Lorentzian and signature changing branes

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General hypersurface layers are considered in order to describe braneworlds and shell cosmologies. No restriction is placed on the causal character of the hypersurface which may thus have internal changes of signature. Strengthening the results in our previous paper [M. Mars, J. M. M. Senovilla, and R. Vera, Phys. Rev. Lett. **86**, 4219 (2001).], we confirm that a good, regular, and consistent description of signature change is achieved in these brane/shells scenarios, while keeping the hypersurface and the bulk completely regular. Our formalism allows for a unified description of the traditional timelike branes/shells together with the signature changing, or pure null, ones. This allows for a detailed comparison of the results in both situations. An application to the case of hypersurface layers in static bulks is presented, leading to the general Robertson-Walker geometry on the layer—with a possible signature change. Explicit examples on anti-de Sitter bulks are then studied. The permitted behaviors in different settings (Z_2-mirror branes, asymmetric shells, signature changing branes) are analyzed in detail. We show, in particular, that (i) in asymmetric shells there is an upper bound for the energy density, and (ii) that the energy density within the brane vanishes when approaching a change of signature. The description of a signature change as a ‘‘singularity’’ seen from within the brane is considered. We also find new relations between the fundamental constants in the brane/shell, its tension, and the cosmological and gravitational constants of the bulk, independently of the existence or not of a change of signature.

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I. INTRODUCTION

In a previous paper [1], we explicitly showed that braneworld models [2–4] constitute a natural scenario for the classical *regular* description of a change of signature in the physical spacetime. This can also be said in general of any higher-dimensional theory (e.g. [5,6] and references therein) admitting domain walls or (hyper)surface layers. The same idea, together with the possibility of topology change, was later advocated in [7] by constructing explicit solutions of an action given by the area/volume functional of the brane.

In this paper we want to elaborate on the ideas put forward in [1], but keeping the full generality so that we can also recover the traditional results usually derived for pure timelike branes or shells, see [8–15]. We also want to present detailed proofs of several interesting results merely announced in [1].

In geometrical terms, branes, or shells are submanifolds of a higher-dimensional spacetime called the bulk. The bulk metric is differentiable everywhere except on the brane where it is only continuous. The jump in the derivatives of the bulk metric is related to the part of the *distributional* energy-momentum tensor with support on the brane. Traditionally, branes have been assumed to be timelike submanifolds so that the induced geometry is Lorentzian and the brane can describe the four-dimensional spacetime where we live. In this case, the precise relation between the jumps in the metric derivatives and the energy-momentum quantities on the brane is given by the so-called Israel conditions [16].

However, the timelike assumption is too strong: on physical grounds, it is enough that there is a region of the brane where it is timelike. *A priori*, there is no physical or mathematical obstruction to the existence of *completely regular* branes—in totally regular bulks—which change its causal character. As a trivial example, consider a circumference centered at the origin of two-dimensional Minkowski spacetime: this has two spacelike regions, two timelike ones, and four points where it is null. Therefore, in these higher-dimensional scenarios with branes or shells, the study of signature change becomes the geometrical analysis of embedded submanifolds in the bulk: a well-posed mathematical problem.

As we shall see, a very interesting property of this type of signature changing branes is that, even though the change of signature may appear as a dramatical event when seen from *within the brane*—specially if the scientists living there believe that their universe is Lorentzian everywhere—both the bulk and the brane can be totally smooth. As a matter of fact, we shall see that the change of signature occurs at a region in the brane which might be *interpreted* as a curvature singularity by those scientists. Of course, this opens the door to explain, or avoid, the classical singularities of general relativity [17,18]. For instance, a past big bang singularity may be replaced by a signature changing set leading to a Euclidean region prior to the birth of time, or a neighborhood of the singularity inside a black hole by a kind of Euclidean core. Explicit examples of these situations were actually built in [1,19], see also [7]. This has natural and obvious links with the no-boundary proposal [20] for the prescription of the wave function of the Universe in quantum cosmology, and with similar ideas of quantum tunneling [21,22] from "nothing" or from instantons.

From a classical standpoint, changes of signature were treated in the literature mainly from the inner point of view, by just considering a manifold with a metric which becomes degenerate somewhere and changes signature, see [23–26] and references therein. There was much debate on whether the transition between the Euclidean and the Lorentzian regions should occur smoothly or with a jump, see [27] and references therein. Both cases can be treated in the brane scenario that we proposed in [1]. The most natural case, though, is when the brane or shell is differentiable and the signature change is therefore smooth.[1] Some explicit signature changing solutions of the field equations for scalar field sources [31,32] and for spinor fields [33,34] have been found, as well as in higher-dimensional cases with compact extra dimensions [35], or for the spherically symmetric case [36]. In any case, the differences between our approach and those intrinsic treatments are radical, as we have a bulk structure available which defines inherited regular structures on the brane. As a matter of fact, we can even prove that some *ad hoc* assumptions in [23–27] become *necessary conditions* in our setting, precisely because of this bulk structure, see Sec. II A.

In order to describe signature changing branes we need to consider hypersurfaces without a fixed causal character, so that they may have timelike, spacelike, and null portions. There are some obvious technical difficulties when dealing with hypersurfaces of such an arbitrary causal character. For instance, the first fundamental form is degenerate somewhere, and also the second fundamental form is no longer *extrinsic* everywhere—it is actually intrinsic at null points. This leads to the most important difficulty: the usual matching conditions are no longer valid for hypersurfaces with changing causal character. In particular, the Israel formulas [16] are not suitable to describe the energy-momentum on the shell or brane, and the appropriate generalization must be used. Probably this has been the reason behind the lack of studies on signature changing branes prior to [1], and also of some misunderstandings[2] in the interesting recent work [37]. Fortunately, the required generalization was already developed in [38] in four spacetime dimensions. The results carry over to any dimension with no essential change and can therefore be used to study signature changing branes. A self-consistent summary of the required results from [38] is presented in Secs. II and III.

When dealing with changes of signature, there always remains the important unsolved question of which physical mechanisms may produce, or induce, them. Several speculative possibilities have appeared in the literature, such as large time fluctuations [39], tachyon condensation and S branes [40–43], dynamical stabilization of extra dimensions by means of scalar fields [44], or emergent spacetimes in Bose-Einstein condensates [45]. In particular, one should be able to devise a physical process stimulating the signature change on a brane of an otherwise innocuous bulk. As far as we are aware, this is an important and fully open question, which we shall not address in this paper.

Plan of the paper and summary

The basics of gluing and how to construct branes or shells by pasting together two spacetimes with boundary is presented in Sec. II. When the boundaries have a non-constant signature, one needs the results of [38,46] to perform the matching correctly. These are described, and corrected, in that section. In particular, we also correct an erroneous statement in our paper [1]. In this section we also prove rigorously that signature changing branes or shells are not compatible with the Z_2-mirror symmetry assumed many times in braneworlds.

The generalized Israel formula is then presented and briefly explained in Sec. III. Section IV is devoted to the field equations on the bulk and their consequences on the brane or shell. In particular, we prove some statements announced in [1]: there cannot be umbilical hypersurfaces changing signature, and the brane tension cannot be constant in signature changing branes.

Readers familiar with these matters may skip the mentioned sections and go directly to the more interesting physical results discussed afterwards.

Sections V and VI deal with the explicit construction of (signature changing or not) branes and shells in static and

[1]By allowing continuous piecewise differentiable hypersurfaces, we could also describe discontinuous changes of signature. This would require a detailed knowledge of the matching conditions within a submanifold which, itself, is a matching hypersurface between the two bulk subregions. The mathematical tools needed for that purpose have only appeared recently in the literature [28]. Incidentally, a similar comment holds for standard timelike branes whenever the energy-momentum tensor within the brane has jumps, as for instance on the surface of a star. A proper mathematical description of such a situation would require the results in [28]. Let us remark that several papers have actually dealt with stars on the brane, e.g. [29,30]. However, in those papers the whole description is made from within the brane, with standard matching conditions across the surface of the star. It would be interesting to perform a full bulk description and compare the results.

[2]In [37] the traditional Israel formula (9) is applied to a Z_2-symmetric brane, and then the brane is claimed to undergo a change of signature. Nevertheless, for a signature changing brane the Israel conditions (9) are not valid, and the appropriate generalized formula (11) should have been used. Observe that, in fact, a Z_2-symmetric brane can never undergo a signature change, according to our general result Corollary 2.

spherically, plane, or hyperbolically symmetric bulks. We prove that, by gluing two such spacetimes across any hypersurface preserving the spatial symmetries, a brane or a shell is obtained which has the general Robertson-Walker line element, with a possible change of signature, as first fundamental form. The physical quantities of these branes are then computed in general. Section VI specializes these results to the case of two (AdS) spacetimes, which produces a bulk with two, different in principle, cosmological constants. The different possibilities are then analyzed in detail, and we recover all previous results on Z_2-symmetric branes and asymmetric shells.

We also derive the corresponding new results for the signature changing branes or shells, and we prove that the boundary of the Lorentzian part of the brane is part of a signature changing set which is completely regular. We further show that the energy density of the matter fields *vanishes* when approaching this set. The possible interpretation of this set as a curvature singularity for observers living within the Lorentzian part of the brane is carefully considered. Finally, old *and new* relations between the fundamental constants in the brane and the parameters of the bulk are derived in some physically motivated limits.

Throughout the paper, we will use units with the speed of light $c = 1$.

II. SIGNATURE CHANGING HYPERSURFACES: BASIC PROPERTIES AND BRANE CONSTRUCTION BY GLUING

In general, branes are submanifolds in a higher-dimensional spacetime $(\mathcal{M}, g)$, which is called the bulk, with g being a Lorentzian metric of signature $(-, +, \ldots, +)$. Such objects had been traditionally known (specially for the case of codimension one) as thin layers or domain walls. The typical branes have a constant causal character, usually timelike. However, the purpose of this paper is to analyze the possibility of having more general branes such that their causal character may change from point to point. Thus, *signature changing branes* are submanifolds of changing causal character in $(\mathcal{M}, g)$. In this paper we will present a unified formalism which allows to deal with general types of branes, signature changing and signature-constant ones, at the same time.

In order to have a topological defect such as a codimension one brane, *and* to have well-defined Einstein's field equations on the entire bulk, the metric g needs to be at least of class C^2 everywhere on $\mathcal{M}$ except on the brane Σ, where it should only be continuous—in a suitable coordinate system. Thus, the differentiability of the manifold $\mathcal{M}$ must be at least C^3, which we will assume from now on. C^3 manifolds with C^2 Lorentzian metrics will be called C^2 spacetimes from now on. The brane Σ is a hypersurface and hence it inherits a first fundamental form h which must also be at least C^2 in order to admit gravitational field equations within the brane.[3] Consequently, as a submanifold Σ must be at least C^3 too. Let $\boldsymbol{N}$ be a normal one-form of Σ, i.e. a nonzero one-form satisfying

$$\boldsymbol{N}(\vec{v}) = 0 \quad \forall\ \vec{v} \in T_p\Sigma, \quad \forall\ p \in \Sigma$$

so that it annihilates all vectors *tangent* to Σ. Raising the index of $\boldsymbol{N}$ we obtain a vector field $\vec{N}$ which may still be called a "normal vector field," but which is not necessarily transverse to Σ everywhere. From our assumptions, $\boldsymbol{N}$ and $\vec{N}$ are differentiable fields. Observe that, if we want to allow for signature changes on the brane, $\boldsymbol{N}$ cannot be globally normalized as it is null somewhere. Thus, $\boldsymbol{N}$ is defined only up to rescaling $\boldsymbol{N} \to A\boldsymbol{N}$, where A is a nowhere vanishing C^2 function on Σ. This "normalization" freedom plays an important role in the physics of the brane and needs to be kept in mind. Since the bulk metric is continuous across Σ, the norm (Greek lowercase indices run from $0, \cdots, n-1$, where n is the dimension of the bulk)

$$\boldsymbol{N}(\vec{N}) \equiv (\boldsymbol{N}, \boldsymbol{N}) \equiv g^{\mu\nu} N_\mu N_\nu$$

is well defined on Σ and of class C^2. For the signature of the brane to change, the set of points where Σ is null must be nonempty. Thus, to fix ideas and notation, we put forward the following.

Definition 1—Let $\Sigma_E \subset \Sigma$, $\Sigma_0 \subset \Sigma$, and $\Sigma_L \subset \Sigma$ be the subsets where the hypersurface Σ is spacelike, null, and timelike, respectively. Equivalently,

$$\Sigma_E \equiv \{p \in \Sigma : (\boldsymbol{N}, \boldsymbol{N})|_p < 0\},$$
$$\Sigma_0 \equiv \{p \in \Sigma : (\boldsymbol{N}, \boldsymbol{N})|_p = 0\},$$
$$\Sigma_L \equiv \{p \in \Sigma : (\boldsymbol{N}, \boldsymbol{N})|_p > 0\}.$$

Accordingly, the induced metric h is positive definite at Σ_E, Lorentzian at Σ_L, and degenerate at Σ_0. Then, Σ_E is called the *Euclidean phase* of the brane, Σ_L its *Lorentzian phase*, and Σ_0 its *null phase*. Finally, the set

$$S \equiv (\overline{\Sigma_L} \cap \Sigma_0) \cup (\overline{\Sigma_E} \cap \Sigma_0)$$

is called the *signature changing set* of Σ.

By definition Σ_0 is a closed subset of Σ. Also by definition we have $S \subset \Sigma_0$. The case when Σ_0 has empty interior is characterized by $S = \Sigma_0$, and will be one of the important cases in our analysis. Note also that $\boldsymbol{N}(\vec{N})|_{\Sigma_0} = 0$ so that $\vec{N}$ is actually tangent to Σ on Σ_0, see [38].

We will implicitly assume that Σ_L is nonempty so that we have at least one region where the brane is timelike and therefore able to describe a real (Lorentzian) world. Notice, though, that it is still possible that both S and Σ_L are nonempty while $\Sigma_E = \varnothing$. In fact it is even possible to

[3] If objects like stars, with discontinuous energy-momentum tensors, are present on the brane, then the C^2 differentiability of h holds only outside the separating surfaces.

have branes which are timelike everywhere except at a single point, where it is null. A simple example is given by the hypersurface

$$\{x = a(t)\cos\theta, y = b\sin\theta, t\}$$

in 3-dimensional Minkowski spacetime with Cartesian coordinates $\{t, x, y\}$, where $b > 0$ is a constant and $a(t)$ is a positive function whose derivative satisfies $|\dot{a}| \leq 1$ with equality at one single value. Similarly, branes which are spacelike everywhere except for a single point are possible, as well as *null branes* so that $\Sigma_L = \Sigma_E = \emptyset$. Most of these situations do not truly describe a signature changing brane, or at least not the one we usually have in mind, which require that both Σ_L and Σ_E—and therefore S too—are nonempty. In this situation it is obvious that S cannot consist of a finite number of points. Even though our main goal in this paper is proper signature changing branes, *all* mentioned cases are included and can be treated within our formalism. In the explicit examples, however, we will mainly deal with proper signature changing branes with $\Sigma_0 = S$, i.e. such that there is no open set where the hypersurface Σ is null.

A. Restrictions on the signature changing set S

Even if one assumed that Σ_0 has empty interior there remains a lot of freedom on the structure of S. In a general setting, not necessarily of brane type, the signature changing set S may have many different structures. Nevertheless, this is no longer true in a brane-in-bulk setting, which is a desirable outcome, because conditions on S which are typically assumed *ad hoc* become predictions in this scenario. As a matter of fact, in our paper [1] we claimed that one of the advantages of studying signature change within the brane scenario is that the structure of S becomes restricted. While this general claim remains true, see Lemma 1 below, the specific result on the structure of S presented in [1] is unfortunately false. We are grateful to E. Aguirre-Dabán and J. Lafuente-López [47] for pointing out that Result 1.1 in [1] is not correct. Let us describe this in detail.

Result 1.1 in [1] states that in the brane scenario, changes of signature occur at a single "instant of time." In other words, that S is a spacelike $(n-2)$-submanifold of the bulk. If we define, as usual, the radical of a degenerate metric h as the set of vectors $\vec{V}$ satisfying $h(\vec{V}, \cdot) = 0$, the claim above amounts to saying that the first fundamental form h of Σ at $p \in S$ has a transverse radical (i.e. that the radical is nowhere tangent to S). A detailed study of signature changes with *tangent* radical (i.e. such that the degeneration vectors are tangent to the signature changing set S) has been performed in [48]. This analysis was done in full generality, without assuming that the signature changing space (Σ, h) is a brane within a bulk. From these general results, explicit examples of signature changes for branes with tangent radical may be derived [47]. One such example is as follows. For signature changes with $\Sigma_0 = S$ and tangent radical there exists [48] a coordinate system $\{y, x^i, v\}$ $(i, j = 3, \cdots, n-1)$ on a neighborhood of any point $p \in S$ such that S: $\{y = 0\}$ and the signature changing "metric" reads

$$ds^2|_\Sigma = dy^2 + y(g_2 dv + g_i dx^i)^2 + g_{ij}dx^i dx^j,$$

where g_2, g_i, and g_{ij} are differentiable functions of (y, x^i, v) such that $g_2(0, x^i, v) = 1$ and (g_{ij}) is positive definite. This tensor can be obtained as the first fundamental form of the hypersurface Σ: $\{t = 0\}$ in an n-dimensional bulk spacetime with metric

$$ds^2 = ydt^2 + 2kdtdv + dy^2 + y(k_2 dv + k_i dx^i)^2 + k_{ij}dx^i dx^j,$$

where k, k_i, k_2, and k_{ij} are functions of (t, y, x^i, v) satisfying $k_2|_{t=0} = g_2$, $k_i|_{t=0} = g_i$, $k_{ij}|_{t=0} = g_{ij}$ and k is chosen so that ds^2 has Lorentzian signature everywhere. To see an explicit example (in four dimensions, for definiteness) consider the metric

$$ds^2 = f(dt^2 + dv^2) + 2(\sqrt{1+f^2})dtdv + dy^2 + dz^2, \tag{1}$$

which is a globally defined, smooth, Lorentzian metric on $\mathbb{R}^4$ for any smooth choice of $f(t, v, y, z)$. Take $f = y$ and the brane defined by Σ: $\{t = 0\}$, which is Lorentzian for $y < 0$ and Riemannian for $y > 0$. The signature changing set $S \subset \Sigma$ is defined by $\{y = t = 0\}$, which is clearly a two-dimensional *null* surface, contradicting Result 1.1 in [1]. In fact, the same example (1) with a different f can be used to show that S needs not even be a differentiable submanifold and that branch points are allowed. Indeed, taking $f = yz$, the signature changing set is located at $yz = 0$, which are two 2-planes intersecting at the branch line $(y = z = 0, v \in \mathbb{R})$. Thus, Result 1.1 in [1] does not hold and the structure of S allows for much more freedom than claimed there.

Despite several efforts, the only restriction on the structure of S, and more generally on the properties of Σ_0, that we have been able to derive from the brane setting is

Lemma 1. At any point $p \in \Sigma_0$ (and therefore at all points of S) of a codimension one brane Σ, the induced metric h of Σ has a *unique* degeneration direction given by $\vec{N}|_p$.

Proof. At $p \in \Sigma_0$ the normal vector $\vec{N}|_p$ is also tangent to Σ. It obviously satisfies $g(\vec{N}, \vec{v})|_p = 0$ for any vector $\vec{v}|_p \in T_p\Sigma$, which clearly implies $h(\vec{N}, \cdot)|_p = 0$, so that $\vec{N}|_p$ is a degeneration vector. To show uniqueness, let us take another degeneration vector $\vec{w}|_p \in T_p\Sigma$. It follows $g(\vec{w}, \vec{w})|_p = 0$ and $g(\vec{N}, \vec{w})|_p = 0$, so that the two null vectors $\vec{N}|_p$ and $\vec{w}|_p$ must be parallel. ■

This Lemma does indeed restrict the structure of S in the brane scenario because more general behaviors can occur for arbitrary signature changes. It may happen, for instance, that the metric changes signature at a smooth hypersurface where the radical is two- or higher-dimensional, or even spans the whole tangent space. A simple example of the latter is given by the following (0, 2)-tensor in $\mathbb{R}^m$,

$$ds^2 = -tdt^2 + t^2(dx_1^2 + dx_2^2 + \cdots dx_{m-1}^2).$$

B. Gluing

Let us next discuss the standard procedure of how to build branes by gluing manifolds with boundary, and the possibility of actually constructing signature changing branes by that method. This is important as most of the standard branes are constructed in this manner. However, for hypersurfaces with changing causal character, the usual matching conditions are no longer valid and an appropriate generalization must be used. Fortunately, such a generalization was already developed in [38] in four dimensions. These results can be readily generalized to arbitrary dimension with no essential change. Since we shall use this matching procedure extensively, let us describe its essential features.

We start from two oriented C^3 n-dimensional manifolds with boundary $\mathcal{M}^\pm$, whose boundaries are $\Sigma^\pm$. These manifolds are endowed with C^2 Lorentzian metrics $g^\pm$. In order to join them across their boundaries we need to identify the boundaries pointwise. This means, in particular, that there must exist a one-to-one correspondence between Σ^+ and Σ^-, which moreover must be a diffeomorphism in order to preserve the differential structure. Both for conceptual and operational reasons, it is convenient to state this condition in the following equivalent manner: there exists an abstract $(n-1)$-dimensional C^3 manifold Σ and two C^3 embeddings

$$\Phi_+ : \Sigma \longrightarrow \mathcal{M}^+, \qquad \Phi_- : \Sigma \longrightarrow \mathcal{M}^-,$$

which satisfy $\Phi_+(\Sigma) = \Sigma^+$ and $\Phi_-(\Sigma) = \Sigma^-$. The identification of the boundaries is then given by $\Phi \equiv \Phi_+ \circ \Phi_-^{-1}|_{\Sigma^-}$. Under these circumstances, and using standard techniques of differential topology, it follows that the space $\mathcal{M} \equiv \mathcal{M}^+ \cup \mathcal{M}^-$, with the boundaries identified, can be endowed with a differential structure [49] so that it becomes a manifold. Our aim is to define a metric g on $\mathcal{M}$ which is continuous everywhere, in particular, across Σ (we shall often abuse notation and identify Σ^+, Σ^-, and Σ when necessary), such that g coincides with the original $g^\pm$ in the interiors of $\mathcal{M}^\pm$, respectively. Demanding continuity is obviously sufficient for having a well-defined induced metric on the brane. It turns out that continuity is in fact the only possibility, as we discuss next.

1. Tangent space identification: Riggings

As pointed out by Clarke and Dray [46], defining a metric on $\mathcal{M}$ requires not only that we identify the points on the boundary but also that the tangent spaces are properly identified. The differential map $d\Phi$ fixes uniquely the way of identifying the tangent vectors of the boundaries. Thus, if we want to define a continuous metric on $\mathcal{M}$ we need to require at least that the first fundamental forms of Σ^+ and Σ^- coincide (via Φ). In other words

$$h^+ \equiv \Phi_+^\star(g^+) = \Phi_-^\star(g^-) \equiv h^-, \tag{2}$$

where $\Phi_\pm^\star$ denote the pullbacks of $\Phi_\pm$ and h^+, h^- are the first fundamental forms of Σ as defined from $\mathcal{M}^+$ and $\mathcal{M}^-$, respectively. Conditions (2) are called preliminary matching conditions. When they hold we write $h = h^+ = h^-$. In local coordinates they read as follows. Let $\{\xi^a\}$ (a, $b = 1, \cdots, n-1$), $\{x_+^\mu\}$ and $\{x_-^\mu\}$ be local coordinate systems on Σ, $\mathcal{M}^+$, and $\mathcal{M}^-$, respectively. Consider the basis vectors $\frac{\partial}{\partial \xi^a}$ and their images by $d\Phi_\pm$

$$\vec{e}_a^{\,\pm} \equiv \left[d\Phi_\pm\left(\frac{\partial}{\partial \xi^a}\right)\right] = \frac{\partial x_\pm^\mu(\xi)}{\partial \xi^a}\frac{\partial}{\partial x_\pm^\mu}, \tag{3}$$

where the functions $x_\pm^\mu(\xi)$ define the embeddings $\Phi_\pm$ in local coordinates, i.e.

$$\Phi_\pm : \xi^a \longrightarrow x_\pm^\mu(\xi^a).$$

Obviously, $\{\vec{e}_a^{\,\pm}\}$ span the tangent planes of the hypersurfaces $\Sigma^\pm$ as embedded in $\mathcal{M}^\pm$. In terms of these objects, the preliminary matching conditions (2) read

$$h_{ab}^+(\xi) = h_{ab}^-(\xi), \tag{4}$$

where

$$h_{ab}^\pm(\xi) \equiv g_{\mu\nu}^\pm(x^\pm(\xi))\frac{\partial x_\pm^\mu(\xi)}{\partial \xi^a}\frac{\partial x_\pm^\nu(\xi)}{\partial \xi^b}.$$

In order to complete the identification of the tangent spaces, we only need to identify one transversal vector on Σ^+ with one transversal vector on Σ^-. Then, the identification of all tangent vectors follows by linearity. To that end, let us choose a C^2 vector field $\vec{\ell}_+$ on Σ^+ which is nowhere tangent to Σ^+. The existence of such a vector field, sometimes called *rigging* [38], is a standard property of manifolds with boundary. Transversality means $N_\mu^+ \ell_+^\mu \neq 0$ where $\boldsymbol{N}^+$ is a normal one-form of Σ^+ in $\mathcal{M}^+$. Furthermore, we choose $\vec{\ell}_+$ pointing towards $\mathcal{M}^+$ everywhere; actually, since $\vec{\ell}_+$ is transversal to Σ^+, it is sufficient to impose that $\vec{\ell}_+$ points towards $\mathcal{M}^+$ at one point of Σ^+. Of course, we could alternatively demand that $\vec{\ell}_+$ points outwards from $\mathcal{M}^+$. This would induce obvious changes in the discussion below with no essential new features.

Now we need to choose another C^2 rigging $\vec{\ell}_-$ on Σ^-. Since we intend to identify $\vec{\ell}_+$ with $\vec{\ell}_-$ and get a continuous metric, we must impose *at least* that their norms and scalar products with arbitrary vectors in $T\Sigma$ coincide. This amounts to requiring that

$$g^+_{\mu\nu}\ell^\mu_+\ell^\nu_+ \stackrel{\Sigma}{=} g^-_{\mu\nu}\ell^\mu_-\ell^\nu_-, \qquad g^+_{\mu\nu}\ell^\mu_+ e_a^{+\nu} \stackrel{\Sigma}{=} g^-_{\mu\nu}\ell^\mu_- e_a^{-\nu}, \quad (5)$$

where the symbol $\stackrel{\Sigma}{=}$ stands for equality using the diffeomorphism Φ. Equations (5) should be interpreted as restrictions on $\vec{\ell}_-$ once $\vec{\ell}_+$ has been chosen, or vice versa. These n conditions are not sufficient to ensure a proper matching, as the rigging $\vec{\ell}_-$ must also satisfy the property of pointing outwards from $\mathcal{M}^-$ everywhere. This is necessary because, after the identification, the vector $\vec{\ell} \equiv \vec{\ell}_+ = \vec{\ell}_-$ points towards $\mathcal{M}^+$ (as $\vec{\ell}_+$ does). When viewed from the glued manifold $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$, this is equivalent to saying that $\vec{\ell}$ points outwards from $\mathcal{M}^-$.

Two important questions arise: (a) are these conditions on the riggings already sufficient for the existence of a matching with continuous metric? and (b) do they introduce any restrictions on the manifolds $\mathcal{M}_\pm$ to be matched? In a remarkable paper [46], Clarke and Dray addressed these questions for the case of constant-signature matching hypersurfaces. Their conclusion was that the answer is affirmative in both cases. Unfortunately, this conclusion is not completely correct as stated, as we shall see presently with examples. Let us discuss this.

The proof given by Clarke and Dray can be divided in two parts. In the first one, question (a) above is addressed and the authors try to prove that a pair of riggings $\vec{\ell}_\pm$ satisfying (5) with the correct orientation does exist. In the second part, which corresponds to question (b) above, the existence of a maximal atlas on $\mathcal{M}$ for which the metric g is continuous is shown, provided the preliminary matching conditions hold and a pair of suitable riggings $\vec{\ell}_\pm$ exist. This second part is correct and, in fact, depends very weakly on the assumption of constant signature of the matching hypersurface. A slight modification of the argument allows one to show that the same result holds for spacetimes with boundaries having varying causal character. The first part of the proof, however, is not correct *for boundaries having null points*, both in the constant null-signature case treated in [46] or in its generalization to signature changing boundaries. Thus, a correct reformulation of Clarke and Dray's result is

Theorem 1. Let $(\mathcal{M}^\pm, g^\pm)$ be two n-dimensional C^2 oriented spacetimes with boundary, with respective C^3 boundaries $\Sigma^\pm$ such the preliminary matching conditions (4) hold on Σ. Assume further that there exist transverse vector fields $\vec{\ell}_\pm$ on $\Sigma^\pm$ satisfying the scalar product conditions (5) and such that $\vec{\ell}_+$ points towards $\mathcal{M}^+$ and $\vec{\ell}_-$ points outwards from $\mathcal{M}^-$. Then, there exists a unique, maximal, C^3 differentiable structure on $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$ (with their points on Σ^+ and Σ^- identified), and a unique continuous metric g which coincides with g^+ on $\mathcal{M}^+$ and with g^- on $\mathcal{M}^-$.

Remark. The hypothesis on the existence of the rigging is necessary only in the case of boundaries which have at least one point of degeneration, i.e. $\Sigma_0 \neq \emptyset$. For everywhere spacelike or everywhere timelike boundaries the unit normal vectors with appropriate orientation fulfil all the requirements.

2. On the existence of riggings for Σ with null points

When Σ has null points existence of the appropriate riggings is not guaranteed, as we show next. We start with a Lemma stating that, at points where the hypersurface is non-null, the solution of (5) with the proper orientation is unique, if it exists.

Lemma 2. Let $\mathcal{M}^\pm$ be two spacetimes with boundary satisfying the preliminary matching conditions (4). Let Σ^- be non-null at $p^- \in \Sigma^-$ and set $p^+ = \Phi(p^-)$. Choose any transverse vector field $\vec{\ell}_+|_{p^+}$ pointing towards $\mathcal{M}^+$. Then there is at most one solution of (5) for $\vec{\ell}_-|_{p^-}$ pointing outwards from $\mathcal{M}^-$.

Proof. Take two solutions $\vec{\ell}_-|_{p^-}$ and $\vec{\hat{\ell}}_-|_{p^-}$ of (5). From the second equation it follows that its difference must be proportional to a normal vector:

$$\vec{\hat{\ell}}_-|_{p^-} = \vec{\ell}_-|_{p^-} + A\vec{N}^-|_{p^-}.$$

Inserting this into the first equation in (5) we obtain

$$0 = A(AN^-_\mu N^{-\mu} + 2N^-_\mu \ell^\mu_-)|_{p^-}, \qquad (6)$$

which admits two solutions. The solution with $A \neq 0$ gives an $\vec{\hat{\ell}}_-$ satisfying $N^-_\mu \hat{\ell}^\mu_-|_{p^-} = -N^-_\mu \ell^\mu_-|_{p^-}$. Thus, $\vec{\ell}$ and $\vec{\hat{\ell}}$ cannot both have the correct orientation. ■

The next Lemma shows that, at null points, uniqueness of $\vec{\ell}_-|_{p^-}$ holds *irrespective* of orientation.

Lemma 3. With the same notation as in Lemma 2 assume now that Σ^- is null at p^-. Then the solution of the algebraic equations (5) at p^- is unique, if it exists.

Proof. As $N^-_\mu N^{-\mu}|_{p^-} = 0$, Eq. (6) simplifies to $0 = 2AN^-_\mu \hat{\ell}^\mu_-|_{p^-}$. Transversality of the rigging immediately implies then that $A = 0$. ■

This Lemma implies that the orientation of $\vec{\ell}_-$ is *fixed* directly by the algebraic conditions (5) at null points. This clearly suggests that there will exist spacetimes with boundaries satisfying all the preliminary matching conditions which, however, cannot be matched continuously.

Before showing this explicitly, we must check that the existence of an $\vec{\ell}_-$ does not depend on the choice of $\vec{\ell}_+$. Assume that a solution of the preliminary matching conditions (4) exists for one choice of rigging $\vec{\ell}_+$ and take any other rigging $\vec{\hat{\ell}}_+$. To show that a solution also exists for the second choice, we only need to decompose $\vec{\hat{\ell}}_+$ in the basis

$\{\vec{e}_a^+, \vec{\ell}_+\}$ and *define* $\vec{\hat{\ell}}_-$ as the same linear combination of $\{\vec{e}_a^-, \vec{\ell}_-\}$ (with $\vec{\ell}_-$ being the solution for $\vec{\ell}_+$ which we assume it exists and which we know it is unique). All the rigging and orientability conditions for $\vec{\hat{\ell}}_\pm$ are automatically satisfied. Thus, existence (or nonexistence) of a suitable pair of riggings is reduced to existence of a solution of (5) for $\vec{\ell}_-$ given *any chosen* rigging $\vec{\ell}_+$.

We can now discuss examples showing that the preliminary matching conditions are not sufficient for the existence of a continuous matching. Let us begin with the simplest possible example, so that the main obstruction to existence becomes clear. Let us consider two identical copies of the submanifold with boundary defined by $t \geq x$ in 2-dimensional Minkowski spacetime in Cartesian coordinates $\{t, x\}$. Let us denote them by (M^+, η) and (M^-, η). Their corresponding boundaries are obviously $\Sigma^\pm$: $\{t = x\}$, see Figure 1. Let us now try to match them by identifying the boundaries in the natural way, i.e. by taking Φ as the identity mapping. Without loss of generality, let the rigging vector $\vec{\ell}_+$ be null and point towards M^+. We know by (5) that $\vec{\ell}_-$ also has to be null. Moreover it has to point outwards from M^- (see Fig. 1). However, with the natural identification we have chosen, if the tangent vector $\vec{e}_1^+$ points in one possible direction, then the tangent vector $\vec{e}_1^-$ to be identified with $\vec{e}_1^+$ must also point in that same direction, see Fig. 1. But then it is clear that the second equality in (5) cannot be satisfied, showing that these two spacetimes cannot be matched across their boundaries by using the natural identification of points. One might still think that the problem arises from the choice of identification of boundaries. This is not the case, however, because generically two manifolds with boundary will have at most *one* diffeomorphism between their boundaries for which the preliminary matching conditions are satisfied. Hence, in general there is no freedom in choosing another identification (see Corollary 1 below).

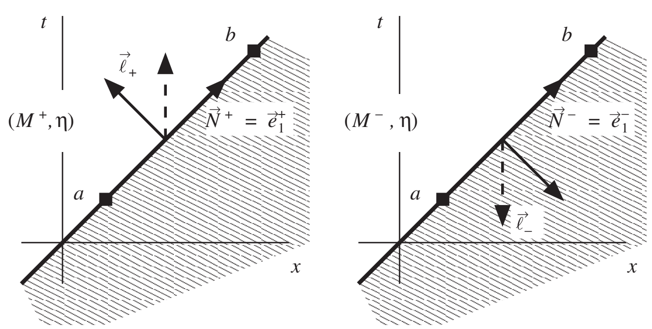

FIG. 1. Two identical copies of the manifold with boundary defined by the region $t \geq x$ of Minkowski spacetime. These are the nonshadowed parts of the picture. The natural identification of boundaries amounts to identifying the two a-points, and the two b-points, and so on. The rigging vectors $\vec{\ell}^\pm$ are chosen to be null and one pointing towards M^+ and the other outwards from M^-, as required. The second vector $\vec{e}_1^\pm$ to complete the bases $\{\vec{\ell}^\pm, \vec{e}_1^\pm\}$ coincides with the corresponding normal vector $\vec{N}^\pm$. Both $\vec{e}_1^\pm$ must point from a to b (or vice versa) due to the chosen identification. But then the tangent spaces cannot be properly identified because the scalar products do not match. A second possible choice of rigging vectors is represented by the broken-line vectors, leading to the same problem.

This example can be generalized to arbitrary spacetimes. First of all let us notice that a natural way of building spacetimes with boundary is picking up an arbitrary spacetime $(\mathcal{V}, \gamma)$ and choosing a hypersurface Σ^- which divides $\mathcal{V}$ into two regions, which we denote by $\mathcal{M}_1^-$ and $\mathcal{M}_2^-$. Clearly both regions have Σ^- as their boundary. Assume now that we have another spacetime with boundary $(\mathcal{M}^+, g^+)$ and assume that $\mathcal{M}^+$ can be matched with $\mathcal{M}_1^-$ (say) to produce a spacetime with continuous metric g. The following proposition ensures that $\mathcal{M}^+$ *cannot* be matched to $\mathcal{M}_2^-$ using the same identification of boundaries if Σ^- has at least one null point.

Proposition 1. Let $(\mathcal{M}^+, g^+)$ be a C^2 spacetime with boundary Σ^+ and $\mathcal{M}_1^-$, $\mathcal{M}_2^-$ be two regions of a C^2 spacetime $(\mathcal{V}, \gamma)$ satisfying

$$\mathcal{M}_1^- \cup \mathcal{M}_2^- = \mathcal{V}, \qquad \mathcal{M}_1^- \cap \mathcal{M}_2^- = \Sigma^-,$$

where Σ^- is a C^3 hypersurface with at least one null point. If there exists a diffeomorphism Φ between Σ^+ and Σ^- such that $(\mathcal{M}^+, g^+)$ can be matched continuously to $(\mathcal{M}_1^-, \gamma|_{\mathcal{M}_1^-})$, then $(\mathcal{M}^+, g^+)$ cannot be matched continuously to $(\mathcal{M}_2^-, \gamma|_{\mathcal{M}_2^-})$ with the same diffeomorphism Φ.

Proof. Take a point $p^- \in \Sigma_0^-$ and define $p^+ = \Phi(p^-)$ as usual. By Lemma 3, for any transverse vector $\vec{\ell}^+|_{p^+}$ pointing towards $\mathcal{M}^+$ there exists exactly one transverse vector $\vec{\ell}^-|_{p^-}$ satisfying the rigging conditions (5). Moreover, we know that $\vec{\ell}^-|_{p^-}$ points outwards from $\mathcal{M}_1^-$ because $(\mathcal{M}^+, g^+)$ can be matched continuously to $(\mathcal{M}_1^-, \gamma|_{\mathcal{M}_1^-})$. Thus, there exists no rigging solving (5) pointing towards $\mathcal{M}_1^-$. Since $\mathcal{M}_1^-$ and $\mathcal{M}_2^-$ can be visualized inside the total spacetime $\mathcal{V}$, it follows that there is no rigging solving (5) pointing outwards from $\mathcal{M}_2^-$. ■

For any hypersurface Σ of arbitrary causal character, with first fundamental form h, a diffeomorphism Ψ: $\Sigma \to \Sigma$ is called an *isometry* if $\Psi^\star(h) = h$. The following corollary follows immediately from Proposition 1, taking into account that if $(\mathcal{M}^+, g^+)$ could still be matched to $(\mathcal{M}_2^-, \gamma|_{\mathcal{M}_2^-})$ through a different diffeomorphism Φ', then $\Phi^{-1} \circ \Phi'$ would constitute an isometry of Σ^- different from the identity.

Corollary 1. With the same assumptions as in Proposition 1, let h^- be the first fundamental form of $\Sigma^- \in \mathcal{V}$. If (Σ^-, h^-) admits no isometries apart from the identity, then $(\mathcal{M}^+, g^+)$ cannot be matched to $(\mathcal{M}_2^-, \gamma|_{\mathcal{M}_2^-})$.

C. Gluing and the Z_2-mirror symmetry

Proposition 1 has another interesting corollary applicable to the case of joining two identical copies of the same C^2 spacetime with boundary: the so-called Z_2-mirror symmetry branes. If the boundary has at least one null point, and if the spacetime is a subregion of a larger C^2 spacetime without boundary, impossibility of matching would follow immediately. It is likely that such a C^2 extension always exists. However, instead of attempting a proof of this fact, let us show directly that, in any case, the two copies cannot be matched continuously, as announced in [1].

Corollary 2. It is impossible to join two identical copies of a spacetime with boundary Σ such that $\Sigma_0 \neq \varnothing$ (in particular with a signature changing boundary Σ), identifying naturally the points on Σ, to produce a bulk with continuous metric.

Proof. Let us call $(\mathcal{M}^\pm, g^\pm)$ the two identical copies and $\Sigma^\pm$ their corresponding boundaries. Let χ be the natural identification of $\mathcal{M}^+$ and $\mathcal{M}^-$. Take a point p where the boundary is null and any rigging vector $\vec{\ell}_+|_p$ pointing towards $\mathcal{M}^+$. Clearly $\chi|_{\Sigma_+}$ is the diffeomorphism we are using to identify the boundaries. By Lemma 3 there is a unique solution $\vec{\ell}_-|_{\chi(p)}$ of the rigging equations (5), and that must be obviously $\vec{\ell}_-|_{\chi(p)} \equiv \mathrm{d}\chi(\vec{\ell}_+|_p)$ which is just a copy of the original rigging. But since $\vec{\ell}_+|_p$ points towards $\mathcal{M}^+$ it follows that $\vec{\ell}_-|_{\chi(p)}$ points towards $\mathcal{M}^-$ and the proposed matching is impossible. ∎

This corollary shows that the usual Z_2-mirror symmetry extensively used in the brane scenario is *incompatible* with signature changing branes, with null branes, and in general with branes having a nonempty Σ_0. Therefore constructing such branes requires more sophisticated methods. For instance, we can try to join two different regions of the same spacetime or two different spacetimes. In general such constructions are more involved than in the mirror symmetric case because the preliminary matching conditions are not automatically satisfied and more equations need to be solved. This does not mean, however, that such constructions are impossible. Explicit examples were in fact given in [1]. We shall go back to these and other examples in Secs. V and VI.

In this section we have seen that in order to construct spacetimes with signature changing branes one must be careful with the existence of suitable riggings at points where the matching hypersurface is null. The results we have presented obviously hold for usual matching conditions in general relativity, but they also hold in any other geometrical theory. When trying to join spacetimes involving matching hypersurfaces with null points (in particular if the matching hypersurface is null everywhere), the equality of the first fundamental forms *is not sufficient* to ensure the existence of a matched spacetime with continuous metric. Existence of suitable riggings must always be checked in those cases.

Having discussed the construction of branes by the method of gluing and its consequences for the signature changing case, we need to analyze the equations relating the jump in the metric derivatives with the singular part of the Einstein tensor on the bulk. Again, the standard Israel conditions do not apply in the signature changing case and need to be generalized. We discuss the results in the next section.

III. GENERALIZED ISRAEL FORMULA

Under the assumptions of Theorem 1 we have an orientable C^3 bulk $\mathcal{M}$ with a continuous, piecewise C^2, metric g. We choose an orientation on $\mathcal{M}$ and denote by $\boldsymbol{\eta}$ its canonical volume n-form. This allows us to define the Riemann, Ricci, and Einstein tensors in a distributional way. Since the definitions of the Riemann and Einstein tensors contain second derivatives of the metric and this is not, in general, C^1 across Σ, one expects that delta-type distributions with support on Σ will arise. Indeed, it can be shown [38,46] that the Einstein tensor of g, viewed as a tensor distribution on $\mathcal{M}$ and denoted by $\underline{G}_{\mu\nu}$, takes the form

$$\underline{G}_{\mu\nu} = \underline{\theta}^+ G^+_{\mu\nu} + \underline{\theta}^- G^-_{\mu\nu} + \delta \mathcal{G}_{\mu\nu}, \qquad (7)$$

where $G^\pm_{\mu\nu}$ are the Einstein tensors of $(\mathcal{M}^\pm, g^\pm)$ and $\mathcal{G}_{\mu\nu}$, which is defined only on Σ, is called *the singular part of the Einstein tensor distribution*. The distributions $\underline{\theta}^\pm$ and δ are defined as follows: for any test function Y (i.e. a C^3 function with compact support on $\mathcal{M}$—note that $\mathcal{M}$ is only C^3 so it makes no sense to assume higher differentiability for Y)—

$$\langle \underline{\theta}^\pm, Y \rangle \equiv \int_{\mathcal{M}^\pm} Y \boldsymbol{\eta}.$$

Regarding δ, we first define a one-form distribution $\boldsymbol{\delta} \equiv \nabla \underline{\theta}^+ = -\nabla \underline{\theta}^-$, see [38]. Explicitly, $\boldsymbol{\delta}$ acts on any test vector field $\vec{Y}$ (C^2 vector field with compact support) as

$$\langle \boldsymbol{\delta}, \vec{Y} \rangle = \int_\Sigma Y^\mu d\sigma_\mu,$$

where $d\sigma_\mu$ is defined as

$$d\sigma_\mu = \eta_{\mu\alpha_1\ldots\alpha_{n-1}} e_1^{\alpha_1} \ldots e_{n-1}^{\alpha_{n-1}} d\xi^1 \wedge \cdots \wedge d\xi^{n-1},$$

where $\vec{e}_a = \vec{e}^{\,+}_a = \vec{e}^{\,-}_a$, and $\{\vec{\ell}, \vec{e}_1, \cdots \vec{e}_{n-1}\}$ is a positively oriented basis, that is $\ell^\mu d\sigma_\mu > 0$ (recall that $\vec{\ell} = \vec{\ell}_+ = \vec{\ell}_-$ after identification).

It is convenient here to choose the normal

$$\boldsymbol{n} \equiv \frac{1}{N_\alpha \ell^\alpha} \boldsymbol{N},$$

which does not depend on the choice of $\boldsymbol{N}$, but does indeed on the choice of rigging vector $\vec{\ell}$: given the rigging, its intrinsic characterization is $\boldsymbol{n} \propto \boldsymbol{N}$ and $n_\alpha \ell^\alpha = 1$. The importance of this normal is that the identification of the tangent vectors and the riggings at both sides induces the identification $\boldsymbol{n} = \boldsymbol{n}^+ = \boldsymbol{n}^-$.

Denoting by $d\sigma$ a volume element on Σ defined by

$$d\sigma_\mu = n_\mu d\sigma \Leftrightarrow d\sigma = \ell^\alpha d\sigma_\alpha$$

the distribution δ is defined by

$$\langle \delta, Y \rangle \equiv \int_\Sigma Y d\sigma,$$

where Y is any test function. δ obviously depends on the choice of rigging via the normal form $\boldsymbol{n}$. From the identity

$$\boldsymbol{\delta} = \boldsymbol{n}\delta$$

and the fact that $\boldsymbol{\delta}$ is intrinsically defined, it follows that a rescaling $\vec{\ell}' = A^{-1}\vec{\ell}$, so that $\boldsymbol{n}' = A\boldsymbol{n}$, induces the transformation

$$\delta' = A^{-1}\delta, \Rightarrow \mathcal{G}'_{\mu\nu} = A\mathcal{G}_{\mu\nu}, \tag{8}$$

after using (7). Observe that both $\boldsymbol{\delta}$ and δ have support on Σ.

We still need to specify the explicit form of $\mathcal{G}_{\mu\nu}$ in expression (7). By construction, the metric g has continuous tangential derivatives at Σ. Therefore, this singular part of the Einstein tensor distribution will be related to the discontinuity of the transversal derivatives of the metric across Σ. In the cases where Σ is timelike (or spacelike) everywhere, the normal vector $\vec{N}$ is transversal to Σ, and therefore we can choose the rigging to be proportional to $\vec{N}$ and unit. This implies that $\boldsymbol{n}$ is also unit and in fact $\ell^\alpha = \text{sign}(\boldsymbol{n}, \boldsymbol{n}) n^\alpha$. Thus, the second fundamental forms inherited by Σ from both sides, which can be promoted to spacetime tensors by means of the definition

$$K^\pm_{\mu\nu} \equiv P^\alpha_\mu P^\beta_\nu \nabla^\pm_\alpha n_\beta,$$

where $\nabla^\pm$ are the Levi-Civita connections of $g^\pm$ and

$$P^\alpha_\mu = \delta^\alpha_\mu - \text{sign}(\boldsymbol{n}, \boldsymbol{n}) n^\alpha n_\mu$$

is the projector orthogonal to Σ, encode properly the jumps of the transversal derivatives of the metric. It is not surprising, therefore, that $\mathcal{G}_{\mu\nu}$ can be written down in terms of the jumps of the second fundamental forms in the non-null case. This is the content of the so-called Israel formula [16] which reads, taking $\text{sign}(\boldsymbol{n}, \boldsymbol{n})\vec{n}$ pointing towards $\mathcal{M}^+$,

$$\mathcal{G}_{\mu\nu} = -[K_{\mu\nu}] + P_{\mu\nu}[K]. \tag{9}$$

Here and in the rest of the paper, the "discontinuity" $[f]$ of any object which has well-defined limits at both sides of Σ is defined as

$$[f](p) \equiv \lim_{\substack{x \to p \\ \mathcal{M}^+}} f^+(x) - \lim_{\substack{x \to p \\ \mathcal{M}^-}} f^-(x) \quad \forall\ p \in \Sigma.$$

In the signature changing case, and in general whenever $\Sigma_0 \neq \varnothing$, the normal vector is no longer transverse to the hypersurface everywhere. Thus, the second fundamental forms of Σ are no longer suitable to measure the jumps in the transversal derivatives of the metric. This makes clear that the Israel formula (9) must be modified in these cases. Taking into account that $\vec{\ell}$ is transverse to the hypersurface, it is natural to substitute the second fundamental forms by the new objects [38][4]

$$\mathcal{H}^\pm_{\mu\nu} \equiv \Pi^\alpha{}_\mu \Pi^\beta{}_\nu \nabla^\pm_\alpha \ell_\beta|_\Sigma, \tag{10}$$

where now the generalized projector $\Pi^\alpha{}_\mu$ reads

$$\Pi^\alpha{}_\mu = \delta^\alpha_\mu - n_\mu \ell^\alpha.$$

Observe that $\Pi_{\mu\nu}$ is no longer symmetric, and that $\ell^\mu \Pi^\alpha{}_\mu = 0$, hence

$$\ell^\mu \mathcal{H}^\pm_{\mu\nu} = 0, \qquad \ell^\nu \mathcal{H}^\pm_{\mu\nu} = 0.$$

We can now write down the expression of $\mathcal{G}_{\mu\nu}$ in terms of jumps of these objects [38]

$$\begin{aligned}\mathcal{G}_{\mu\nu} = {}& n^\alpha[\mathcal{H}_{\alpha\mu}]n_\nu + n^\alpha[\mathcal{H}_{\alpha\nu}]n_\mu - n^\alpha n_\alpha[\mathcal{H}_{\mu\nu}] \\ & - n_\mu n_\nu[\mathcal{H}^\alpha{}_\alpha] - g_{\mu\nu}|_\Sigma(n^\alpha n^\beta[\mathcal{H}_{\alpha\beta}] \\ & - n^\alpha n_\alpha[\mathcal{H}^\beta{}_\beta]).\end{aligned} \tag{11}$$

This is the generalization of Israel's formula (9) to arbitrary hypersurfaces. The transformation (8) under change of rigging can be directly checked in this expression, taking into account that $[\mathcal{H}'_{\alpha\beta}] = A^{-1}[\mathcal{H}_{\alpha\beta}]$. Note that [38]:

(i) $\mathcal{H}^\pm_{\mu\nu}$ are not symmetric, but $[\mathcal{H}_{\mu\nu}]$ is symmetric;

(ii) $[\mathcal{H}_{\mu\nu}]$ does not depend on a change of rigging $\vec{\ell}' = \vec{\ell} + f^a \vec{e}_a$ for any functions f^a defined on Σ. Neither $\boldsymbol{n}$ does, and therefore $\mathcal{G}_{\mu\nu}$ does not depend on a change of rigging of that kind. Thence, the only transformation of $\mathcal{G}_{\mu\nu}$ under a change of rigging is through the rescaling (8) discussed above.

These are of course necessary consistency properties of the final expression (11). It should also be remarked that, for nonnull branes, this expression reduces to the usual Israel formula by taking $\vec{\ell} = \text{sign}(\boldsymbol{n}, \boldsymbol{n})\vec{n}$ as before. Note that then $n^\alpha \mathcal{H}^\pm_{\alpha\beta} = \ell^\alpha \mathcal{H}^\pm_{\alpha\beta} = 0$ and $\mathcal{H}^\pm_{\alpha\beta} = \text{sign}(\boldsymbol{n}, \boldsymbol{n}) K^\pm_{\alpha\beta}$.

The generalized expression (11) satisfies

$$n^\mu \mathcal{G}_{\mu\nu} \equiv 0$$

as one can immediately check. Thus, at points where Σ is not null only the *tangential* components $\mathcal{G}_{ab} = e_a{}^\alpha e_b{}^\beta \mathcal{G}_{\alpha\beta}$ are present, and they contain *all* the informa-

[4]It must be remarked that, in purity, the second fundamental form of a hypersurface is a tensor field defined only on the hypersurface. Thus, the rigorously defined object is in fact $K_{ab} = -g(\vec{n}, \nabla_{\vec{e}_a} \vec{e}_b)$, which is symmetric. One can however use any unit extension of $\boldsymbol{n}$ outside Σ to define $K_{\mu\nu}$ and then, in fact, $K_{ab} = K_{\mu\nu} e^\mu_a e^\nu_b$. Similarly, the rigorously defined object using the rigging is $\mathcal{H}_{ab} = -g(\vec{\ell}, \nabla_{\vec{e}_a} \vec{e}_b)$, which in this case is not necessarily symmetric. Extensions of $\boldsymbol{n}$ and $\vec{\ell}$ outside Σ keeping $n_\mu \ell^\mu = 1$ permit then to define $\mathcal{H}_{\mu\nu}$ and, again, $\mathcal{H}_{ab} = \mathcal{H}_{\mu\nu} e^\mu_a e^\nu_b$.

tion carried by $\mathcal{G}_{\mu\nu}$. For a Σ with a nonempty Σ_0, though, one should bear in mind that the normal vector $\vec{n}$ is tangent to Σ at the null phase Σ_0, so that the geometrical interpretation of this vanishing contraction is not so straightforward.

IV. FIELD EQUATIONS: THE ENERGY-MOMENTUM TENSOR ON THE BRANE

We are now in a position where the Einstein equations on the bulk can be discussed. Because of the structure of the Einstein tensor distribution of the bulk (7), the corresponding energy-momentum tensor on the bulk $\underline{T}^{\mathcal{M}}{}_{\mu\nu}$ will also be a tensor distribution and will consist of three parts: the tensor fields $T^{\pm}_{\mu\nu}$ defined on each region $\mathcal{M}^{\pm}$, at each side of Σ, plus a singular part with support on Σ proportional to δ,

$$\underline{T}^{\mathcal{M}}{}_{\mu\nu} = \underline{\theta}^{+} T^{+}_{\mu\nu} + \underline{\theta}^{-} T^{-}_{\mu\nu} + \delta\tau_{\mu\nu}.$$

Notice that, again, $\tau_{\mu\nu}$ does not have intrinsic meaning because $\vec{n}$ cannot be canonically normalized on a signature changing brane. Only the product $\delta\tau_{\mu\nu}$ is independent of the normalization. For the individual term $\tau_{\mu\nu}$ to become meaningful, a volume element must be fixed once and for all on Σ. Equivalently, one must choose a given rigging, which in turn determines a unique normal one-form. Despite these issues, we will refer to $\tau_{\mu\nu}$ as the *energy-momentum tensor on the brane* Σ.

Keeping this in mind, the Einstein equations on the bulk are given by

$$\underline{G}_{\mu\nu} + \underline{\Lambda}_{\mu\nu} = \kappa_n^2 \underline{T}^{\mathcal{M}}{}_{\mu\nu}, \tag{12}$$

where κ_n is the n-dimensional gravity coupling constant and we have set

$$\underline{\Lambda}_{\mu\nu} = \underline{\theta}^{+}\Lambda_n^{+} g^{+}_{\mu\nu} + \underline{\theta}^{-}\Lambda_n^{-} g^{-}_{\mu\nu},$$

where $\Lambda_n^{\pm}$ are the cosmological constants corresponding to $\mathcal{M}^{\pm}$. Observe that we are allowing for *different* values of the cosmological constant at each side of the brane Σ.

The Einstein equations (12) decompose then as

$$G^{\pm}_{\mu\nu} + \Lambda_n^{\pm} g^{\pm}_{\mu\nu} = \kappa_n^2 T^{\pm}_{\mu\nu}$$

on each of $\mathcal{M}^{\pm}$ plus

$$\mathcal{G}_{\mu\nu} = \kappa_n^2 \tau_{\mu\nu} \tag{13}$$

at points on Σ. Let us insist once more that this last equation is intrinsic only when multiplied by the distribution δ. However, one can still write (13) as it stands because both sides of the equation are affected by exactly the same normalization freedom. Furthermore, note that (13) together with (11) constitute the generalization of the Israel equations to general hypersurfaces in terms of the energy-momentum of the (hyper)surface layer.

The geometrical property $n^{\mu}\mathcal{G}_{\mu\nu} = 0$ implies then that

$$n^{\mu}\tau_{\mu\nu} = 0. \tag{14}$$

As discussed above, at points where Σ is not null, in particular, on its Lorentzian part Σ_L, Eqs. (13) are equivalent to the $n(n-1)/2$ projected equations

$$\mathcal{G}_{ab} = \kappa_n^2 \tau_{ab}, \tag{15}$$

which are defined on the brane, where as usual

$$\tau_{ab} = e_a{}^{\mu} e_b{}^{\nu} \tau_{\mu\nu}.$$

Nevertheless, for general branes the $n(n-1)/2$ independent relations contained in (13) are not so simply interpreted, and in fact the meaning of (14) on the null phase Σ_0 and the signature changing set $S \subset \Sigma_0$ is that any *tangential* component of $\tau_{\mu\nu}$ along the unique null degeneration direction must vanish.

It is customary to decompose the total energy-momentum tensor on the Lorentzian part Σ_L of the brane into two parts [14]: the *brane tension* which takes the form $-\Lambda h_{ab}$ of a cosmological constant term given by some effective theory defining the brane, and the energy-momentum tensor τ^{m}_{ab} of the particles and fields confined to the brane. Following the same idea, sometimes we will consider a similar decomposition *all over* Σ

$$\tau_{ab} = -\Lambda h_{ab} + \tau^{m}_{ab}. \tag{16}$$

It must be remarked then that, at points in the signature changing set S, τ^{m}_{ab} does not contain the full information of the energy-momentum tensor of fields "confined" on the brane.

Regarding specific energy-momentum tensors on the brane, much attention has been focused to the case where the total energy-momentum tensor on the brane is of "cosmological constant type," probably for simplicity. From relation (16) it follows that this case corresponds to a brane with nonvanishing tension but no matter content, so that $\tau^{m}_{ab} = 0$. In the final part of this section we will show that for signature changing branes the energy-momentum tensor *cannot* be of this type near S. We do this in two steps: for so-called umbilical branes, and in the general case.

A. Umbilical branes

Recall that a hypersurface is called *umbilical* whenever its second fundamental form is proportional to the first fundamental form: $K_{ab} \propto h_{ab}$. In the constant signature case, the most simple way of obtaining $\tau_{ab} = \alpha h_{ab}$ for some scalar field α consists on gluing two umbilical hypersurfaces $\Sigma^{\pm}$. This follows immediately from the standard Israel formula (9). As a matter of fact, this procedure is *exclusive* of constant signature branes, because signature changing branes cannot be umbilical, as we show next.

To that end, let us decompose the normal vector $\vec{n}$ in the basis $\{\vec{\ell}, \vec{e}_a\}$. Since the contraction of $\vec{n} - (\boldsymbol{n}, \boldsymbol{n})\vec{\ell}$ with $\boldsymbol{n}$ vanishes, it follows that this vector must be a linear combination of the tangent vectors $\vec{e}_a$. Denoting the coefficients by n^a we have

$$\vec{n} = (\boldsymbol{n}, \boldsymbol{n})\vec{\ell} + n^a \vec{e}_a. \tag{17}$$

Recall also that the second fundamental form defined as an object in Σ reads $K_{ab} = e_a{}^\mu e_b{}^\nu K_{\mu\nu} = e_a{}^\mu e_b{}^\nu \nabla_\mu n_\nu$. Then, we have the following important result, which was advanced in [1].

Proposition 2. A C^3 umbilical hypersurface of a C^2 spacetime must have constant signature.

Proof. Let Σ be a C^3 hypersurface and $\boldsymbol{n}$ a C^2 normal one-form. Multiplying $\vec{n}$ in (17) by $\vec{e}_b$ and using $h_{ab} = e_a{}^\mu e_b{}^\nu g_{\mu\nu}|_\Sigma$, it follows

$$n^a h_{ab} = -(\boldsymbol{n}, \boldsymbol{n})\ell_\beta e_b{}^\beta.$$

Defining [38]

$$\varphi_a \equiv -\ell^\nu e_a{}^\mu \nabla_\mu n_\nu,$$

it is straightforward to obtain ($\partial_a = e_a{}^\mu \partial_\mu = \partial/\partial\xi^a$)

$$\partial_a(\boldsymbol{n}, \boldsymbol{n}) = -2(\boldsymbol{n}, \boldsymbol{n})\varphi_a + 2K_{ab}n^b. \tag{18}$$

Let us assume now that Σ is umbilical, i.e.

$$K_{ab} = F h_{ab},$$

for some function F on Σ. F is at least C^1, because the second fundamental form is C^1 and h is C^2. Equation (18) becomes

$$\partial_a(\boldsymbol{n}, \boldsymbol{n}) = -2(\varphi_a + F\ell_\beta e_a{}^\beta)(\boldsymbol{n}, \boldsymbol{n}), \tag{19}$$

which can be viewed as a differential equation for $(\boldsymbol{n}, \boldsymbol{n})$. Uniqueness of the solution follows because the first factor on the right-hand side is at least C^1 (notice that φ_a is C^1 from its definition). Thus, if $(\boldsymbol{n}, \boldsymbol{n})$ vanishes somewhere, then it mush vanish everywhere on Σ. This proves the claim. ■

Observe that the door for umbilical hypersurfaces which are null everywhere is still open. In this case both the second and first fundamental forms are degenerate and share the null degeneration direction. Thus, one can also try to glue two spacetimes across umbilical null branes.

B. The brane tension

Let us finally address the question of whether there can be general branes with only brane tension. Proposition 2 is a preliminary no-go result along that direction. Nevertheless, in principle one could still try to obtain $\tau_{ab} = \alpha h_{ab}$ by gluing two nonumbilical hypersurfaces. The following result, already announced in [1], proves that such a brane cannot undergo a change of signature unless α vanishes somewhere on Σ.

Theorem 2. Let Σ be a brane constructed under the assumptions of Theorem 1. If $\mathcal{G}_{ab} = \beta h_{ab}$ for a function β which is nonzero everywhere on Σ, then Σ cannot change its causal character.

Proof. Projecting (11) onto Σ with $e_a{}^\mu e_b{}^\nu$ and using $\mathcal{G}_{ab} = \beta h_{ab}$, we get

$$\beta h_{ab} = -(\boldsymbol{n}, \boldsymbol{n})[\mathcal{H}_{ab}] - h_{ab}(n^\alpha n^\beta [\mathcal{H}_{\alpha\beta}] - (\boldsymbol{n}, \boldsymbol{n})[\mathcal{H}^\alpha{}_\alpha]). \tag{20}$$

Expression (17) and $\ell^\alpha \mathcal{H}^\pm_{\alpha\beta} = 0$ implies $n^\alpha n^\beta [\mathcal{H}_{\alpha\beta}] = n^a n^b [\mathcal{H}_{ab}]$. Using also $n^a n^b h_{ab} = (\boldsymbol{n}, \boldsymbol{n})((\boldsymbol{n}, \boldsymbol{n})\ell^\alpha \ell_\alpha - 1)$ which follows by squaring $(\boldsymbol{n}, \boldsymbol{n})\ell^\alpha$ in (17), the contraction of (20) with $n^a n^b$ gives

$$(\boldsymbol{n}, \boldsymbol{n})\{(\beta - (\boldsymbol{n}, \boldsymbol{n})[\mathcal{H}^\alpha{}_\alpha])((\boldsymbol{n}, \boldsymbol{n})\ell_\mu \ell^\mu - 1) + n^\alpha n^\beta [\mathcal{H}_{\alpha\beta}](\boldsymbol{n}, \boldsymbol{n})\ell_\mu \ell^\mu\} = 0. \tag{21}$$

Thus, the expression between braces must vanish on $\Sigma_L \cup \Sigma_E$ which readily implies

$$\lim_{p \to S} \beta = 0.$$

Since β is at least C^1, hence continuous, we have $\beta|_S = 0$ and the result follows. ■

Evaluating (20) on S and using that β vanishes there, we obtain

$$n^\alpha n^\beta [\mathcal{H}_{\alpha\beta}]|_S = n^a n^b [\mathcal{H}_{ab}]|_S = 0.$$

In addition to this result, let us note that the identity (see [38] for a proof)

$$[K_{ab}] = (\boldsymbol{n}, \boldsymbol{n})[\mathcal{H}_{ab}]$$

clearly implies that $[K_{ab}]|_{\Sigma_0} = 0$ on the null phase Σ_0; so, if we demand $[K_{ab}] = F h_{ab}$ on Σ, then F must vanish at the null phase Σ_0 too.

An important corollary follows from Theorem 2.

Corollary 3. For any choice of normalization, the condition $\tau_{ab} = -\Lambda h_{ab}$ for a constant brane tension $\Lambda \neq 0$ is incompatible with a change of signature on Σ.

A physical interpretation of this result is that a change of signature on the brane requires that some matter fields become excited, or equivalently that a signature change cannot occur just spontaneously. Let us remark that the possibility of having $\tau_{ab} = \alpha h_{ab}$ for some *function* α has not been ruled out, but this function must necessarily vanish at the signature changing set S.

V. GENERAL BRANES IN STATIC AND SPHERICALLY, PLANE, OR HYPERBOLICALLY SYMMETRIC BULKS

Our aim now is to provide examples of sufficient generality for the construction described in the previous sections. More particular examples on anti-de Sitter bulks will be then considered in the next section. We will put particular emphasis on the possibility of signature changing or null branes, but we will also compare these cases with the standard timelike branes.

In this section, we treat the case of general n-dimensional static spacetimes $(\mathcal{N}^{\pm}, g^{\pm})$ (with $n > 2$) admitting an isometry group G_k of dimension $k = (n-1)(n-2)/2$ acting on the hypersurfaces orthogonal to the static Killing vector and containing an isotropy group I_s with $s = (n-2)(n-3)/2$ parameters. We will restrict to branes preserving the G_k symmetries, which leads to a symmetry-preserving matching of spacetimes, see [50].

In appropriate adapted coordinates, the most general such spacetimes have line elements

$$ds^{2^+} = -A^2(r)dt^2 + B^2(r)dr^2 + C^2(r)d\Omega^2_{\Upsilon^{n-2}_{\phi}},$$
$$ds^{2^-} = -\tilde{A}^2(\tilde{r})d\tilde{t}^2 + \tilde{B}^2(\tilde{r})d\tilde{r}^2 + \tilde{C}^2(\tilde{r})d\Omega^2_{\Upsilon^{n-2}_{\tilde{\phi}}},$$

where $d\Omega^2_{\Upsilon^{n-2}_{\phi}}$ is the "unit" metric on the $(n-2)$-dimensional Riemannian space Υ^{n-2} of constant curvature, written in standard coordinates denoted by ϕ (and analogously for $d\Omega^2_{\Upsilon^{n-2}_{\tilde{\phi}}}$). The functions A, B, and C depend only on r and are taken to be positive without loss of generality. The range of the coordinates t and r may vary from case to case, and thus it is left free in principle. The same comments apply to $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{t}$, and $\tilde{r}$.

Let us consider the G_k-symmetric hypersurfaces $\Sigma^{\pm}$ in $\mathcal{N}^{\pm}$. They can be defined via C^3 embedding maps $\Phi_{\pm}$: $\Sigma \to \mathcal{N}^{\pm}$. Taking local coordinates $\{\xi, \varphi^M\}$ on the abstract matching hypersurface Σ $(M, N, \ldots = 2 \ldots n-1)$, where $\{\varphi^M\}$ are standard coordinates on Υ^{n-2}, the embeddings $\Phi_{\pm}$ can be written in local form as

$$\Phi_+(\xi, \varphi^M) \equiv \{t = t(\xi), r = r(\xi), \phi^M = \varphi^M\} \quad (\Sigma^+),$$
$$\Phi_-(\xi, \varphi^M) \equiv \{\tilde{t} = \tilde{t}(\xi), \tilde{r} = \tilde{r}(\xi), \tilde{\phi}^M = \varphi^M\} \quad (\Sigma^-).$$

The images under the differential maps $d\Phi_{\pm}$ of the tangent space basis $\{\partial_{\xi}, \partial_{\varphi^M}\}$ on Σ are of course bases of the tangent spaces on $\Sigma^{\pm}$. They read explicitly

$$\vec{e}^{\,+}_{\xi} = \dot{t}\partial_t + \dot{r}\partial_r|_{\Sigma^+}, \qquad \vec{e}^{\,+}_{\varphi^M} = \partial_{\phi^M}|_{\Sigma^+},$$
$$\vec{e}^{\,-}_{\xi} = \dot{\tilde{t}}\partial_{\tilde{t}} + \dot{\tilde{r}}\partial_{\tilde{r}}|_{\Sigma^-}, \qquad \vec{e}^{\,-}_{\varphi^M} = \partial_{\tilde{\phi}^M}|_{\Sigma^-},$$

where the dot means differentiation with respect to ξ. Defining the functions

$$N^+ \equiv -A^2\dot{t}^2 + B^2\dot{r}^2|_{\Sigma^+}, \tag{22a}$$
$$N^- \equiv -\tilde{A}^2\dot{\tilde{t}}^2 + \tilde{B}^2\dot{\tilde{r}}^2|_{\Sigma^-}, \tag{22b}$$

a simple calculation shows that the two first fundamental forms inherited by Σ from $\mathcal{N}^{\pm}$ coincide if and only if

$$N^+ = N^- \equiv N, \qquad C \stackrel{\Sigma}{=} \tilde{C} \equiv a(\xi), \tag{23}$$

so that the induced metric on the brane takes the form

$$ds^2|_{\Sigma} = N(\xi)d\xi^2 + a^2(\xi)d\Omega^2_{\Upsilon^{n-2}_{\varphi}}. \tag{24}$$

Thus, the brane Σ will have in general a Lorentzian phase Σ_L where $N < 0$, a Euclidean phase Σ_E defined by $N > 0$, and a null phase Σ_0 where $N = 0$. The Lorentzian part Σ_L describes a Robertson-Walker (RW) spacetime with ξ related to the standard cosmic time $T(\xi)$ by

$$\dot{T} = \sqrt{-N} \quad \text{on } \Sigma_L. \tag{25}$$

The whole brane is foliated by homogeneous and isotropic (maximally symmetric) spacelike hypersurfaces. Changes of signature occur at given "instants of time" corresponding to the values ξ_m of ξ where N vanishes but is not identically zero in any neighborhood of ξ_m. The set of all such ξ_m define the signature changing set S of Σ.

From the point of view of the Lorentzian part of the brane the Lorentzian geometry becomes singular at S. We shall describe later the type of singularity that any observers living on Σ_L will see there. We must emphasize, however, that this singularity exists *only* from the *inner point of view* of the Lorentzian part Σ_L, and concerns *only* the brane's "Lorentzianity." *Neither the bulk nor the hypersurface* Σ *defining the brane have any singularity anywhere* for regular functions $N(\xi)$ and $a(\xi)$.

In order to complete the matching and have a well-defined bulk and brane, we need to choose a rigging and solve the algebraic equations (5). For convenience we choose normal one-forms of $\Sigma^{\pm}$ with the same norm at points $\Phi_{\pm}(p)$, $p \in \Sigma$. One possibility (not unique, of course) is

$$\boldsymbol{N}^+ = AB(-\dot{r}\mathrm{d}t + \dot{t}\mathrm{d}r)|_{\Sigma^+},$$
$$\boldsymbol{N}^- = \tilde{A}\tilde{B}(-\dot{\tilde{r}}\mathrm{d}\tilde{t} + \dot{\tilde{t}}\mathrm{d}\tilde{r})|_{\Sigma^-}.$$

Note that $(\boldsymbol{N}^+, \boldsymbol{N}^+) = (\boldsymbol{N}^-, \boldsymbol{N}^-) = -N$ provided that the preliminary matching conditions (23) hold. A suitable rigging on Σ^+ is

$$\vec{\ell}_+ = \epsilon_1(-A^{-2}\dot{r}\partial_t + B^{-2}\dot{t}\partial_r)|_{\Sigma^+},$$

where ϵ_1 selects the subregion of the spacetime we are choosing; see subsection V B below. Note that $N^+_{\alpha}\ell^{\alpha}_+ \neq 0$ everywhere on Σ^+, as required. To find the rigging $\vec{\ell}_-$ satisfying (5), observe that $\vec{\ell}_+$ is orthogonal to the tangent vectors of Υ^{n-2}, which implies that $\vec{\ell}_-$ must be a linear combination of $\partial_{\tilde{t}}$ and $\partial_{\tilde{r}}$. Thus, we can write without loss of generality

$$\vec{\ell}_- = \epsilon_1 L(-\alpha^2\tilde{A}^{-2}\dot{\tilde{r}}\partial_{\tilde{t}} + \tilde{B}^{-2}\dot{\tilde{t}}\partial_{\tilde{r}})|_{\Sigma^-},$$

where $L \neq 0$ and α^2 are coefficients fulfilling the equations

$$\begin{aligned} \epsilon_1 g^+_{\mu\nu}\ell^{\mu}_+ e^{+\nu}_{\xi} &\stackrel{\Sigma}{=} \epsilon_1 g^-_{\mu\nu}\ell^{\mu}_- e^{-\nu}_{\xi}: \\ 2\dot{r}\,\dot{t} &\stackrel{\Sigma}{=} L(\alpha^2+1)\dot{\tilde{t}}\,\dot{\tilde{r}}, \end{aligned} \tag{26}$$

$$g^{+}_{\mu\nu}\ell^{\mu}_{+}\ell^{\nu}_{+} \stackrel{\Sigma}{=} g^{-}_{\mu\nu}\ell^{\mu}_{-}\ell^{\nu}_{-}: \\ -\frac{\dot{r}^2}{A^2}+\frac{\dot{t}^2}{B^2} \stackrel{\Sigma}{=} L^2\left(-\frac{\alpha^4\dot{\tilde{r}}^2}{\tilde{A}^2}+\frac{\dot{\tilde{t}}^2}{\tilde{B}^2}\right). \tag{27}$$

The second equation involves L quadratically. In order to obtain a linear equation in L which will be useful below, let us consider the linear combination of (27) times N minus the square of (26). The resulting expression is a perfect square. Taking its square root, which introduces an extra sign ϵ, we get

$$\epsilon_1(N^{+}_{\alpha}\ell^{\alpha}_{+})|_{\Sigma^+} \stackrel{\Sigma}{=} \epsilon\epsilon_1(N^{-}_{\alpha}\ell^{\alpha}_{-})|_{\Sigma^-}: \\ \frac{A}{B}\dot{t}^2+\frac{B}{A}\dot{r}^2 \stackrel{\Sigma}{=} \epsilon L\left(\frac{\tilde{A}}{\tilde{B}}\dot{\tilde{t}}^2+\alpha^2\frac{\tilde{B}}{\tilde{A}}\dot{\tilde{r}}^2\right). \tag{28}$$

Because of the positivity of the rest of the factors, this equation readily implies that $\epsilon = \mathrm{sign}(L)$. The fact that the above combinations can be written in the covariant form (28) is not by chance. It simply accounts for the *a posteriori* identification (after the matching is completed) of $\boldsymbol{n}^+$ with $\boldsymbol{n}^-$ (see Sec. III): this trivially implies $(\boldsymbol{n}^+, \boldsymbol{n}^+) = (\boldsymbol{n}^-, \boldsymbol{n}^-)$, which thanks to choosing $\boldsymbol{N}^+$ and $\boldsymbol{N}^-$ with the same norm yields $(N^{+}_{\alpha}\ell^{\alpha}_{+})^2 = (N^{-}_{\alpha}\ell^{\alpha}_{-})^2$. Thus, (28) follows for a certain sign ϵ. Moreover, as a result, the identification of $\boldsymbol{n}^+$ with $\boldsymbol{n}^-$ clearly leads now to the identification of $\boldsymbol{N}^+$ with $\epsilon\boldsymbol{N}^-$. In fact, it turns out that the first equation in (5) can be substituted by this relation (28)—whenever the normal one-forms $\boldsymbol{N}^{\pm}$ have the same norm—provided that the set Σ_0 has empty interior.

A. The energy-momentum tensor on the brane

In order to calculate the singular part of the Einstein tensor distribution, and thereby the energy-momentum tensor on the brane, we need to know $[\mathcal{H}_{ab}]$. After a straightforward calculation using the definition (10) we obtain

$$\epsilon_1[\mathcal{H}_{\xi\xi}] = -\dot{r}\ddot{t}-\dot{t}\ddot{r}+L(\alpha^2\dot{\tilde{r}}\ddot{\tilde{t}}+\dot{\tilde{t}}\ddot{\tilde{r}})-\dot{r}^2\dot{t}\left(2\frac{A_{,r}}{A}+\frac{B_{,r}}{B}\right) \\ -\dot{t}^3\frac{AA_{,r}}{B^2}+L\left[\dot{\tilde{r}}^2\dot{\tilde{t}}\left(2\alpha^2\frac{\tilde{A}_{,\tilde{r}}}{\tilde{A}}+\frac{\tilde{B}_{,\tilde{r}}}{\tilde{B}}\right) \\ +\dot{\tilde{t}}^3\frac{\tilde{A}\tilde{A}_{,\tilde{r}}}{\tilde{B}^2}\right]\Bigg|_{\Sigma}, \tag{29}$$

$$[\mathcal{H}_{MN}]\mathrm{d}\varphi^M\mathrm{d}\varphi^N = \epsilon_1\left(\dot{t}\frac{CC_{,r}}{B^2}-L\dot{\tilde{t}}\frac{\tilde{C}\tilde{C}_{,\tilde{r}}}{\tilde{B}^2}\right)\Bigg|_{\Sigma}\mathrm{d}\Omega^2_{Y^{n-2}_{\phi}} \\ \equiv [\mathcal{H}]\mathrm{d}\Omega^2_{Y^{n-2}_{\phi}}, \tag{30}$$

$$[\mathcal{H}_{\xi M}] = 0,$$

where, for later convenience, we have defined

$$[\mathcal{H}] \equiv \mathcal{H}^+ - \mathcal{H}^-$$

with

$$\mathcal{H}^+ \equiv \epsilon_1\dot{t}\frac{CC_{,r}}{B^2}\Bigg|_{\Sigma}, \qquad \mathcal{H}^- \equiv \epsilon_1 L\dot{\tilde{t}}\frac{\tilde{C}\tilde{C}_{,\tilde{r}}}{\tilde{B}^2}\Bigg|_{\Sigma}. \tag{31}$$

Next, we must use expression (11) to obtain the tensor $\mathcal{G}_{\mu\nu}$. Obviously, the explicit form of this tensor depends on the coordinate system used to describe the spacetime. Since the matching procedure allows for different coordinate systems on each side of the matching hypersurface we need to choose one of them. For definiteness we choose the coordinate system on $\mathcal{N}^+$. Using the explicit expressions (29)–(31) for $[\mathcal{H}_{ab}]$ together with the fact that $\ell^\mu[\mathcal{H}_{\mu\nu}] = 0$, and after some calculations, the final result can be conveniently written as

$$\mathcal{G}_{\mu\nu}dx^\mu dx^\nu = -\frac{(n-2)[\mathcal{H}]}{C^2(\ell^{\alpha}_{+}N^{+}_{\alpha})^2}(A^2\dot{t}dt - B^2\dot{r}dr)^2 \\ -\frac{C^2[\mathcal{H}_{\xi\xi}]+(n-3)N[\mathcal{H}]}{(\ell^{\alpha}_{+}N^{+}_{\alpha})^2}\mathrm{d}\Omega^2_{Y^{n-2}_{\phi}}|_{\Sigma^+}. \tag{32}$$

As expected, $\mathcal{G}_{\mu\nu}$ is directly related to the quantities $[\mathcal{H}_{\xi\xi}]$ and $[\mathcal{H}]$. However, expressions (29) and (30) for these two quantities are not quite satisfactory yet because they involve L and α which are the solutions of the algebraic equations (26) and (28). Solving directly for L and α and substituting into (29) and (30) is not convenient since the preliminary matching conditions must also be taken into account. We leave the details of this somewhat tricky calculation to the appendix and quote here the final results. It turns out that, at points where $N \neq 0$, $[\mathcal{H}]$ can be written in the symmetric form

$$[\mathcal{H}] = \frac{a(\ell^{\alpha}_{+}N^{+}_{\alpha})}{N}\left(\epsilon\frac{\tilde{A}}{\tilde{B}}\tilde{C}_{,\tilde{r}}\dot{\tilde{t}} - \frac{A}{B}C_{,r}\dot{t}\right)\Bigg|_{\Sigma} \tag{33}$$

while $[\mathcal{H}_{\xi\xi}]$ reads

$$\frac{1}{(\ell^{\alpha}_{+}N^{+}_{\alpha})}[H_{\xi\xi}]\dot{t}\dot{\tilde{t}} = \epsilon\dot{t}\left[\frac{\tilde{A}_{,\tilde{r}}}{\tilde{B}}\dot{\tilde{t}}^2+\frac{\tilde{B}_{,\tilde{r}}}{\tilde{A}}\dot{\tilde{r}}^2-\frac{\tilde{B}}{\tilde{A}}\left(\frac{\dot{N}}{2N}\dot{\tilde{r}}-\ddot{\tilde{r}}\right)\right] \\ -\dot{\tilde{t}}\left[\frac{A_{,r}}{B}\dot{t}^2+\frac{B_{,r}}{A}\dot{r}^2-\frac{B}{A}\left(\frac{\dot{N}}{2N}\dot{r}-\ddot{r}\right)\right]\Bigg|_{\Sigma}. \tag{34}$$

Because of the presence of N in the denominator it may seem at first sight that the expressions (33) and (34) diverge when we approach the null phase Σ_0. This is however not the case because $[\mathcal{H}_{\mu\nu}]$ is by construction well defined everywhere on Σ. This also follows directly from expressions (29) and (30), which are regular on Σ_0.

For completeness, let us include here an expression for $[\mathcal{H}]$ at points on Σ_0. Equations (22) become

$$\dot{r}^2|_{\Sigma_0} = \frac{A^2}{B^2}\dot{t}^2|_{\Sigma_0}, \qquad \dot{\tilde{r}}^2|_{\Sigma_0} = \frac{\tilde{A}^2}{\tilde{B}^2}\dot{\tilde{t}}^2|_{\Sigma_0}.$$

This implies that neither $\dot{r}$, $\dot{t}$, $\dot{\tilde{r}}$, nor $\dot{\tilde{t}}$ can vanish on Σ_0 (otherwise $\Phi_\pm$ would not be embeddings). Then, Eq. (27) implies that

$$\alpha^2|_{\Sigma_0} = 1,$$

which inserted in (26) gives

$$L|_{\Sigma_0} = \frac{\dot{r}\,\dot{t}}{\dot{\tilde{r}}\,\dot{\tilde{t}}}\bigg|_{\Sigma_0}. \tag{35}$$

Using all this in (30) and recalling $\dot{a} = C_{,r}\dot{r}|_\Sigma = C_{,\tilde{r}}\dot{\tilde{r}}|_\Sigma$, we finally obtain

$$[\mathcal{H}]|_{\Sigma_0} = \mathrm{sign}(\dot{r}\,\dot{t})a\dot{a}\frac{\epsilon_1}{AB}\left(1 - \frac{\dot{r}^2B^2}{\dot{\tilde{r}}^2\tilde{B}^2}\right)\bigg|_{\Sigma_0}.$$

Once we have computed the singular part $\mathcal{G}_{\mu\nu}$ of the Einstein tensor, given by (32), the energy-momentum tensor on the brane follows directly from (13). A convenient way of describing this object is via its eigenvalues. Since $\mathcal{G}_{\mu\nu}n^\mu = 0$ holds identically, the rank of the tensor $\mathcal{G}_{\mu\nu}$, and hence of $\tau_{\mu\nu}$ is at most $n-1$ and 0 is always one of its eigenvalues. In order to evaluate the remaining eigenvalues of $\tau_{\mu\nu}$—which correspond to the eigenvalues of τ_{ab} wherever Σ is not null—let us rewrite (32) as

$$\tau_{\mu\nu}dx^\nu dx^\mu \stackrel{\Sigma}{=} -N^{-1}\hat{\varrho}(A^2\dot{t}dt - B^2\dot{r}dr)^2 + \hat{p}a^2d\Omega^2_{Y_\phi^{n-2}}, \tag{36}$$

where we have defined

$$\kappa_n^2\hat{\varrho} \equiv \frac{(n-2)N[\mathcal{H}]}{a^2(\ell_+^\alpha N_\alpha^+)^2}\bigg|_\Sigma = \frac{(n-2)}{a(\ell_+^\alpha N_\alpha^+)}\left(\epsilon\frac{\tilde{A}}{\tilde{B}}\tilde{C}_{,\tilde{r}}\dot{\tilde{t}} - \frac{A}{B}C_{,r}\dot{t}\right)\bigg|_\Sigma,$$

$$\kappa_n^2\hat{p} \equiv -\frac{a^2[\mathcal{H}_{\xi\xi}] + (n-3)N[\mathcal{H}]}{a^2(\ell_+^\alpha N_\alpha^+)^2} = -\frac{[\mathcal{H}_{\xi\xi}]}{(\ell_+^\alpha N_\alpha^+)^2} - \frac{n-3}{n-2}\kappa_n^2\hat{\varrho}.$$

Since the one-form $-A^2\dot{t}dt + B^2\dot{r}dr$ appearing in (36) is precisely the tangent vector $\vec{e}_\xi$ with index down and that its norm is simply N, it follows easily that the remaining eigenvalues of $\tau_{\mu\nu}$ are precisely $-\hat{\rho}$ and $\hat{p}$.

The explicit expression for $\hat{p}$ can be read off directly from the previous formula and the use of (34). However, it is simpler and more convenient to note the following identity which follows after a straightforward, if somewhat long, calculation

$$\dot{\hat{\varrho}} + \left(\frac{(\ell_+^\alpha N_\alpha^+)^{\cdot}}{(\ell_+^\alpha N_\alpha^+)} - \frac{\dot{N}}{2N}\right)\hat{\varrho} + (n-2)\frac{\dot{a}}{a}(\hat{\varrho} + \hat{p}) + \frac{(n-2)}{\kappa_n^2(\ell_+^\alpha N_\alpha^+)}\left\{\frac{A^2\dot{t}\,\dot{r}}{a}\left(\frac{C_{,r}}{AB}\right)_{,r} - \epsilon\frac{\tilde{A}^2\dot{\tilde{t}}\,\dot{\tilde{r}}}{a}\left(\frac{\tilde{C}_{,\tilde{r}}}{\tilde{A}\,\tilde{B}}\right)_{,\tilde{r}}\right\}\bigg|_\Sigma = 0. \tag{37}$$

This identity clearly resembles a continuity equation. We shall see that this is exactly the case, with explicit applications for anti-de Sitter bulks.

B. The meaning of the signs

Since our convention is that the rigging $\vec{\ell}_+$ of Σ^+ points towards the submanifold $\mathcal{M}^+ \subset \mathcal{N}^+$ and that $\vec{\ell}_-$ points outward from the submanifold $\mathcal{M}^- \subset \mathcal{N}^-$ it follows that choosing the signs ϵ and ϵ_1 amounts to selecting which subsets $\mathcal{M}^\pm \subset \mathcal{N}^\pm$ are taken to perform the matching.

Hitherto everything is valid for general branes. However, if Σ is non-null everywhere, the algebraic equations (26) and (27) admit two different solutions for L for each choice of $\vec{\ell}_+$, and these two solutions have a different sign ϵ, according to (28). On the other hand, if there is a point p where Σ becomes null, from Lemma 3 there is at most one solution for ϵ. Let us determine its value. We already know that $\epsilon = \mathrm{sign}(L)$, but $L|_{\Sigma_0}$ has been already computed on (35), and consequently

$$\epsilon = \mathrm{sign}(L) = \mathrm{sign}(\dot{r}\,\dot{t}\,\dot{\tilde{r}}\,\dot{\tilde{t}})|_{\Sigma_0} \qquad (\Sigma_0 \neq \varnothing). \tag{38}$$

Therefore, if Σ_0 is not empty then ϵ is unique and explicitly determined by the two embeddings. Since in the purely Lorentzian (or Euclidean) case ϵ is free, we shall also keep ϵ free in order to compare our general results with previous works on Lorentzian branes.

With regard to the remaining sign ϵ_1, this has not been fixed so far. Observe that $\epsilon_1 = \mathrm{sign}(\ell_+^\alpha N_\alpha^+)$, as follows from (28) and the fact that A, B, C have been chosen to be positive. The interpretation of this sign is, therefore, as follows. In the construction above, we use *two* spacetimes, each of which contains a hypersurface that separates each spacetime into two regions. So we have *four* regions to play with. Fixing one of the regions in one spacetime, this may be matchable to none, one, or both of the regions in the second spacetime—if Σ has a nonempty null phase, there is at most one possibility as follows from Proposition 1. But, can the left-out region of the first spacetime be matched to any of the regions in the second? The answer is yes *if the originally chosen region in the first spacetime was matchable* to one of the regions in the second; and actually the region that now matches with it is precisely the *complementary* part of the one that matched with the first region of the first spacetime. In short, given two matchable spacetimes there *always* are two complementary matchings, as discussed in detail in [51]. This provides an interpretation for ϵ_1: it selects which region at both sides of Σ in

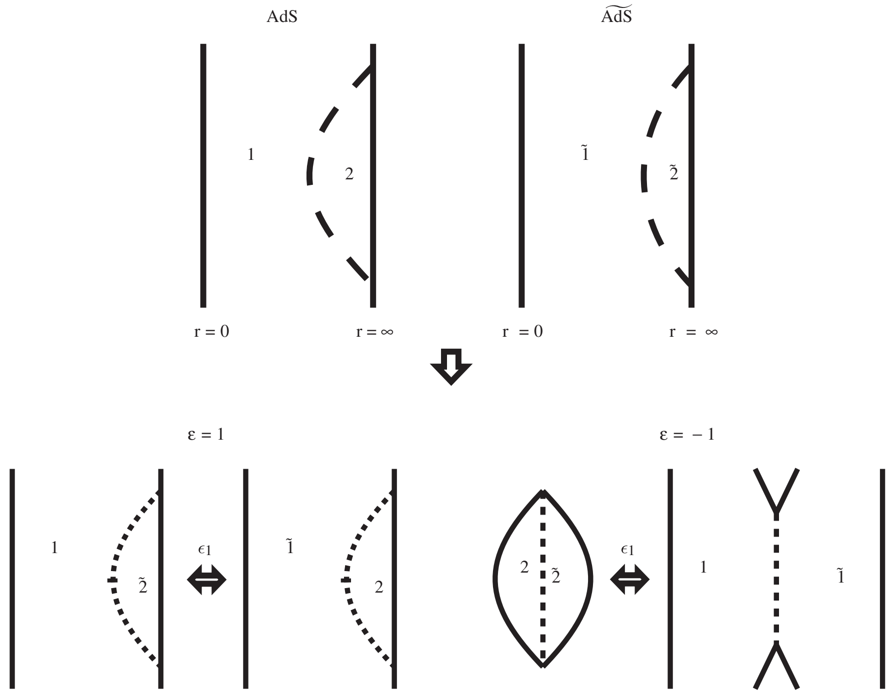

FIG. 2. Four different possible matchings between AdS and $\widetilde{\text{AdS}}$ in the case $k = 1$ driven by the signs ε and ϵ_1 (fixed σ and $\tilde{\sigma}$). The anti-de Sitter diagrams for $k = 1$ are drawn at the top. The slashed curves represent Σ^+ and Σ^-, that divide AdS and $\widetilde{\text{AdS}}$ into two parts, respectively. The pairs of choices of regions (halves) to be joined depend on the relative signs of the rigging vectors, ε, and on the orientation of $\vec{\ell}$, which is determined by ϵ_1. For $\varepsilon = 1$ we obtain one of two possibilities on the bottom left, which differ on the sign of ϵ_1, and for $\varepsilon = -1$, the two on the bottom right.

the first spacetime is taken to perform the matching. A scheme of the four possible different cases discussed in this paragraph for the particular case of AdS bulks is shown in Fig. 2 (see Sec. VI for the notation).

VI. SIGNATURE CHANGING BRANES IN AdS_n BULKS

Let us now specialize to the case where $\mathcal{N}^+$ and $\mathcal{N}^-$ are anti-de Sitter spaces of dimension n, usually denoted by AdS_n. For that, we choose the metric functions to be

$$A^2 = B^{-2} = k + \lambda^2 r^2, \qquad C = r, \tag{39}$$

$$\tilde{A}^2 = \tilde{B}^{-2} = k + \tilde{\lambda}^2 \tilde{r}^2, \qquad \tilde{C} = \tilde{r}, \tag{40}$$

where λ and $\tilde{\lambda}$ are non-negative constants related to the cosmological constant by means of $2\Lambda_n = -(n-1)(n-2)\lambda^2$, and analogously for the tilded ones. Here $k = -1, 0, 1$ corresponding to three possible coordinate systems to describe the AdS_n spacetime. k coincides with the sectional curvature of $d\Omega_{\Upsilon_\phi^{n-2}}$.

The case of a flat bulk is included here for the values $\lambda = 0$ and $k = 1$. When $k = 0, 1$, the ranges of the non-angular coordinates are $-\infty < t < \infty$ and $r > 0$. In the $k = -1$ case, though, the range of r is further restricted to $r > 1/\lambda$.

Because of Corollary 2 we cannot construct the bulk by gluing together two copies of a submanifold with boundary of AdS_n if the boundary has a nonempty null phase. However, there is no *a priori* obstruction to consider two different submanifolds with boundary of AdS_n or, more generally, to try and paste a region of AdS_n with another region of a possibly different anti-de Sitter space, $\widetilde{\text{AdS}}_n$, with another cosmological constant. For simplicity, and as in the previous section, we will only consider branes Σ with spherical, plane, or hyperboloidal symmetry.

Particularizing the equations of the previous section to the explicit functions (39) and (40), we get

$$a(\xi) = r(\xi) = \tilde{r}(\xi) \tag{41}$$

while (22) and (23) yield ordinary differential equations for $t(\xi)$ and $\tilde{t}(\xi)$ in terms of $N(\xi)$

$$\dot{t} = \frac{\sigma a}{k + \lambda^2 a^2} \sqrt{\frac{\dot{a}^2}{a^2} - N\left(\frac{k}{a^2} + \lambda^2\right)}, \tag{42a}$$

$$\dot{\tilde{t}} = \frac{\tilde{\sigma} a}{k + \tilde{\lambda}^2 a^2} \sqrt{\frac{\dot{a}^2}{a^2} - N\left(\frac{k}{a^2} + \tilde{\lambda}^2\right)}, \tag{42b}$$

where σ and $\tilde{\sigma}$ are two signs. For compactness, it is convenient to define

$$\varepsilon \equiv \epsilon\sigma\tilde{\sigma}, \qquad (43)$$

which will in fact substitute $\tilde{\sigma}$.

With these expressions we can write down the explicit form for $\hat{\varrho}$ and $\hat{p}$ in the present case:

$$\frac{\kappa_n^2\hat{\varrho}}{n-2} = \frac{\sigma}{(\ell_+^\alpha N_\alpha^+)}\left(\varepsilon\sqrt{\frac{\dot{a}^2}{a^2} - N\left(\frac{k}{a^2}+\tilde{\lambda}^2\right)} - \sqrt{\frac{\dot{a}^2}{a^2} - N\left(\frac{k}{a^2}+\lambda^2\right)}\right),$$

$$\kappa_n^2(\ell_+^\alpha N_\alpha^+)\left(\hat{p} + \frac{n-3}{n-2}\hat{\varrho}\right) = \varepsilon\sigma\frac{(\tilde{\lambda}^2 N + \frac{\dot{N}}{2N}\frac{\dot{a}}{a} - \frac{\ddot{a}}{a})}{\sqrt{\frac{\dot{a}^2}{a^2} - N(\frac{k}{a^2}+\tilde{\lambda}^2)}} - \sigma\frac{(\lambda^2 N + \frac{\dot{N}}{2N}\frac{\dot{a}}{a} - \frac{\ddot{a}}{a})}{\sqrt{\frac{\dot{a}^2}{a^2} - N(\frac{k}{a^2}+\lambda^2)}},$$

where

$$\ell_+^\alpha N_\alpha^+ = \epsilon_1\left(2\frac{\dot{a}^2}{k+a^2\lambda^2} - N\right).$$

Regarding the identity (37), it simplifies to

$$\dot{\hat{\varrho}} + \frac{d}{d\xi}\left(\log\frac{|\ell_+^\alpha N_\alpha^+|}{\sqrt{|N|}}\right)\hat{\varrho} + (n-2)\frac{\dot{a}}{a}(\hat{\varrho}+\hat{p}) = 0. \qquad (44)$$

At points where $N \neq 0$ (i.e. outside the null phase Σ_0) we can define

$$\varrho \equiv \hat{\varrho}\frac{|\ell_+^\alpha N_\alpha^+|}{\sqrt{|N|}}, \qquad p \equiv \hat{p}\frac{|\ell_+^\alpha N_\alpha^+|}{\sqrt{|N|}}, \qquad (45)$$

so that the conservation law is obtained from (44) in its standard form

$$\dot{\varrho} + (n-2)\frac{\dot{a}}{a}(\varrho+p) = 0. \qquad (46)$$

This choice of normalization may seem artificial, but it corresponds precisely to the choice of the unit normal vector as the rigging vector on Σ_L. Therefore, ϱ and p are functions that correspond to *the energy density and pressure measured within the Lorentzian part* Σ_L *of the brane* Σ. These nonhatted functions are then relevant physical quantities one has to analyze.

To start with, recall that $\hat{\varrho}$ and $\hat{p}$ *are regular* everywhere on Σ. Thus, from (45) one could be misled to think that the energy density ϱ and pressure p blow up when approaching a change of signature $S \cap \overline{\Sigma_L}$. Nevertheless, we are going to prove in what follows that, actually, ϱ *vanishes* at the signature change, and that p can also be regular in many cases; see subsection VI B.

To show this and to compare with previous works on purely Lorentzian branes in AdS (see e.g. [11]), let us perform the change (25) from the timelike coordinate ξ to the cosmic time T, which is suitable at points where $N < 0$, so that the line element (24) reads on Σ_L

$$ds^2|_{\Sigma_L} = -dT^2 + a^2 d\Omega^2_{\Upsilon_\varphi^{n-2}}.$$

Using the notation $' = d/dT$ we have

$$a' = \frac{\dot{a}}{\sqrt{-N}}, \qquad a'' = -\frac{\ddot{a}}{N} + \dot{a}\frac{\dot{N}}{2N^2}, \qquad (47)$$

which we use to obtain

$$\varrho' + (n-2)\frac{a'}{a}(\varrho+p) = 0, \qquad (48)$$

$$\frac{\kappa_n^2}{n-2}\varrho = \sigma\epsilon_1\left[\varepsilon\sqrt{\frac{a'^2}{a^2}+\frac{k}{a^2}+\tilde{\lambda}^2} - \sqrt{\frac{a'^2}{a^2}+\frac{k}{a^2}+\lambda^2}\right]. \qquad (49)$$

Defining the Hubble function $H \equiv a'/a$ as usual, Eq. (48) yields

$$p = \varrho\left[\frac{\varepsilon}{n-2}\left(H' - \frac{k}{a^2}\right)\left(H^2+\frac{k}{a^2}+\tilde{\lambda}^2\right)^{-1/2} \times\left(H^2+\frac{k}{a^2}+\lambda^2\right)^{-1/2} - 1\right]. \qquad (50)$$

Passing any of the square roots of (49) to the left and squaring we obtain the following respective two expressions

$$\varrho\sqrt{\frac{a'^2}{a^2}+\frac{k}{a^2}+\tilde{\lambda}^2} = \sigma\epsilon_1\varepsilon\frac{(n-2)}{2\kappa_n^2}\times\left(\tilde{\lambda}^2-\lambda^2+\frac{\kappa_n^4}{(n-2)^2}\varrho^2\right), \qquad (51)$$

$$\varrho\sqrt{\frac{a'^2}{a^2}+\frac{k}{a^2}+\lambda^2} = -\sigma\epsilon_1\frac{(n-2)}{2\kappa_n^2}\times\left(\lambda^2-\tilde{\lambda}^2+\frac{\kappa_n^4}{(n-2)^2}\varrho^2\right). \qquad (52)$$

Now, squaring any of these two expressions, and provided $\varrho \neq 0$, we obtain the following condition

$$\frac{a'^2}{a^2}+\frac{k}{a^2} = \frac{(n-2)^2}{4\kappa_n^4\varrho^2}\left[\left(\tilde{\lambda}^2+\lambda^2-\frac{\kappa_n^4}{(n-2)^2}\varrho^2\right)^2 - 4\tilde{\lambda}^2\lambda^2\right], \qquad (53)$$

which is usually referred to as "the modified Friedmann equation" for braneworld cosmologies.

Let us discuss these relations (48)–(53) in detail.

(i) An important remark is that (48)–(53) hold only on Σ_L.

(ii) Equation (48) is the usual continuity equation in $(n-1)$-dimensional RW spacetimes. The traditional 4-dimensional case is recovered by assuming $n = 5$, that is, a 5-dimensional bulk.

(iii) Equation (50) can be regarded as a Raychaudhuri-like equation on the brane.

(iv) Concerning (49), let us first of all stress the fact that the modified Friedmann equation (53), which is the equation usually found in the literature as a consequence of using the formalism in [9], is just one of its consequences. In other words, (53) is only a *necessary* quadratic condition, and its solutions still have to satisfy (49). Thus, *the truly relevant equation, containing all the information, is* (49).
To see this in more detail, and its consequences, let us focus on (51) and (52). By multiplying these two equations we obtain

$$0 \leq -\varepsilon\left(\left(\frac{\kappa_n^2}{n-2}\varrho\right)^4 - (\tilde{\lambda}^2 - \lambda^2)^2\right),$$

from where

$$\frac{\kappa_n^2}{n-2}|\varrho| \leq \sqrt{|\tilde{\lambda}^2 - \lambda^2|} \quad \text{if } \varepsilon = 1, \tag{54}$$

$$\frac{\kappa_n^2}{n-2}|\varrho| \geq \sqrt{|\tilde{\lambda}^2 - \lambda^2|} \quad \text{if } \varepsilon = -1. \tag{55}$$

Actually, these limits can be strengthened when $k = 0, 1$, for which the inequalities are strict. In fact, for $k = 0$ and $\varepsilon = 1$ we obtain $\frac{\kappa_n^2}{n-2}|\varrho| \leq |\lambda - \tilde{\lambda}|$.

The important expression (49) appears in full form in [8,15], and partially in [10,11] for the so-called "shell cosmologies."

In expression (49) (for $n = 5$), *both* the usual "brane," i.e. using Z_2-mirror symmetry in AdS_5, and the "shell" cosmologies are naturally recovered. The Z_2-mirror branes require $\lambda = \tilde{\lambda}$ and, as we already know, are incompatible with signature changes or null phases. The shell cosmologies, also referred to as "asymmetric" brane cosmologies in [13], require on the contrary that $\lambda \neq \tilde{\lambda}$, and they are compatible, in principle, with the existence of null phases Σ_0 and signature changing sets S.

Next, we discuss all these different possibilities.

A. Constant signature branes or shells in AdS_5

These cases are characterized by having only one of the possible phases, and thus $S = \emptyset$. The relevant physical case is the Lorentzian one, that is, when $\Sigma = \Sigma_L$. Then, relations (48)–(53) hold on the entire Σ. The other two cases $\Sigma = \Sigma_E$ and $\Sigma = \Sigma_0$ can also be treated in the formalism, but they have no direct physical interpretation apart from possible topological defects.

1. Z_2-mirror Lorentzian branes

For Lorentzian branes $\Sigma = \Sigma_L$ with Z_2-mirror symmetry one only has to take

$$\lambda = \tilde{\lambda}$$

and $\varepsilon = -1$. The latter is necessary because for a Z_2 matching, $t = \tilde{t}$ and $r = \tilde{r}$ and (42) implies $\sigma = \tilde{\sigma}$. Moreover, $\epsilon = -1$ because N^+ must be identified to $-N^-$, c.f. the discussion after (28). Thus $\varepsilon = -1$ follows from (43). Notice that $\varepsilon = 1$ corresponds to a matching that recovers the original AdS_5 spacetime (in particular $\varrho = 0$ in that case, as follows from (49)). In order to have a positive ϱ, we have to choose the matching such that $\sigma\epsilon_1 = -1$. The geometrical view of different possible matchings depending on the values of $\sigma\epsilon_1$ and ε are shown in Fig. 2.

For these Z_2-symmetric branes in AdS bulks, the big bang singularity on the brane is characterized by the divergence of ϱ and p. In the cases $k = 0, 1$, since the brane is assumed to be regular ($r > 0$) and Lorentzian ($N < 0$) everywhere, the only possibility is that the big bang coincides with the vanishing of a. This big bang is therefore located at $r \to 0$ in the AdS bulk. In fact, the brane cannot be regular there, because it is forced to be Lorentzian. As for the cases with $k = -1$, the range for r is restricted to $r > 1/\lambda$ and therefore the description of Σ in those coordinates obviously fails at $a \leq 1/\lambda$.

2. Shells, or asymmetric Lorentzian branes

The asymmetric case is characterized by

$$\lambda \neq \tilde{\lambda}.$$

Observe that then, *both possible signs* $\varepsilon = \pm 1$ are feasible. This has been correctly stated in [8,15] but, for unclear reasons, only the case $\varepsilon = 1$ was considered in [10,11]. By setting $\varepsilon = 1$ in our formulae and using the freedom in interchanging λ and $\tilde{\lambda}$ one can set $\sigma\epsilon_1 = 1$ without loss of generality. This implies $\tilde{\lambda} > \lambda$ for a positive ϱ. Notice that there is an *upper bound* for the energy density ϱ given by (54). As far as we know, this upper limit had not been noticed before.

On the other hand, the case $\varepsilon = -1$ requires from (49) that $\sigma\epsilon_1 = -1$ if ϱ is to be positive. Acceptable matchings are hence possible, and both signs of $\tilde{\lambda}^2 - \lambda^2$ are allowed. In this case, there is a *lower bound* for the energy density given by (55).

B. Signature changing branes in AdS_5

It follows from Lemma 3 that only one of the values of ϵ allows for a nonempty signature changing set S. This was identified in (38) as $\epsilon = \text{sign}(\dot{r}\dot{\tilde{r}}\dot{t}\dot{\tilde{t}})|_{\Sigma_0}$. Using (41) and (42) we get $\epsilon = \sigma\tilde{\sigma}$, so that in this subsection we must set

$$\varepsilon = 1.$$

This implies, on using (49), that we must necessarily require

$$\lambda \neq \tilde{\lambda}$$

so that signature changing branes must be of "asymmetric" type. This, of course, is nothing but a direct consequence of the general Corollary 2. Furthermore, the possible matchings for a positive ϱ are identified by the necessary condition

$$\sigma\epsilon_1 = \mathrm{sign}(\tilde{\lambda}^2 - \lambda^2),$$

as follows from the discussion after (55). Again, ϱ is upper bounded by (54).

Nevertheless, things can behave quite differently now in comparison with the typical, purely Lorentzian, "asymmetric" case studied above. For instance, new types of "big bangs"—in the sense of the beginning of time—can appear at points where a is not zero, $\dot{a}$ and $\ddot{a}$ are well behaved, but where a' and/or a'' diverge. Actually, that happens precisely at the signature changing set S due to the vanishing of N there. This type of behavior simply cannot be found in pure Lorentzian brane cosmologies, be them Z_2-symmetric or asymmetric.

Moreover, in the signature changing case one can further prove that (49), or its consequence the modified Friedmann relation (53), allows us to avoid the presence of truly singular big bangs *even from the point of view of the observers in the brane*. To show this, we first note that

$$\dot{a}|_S \neq 0$$

as otherwise, since $N = 0$ on S, from (42) we would have that $\dot{t}|_S = \dot{\tilde{t}}|_S = 0$, which we know is impossible on S, cf. Sec. V B. Thus, from the definition of a' we have the following

Lemma 4. On a signature changing brane with $\Sigma_L \neq \emptyset$, a' diverges necessarily when approaching the signature changing set $S \cap \overline{\Sigma_L}$. Hence, H also diverges there.

Now, since H is unbounded when approaching $S \cap \overline{\Sigma_L}$, (49) easily implies that ϱ vanishes there:

$$\lim_{x \to S\cap\overline{\Sigma_L}} \varrho = \lim_{x \to S\cap\overline{\Sigma_L}} \frac{3\sigma\epsilon_1}{2\kappa_5^2}\frac{\tilde{\lambda}^2 - \lambda^2}{|H|} = 0. \qquad (56)$$

Collecting the results we have thus proven the following:

Theorem 3. In a signature changing brane produced by joining AdS_5 and $\widetilde{\mathrm{AdS}}_5$ preserving the spatial symmetries, the total energy density ϱ on the Lorentzian part Σ_L of the brane is bounded above by (54) and vanishes at the set of signature changing points $S \cap \overline{\Sigma_L}$.

As a remark, observe that linear equation of states of type $p = \gamma\varrho$ with constant γ are not allowed in this signature changing case, for this would imply from (48) that $\varrho a^{3(1+\gamma)} = \mathrm{const}$, which is not compatible with the vanishing of ϱ at $S \cap \overline{\Sigma_L}$ (where a must be finite.) Nevertheless, general linear equations of state of type $p = p_0 + \gamma\varrho$ are possible, as (48) gives now $(p_0 + (1 + \gamma)\varrho)a^{3(1+\gamma)} = \mathrm{const}$, which has no problems at $S \cap \overline{\Sigma_L}$. Observe, however, that this particular equation of state implies clearly that p must be finite at $S \cap \overline{\Sigma_L}$.

Thus, to study the behavior of p close to the change of signature we use (50) for $\varepsilon = 1$, together with (56), to obtain the following limit

$$\begin{aligned}\lim_{x \to S\cap\overline{\Sigma_L}} p &= \lim_{x \to S\cap\overline{\Sigma_L}} \varrho\left(\frac{H'}{3H^2} - 1\right)\\ &= \lim_{x \to S\cap\overline{\Sigma_L}} \frac{3\sigma\epsilon_1}{2\kappa_5^2}\frac{\tilde{\lambda}^2 - \lambda^2}{|H|}\left(\frac{H'}{3H^2} - 1\right)\\ &= \lim_{x \to S\cap\overline{\Sigma_L}} \frac{\sigma\epsilon_1}{2\kappa_5^2}(\tilde{\lambda}^2 - \lambda^2)\sqrt{-N}\frac{a^2}{\dot{a}|\dot{a}|}\\ &\quad\times\left(\frac{\ddot{a}}{\dot{a}} - \frac{\dot{N}}{2N} - \frac{4\dot{a}}{a}\right).\end{aligned}$$

Thus, the actual value of this limit will depend on the particular choice of the function $N(\xi)$: for regular branes, p diverges at $S \cap \overline{\Sigma_L}$ if $\dot{N}|_S \neq 0$, while p may remain regular if $\dot{N}|_S = 0$. Observe that changing the function $N(\xi)$ does not necessarily mean a change of cosmological model.

Therefore, by choosing appropriately the hypersurface Σ in AdS_5, *signature changing branes such that both ϱ and p remain finite and well behaved everywhere on $\overline{\Sigma_L}$ are feasible*. Recall that ϱ always vanishes at the change of signature.

We would like to stress that this conclusion and theorem 3 are very satisfactory results: the Hubble parameter H—an observable quantity—diverges when approaching the change of signature, yet the whole geometrical structure remains unhurt and the relevant physical quantities, such as ϱ and p, are regular there.

Fully explicit examples of signature changing branes, with particular known functions $a(\xi)$ and $N(\xi)$, were presented in [1]. We refer to this letter for some discussion and extra comments of physical interest.

The Lorentzian phase Σ_L considered as a classical spacetime in general relativity

An observer living on the Lorentzian part Σ_L of the brane might interpret, in principle, that a change of signature would correspond to a singularity in a RW spacetime. If this "singularity" is in the past, it could represent a big bang from the inner point of view of Σ_L. We would like to discuss this now in detail.

To begin with, it may seem contrary to our physical intuition that $\varrho \to 0$ at the signature changing set $S \cap \overline{\Sigma_L}$, which plays the role of such a "singularity" from the inner point of view of the Lorentzian phase Σ_L. The meaning of this is that the total energy density ϱ (the matter and radiation energy density plus the brane tension) "starts" at $S \cap \overline{\Sigma_L}$, which is the origin of time in Σ_L, with a vanishing value which increases from then on but is always bounded by (54). We must remark, however, that the usual 4-dimensional Einstein equations do *not* apply anywhere on Σ, and that ϱ and p are (normalized) quantities associated to the singular part $\tau_{\mu\nu}$, with support on Σ, of the energy-momentum distribution $\underline{T}^{\mathcal{M}}{}_{\mu\nu}$.

But what would an uninformed scientist, confined to live within Σ_L, interpret about these facts? If this scientist believes that general relativity (GR) is the correct theory describing the universe (i.e., Σ_L for him/her), he/she would rather try to compute the eigenvalues of the *Einstein tensor* within the brane, that is to say, the Einstein tensor of the first fundamental form h_{ab} of Σ_L. The eigenvalues of this tensor are, of course,

$$8\pi G\varrho^{(\mathrm{GR})} + \Lambda_4 = \frac{3}{a^2}(a'^2 + k) = \frac{3}{a^2}\left(-\frac{\dot{a}^2}{N} + k\right), \quad (57)$$

$$8\pi G p^{(\mathrm{GR})} - \Lambda_4 = -2\frac{a''}{a} - \frac{1}{a^2}(a'^2 + k) = \frac{1}{N}\left(2\frac{\ddot{a}}{a} - \frac{\dot{N}}{N}\frac{\dot{a}}{a}\right) + \frac{1}{a^2}\left(\frac{\dot{a}^2}{N} - k\right)$$

which obviously diverge at the "singularity" placed on the signature changing set $S \cap \overline{\Sigma_L}$. Here, Λ_4 is the GR cosmological constant as computed by that scientist.

More importantly, let us stress the fact that there is a relation between these GR quantities and the actual energy density and pressure on the brane according to the real 5-dimensional field equations. For instance, from (49) we derive

$$\frac{\kappa_5^2}{3}\varrho = \sigma\epsilon_1\left\{\sqrt{\frac{8\pi G}{3}\varrho^{(\mathrm{GR})} + \frac{\Lambda_4}{3} + \tilde{\lambda}^2} - \sqrt{\frac{8\pi G}{3}\varrho^{(\mathrm{GR})} + \frac{\Lambda_4}{3} + \lambda^2}\right\}$$

while (51) and (52) give the inverse relations

$$8\pi G\varrho^{(\mathrm{GR})} + \Lambda_4 + 3\tilde{\lambda}^2 = \frac{27}{4\kappa_5^4\varrho^2}\left(\tilde{\lambda}^2 - \lambda^2 + \frac{\kappa_5^4}{9}\varrho^2\right)^2$$

and the one obtained by interchanging $\tilde{\lambda} \leftrightarrow \lambda$.

These formulas patently show that the GR "singularity" where $\varrho^{(\mathrm{GR})} \longrightarrow \infty$, which corresponds to the signature change, is simply a manifestation of the fact that the proper energy density on the Lorentzian phase of the brane actually vanishes there.

C. Recovering the Friedmann equation at different limits: Effective 4-dimensional fundamental constants

A well-known fact in the Z_2-symmetric brane cosmologies, as well as in the asymmetric "shell" cosmologies, is that the usual Friedmann equation (57) of general telativity can be recovered from the equation on the brane (49) at the limit when the matter density is small compared to Λ, once a nonvanishing tension Λ has been introduced in an appropriate manner. This limit is, in fact, the one used to recover the full 4-dimensional Einstein equations in GR, see [9,14], and to relate Λ_5 and Λ with effective 4-dimensional gravitational and cosmological constants (see (65) below).

Nevertheless, that limit relies on the existence of a nonvanishing tension. In the present case there is another limit, both natural and convenient, for which no tension is needed. Such a limit corresponds to large values of a while keeping a finite a', so that $H^2 + k/a^2$ is small. Another characterization of this limit is that $8\pi G\varrho^{(\mathrm{GR})} + \Lambda_4$ is small. One appropriate dimensionless quantity to perform rigorously this limit is

$$\frac{H^2 + k/a^2}{\lambda^2} \approx \frac{8\pi G\varrho^{(\mathrm{GR})} + \Lambda_4}{\Lambda_5},$$

where $\approx$ stands for equality except for a constant of order one. $\tilde{\lambda}$ or $\tilde{\Lambda}_5$ could also be used to define the dimensionless parameter.

Also worth mentioning is the fact that in many papers (see [9,13,14]) the limits of the modified Friedmann equation have been taken starting from the quadratic equation (53), instead of the original (49) which contains more information, thus missing the meaning of the signs ε and $\epsilon_1\sigma$. An exception is [11] where the authors considered Eq. (49), but as mentioned before not all the possible signs were taken into account. Therefore, for the sake of completeness, let us derive the limits keeping those signs free.

1. Large values of a with small values of H

Let us start by considering the limit for large values of a while keeping H small. Equation (49) for $n = 5$ can be approximated to

$$\sigma\epsilon_1\frac{\kappa_5^2}{3}\varrho = \varepsilon\tilde{\lambda} - \lambda + \frac{1}{2}\frac{a'^2 + k}{a^2}\left(\frac{\varepsilon}{\tilde{\lambda}} - \frac{1}{\lambda}\right) + O(a^{-4}).$$

Since $\varepsilon\lambda - \tilde{\lambda} \neq 0$ in order to have a brane or shell at all, this expression can be rearranged as

$$3\left(\frac{a'^2}{a^2} + \frac{k}{a^2}\right) = \sigma\epsilon_1\kappa_5^2\frac{2\tilde{\lambda}\lambda}{\varepsilon\lambda - \tilde{\lambda}}\varrho + 6\varepsilon\tilde{\lambda}\lambda + O(a^{-4}). \quad (58)$$

Let us consider now the tension of the brane as a contributing part of τ_{ab}, so that (16) holds. Then, ϱ and p decompose as $\varrho = \varrho_m + \Lambda$ and $p = p_m - \Lambda$, where ϱ_m and p_m correspond to τ^m_{ab}. Using this together with (57) in (58), we derive

$$8\pi G\varrho^{(\mathrm{GR})} + \Lambda_4 = \sigma\epsilon_1\kappa_5^2\frac{2\tilde{\lambda}\lambda}{\varepsilon\lambda - \tilde{\lambda}}(\varrho_m + \Lambda) + 6\varepsilon\tilde{\lambda}\lambda + O(a^{-4}).$$

There are many ways to interpret this relation. In principle, it simply determines the value of $\varrho^{(\mathrm{GR})}$ in terms of ρ_m, Λ and the constants G, κ_5, λ, $\tilde{\lambda}$, and Λ_4. It seems natural, however, to identify the constant terms at both sides of this relation, and therefore the remaining terms too. Identifying

$$\varrho^{(\mathrm{GR})} \leftrightarrow \varrho_m \tag{59}$$

we obtain the following relations between the fundamental constants

$$8\pi G = \sigma \epsilon_1 \kappa_5^2 \frac{2\tilde{\lambda}\lambda}{\varepsilon\lambda - \tilde{\lambda}}, \tag{60}$$

$$\Lambda_4 = \sigma \epsilon_1 \kappa_5^2 \frac{2\tilde{\lambda}\lambda}{\varepsilon\lambda - \tilde{\lambda}} \Lambda + 6\varepsilon\tilde{\lambda}\lambda. \tag{61}$$

As far as we are aware, these relations were previously unknown.

Relations (60) and (61) can be particularized to the case of Z_2-symmetric Lorentzian branes, for which $\lambda = \tilde{\lambda}$ and $\varepsilon = -1$, so that

$$8\pi G = -\sigma\epsilon_1 \kappa_5^2 \lambda, \qquad \Lambda_4 = -\sigma\epsilon_1\kappa_5^2 \lambda\Lambda - 6\lambda^2.$$

In view that we need $\sigma\epsilon_1 = -1$ for a positive gravitational constant these equations can be rearranged as

$$8\pi G = \frac{\kappa_5^4}{6}\left(\Lambda - \frac{\Lambda_4}{8\pi G}\right), \qquad \Lambda_4 = 8\pi G\Lambda + \Lambda_5. \tag{62}$$

These expressions differ from the ones usually obtained in the literature, involving a different limit—given by (65) below—, and seem to be new.

2. Small values of ϱ_m/Λ

As for the usual limit $\varrho_m/\Lambda \to 0$ with a nonvanishing Λ it is convenient to start from the quadratic equation (53). Using $\varrho = \varrho_m + \Lambda$ and defining $\beta \equiv 3\lambda/(\Lambda\kappa_5^2)$ and $\tilde{\beta} \equiv 3\tilde{\lambda}/(\Lambda\kappa_5^2)$, it can be expressed as

$$\frac{a'^2}{a^2} + \frac{k}{a^2} = \frac{\kappa_5^4}{36}\Lambda^2[1 - 2(\tilde{\beta}^2 + \beta^2) + (\tilde{\beta}^2 - \beta^2)^2] + \frac{\kappa_5^4}{18}\Lambda^2[1 - (\tilde{\beta}^2 - \beta^2)^2]\frac{\varrho_m}{\Lambda} + O[(\varrho_m/\Lambda)^2].$$

Comparing with (57) we get

$$8\pi G\varrho^{(\mathrm{GR})} + \Lambda_4 = \frac{\kappa_5^4}{12}\Lambda^2[1 - 2(\tilde{\beta}^2 + \beta^2) + (\tilde{\beta}^2 - \beta^2)^2] + \frac{\kappa_5^4}{6}\Lambda^2[1 - (\tilde{\beta}^2 - \beta^2)^2]\frac{\varrho_m}{\Lambda} + O[(\varrho_m/\Lambda)^2],$$

which, as before, provides an expression for $\varrho^{(\mathrm{GR})}$ and can be resolved in many different ways. Using again the natural identification (59), a different set of relations for the effective fundamental constants is obtained:

$$8\pi G = \frac{1}{6}\kappa_5^4\Lambda\left[1 - \frac{81}{\kappa_5^8\Lambda^4}(\tilde{\lambda}^2 - \lambda^2)^2\right], \tag{63}$$

$$\Lambda_4 = \frac{1}{12}\kappa_5^4\Lambda^2\left[1 - \frac{18}{\kappa_5^4\Lambda^2}(\tilde{\lambda}^2 + \lambda^2) + \frac{81}{\kappa_5^8\Lambda^4}(\tilde{\lambda}^2 - \lambda^2)^2\right]. \tag{64}$$

For the particular Z_2-symmetric branes, for which $\lambda = \tilde{\lambda}$, and recalling that $\lambda^2 = -\Lambda_5/6$, these two relations simplify to

$$8\pi G = \tfrac{1}{6}\kappa_5^4\Lambda, \qquad \Lambda_4 = \tfrac{1}{2}(8\pi G\Lambda + \Lambda_5), \tag{65}$$

which correspond to the usual relations found in the literature [14].

3. Relationship between the two limits

We have seen that the usual relations (65) are not unique, since they depend crucially on the kind of limit taken. Another limit of physical interest, with no need of a tension Λ, leads for instance to the alternative relations (62). These two sets (62) and (65) only coincide when one demands a vanishing effective four-dimensional cosmological constant, this is, if the tension of the brane is fine-tuned in order to have $\Lambda_4 = 0$. In that case both sets contain the same information, given by $\kappa_5^4\Lambda^2/6 = -\Lambda_5$ (the fine-tuning of the tension) and $8\pi G = \kappa_5^4\Lambda/6$. This was to be expected, because the limit at large a implies that $\varrho^{(\mathrm{GR})}$ tends to the constant $-\Lambda_4/8\pi G$, and therefore if (and only if) Λ_4 vanishes then the limit $\varrho_m/\Lambda \to 0$ is recovered by means of the identification (59).

It is worth mentioning here that in Ref. [11], despite the use of the limit of a particular case of expression (48), the relations found for the fundamental constants are the usual ones (65) precisely because it was assumed that $\Lambda_4 = 0$.

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APPENDIX

The aim of this appendix is to present the intermediate steps leading from the expressions (29) and (30) involving L and α to the final result (33) and (34) which is independent of L and α and symmetric under the interchange of $\mathcal{N}^+$ by $\mathcal{N}^-$.

Regarding $[\mathcal{H}]$, it turns out to be convenient to work with $N[\mathcal{H}]$. Directly from their definitions (22) and (31) we have, using $C \stackrel{\Sigma}{=} \tilde{C} \equiv a(\xi)$,

$$
\begin{aligned}
N[\mathcal{H}] &= N^+\mathcal{H}^+ - N^-\mathcal{H}^- \\
&= a\epsilon_1\bigg\{\dot t\frac{C_{,r}}{B^2}(-A^2\dot t^2 + B^2\dot r^2) \\
&\quad - L\dot{\tilde t}\frac{\tilde C_{,\tilde r}}{\tilde B^2}(-\tilde A^2\dot{\tilde t}^2 + \tilde B^2\dot{\tilde r}^2)\bigg\}\bigg|_\Sigma. \qquad \text{(A1)}
\end{aligned}
$$

Taking now the derivative of $a(\xi)$ we get $\dot a = C_{,r}\dot r|_{\Sigma^+} = \tilde C_{,\tilde r}\dot{\tilde r}|_{\Sigma^-}$, which allows us to build the following chain of equalities

$$
\begin{aligned}
C_{,r}\dot r^2\dot t - \tilde C_{,\tilde r}L\dot{\tilde r}^2\dot{\tilde t} &= \dot a(\dot r\dot t - L\dot{\tilde r}\dot{\tilde t}) = \dot a(\alpha^2 L\dot{\tilde t}\dot{\tilde r} - \dot t\dot r) \\
&= \tilde C_{,\tilde r}L\alpha^2\dot{\tilde t}\dot{\tilde r}^2 - C_{,r}\dot t\dot r^2|_\Sigma,
\end{aligned}
$$

where in the second equality we used (26). Substituting now the term $C_{,r}\dot r^2\dot t - \tilde C_{,\tilde r}L\dot{\tilde r}^2\dot{\tilde t}$ appearing in (A1) by this expression, we find

$$
\begin{aligned}
N[\mathcal{H}] = a\epsilon_1\bigg\{&-C_{,r}\dot t\frac{A}{B}\left(\frac{A}{B}\dot t^2 + \frac{B}{A}\dot r^2\right) \\
&+ \tilde C_{,\tilde r}\dot{\tilde t}L\frac{\tilde A}{\tilde B}\left(\frac{\tilde A}{\tilde B}\dot{\tilde t}^2 + \alpha^2\frac{\tilde B}{\tilde A}\dot{\tilde r}^2\right)\bigg\}\bigg|_\Sigma.
\end{aligned}
$$

It only remains to use (28) in the two terms in parenthesis in order to get the final result (33).

Let us now rewrite (29) in a symmetric manner. To that aim, it is convenient to consider $[\mathcal{H}_{\xi\xi}]\dot t\dot{\tilde t}$ and try to get common factors $\ell_+^\alpha N_\alpha^+$ as we did before. Rearranging terms in (29) yields

$$
\begin{aligned}
\epsilon_1[\mathcal{H}_{\xi\xi}]\dot t\dot{\tilde t} \overset{\Sigma}{=} \dot{\tilde t}\bigg\{&-\frac{A_{,r}}{A}\dot r^2\dot t^2 + \frac{BB_{,r}}{A^2}\dot r^4 - \dot r\dot t\ddot t - \dot t^2\ddot r \\
&- \left(\frac{A_{,r}}{B}\dot t^2 + \frac{B_{,r}}{A}\dot r^2\right)\left(\frac{A}{B}\dot t^2 + \frac{B}{A}\dot r^2\right)\bigg\} \\
&- \dot t L\bigg\{-\alpha^2\frac{\tilde A_{,\tilde r}}{\tilde A}\dot{\tilde r}^2\dot{\tilde t}^2 + \alpha^2\frac{\tilde B\tilde B_{,\tilde r}}{\tilde A^2}\dot{\tilde r}^4 \\
&- \alpha^2\dot{\tilde r}\dot{\tilde t}\ddot{\tilde t} - \dot{\tilde t}^2\ddot{\tilde r} \\
&- \left(\frac{\tilde A_{,\tilde r}}{\tilde B}\dot{\tilde t}^2 + \frac{\tilde B_{,\tilde r}}{\tilde A}\dot{\tilde r}^2\right)\left(\frac{\tilde A}{\tilde B}\dot{\tilde t}^2 + \alpha^2\frac{\tilde B}{\tilde A}\dot{\tilde r}^2\right)\bigg\}. \qquad \text{(A2)}
\end{aligned}
$$

Now, evaluating $\dot N^\pm$ allows us to write the identities

$$
\begin{aligned}
&-\frac{A_{,r}}{A}\dot r^2\dot t^2 + \frac{BB_{,r}}{A^2}\dot r^4 - \dot r\dot t\ddot t - \dot t^2\ddot r \\
&\qquad \overset{\Sigma}{=} \frac{1}{2A^2}\dot r\dot N - \frac{B}{A}\ddot r\left(\frac{A}{B}\dot t^2 + \frac{B}{A}\dot r^2\right), \\
&-\alpha^2\frac{\tilde A_{,\tilde r}}{\tilde A}\dot{\tilde r}^2\dot{\tilde t}^2 + \alpha^2\frac{\tilde B\tilde B_{,\tilde r}}{\tilde A^2}\dot{\tilde r}^4 - \alpha^2\dot{\tilde r}\dot{\tilde t}\ddot{\tilde t} - \dot{\tilde t}^2\ddot{\tilde r} \\
&\qquad \overset{\Sigma}{=} \frac{\alpha^2}{2\tilde A^2}\dot{\tilde r}\dot N - \frac{\tilde B}{\tilde A}\ddot{\tilde r}\left(\frac{\tilde A}{\tilde B}\dot{\tilde t}^2 + \alpha^2\frac{\tilde B}{\tilde A}\dot{\tilde r}^2\right),
\end{aligned}
$$

which substituted in (A2) yields

$$
\begin{aligned}
\epsilon_1[\mathcal{H}_{\xi\xi}]\dot t\dot{\tilde t} \overset{\Sigma}{=} \dot{\tilde t}\bigg\{&\frac{1}{2A^2}\dot r\dot N - \left(\frac{B}{A}\ddot r + \frac{A_{,r}}{B}\dot t^2 + \frac{B_{,r}}{A}\dot r^2\right) \\
&\times\left(\frac{A}{B}\dot t^2 + \frac{B}{A}\dot r^2\right)\bigg\} \\
&- \dot t L\bigg\{\frac{\alpha^2}{2\tilde A^2}\dot{\tilde r}\dot N - \left(\frac{\tilde B}{\tilde A}\ddot{\tilde r} + \frac{\tilde A_{,\tilde r}}{\tilde B}\dot{\tilde t}^2 + \frac{\tilde B_{,\tilde r}}{\tilde A}\dot{\tilde r}^2\right) \\
&\times\left(\frac{\tilde A}{\tilde B}\dot{\tilde t}^2 + \alpha^2\frac{\tilde B}{\tilde A}\dot{\tilde r}^2\right)\bigg\}. \qquad \text{(A3)}
\end{aligned}
$$

In this expression the only terms that require extra treatment are $A^{-2}\dot r\dot{\tilde t} - L\alpha^2\tilde A^{-2}\dot{\tilde r}\dot t$. Multiplying the first summand by N^+ and the second by N^- we get, after adding zero in the form of $2\dot{\tilde t}\dot r\dot t^2 - L(1+\alpha^2)\dot{\tilde r}\dot{\tilde t}^2\dot t = 0$, see (26),

$$
\begin{aligned}
N\left(\frac{1}{A^2}\dot r\dot{\tilde t} - L\alpha^2\frac{1}{\tilde A^2}\dot{\tilde r}\dot t\right) \overset{\Sigma}{=} &\frac{B}{A}\dot{\tilde t}\dot r\left(\frac{A}{B}\dot t^2 + \frac{B}{A}\dot r^2\right) \\
&- L\frac{\tilde B}{\tilde A}\dot t\dot{\tilde r}\left(\frac{\tilde A}{\tilde B}\dot{\tilde t}^2 + \alpha^2\frac{\tilde B}{\tilde A}\dot{\tilde r}^2\right).
\end{aligned}
$$

Inserting this into (A3) we finally find

$$
\begin{aligned}
\epsilon_1[\mathcal{H}_{\xi\xi}]\dot t\dot{\tilde t} \overset{\Sigma}{=} \dot{\tilde t}&\left(\frac{\dot N}{2N}\frac{B}{A}\dot r - \frac{B}{A}\ddot r - \frac{A_{,r}}{B}\dot t^2 - \frac{B_{,r}}{A}\dot r^2\right) \\
&\times\left(\frac{A}{B}\dot t^2 + \frac{B}{A}\dot r^2\right) - \dot t L\left(\frac{\dot N}{2N}\frac{\tilde B}{\tilde A}\dot{\tilde r} - \frac{\tilde B}{\tilde A}\ddot{\tilde r} - \frac{\tilde A_{,\tilde r}}{\tilde B}\dot{\tilde t}^2\right. \\
&\left.- \frac{\tilde B_{,\tilde r}}{\tilde A}\dot{\tilde r}^2\right)\left(\frac{\tilde A}{\tilde B}\dot{\tilde t}^2 + \alpha^2\frac{\tilde B}{\tilde A}\dot{\tilde r}^2\right),
\end{aligned}
$$

which becomes exactly (34) after using (28).

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