4]

$$W_{\text{Yukawa}} = \tilde{h}_{Ak} h^{kB} M_B^A. \tag{3.2}$$

The mass term $m_q \tilde{Q}_{\check{K}} Q^{\check{K}}$ of the quark theory takes the form of a linear in M superpotential

$$W_{\text{linear}} = -\tfrac{1}{2} \xi M_{\check{K}}^{\check{K}}. \tag{3.3}$$

From Eq. (3.1) we find the following relation between the parameter ξ in the linear superpotential (3.3) and the quark masses in the original theory:

$$\xi = -2\kappa m_q. \tag{3.4}$$

The bosonic part of the action of the monopole theory (in the limit $m_q = \mathcal{Q} = 0$) is

$$S = \int d^4 x \left\{ \frac{1}{4(g_{M2})^2} (F_{\mu\nu}^a)^2 + \frac{1}{4(g_{M1})^2} (F_{\mu\nu})^2 + \text{Tr}|\nabla_\mu h|^2 \right.$$
$$+ \text{Tr}|\nabla_\mu \bar{\tilde{h}}|^2 + \frac{2}{\gamma} \text{Tr}|\partial_\mu M|^2 + \frac{g_{M2}^2}{2} (\text{Tr}\bar{h} T^a h$$
$$- \text{Tr}\tilde{h} T^a \bar{\tilde{h}})^2 + \frac{g_{M1}^2}{8} (\text{Tr}\bar{h}h - \text{Tr}\tilde{h}\bar{\tilde{h}})^2 + \text{Tr}|hM|^2$$
$$\left. + \text{Tr}|\bar{\tilde{h}}M|^2 + \frac{\gamma}{2} \left| \tilde{h}_A h^B - \frac{1}{2} \delta_A^{\check{K}} \delta_{\check{K}}^B \xi \right|^2 \right\}, \tag{3.5}$$

where the covariant derivatives are defined as[4]

$$\nabla_\mu = \partial_\mu - \frac{i}{2} A_\mu - i A_\mu^a T^a. \tag{3.6}$$

The trace in (3.5) runs over all flavor indices. In addition to the dual gauge couplings g_{M1} and g_{M2} for the U(1) and

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[4]For further explanations on our notation, see Appendix A.

SU(N) factors, respectively, we introduced the coupling constant γ for the M field.

Let us note that the duality pairs were found in Ref. [1] for theories with the gauge groups SU(N) rather than U(N). To generalize Seiberg's duality to the latter case we gauge the U(1) global baryon symmetry present on both sides of Seiberg's duality.

If in the original quark theory the quark fields have the baryon charge $1/2$, then the baryon charges of the dual quark fields in the monopole theory are $N_c/(2N)$ (see Refs. [1,4]). If we still want to keep the corresponding couplings identical we must choose

$$g_{M1} = \frac{N_c}{N} g_{Q1}. \tag{3.7}$$

Then on both sides of Seiberg's duality the U(1) charge is $1/2$ being measured in the units of the appropriate gauge coupling. Then the definition (3.6) stays intact.

The SU(N) gauge coupling constant g_{M2} in (3.5) is determined by the scale Λ_M of the monopole theory. The latter is related to the scale Λ_Q of the quark theory as [1,4]

$$\Lambda_Q^{2N_c - N} \Lambda_M^{2N - N_c} = (-1)^N \kappa^{N_c + N}. \tag{3.8}$$

As we will see shortly, Λ_M is the largest parameter in our analysis, $\Lambda_M \gg \Lambda_Q$, but it will play no role since the monopole theory in the limit $m_q = \mathcal{Q} = 0$ (and with N satisfying the condition (2.16)) is infrared rather than asymptotically free; it lies to the right of the right edge of the conformal window. What will be important for dynamical considerations is an effective low-energy parameter $\Lambda_{M,le}$ which will emerge after $m_q \neq 0$ and $\mathcal{Q} \neq 0$ are taken into account.

It is natural to assume that

$$\gamma \sim 1. \tag{3.9}$$

Now let us switch on $m_q \neq 0$, $\mathcal{Q} \neq 0$ in the quark theory and discuss the vacuum structure of the monopole theory. The linear in M superpotential which reflects $m_q \neq 0$ triggers spontaneous breaking of the U(N) gauge symmetry. The vacuum expectation values of the h fields can be chosen as

$$\langle h^{k\check{K}} \rangle = \langle \bar{\tilde{h}}^{k\check{K}} \rangle = \sqrt{\frac{\xi}{2}} \begin{pmatrix} 1 & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & 0 & 1 \end{pmatrix}, \tag{3.10}$$

$$\langle h^{kP} \rangle = \langle \bar{\tilde{h}}^{kP} \rangle = 0,$$

up to gauge rotations.

Furthermore, Higgsing the quark theory (2.7) and (2.8) manifests itself in the monopole theory as nonvanishing VEVs of the M_S^P fields related to VEVs of $\tilde{Q}_S Q^P$ via (3.1),

$$\langle M_S^P \rangle = \begin{pmatrix} \mathcal{M}_1 & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & 0 & \mathcal{M}_{N_c} \end{pmatrix}, \tag{3.11}$$

where $\mathcal{M}_1 \sim \mathcal{M}_2 \sim \ldots \sim \mathcal{M}_{N_c} \sim \mathcal{M}$ and we introduced a common scale $\mathcal{M}$. The above nonvanishing VEVs make the first N_c monopole flavors massive by virtue of W_{Yukawa}. If we descend below $\mathcal{M}$ the massive flavors can be integrated out. What remains is a U(N) gauge theory with N flavors which is asymptotically free. The scale of this theory $\Lambda_{M,le}$ is defined via the relation

$$\Lambda_{M,le}^{2N} = \Lambda_M^{2N-N_c} \mathcal{M}^{N_c}, \tag{3.12}$$

which, in turn, can be expressed in terms of the quark-theory scale,

$$\Lambda_Q^{2N_c-N} \Lambda_{M,le}^{2N} = (-1)^N \mathcal{M}^{2N_c+N} \tag{3.13}$$

by invoking Eq. (3.8). As we will see shortly, the scale $\Lambda_{M,le}$ lies between m_q and $\sqrt{\xi}$. This guarantees that the monopole theory is weakly coupled.

Other M fields do not condense,

$$\langle M_L^{\dot{K}} \rangle = \langle M_L^P \rangle = \langle M_S^{\dot{K}} \rangle = 0. \tag{3.14}$$

The color-flavor locked form of the quark VEVs in Eq. (3.10) and the vanishing of VEVs in (3.14) result in the fact that, while the theory is fully Higgsed, a diagonal SU(N)$_{C+F}$ symmetry survives as a global symmetry. Namely, the global rotation

$$h \to UhU^{-1}, \qquad \tilde{\bar{h}} \to U\tilde{\bar{h}}U^{-1}, \qquad M \to U^{-1}MU \tag{3.15}$$

is not broken by the VEVs (3.10) and (3.11). Here U is an arbitrary matrix from SU(N). We write the dual quark (monopole) fields $h^{k\dot{K}}$, $\tilde{h}_{\dot{K}k}$, and $M_{\dot{L}}^{\dot{K}}$ with indices $k = 1, \ldots, N$ and $\dot{K}, \dot{L} = N_c + 1, \ldots, N_c + N$ as $N \times N$ matrices; the matrices U act on these indices. This is a particular case of the Bardakçı-Halpern mechanism [18].

Classically, in addition to the unbroken global SU(N)$_{C+F}$ symmetry, there is also a chiral U(1)$_{R''}$ symmetry which survives the breaking induced by VEVs (3.10) and (3.11), see Appendix B. However, it turns out to be anomalous with respect to the U(1) gauge fields. Therefore, the global unbroken symmetry of the monopole theory

$$\text{SU}(N)_{C+F} \tag{3.16}$$

is the same as the symmetry of the quark theory (2.12). In Appendix B we check duality by demonstrating that the 't Hooft anomaly matching conditions in the quark and monopole theories are satisfied. There are certain peculiarities since the matching looks different at "high energies" (i.e. when the momentum q flowing in the axial current satisfies $q^2 \gg \xi$) and "low energies" ($q^2 \ll \xi$). We check both limits.

The SU(N)$_{C+F}$ global symmetry of the theory is spontaneously broken on strings, which gives rise to the orientational zero modes [12] of the Z_N strings in the model (3.5).

Below we assume that the original quark theory has the same low-energy physics as the monopole theory (3.5). By low energies we mean scales $\sqrt{\xi}$ and below.

IV. ELEMENTARY EXCITATIONS IN THE MONOPOLE THEORY

First we observe that the fields M_S^P can develop VEVs; thus, the dimension of the Higgs branch in the monopole theory

$$\dim(\mathcal{H}_M) = 2N_c^2 \tag{4.1}$$

agrees with the one in the quark theory (2.2). As a result, $2N_c^2$ (real) fields M_S^P are massless.

Now we will demonstrate that the scale of VEVs of the M_S^P fields is the largest relevant parameter in the monopole theory. In particular, it is much larger than the effective scale $\Lambda_{M,le}$ of the monopole theory. Equation (2.8) implies that all parameters $\mathcal{M}_S$ in Eq. (3.11) are of the same order,

$$\mathcal{M}_1 \sim \mathcal{M}_2 \sim \ldots \sim \mathcal{M}_{N_c} \sim \mathcal{M}. \tag{4.2}$$

Because of the Yukawa interactions (3.2) the flavors h^{kP} ($\tilde{h}_{Pk}$) become massive, with masses $m(h^P) \sim \mathcal{M}$, and decouple below this scale. Integrating them out in the superpotential (3.2) produces an effective low-energy superpotential

$$W_{M,le} = \tilde{h}_{\dot{K}k} h^{k\dot{L}} \left[M_{\dot{L}}^{\dot{K}} - \frac{M_P^{\dot{K}} M_{\dot{L}}^P}{\mathcal{M}} \right]. \tag{4.3}$$

This superpotential gives small masses to $4NN_c$ (real) off-diagonal fields $M_P^{\dot{K}}, M_{\dot{K}}^P$. To see that this is indeed the case please observe that the fields $h^{k\dot{K}}$ develop VEVs $\sim \sqrt{\xi}$ (see Eq. (3.10)). Then the second term in (4.3) implies

$$m(M_{\dot{K}}^P) \sim m(M_P^{\dot{K}}) \sim \frac{\xi}{\mathcal{M}}. \tag{4.4}$$

We see that the number of the off-diagonal M fields that are light coincides with the number of the pseudo-Goldstone fields $\tilde{q}_L q^P$ in the quark theory, see (2.13). Requiring their masses to be the same we get

$$\kappa = \mathcal{M}, \tag{4.5}$$

where we used (2.14), (3.4), and (4.4). This fixes the so-far unknown coefficient κ. All three scales of the monopole theory, namely Λ_M, ξ, and $\mathcal{M}$ are now fixed in terms of three scales of the quark theory Λ_Q, m_q, and

$$\langle \tilde{Q}Q \rangle \sim \mathcal{Q} \sim \mathcal{M}^2. \tag{4.6}$$

As was mentioned, below the scale $\mathcal{M}$ the effective low-energy theory is the U(N) gauge theory with N flavors of the $h^{k\dot{K}}$ fields supplemented by the mesonic (gauge-singlet) field M, out of which M_S^P are massless, $M_P^{\dot{K}}$ and $M_{\dot{K}}^P$ have masses $\sim m_q$, and $M_{\dot{L}}^{\dot{K}}$ have masses $\sim \sqrt{\xi}$, see below. We consider this theory in the weak coupling regime imposing

the condition

$$\xi \gg \Lambda_{M,le}^2. \tag{4.7}$$

In terms of the quark-theory scales this condition can be rewritten by virtue of Eq. (3.13) as

$$\mathcal{M}^{2N_c} \ll \Lambda_Q^{2N_c-N} m_q^N. \tag{4.8}$$

We see that in order to keep the monopole theory at weak coupling, the scale $\mathcal{M}$ cannot be too large.

To elucidate the meaning of this condition, following [4] we relate the baryon operators in the quark theory B and $\tilde{B}$ to the baryon operators in the monopole theory

$$B = q^1 \dots q^{N_c} = [-(-1)^N \kappa^{-N} \Lambda_Q^{2N_c-N}]^{1/2} h^1 \dots h^N,$$
$$\tilde{B} = \tilde{q}_1 \dots \tilde{q}_{N_c} = [-(-1)^N \kappa^{-N} \Lambda_Q^{2N_c-N}]^{1/2} \tilde{h}^1 \dots \tilde{h}^N. \tag{4.9}$$

The right-hand sides of these expressions have nonvanishing VEVs in the vacuum of the monopole theory, see (3.10). Note, however, that the h field VEVs are given by (3.10) only classically. We expect corrections in $\langle h \dots h \rangle$ of order of $\Lambda_{M,le}$ to the product of the classical expectation values (3.10). These corrections are small provided (4.7) is satisfied.

Substituting VEVs (3.10) in (4.9) and ignoring the above corrections we get for the baryon operator VEV in the quark theory

$$\langle \tilde{B}B \rangle = -\Lambda_Q^{2N_c-N} m_q^N, \tag{4.10}$$

where we used Eq. (3.4). We see that the relation (2.4) of the quark theory is saturated to the leading order by the baryon operator. Quantum corrections to VEVs (3.10) substituted into (2.4) allow for

$$\det\langle \tilde{Q}_P Q^S \rangle \sim \mathcal{M}^{2N_c} \ll \Lambda_Q^{2N_c-N} m_q^N = -\langle \tilde{B}B \rangle. \tag{4.11}$$

Thus, the weak coupling regime in the monopole theory corresponds to the baryon dominated vacuum in the quark theory.

Relevant scales of the quark and monopole theories are shown in Fig. 1.

Now let us prove the statement made above regarding the states with masses of the order of $\sqrt{\bar{\xi}}$. Since both the U(1) and SU(N) gauge groups are broken by the $h^{\check{K}}$ field condensation, see Eq. (3.10), all gauge bosons become massive. From (3.5) we get for the U(1) gauge boson mass (dual "photon")

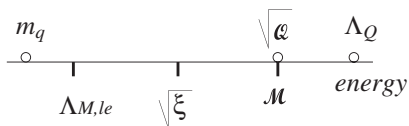
FIG. 1. Scales of the quark (open points) and monopole (dashes) theories.

$$m_{\text{ph}} = g_{M1} \sqrt{\frac{N}{2}} \sqrt{\xi}. \tag{4.12}$$

The $(N^2 - 1)$ gauge bosons of the SU(N) group (dual "W bosons") acquire a common mass

$$m_W = g_{M2} \sqrt{\xi}. \tag{4.13}$$

This is typical of the Bardakçı-Halpern mechanism. Needless to say, N^2 vector states with masses $\sim\sqrt{\xi}$ must appear in the quark theory too.

To get the scalar boson masses we expand the potential in (3.5) near the vacuum (3.10) and (3.11) and diagonalize the corresponding mass matrix. The N^2 components of the $2N^2$-component[5] scalar $(h - \bar{\tilde{h}})^{k\check{K}}/\sqrt{2}$ are eaten by the Higgs mechanism for the U(1) and SU(N) gauge groups. Another N^2 components are split as follows: one component acquires the mass (4.12). It becomes a scalar component of a massive $\mathcal{N} = 1$ vector U(1) gauge multiplet. The remaining $N^2 - 1$ components acquire masses (4.13) and become scalar superpartners of the SU(N) gauge bosons in the $\mathcal{N} = 1$ massive gauge supermultiplet.

The fields $M_{\check{L}}^{\check{K}}$ and $(h + \bar{\tilde{h}})^{k\check{K}}/\sqrt{2}$ form chiral multiplets. Namely, the states proportional to the unit $N \times N$ matrix (associated with U(1)) acquire masses

$$m_{\text{U}(1)}^M = \sqrt{\frac{\gamma N \xi}{4}}, \qquad m_{\text{U}(1)}^h = \sqrt{\frac{\gamma N \xi}{4}}, \tag{4.14}$$

respectively, while the traceless parts of $M_{\check{L}}^{\check{K}}$ and $(h + \bar{\tilde{h}})^{k\check{K}}/\sqrt{2}$ (associated with the SU(N) sector) have masses

$$m_{\text{SU}(N)}^M = \sqrt{\frac{\gamma \xi}{2}}, \qquad m_{\text{SU}(N)}^h = \sqrt{\frac{\gamma \xi}{2}}. \tag{4.15}$$

Other states with masses of the order of $\mathcal{M}$ are much heavier and we do not include them in the low-energy spectrum. To reiterate, in the monopole theory the low-energy spectrum includes the Goldstone and pseudo-Goldstone states (with masses (4.4)), as well as the states with masses $\sim\sqrt{\xi}$, "elementary" states discussed above, and composite states to be discussed below. In Sec. V we will discuss formation of non-Abelian strings in the monopole theory. These strings produce (extra) nonperturbative states in the monopole theory with masses $\sim\sqrt{\xi}$. Needless to say, all these states from the low-energy sector of the monopole theory are presumed to exactly match the low-energy spectrum of the quark theory. This is the statement of extended duality.

V. NON-ABELIAN STRINGS

Non-Abelian strings were shown to emerge at weak coupling in $\mathcal{N} = 2$ supersymmetric gauge theories [9–

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[5]We mean here *real* components.

12]. Recently they were also found [6] in $\mathcal{N} = 1$ super-symmetric theory with the U(N) gauge group and N fundamental matter multiplets supplemented by a mesonic field $M_{\dot{L}}^{\dot{K}}$ ($\dot{K}, \dot{L} = N_c + 1, \ldots, N_c + N$).

The M model of Ref. [6] is a close relative of the monopole theory we consider here. The two models differ in that there are extra components of the M fields in the monopole theory, namely, the Goldstones M_S^P and pseudo-Goldstones $M_P^{\dot{K}}$, $M_{\dot{L}}^P$. The impact of their presence is rather unimportant, as we will see below. Extra monopole fields h^{kP} present in the monopole theory are heavy and can be ignored.

A. Non-BPS strings

Since the M field does not enter the classical string solution, the explicit solution for the non-Abelian strings found in the M model [6] can be readily adjusted to fit the monopole theory at hand. The light components of the M field which have no counterparts in the M model will show up only at the quantum level.

A consequential distinction of the monopole theory from the M model is that the ξ parameter which triggers the h-field condensation is introduced via superpotential (3.3) rather than through the FI D term as in the original M model. Introducing ξ through the superpotential is a viable alternative. In this case we will speak of M_F model while the original M model can be termed M_D.

Note that the M_F model strings are necessarily non-BPS [19], while in the M_D model we deal with the BPS-saturated strings. This means that the world sheet theory on the strings will be nonsupersymmetric in the case at hand.

The bosonic part of the action of the M_F model is given by (3.5), with the flavor indices running only over N values $A, B \to \dot{K}, \dot{L} = N_c + 1, \ldots, N_c + N$. This is shown in more detail in Appendix A.

The scalar fields involved in the string solution are

$$h^{k\dot{K}} = \bar{\tilde{h}}^{k\dot{K}} \equiv \frac{1}{\sqrt{2}} \varphi^{k\dot{K}}. \tag{5.1}$$

With this substitution the *ansatz* for the solution for the elementary non-Abelian strings (in the singular gauge) becomes [6]

$$\varphi = \frac{1}{N}[(N-1)\phi_2 + \phi_1] + (\phi_1 - \phi_2)\left(n \cdot n^* - \frac{1}{N}\right),$$

$$A_i^{\text{SU}(N)} = \left(n \cdot n^* - \frac{1}{N}\right)\varepsilon_{ij}\frac{x_i}{r^2}f_{NA}(r),$$

$$A_i^{\text{U}(1)} = \frac{1}{N}\varepsilon_{ij}\frac{x_i}{r^2}f(r), \qquad M_{\dot{L}}^{\dot{K}} = M_P^{\dot{K}} = M_{\dot{L}}^P = 0, \tag{5.2}$$

where r is the distance from the string axis to the given point in the orthogonal plane. Moreover, n^l is a set of complex scalar fields forming the fundamental representation of SU(N) subject to the constraint

$$n_l^* n^l = 1 \tag{5.3}$$

($l = 1, \ldots, N$ is the SU(N) index). In Eq. (5.2) for brevity we suppress all SU(N) indices. Varying n^l we change the orientation of the string flux in the non-Abelian color subgroup SU(N). It is associated with the orientational zero modes of the non-Abelian string.

The profile functions for the scalar fields $\phi_1(r)$, $\phi_2(r)$ and for the gauge fields $f(r)$, $f_{NA}(r)$ satisfy the following boundary conditions:

$$\phi_1(0) = 0, \qquad f_{NA}(0) = 1, \qquad f(0) = 1, \tag{5.4}$$

at $r = 0$, and

$$\phi_1(\infty) = \sqrt{\xi}, \qquad \phi_2(\infty) = \sqrt{\xi},$$
$$f_{NA}(\infty) = 0, \qquad f(\infty) = 0 \tag{5.5}$$

at $r = \infty$.

To see that the strings in the M_F model are not BPS it is sufficient to note that the masses of the scalar field φ given in the second lines in Eqs. (4.14) and (4.15) are not the same as the gauge boson masses (4.12) and (4.13) (for generic values of the coupling constant γ). In the M_D model the scalars involved in the string solution are in fact the scalar superpartners of the gauge bosons from $\mathcal{N} = 1$ massive vector supermultiplets. The equality of masses of the scalar and gauge fields is ensured and protected by supersymmetry. This is the reason why the M_D model strings are BPS saturated [6].

If we considered a special set of the coupling constants,

$$g_{M1}^2 = g_{M2}^2 = \gamma/2 \equiv g^2, \tag{5.6}$$

the equality of masses of the scalar fields (4.14) and (4.15) and the gauge fields (4.12) and (4.13) would be guaranteed at the classical level. In this case the string profile functions would satisfy the following first-order equations [6,12]:

$$r\frac{d}{dr}\phi_1(r) - \frac{1}{N}(f(r) + (N-1)f_{NA}(r))\phi_1(r) = 0,$$

$$r\frac{d}{dr}\phi_2(r) - \frac{1}{N}(f(r) - f_{NA}(r))\phi_2(r) = 0,$$

$$-\frac{1}{r}\frac{d}{dr}f(r) + \frac{g^2 N}{4}[(\phi_1(r))^2 + (N-1)(\phi_2(r))^2 - N\xi] = 0,$$

$$-\frac{1}{r}\frac{d}{dr}f_{NA}(r) + \frac{g^2}{2}[(\phi_1(r))^2 - (\phi_2(r))^2] = 0. \tag{5.7}$$

For generic values of the coupling constants the string profile functions satisfy the second-order equations. The condition (5.6), even being imposed at the classical level, will be certainly destroyed by loop corrections.

Assuming (5.6) we would find the tension of the elementary string

$$T_{\text{string}} = 2\pi\xi + \text{quantum corrections}. \tag{5.8}$$

The $2\pi\xi$ is no longer exact. Without (5.6), and with no explicit solution of the second-order equations we can only say that the string tension $T_{\text{string}} \propto \xi$. The non-Abelian string discussed above presents an SU(N) rotation of the Z_N strings [12]. The rotation parameters are determined by n_l, ($l = 1, 2, \ldots N$), see Eq. (5.2). The Z_N string carries a combination of magnetic fluxes: the magnetic flux of the U(1) field as well as that of the non-Abelian fields. At the classical level the color orientation of the non-Abelian fluxes is fixed in the Cartan subgroup, in a well-defined manner. At the quantum level the color orientation of the non-Abelian fluxes strongly fluctuates, in accordance with dynamics of the world sheet $CP(N - 1)$ model, so that the entire SU(N) group space is spanned.

To conclude this section let us note a somewhat related development: *non*-BPS non-Abelian strings were recently considered in metastable vacua of a dual description of $\mathcal{N} = 1$ SQCD at $N_f > N$ in Ref. [20].

B. World sheet theory

To derive a world sheet theory describing orientational moduli n^l of the non-Abelian string we follow Refs. [9,12,21], see also the review paper [22].

Assume that the orientational collective coordinates n^l are slowly varying functions of the string world sheet coordinates x_k, $k = 0, 3$. Then the moduli n^l become fields of a $(1 + 1)$-dimensional sigma model on the world sheet. Since the vector n^l parametrizes the string zero modes, there is no potential term in this sigma model.

To obtain the kinetic term we substitute our solution, which depends on the moduli n^l, in the action (3.5), assuming that the fields acquire a dependence on the coordinates x_k via $n^l(x_k)$. Then we arrive at the $CP(N - 1)$ sigma model (for details see the review paper [22]),

$$S_{CP(N-1)}^{(1+1)} = 2\beta \int dt dz \{(\partial_k n^* \partial_k n) + (n^* \partial_k n)^2\}, \quad (5.9)$$

where the coupling constant β is given by a normalizing integral I defined in terms of the string profile functions,

$$\beta = \frac{2\pi}{g_{M2}^2} I. \quad (5.10)$$

The two-dimensional coupling constant is determined by the four-dimensional non-Abelian coupling.

Using the first-order equations for the string profile functions (5.7) one can see that the integral I reduces to a total derivative and is given by the string flux determined by $f_{NA}(0) = 1$, namely $I = 1$. However, for the non-BPS string in the problem at hand we certainly expect corrections to this classical BPS result. In particular, we expect that generically I acquires a dependence on N and coupling constants.

The relation between the four-dimensional and two-dimensional coupling constants (5.10) is obtained at the classical level. In quantum theory both couplings run. So

we have to specify a scale at which the relation (5.10) takes place. The two-dimensional $CP(N - 1)$ model is an effective low-energy theory appropriate for describing string dynamics at low energies, much lower than the inverse string thickness which, in turn, is given by $g_{M2}\sqrt{\xi}$. Thus, $g_{M2}\sqrt{\xi}$ plays the role of a physical ultraviolet (UV) cutoff in (5.9). This is the scale at which Eq. (5.10) holds. Below this scale, the coupling β runs according to its two-dimensional renormalization-group flow.

The sigma model (5.9) presents a nontrivial part of the world sheet dynamics. It is *not* supersymmetric. Besides (5.9), there are translational and supertranslational moduli; they are represented by free bosonic and fermionic fields (2 and 4 degrees of freedom, respectively). Since these are free fields they are not so important in what follows.

The sigma model (5.9) is asymptotically free [23]; at large distances (low energies) it gets into the strong coupling regime. The running coupling constant as a function of the energy scale E at one loop is given by

$$4\pi\beta = N \ln\left(\frac{E}{\Lambda_{CP(N-1)}}\right) + \cdots, \quad (5.11)$$

where $\Lambda_{CP(N-1)}$ is the dynamical scale of the $CP(N - 1)$ model. As was mentioned above, the UV cutoff of the sigma model at hand is determined by $g_{M2}\sqrt{\xi}$. Hence,

$$\Lambda_{CP(N-1)}^N = g_{M2}^N \xi^{N/2} \exp\left(-\frac{8\pi^2}{g_{M2}^2} I\right). \quad (5.12)$$

Note that in the bulk theory, due to the squark field VEVs, the coupling constant is frozen at $g_{M2}\sqrt{\xi}$.

The coupling constant g_{M2} is determined by the scale $\Lambda_{M,le}$ (see Eq. (3.12)) of the bulk monopole theory (3.5). Then Eq. (5.12) implies

$$\Lambda_{CP(N-1)} = \frac{\Lambda_{M,le}^{2I}}{(g_{M2}\sqrt{\xi})^{2I-1}} \ll \Lambda_{M,le}, \quad (5.13)$$

where we take into account that the first coefficient of the β function in (3.5) is $2N$.

Concluding this section let us add a few words on the fermion zero modes on the non-Abelian string. We first note that the index theorem presented in Ref. [6] is valid only in the M_D model. It cannot be generalized to the M_F model. Therefore, we do not expect any superorientational fermion zero modes of the string. Of course, four supertranslational fermion zero modes are guaranteed by "non-BPSness" of the string at hand. They are "Goldstinos" of the $\mathcal{N} = 1$ bulk supersymmetry broken by the string solution. They decouple from the world sheet $CP(N - 1)$ model. We have mentioned this fact above.

C. Higher-derivative corrections

The $CP(N - 1)$ model (5.9) is an effective theory which describes the non-Abelian string dynamics only at low energies. It has higher-derivative corrections which be-

come important at higher energies. Higher-derivative corrections run in powers of

$$\Delta \partial_k, \qquad (5.14)$$

where Δ is a string transverse size.

In the M_F model the string size is determined by the inverse mass of the bulk states,

$$\Delta \sim \frac{1}{g_{M2}\sqrt{\xi}}.$$

A typical energy scale on the string world sheet is $\Lambda_{CP(N-1)}$, see (5.13). Thus,

$$\partial \to \Lambda_{CP(N-1)},$$

and higher-derivative corrections can be ignored.

However, the monopole theory (3.5) is not quite the M model. In addition to the field content of the M model we have more light M fields in the bulk, namely, the Goldstone (M_S^P) and pseudo-Goldstone ($M_P^{\dot{K}}, M_L^P$) states. The fact of their presence entails that the string profile functions acquire long-range tails at the quantum level [24] (classically, as we have already mentioned, the fields M_B^A vanish on the string solution). This means that an effective string thickness grows, and higher-derivative corrections to the basic $CP(N-1)$ model on the string world sheet *might* become important.

Let us show that this does not happen. First note that the Goldstone states (M_S^P) are singlets with respect to the $SU(N)_{C+F}$ global symmetry. This means that the *orientational* zero modes of the string have no long-range tails associated with the M_S^P fields and are perfectly normalizable.

As for the pseudo-Goldstone states $M_P^{\dot{K}}$ and M_L^P, they are not singlets with respect to $SU(N)_{C+F}$. Therefore, long-range profile functions of $M_P^{\dot{K}}$ and M_L^P can acquire n^l dependence at the quantum level. Then the higher-derivative corrections associated with the pseudo-Goldstone fields will run in powers of

$$\Delta_{\mathrm{pG}} \frac{\sqrt{\xi}}{\mathcal{M}} \partial_k \sim \frac{\sqrt{\xi}\Lambda_{CP(N-1)}}{m_{\mathrm{pG}}\mathcal{M}}, \qquad (5.15)$$

where we take into account that the coupling of the pseudo-Goldstone fields to the classical profile functions of the string is suppressed, see Eq. (4.3). Since

$$\xi/(m_{\mathrm{pG}}\mathcal{M}) \sim 1$$

we conclude that the higher-derivative corrections remain to be negligible.

VI. IMPLICATIONS OF STRINGS IN THE MONOPOLE THEORY

We begin from a few technical remarks. The strings we found at weak coupling in the monopole theory are in one-

to-one correspondence with the vacua of the world sheet theory. In reviewing this correspondence we will be brief since our discussion will run parallel to that of Ref. [21] which presents the issue in great detail. The nonsupersymmetric $CP(N-1)$ model was solved by Witten in the large-N limit [25]. Interpretation of Witten's results in terms of non-Abelian strings in four dimensions can be found also in the review paper [22].

The model (5.9) can be understood as a strong coupling limit of a U(1) gauge theory. The action has the form

$$S = \int d^2x \left\{ 2\beta |\nabla_k n^l|^2 + \frac{1}{4e^2} F_{kp}^2 + 2e^2\beta^2(|n^l|^2 - 1)^2 \right\}, \qquad (6.1)$$

where $\nabla_k = \partial_k - iA_k$. In the limit $e^2 \to \infty$ the U(1) gauge field A_k can be eliminated via the (algebraic) equation of motion which leads to the theory (5.9). Moreover, the condition (5.3) is implemented in the limit $e^2 \to \infty$.

The nonsupersymmetric $CP(N-1)$ model is asymptotically free and develops its own dynamical scale $\Lambda_{CP(N-1)}$. Classically the field n^l can have arbitrary direction; therefore, one might naively expect spontaneous breaking of $SU(N)_{C+F}$ and the occurrence of massless Goldstone modes. This cannot happen in two dimensions. Quantum effects restore the full symmetry making the vacuum unique. Moreover, the condition $|n^l|^2 = 1$ gets in effect relaxed. Because of strong coupling we have more degrees of freedom than in the original Lagrangian, namely, all N fields n become dynamical and acquire masses $\Lambda_{CP(N-1)}$. They become N-plets of $SU(N)$.

The modern understanding of the vacuum structure of the $CP(N-1)$ model [26] (see also [27]) is as follows. At large N, along with the unique ground state, the model has $\sim N$ quasistable local minima, quasivacua, which become absolutely stable at $N = \infty$, see Fig. 2. The relative splittings between the values of the energy density in the adjacent minima is of the order of $1/N$, while the probability of the false vacuum decay is proportional to $N^{-1}\exp(-N)$ [26,27]. The n quanta are in fact n kinks interpolating between the genuine vacuum and the adjacent minimum. The spatial domain inside the $\bar{n}n$ meson is a
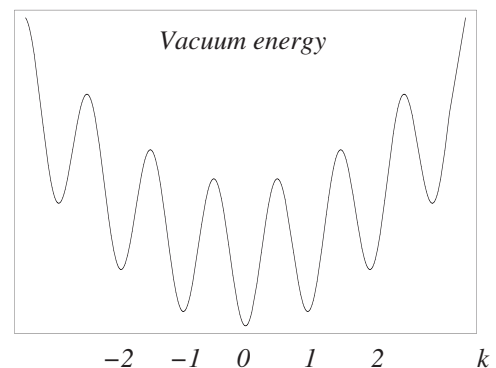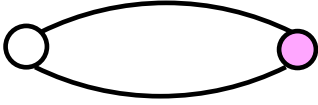
FIG. 2. The vacuum structure of the $CP(N-1)$ model.

FIG. 3 (color online). The junction-antijunction meson. The binding is due to strings.

"bubble" of an excited quasivacuum state inside the true vacuum—that's why the n kinks are confined along the string.

In the four-dimensional bulk theory the above vacua correspond to a variety of non-Abelian strings. Classically all these strings have the same tension. Because of quantum effects in the world sheet theory the degeneracy is slightly lifted. Excited strings can in principle decay into the ground-state string, but at large N their lifetimes tend to infinity.

Now, let us ask ourselves: what is the physical meaning of these strings?

Non-Abelian strings are formed in the monopole theory (3.5) upon the h-field condensation, see (3.10). The dual quark field h represents monopoles of the quark theory. Thus, from the standpoint of the original quark theory the strings must be interpreted as some flux tubes filled in by a chromoelectric field in a highly quantum regime. The string junctions of different elementary strings in the monopole theory—"monopoles" of the monopole theory—are seen as kinks n in the effective theory on the string world sheet, see [22] for details. They must act as some quarklike objects in the original theory. These objects transform according to the representation $\underline{N}$ of unbroken global SU$(N)_{C+F}$.

The monopole-theory strings can form closed curves (e.g. tori) stabilized by angular momentum. They are to be interpreted as sort of glueballs. In addition, there are meson states formed by junctions connected by non-Abelian strings, see Fig. 3. These mesons can belong either to the singlet or to the adjoint representations of the global unbroken SU$(N)_{C+F}$ symmetry. Both types of objects have masses $\sim \sqrt{\xi}$.

VII. LOW-ENERGY SPECTRA AND DUALITY

First let us summarize the low-energy spectrum of the bulk theory as it is seen from the perspective of the monopole theory. The lowest are the Goldstone and pseudo-Goldstone states, M_S^P and $\{M_P^{\check{K}}, M_{\check{K}}^P\}$, respectively. The latter have masses determined by $m_q = \xi/\mathcal{M}$, see (4.4). From the point of view of the quark theory these states can be understood as quarks screened by the condensate of massless squarks. The Goldstone states are singlets with respect to the unbroken SU$(N)_{C+F}$ while the pseudo-Goldstone states transform in the fundamental representation of this group.

Next come states with masses of the order of $\sqrt{\xi}$. This set includes elementary excitations: gauge and h multiplets

as well as the fields $M_{\check{K}}^{\check{L}}$. Their masses are determined by Eqs. (4.12), (4.13), (4.14), and (4.15), see Sec. IV. These states transform in the singlet or adjoint representations of SU$(N)_{C+F}$. In addition, the set includes composite nonperturbative states of the type we discussed at the end of Sec. VI. The latter are believed to be metastable (rather than stable) as they can decay into the massive gauge/monopole multiplets (with masses (4.12) or (4.14)) with the appropriate quantum numbers with respect to the global SU(N).

In addition to mesons in the low-energy part of the spectrum one can speak of "baryons" built of N junctions cyclically connected to each other by elementary strings which form a closed "necklace configuration." (Here N is not treated as an infinitely large parameter. Of course, if $N \to \infty$, the baryons are out of the game.) The baryon is in the $\prod_1^N(N)$ representation of SU$(N)_{C+F}$.

Note that both quarks and monopoles do not carry the baryon numbers. Therefore, our baryon has no baryon number either. The reason for this is that the U(1) baryon current is coupled to a gauge boson (photon) in the U(N) gauge theory considered here. Moreover, the U(1) gauge symmetry is spontaneously broken in the quark and the monopole theories by condensation of quarks and dual quarks, respectively. Thus, the baryon charges are screened. This means, in particular, that baryons can decay into mesons or gauge/monopole multiplets and are in fact unstable.

All these nonperturbative states reflect the existence of "thick" strings with tension scaling as ξ and thickness proportional to $\xi^{-1/2}$.

All states with masses of the order of $\sqrt{\xi}$ will eventually decay into the Goldstone or pseudo-Goldstone mesons. Say, a meson in Fig. 3 in the adjoint representation with respect to the global SU(N) can decay into a pair of pseudo-Goldstone states. However, these decays are suppressed by the smallness of the ratio $\sqrt{\xi}/\mathcal{M}$.

Now it is time to discuss the most interesting question: if the low-energy states in the quark and monopole theories are connected by duality how can one interpret the set of states we uncovered in the monopole theory in the language of the quark theory?

The quark theory is strongly coupled. Quantitative predictions are virtually impossible. Still we do have some qualitative knowledge of this theory. In the quark theory color is screened since the theory is fully Higgsed. There are matter fields in the fundamental representation. Therefore long strings cannot exist. They are screened/ruptured immediately. On the dual side we do see strings, however. The scale of the string-induced confinement $\sqrt{\xi}$ is small in the original quark theory, much smaller than its dynamical scale, $\xi \ll \Lambda_Q^2$.

This apparent puzzle can be resolved if we assume that a "secondary" gauge theory (or a "gauge cascade") develops in the original quark theory. Assume that massless

composite "ρ mesons" whose size is $\sim \Lambda_Q^{-1}$ are formed in the quark theory which interact with each other via a secondary gauge theory whose scale parameter is $\sqrt{\bar{\xi}}$. At distances $\sim 1/\sqrt{\bar{\xi}}$ the above "ρ mesons" must be viewed as massless gluons. It is conceivable that they are coupled to massless secondary quarks which, in addition to their gauge coupling to ρ mesons, have nontrivial quantum numbers with respect to the global SU(N). With respect to the original quark theory the secondary quarks are colorless ("bleached") bound states which include the original quarks at their core. Their sizes are proportional to $\sim \Lambda_Q^{-1}$ and, hence, they are pointlike on the scale of $\sim 1/\sqrt{\bar{\xi}}$, much in the same way as ρ mesons-gluons.

Alternatively one can adopt a more pragmatic albeit less explicit point of view. If we trust duality we can view the predictions derived in the monopole theory as certain data to be interpreted in terms of the quark theory. One can think that these predictions for the quark theory are similar to experimental data for QCD.

Following this line of thought we can interpret the light spectrum seen from the monopole theory as follows. First of all, we note that all states in the physical spectrum of the monopole theory are colorless. We interpret this as confinement in the quark theory. Next, besides Goldstone and pseudo-Goldstone states (which we identify with $\tilde{Q}_S Q^P$ and $\tilde{Q}_S Q^{\dot{K}}$ states) the monopole theory predicts the occurrence of a set of light states with masses of the order of $\sqrt{\bar{\xi}}$. We interpret them in terms of the quark theory according to their global flavor quantum numbers. Say, we interpret singlets with respect to the global SU(N) as glueballs with possible admixture of quark-antiquark states in the singlet representation. Next, we interpret perturbative and nonperturbative states of the monopole theory in the adjoint representation of SU(N) as adjoint quark-antiquark mesons or more complicated "exotic" multiquark states.

Note, however, that we do not attempt to interpret the monopole-theory string junctions as the fundamental quarks $Q^{\dot{K}}$ of the quark theory. There is a number of reasons why this identification does not work. The monopole-theory string junctions should be rather understood as a kind of "constituent quarks" which form mesons in Fig. 3. Although these constituent quarks are in the fundamental representation of the global flavor SU(N) their relation to the fundamental quarks $Q^{\dot{K}}$ of the quark-theory Lagrangian remains unclear.

VIII. CONCLUSIONS

Our starting point is a Seiberg's dual pair with electric theory lying to the left of the conformal window and the magnetic theory to the right. The electric theory is strongly coupled while the magnetic one is infrared free. Our basic idea is to deform the electric theory very weakly—with all deformations being very small in its natural scale Λ_Q— and, nevertheless, they are sufficient to drastically change the infrared behavior of the magnetic dual. It switches from infrared free to asymptotically free. Seiberg's infrared duality now extends beyond purely massless states; it connects with each other light states on both sides of duality.

Upon condensation of the dual quarks, the dual theory supports non-Abelian flux tubes (strings). Being interpreted in terms of the quark theory these flux tubes are supposed to carry chromoelectric fields. The string junctions then can be viewed as constituent quarks of the original quark theory. We interpret closed strings as glueballs of the original quark theory. Moreover, there are string configurations formed by two junctions connected by a pair of different non-Abelian strings. These can be considered as constituent quark mesons of the quark theory. Most of these states are quasistable rather than stable. They can cascade into the lightest Goldstones and pseudo-Goldstones.

The constituent quarks could result from emergent "secondary gauge theory" on the electric side.

APPENDIX A: NOTATION

Flavor indices from SU($N_c + N$) are denoted by capital letters from the beginning of the Latin alphabet,

$$A, B, \ldots = 1, 2, \ldots, N_c, \ldots, N_c + N. \qquad \text{(A1)}$$

Flavor indices from SU(N_c) are denoted by capital letters from the middle of the Latin alphabet,

$$P, S, \ldots = 1, 2, \ldots, N_c. \qquad \text{(A2)}$$

Flavor indices from SU($N_c + N$)/SU(N_c) are denoted by overdotted capital letters from the middle of the Latin alphabet,

$$\dot{K}, \dot{L}, \ldots = N_c + 1, \ldots, N_c + N. \qquad \text{(A3)}$$

Color indices of the fundamental representation of SU(N_c) (and SU(N) in the monopole theory) are denoted by lowercase letters from the middle of the Latin alphabet,

$$k, l, \ldots = 1, 2, \ldots N_c \quad \text{or} \quad k, l, \ldots = 1, 2, \ldots N. \qquad \text{(A4)}$$

Color indices of the adjoint representation of SU(N_c) (and SU(N) in the monopole theory) are denoted by lowercase letters from the beginning of the Latin alphabet,

$$a, b, \ldots = 1, 2, \ldots, N_c^2 - 1 \quad \text{or}$$
$$a, b, \ldots = 1, 2, \ldots, N^2 - 1. \tag{A5}$$

The bosonic part of the action of the M_F model is

$$S = \int d^4x \left[\frac{1}{4(g_{M2})^2}(F_{\mu\nu}^a)^2 + \frac{1}{4(g_{M1})^2}(F_{\mu\nu})^2 + |\nabla_\mu h^{\dot{K}}|^2 \right.$$
$$+ |\nabla_\mu \bar{\tilde{h}}^{\dot{K}}|^2 + \frac{2}{\gamma} \operatorname{Tr}|\partial_\mu M|^2 + \frac{g_2^2}{2}(\bar{h}_{\dot{K}} T^a h^{\dot{K}}$$
$$- \tilde{h}_{\dot{K}} T^a \bar{\tilde{h}}^{\dot{K}})^2 + \frac{g_1^2}{8}(\bar{h}_{\dot{K}} h^{\dot{K}} - \tilde{h}_{\dot{K}} \bar{\tilde{h}}^{\dot{K}})^2 + \operatorname{Tr}|hM|^2$$
$$\left. + \operatorname{Tr}|\bar{\tilde{h}} M|^2 + \frac{\gamma}{2} \left| \tilde{h}_{\dot{K}} h^{\dot{M}} - \frac{1}{2} \delta_{\dot{K}}^{\dot{L}} \delta_{\dot{L}}^{\dot{M}} \xi \right|^2 \right\}. \tag{A6}$$

APPENDIX B: THE 'T HOOFT ANOMALY CONDITIONS

That the quark and monopole theories are dual to each other in the limit $\mathcal{Q} = m_q = 0$ follows from Seiberg's construction. Here we reiterate the analysis of the 't Hooft anomaly matching relevant to our particular deformation of the theory. There are certain peculiarities since the matching looks different at "high energies" (i.e. when the momentum q flowing in the axial current is "large," $q^2 \gg \xi$) and "low energies" ($q^2 \ll \xi$). We check both limits.

As was already mentioned in Sec. II our theory has the following global symmetry at the Lagrangian level

$$\mathrm{SU}(N_c + N)_L \times \mathrm{SU}(N_c + N)_R \times \mathrm{U}(1)_R \times \mathrm{U}(1)_B, \tag{B1}$$

where we also included the vectorlike baryon $\mathrm{U}(1)_B$ symmetry which is in fact gauged in our model. The above symmetry is nonanomalous with respect to the non-Abelian gauge currents [1,4]. The fields of the quark and monopole theories transform as

$$Q: \left(N_c + N, 1, \frac{N}{N_c + N}, \frac{1}{2} \right),$$
$$\tilde{Q}: \left(1, N_c + N, \frac{N}{N_c + N}, -\frac{1}{2} \right),$$
$$h: \left(\bar{N}_c + \bar{N}, 1, \frac{N_c}{N_c + N}, \frac{N_c}{2N} \right), \tag{B2}$$
$$\tilde{h}: \left(1, N_c + N, \frac{N_c}{N_c + N}, -\frac{N_c}{2N} \right)$$

under this symmetry, while the Grassmann θ parameters have the unit charge under $\mathrm{U}(1)_R$ [1,4].

Consider first the quark theory. Classically the tree-level symmetry is broken down to

$$\mathrm{SU}(N) \times \mathrm{U}(1)_{R'} \tag{B3}$$

by the condensation of the Q^P fields and masses m_q for $Q^{\dot{K}}$ fields. Note that the R symmetry $\mathrm{U}(1)_{R'}$ classically survives

the breaking. It is a combination of $\mathrm{U}(1)_R$ and an axial subgroup of non-Abelian factors in (B1) which do not transform quark fields Q^P. It turns out that the fermion superpartners of squarks $(\psi_Q)^{\dot{K}}$ and $(\tilde{\psi}_Q)_{\dot{K}}$ have zero charges with respect to this symmetry. Therefore, it is not broken by masses m_q.

Quark fields of the quark theory transform as

$$\begin{aligned} Q^{\dot{K}}&: (N, 1); & (\psi_Q)^{\dot{K}}&: (N, 0), \\ Q^P&: (1, 0); & (\psi_Q)^P&: (1, -1), \\ \tilde{Q}_{\dot{K}}&: (\bar{N}, 1); & (\tilde{\psi}_Q)_{\dot{K}}&: (\bar{N}, 0), \\ \tilde{Q}_P&: (1, 0); & (\tilde{\psi}_Q)_P&: (1, -1) \end{aligned} \tag{B4}$$

under the classically unbroken symmetry (B3), while the gauginos of the quark theory (λ_Q) transform as (1, 1).

In quantum theory, however, the $\mathrm{U}(1)_{R'}$ is anomalous with respect to the Abelian $\mathrm{U}(1)$ gauge currents (baryonic $\mathrm{U}(1)_B$ currents). At high energies, well above the scale of the $\mathrm{U}(1)_B$ symmetry breaking, the anomaly $\mathrm{U}(1)_{R'}\mathrm{U}(1)_B^2$ is proportional to

$$-2(\tfrac{1}{2})^2 N_c^2 = -\tfrac{1}{2} N_c^2, \tag{B5}$$

which comes from the contribution of Q^P fermions. Here we take into account that we have N_c colors and N_c flavors of these fermions. At low energies the $\mathrm{U}(1)_B$ charges are screened by the Higgs mechanism and the anomaly effectively disappears.

Now consider the monopole theory. It has classically unbroken

$$\mathrm{SU}(N)_{C+F} \times \mathrm{U}(1)_{R''} \tag{B6}$$

symmetry, where $\mathrm{U}(1)_{R''}$ is a combination of the original $\mathrm{U}(1)_R$ and axial subgroup of non-Abelian factors in (B1) which do not transform $h^{\dot{K}}$ and $\tilde{h}_{\dot{K}}$ fields.

Now, duality suggests to us to identify two global classically unbroken groups (B3) and (B6) of the quark and the monopole theories. The monopole fields h transform as

$$\begin{aligned} h^{\dot{K}}&: (\bar{N}, 0); & \psi^{\dot{K}}&: (\bar{N}, -1), \\ h^P&: (1, 1); & \psi^P&: (1, 0), \\ \tilde{h}_{\dot{K}}&: (N, 0); & \tilde{\psi}_{\dot{K}}&: (N, -1), \\ \tilde{h}_P&: (1, 1); & \tilde{\psi}_P&: (1, 0) \end{aligned} \tag{B7}$$

under the classically unbroken symmetry (B6). The M fields of the monopole theory transform as

$$\begin{aligned} M_{\dot{L}}^{\dot{K}}&: (N^2, 2); & (\psi_M)_{\dot{L}}^{\dot{K}}&: (N^2, 1); & M_P^{\dot{K}}&: (N, 1); \\ (\psi_M)_P^{\dot{K}}&: (N, 0); & M_S^P&: (1, 0); & (\psi_M)_S^P&: (1, -1) \end{aligned} \tag{B8}$$

where ψ_M are fermion superpartners of the scalar fields M. The λ fermions transform as (1, 1). Note that the $\mathrm{U}(1)_{R''}$ symmetry is not broken by the condensation of the M_S^P fields because these fields are neutral under this symmetry.

However, much in the same way as in the quark theory, the R symmetry $U(1)_{R''}$ is anomalous with respect to the U(1) gauge currents. At high energies, above the scale of the $U(1)_B$ symmetry breaking, the anomaly $U(1)_{R''}U(1)_B^2$ is proportional to

$$-2\left(\frac{N_c}{2N}\right)^2 N^2 = -\frac{1}{2}N_c^2, \tag{B9}$$

which comes from the contribution of $h^{\check{K}}$ fermions. Here we take into account that we have N colors and N flavors of $h^{\check{K}}$ fermions as well as their baryon charges, see (B2). We see that the anomaly in the monopole theory matches with the one in the quark theory (B5). At low energies the $U(1)_B$ charges are screened and the anomaly effectively disappears.

Since the R symmetry is classically unbroken we can check the 't Hooft anomaly matching conditions for the quark and monopole theories. This calculation is quite similar to that reported in [1]. The first anomaly to check is

$$SU(N)^2 U(1)_R: \quad 0 = -N + N, \tag{B10}$$

where the left-hand side and the right-hand side at high energies are given by the quark and monopole theories, respectively. On the monopole-theory side we take into account the contributions of the h and M fermions. At low energies both h and M fermions become massive and do not contribute to the anomaly. The matching condition is trivially satisfied.

Next we check

$$U(1)_R^3: \quad (-1)^3 2N_c^2 + (N_c^2 - 1)$$
$$= (-1)^3 2N^2 + (N^2 - 1) + N^2 + (-1)^3 N_c^2, \tag{B11}$$

where at high energies on the quark-theory side we take into account the contributions of the Q^P fermions and λ_Q fermions, while four contributions on the monopole-theory side are associated with the h fermions, λ's, and M_L^K and M_S^P fermions, respectively.

At low energies this anomaly matching becomes

$$U(1)_R^3: \quad (-1)^3 N_c^2 = (-1)^3 N_c^2, \tag{B12}$$

where we take into account that on the quark-theory side half of the Q^P fermions[6] and all λ_Q fermions become massive and do not contribute to the anomaly, while on the monopole-theory side the only contribution comes from the massless M_S^P fermions.

Finally, the last anomaly matching to check is

$$U(1)_R: \quad -2N_c^2 + (N_c^2 - 1) = -2N^2 + (N^2 - 1)$$
$$+ N^2 - N_c^2, \tag{B13}$$

where at high energies the quark-theory contribution comes from the Q^P fermions and λ_Q's, while the monopole-theory contribution comes from the h fermions, λ's, and M_L^K and M_S^P fermions, respectively. At low energies we have

$$U(1)_R: \quad -N_c^2 = -N_c^2, \tag{B14}$$

where the contribution on the quark-theory side comes from a half of the Q^P fermions (those which are massless), while the contribution on the monopole-theory side comes from the M_S^P fermions.

We see that all anomalies match.

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[6]In the bosonic sector only $2N_c^2$ squarks Q^P (out of $4N_c^2$ squark fields) are massless, see (2.2). A similar reduction happens in the fermionic sector by supersymmetry.

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