New infinite-dimensional hidden symmetries for heterotic string theory

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The symmetry structures of two-dimensional heterotic string theory are studied further. A $(2d+n)\times(2d+n)$ matrix complex H-potential is constructed and the field equations are extended into a complex matrix formulation. A pair of Hauser-Ernst-type linear systems are established. Based on these linear systems, explicit formulations of new hidden symmetry transformations for the considered theory are given and then these symmetry transformations are verified to constitute infinite-dimensional Lie algebras: the semidirect product of the Kac-Moody $o(d,\widehat{d+n})$ and Virasoro algebras (without center charges). These results demonstrate that the heterotic string theory under consideration possesses more and richer symmetry structures than previously expected.

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I. INTRODUCTION

Due to their importance in theoretical and mathematical physics, the studies of symmetries for the dimensionally reduced low energy effective (super)string theories have attracted much attention in the recent past [e.g. [1–20]]. In addition, it was also found [21] that there exists a remarkable link between the asymptotic cosmological behavior and the properties of symmetry structures for the stringy cosmological models, while string cosmology is believed to be significant for the problem of vacuum selection in string theory. Effective string theories describe various interacting matter fields coupled to gravity, the dimensionally reduced heterotic string theory [e.g. [1,2,8,13–15,17,19,20]] is a typical and very important model of this kind. Some symmetries of this theory have been found and some analogies between it and the reduced Einstein-Maxwell theory have been noted. However, many *scalar* functions in Einstein gravity correspond, formally, to *matrix* ones in the string theory, thus the noncommuting property of the matrices gives rise to essential complications for the further study of the latter. Moreover, some particular relations, such as for any 2×2 matrix A: $A^\top\epsilon A=(\det A)\epsilon$, $A^\top\epsilon+\epsilon A=(\mathrm{tr}A)\epsilon$ with

$$\epsilon=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

have no general analogues for higher dimensional $m\times m$ ($m\geq 3$) matrices, while these relations are important and useful in some studies of the reduced Einstein gravity [e.g. [22–26]]. Since in this paper we deal mainly with $(2d+n)\times(2d+n)$ matrix functions and typically $d=8$, $n=16$, some deeper research and further extended studying methods are needed.

In the present paper, we further study the symmetries of two-dimensional heterotic string theory. We find that we can construct a $(2d+n)\times(2d+n)$ matrix complex H-potential and establish a pair of Hauser-Ernst (HE)-type linear systems. Based on these linear systems, new infinitesimal symmetry transformations for the considered theory are explicitly constructed and then these symmetry transformations are verified to constitute infinite-dimensional Lie algebras: the semidirect product of the Kac-Moody $o(d,\widehat{d+n})$ and Virasoro algebras (without center charges). These results demonstrate that the theory under consideration possesses more and richer symmetry structures than previously expected.

In Sec. II, a $(2d+n)\times(2d+n)$ matrix complex H-potential for the two-dimensional heterotic string theory is constructed, the motion equations are written as a complex matrix formulation in terms of this H-potential, and a pair of associated HE-type linear systems are established. In Sec. III, by virtue of these linear systems, we give explicit expressions of some infinitesimal transformations for the studied theory and then verify that they are all hidden symmetries leaving the motion equations and related conditions invariant. The infinite-dimensional Lie algebra structure of these hidden symmetries is calculated out in Sec. IV. Finally, Sec. V gives some summary and discussions.

II. MATRIX COMPLEX *H*-POTENTIAL AND HE-TYPE LINEAR SYSTEMS

We start with the action describing the massless sector of heterotic string theory as follows:

$$\mathcal{S}=\int d^{2+d}x\sqrt{|\mathcal{G}|}e^{-\Phi}[\mathcal{R}+\mathcal{G}^{LN}\partial_L\Phi\partial_N\Phi-\tfrac{1}{12}\mathcal{H}_{LNP}\mathcal{H}^{LNP}-\tfrac{1}{4}\mathcal{F}^K_{LN}\mathcal{F}^{KLN}],\tag{2.1}$$

where $\mathcal{R}$ is the Ricci scalar for the metric $\mathcal{G}_{LN}$ ($L,N=1,2,\cdots,2+d$), Φ is the dilaton field, and

$$\mathcal{F}^K_{LN}=\partial_L\mathcal{A}^K_N-\partial_N\mathcal{A}^K_L,\qquad \mathcal{H}_{LNP}=(\partial_L\mathcal{B}_{NP}-\tfrac{1}{2}\mathcal{A}^K_L\mathcal{F}^K_{NP})+\text{cyclic},\tag{2.2}$$

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while $\mathcal{B}_{LN}$ and $\mathcal{A}_L^K$ ($K = 1, 2, \cdots, n$) denote the antisymmetric tensor field and $U(1)^n$ gauge fields, respectively. For the heterotic string $d = 8$, $n = 16$, but we keep them arbitrary in the present discussion.

Now following Maharana and Schwarz [1] and Sen [2,3], when dimensionally reducing the above theory from $2 + d$ to 2 dimensions by compactification on a d-dimensional torus and using the fact that the two-dimensional antisymmetry tensor field and two-dimensional gauge fields have no dynamics, then (2.1) can be reduced to the following effective action [2,8,17]

$$S = \int d^2x\sqrt{g}e^{-\phi}[R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{8}g^{\mu\nu}\,\mathrm{Tr}(\partial_\mu\mathcal{M}^{-1}\partial_\nu\mathcal{M})], \tag{2.3}$$

where $g^{\mu\nu}$ ($\mu, \nu = 1, 2$) denotes the inverse of the two-dimensional metric $g_{\mu\nu}$ (in this paper we choose the signature of $g_{\mu\nu}$ to be $++$), $g = \det(g_{\mu\nu})$, R is the Ricci scalar for $g_{\mu\nu}$, ϕ is the shifted dilaton field, and the $(2d + n) \times (2d + n)$ matrix $\mathcal{M}$, representing the moduli G, B, and A, is parametrized as

$$\mathcal{M} = \begin{pmatrix} G^{-1} & G^{-1}(B + C) & G^{-1}A \\ (-B + C)G^{-1} & (G - B + C)G^{-1}(G + B + C) & (G - B + C)G^{-1}A \\ A^\top G^{-1} & A^\top G^{-1}(G + B + C) & I_n + A^\top G^{-1}A \end{pmatrix}, \tag{2.4}$$

in which G, B, and A are, respectively, $d \times d$ symmetric, antisymmetric, and $d \times n$ matrix-valued fields coming from the fields of the $(2 + d)$-dimensional heterotic strings, "$\top$" denotes the transposition, $C = \frac{1}{2}AA^\top$ is a $d \times d$ matrix, and I_n denotes the $n \times n$ unit matrix. All of the above fields are assumed to depend only on x^1, x^2. The motion equations for the moduli $\mathcal{M}$ and dilaton $e^{-\phi}$ can be written as [2,8,17]

$$\partial_\mu(\sqrt{g}g^{\mu\nu}e^{-\phi}\mathcal{M}^{-1}\partial_\nu\mathcal{M}) = 0, \tag{2.5}$$

$$\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu e^{-\phi}) = 0. \tag{2.6}$$

From Eq. (2.6), we see that $M := e^{-\phi}\mathcal{M}$ satisfies the same Eq. (2.5) as $\mathcal{M}$ does. For our purpose, in this paper we shall equivalently use M instead of $\mathcal{M}$. In the conformal gauge $g_{\mu\nu} = e^{2\gamma}\delta_{\mu\nu}$, denoting x^1, x^2 by x, y and $e^{-\phi}$ by ρ for simplicity, the motion equations (2.5) and (2.6) can be equivalently written as

$$d(\rho^{-1}M\mathcal{L}^*dM) = 0, \tag{2.7}$$

$$d^*d\rho = 0 \tag{2.8}$$

with conditions

$$M^\top = M, \tag{2.9a}$$

$$M\mathcal{L}M = \rho^2\mathcal{L}, \tag{2.9b}$$

$$\mathcal{L} := \begin{pmatrix} 0 & I_d & 0 \\ I_d & 0 & 0 \\ 0 & 0 & I_n \end{pmatrix}, \tag{2.9c}$$

where the notations of differential form are adopted, "$*$" is the dual operation of two-dim Euclidian space $\{x, y\}$. Moreover, according to the Einstein equations derived from the action (2.3), the field $\gamma(x, y)$ in the conformal metric can be obtained by a simple integration provided M is known [2,8], so we shall focus our attention on Eqs. (2.7), (2.8), and (2.9) in the following.

Equation (2.7) implies that we can introduce a $(2d + n) \times (2d + n)$ matrix twist potential $Q(J) = Q(x, y)$ such that

$$dQ = -\rho^{-1}M\mathcal{L}^*dM. \tag{2.10}$$

Using (2.9), we obtain from (2.10)

$$dM = \rho^{-1}M\mathcal{L}^*dQ. \tag{2.11}$$

Now introducing a $(2d + n) \times (2d + n)$ matrix complex H-potential

$$H := M + iQ \tag{2.12}$$

and denoting $\Omega := i\mathcal{L}$, then Eqs. (2.10) and (2.11) can be equivalently written as a single complex matrix equation

$$dH = -\rho^{-1}M\Omega^*dH. \tag{2.13}$$

Furthermore, from (2.9) and (2.10) we have $d(Q + Q^\top) = -2^*d\rho\mathcal{L}$. Thus, from (2.8), we can introduce another real field $z = z(x, y)$ such that $*d\rho = dz$ and obtain

$$Q + Q^\top = -2z\mathcal{L}. \tag{2.14}$$

This relation and Eqs. (2.9) and (2.12) imply that we can express Eq. (2.13) as

$$2(z + \rho^*)dH = (H + H^\top)\Omega dH, \tag{2.15}$$

with (2.8) and (2.9), this is equivalent to (2.7). In addition, from (2.15) we can obtain

$$dH^\top\Omega dH = dH^\top\Omega^*dH = 0, \tag{2.16}$$

where we omit the wedge symbol "$\wedge$" in exterior products of differential forms for simplicity.

Now we introduce a complex parameter t and define

$$A(t) := I - t(H + H^\top)\Omega, \quad (I \text{ is the } (2d + n)\text{-dim unit matrix}), \tag{2.17}$$

$$\Gamma(t) := t\Lambda(t)^{-1}dH, \tag{2.18}$$

$$\Lambda(t) := 1 - 2t(z + \rho^*),\quad \Lambda(t)^{-1} = \lambda(t)^{-2}[1 - 2t(z - \rho^*)], \tag{2.19}$$

$$\lambda(t) := [(1 - 2zt)^2 + (2\rho t)^2]^{1/2}, \tag{2.20}$$

then Eq. (2.15) can be rewritten as

$$tdH = A(t)\Gamma(t). \tag{2.21}$$

From Eqs. (2.16), (2.17), and (2.21), we can obtain $d\Gamma(t) = \Gamma(t)\Omega\Gamma(t)$; this is just the complete integrability condition of the following linear differential equation:

$$dF(t) = \Gamma(t)\Omega F(t), \tag{2.22}$$

where $F(t) = F(x, y, t)$ is a $(2d + n) \times (2d + n)$ matrix complex function of x, y, and t.

Equation (2.22) does not define $F(t)$ uniquely, so we suppress some subsidiary conditions consistent with the above equations and the requirement that $F(t)$ be holomorphic in a neighborhood of $t = 0$. From (2.21) and (2.22) and the relation $2t\Lambda^{-1}dz = -\lambda(t)^{-1}d\lambda(t)$ we have

$$dF(0) = 0, \qquad d[\dot{F}(0) - H\Omega F(0)] = 0,$$
$$d[\lambda(t)F(t)^+\Omega F(t)] = 0, \qquad d[F(t)^\top\Omega A(t)F(t)] = 0,$$

where $\dot{F}(t) := \partial F(t)/\partial t$, $F(t)^+ := F(\bar{t})^\dagger$, "$\dagger$" denotes the Hermitian conjugation, and $\bar{t}$ is the complex conjugation of t. These equations and (2.22) determine $F(t)$ up to right-multiplication by an arbitrary nondegenerate $(2d + n) \times (2d + n)$ matrix function of t, so we can use this freedom and choose the integral constants consistently such that

$$F(0) = I, \tag{2.23a}$$
$$\dot{F}(0) = H(J)\Omega, \tag{2.23b}$$

$$\lambda(t)F(t)^+\Omega F(t) = \Omega, \tag{2.24a}$$
$$F(t)^\top\Omega A(t)F(t) = \Omega. \tag{2.24b}$$

We call Eqs. (2.22), (2.23), and (2.24) an HE-type linear system for the two-dimensional heterotic string theory. The F-potential $F(t)$ is essentially a generating function for the hierarchies of potentials given in Ref. [8].

Besides, we can establish another linear system of the two-dimensional string theory. Now, for another complex parameter w, we define

$$\tilde{A}(w) := w - (H + H^\top)\Omega, \tag{2.25}$$

$$\tilde{\Gamma}(w) := \tilde{\Lambda}(w)^{-1}dH, \tag{2.26}$$

$$\tilde{\Lambda}(w) := w - 2(z + \rho^*),\quad \tilde{\Lambda}(w)^{-1} = \tilde{\lambda}(w)^{-2}[w - 2(z - \rho^*)], \tag{2.27}$$

$$\tilde{\lambda}(w) := [(w - 2z)^2 + (2\rho)^2]^{1/2}. \tag{2.28}$$

Then Eq. (2.15) can be rewritten as

$$dH = \tilde{A}(w)\tilde{\Gamma}(w), \tag{2.29}$$

by derivations similar to the above, we have

$$d\tilde{F}(w) = \tilde{\Gamma}(w)\Omega\tilde{F}(w), \tag{2.30}$$

and require consistently that $\tilde{F}(w)$ be analytic around $w = 0$ and satisfy

$$\tilde{\lambda}(w)\tilde{F}(w)^+\Omega\tilde{F}(w) = \Omega, \tag{2.31a}$$
$$\tilde{F}(w)^\top\Omega\tilde{A}(w)\tilde{F}(w) = \Omega, \tag{2.31b}$$

where $\tilde{F}(w) = \tilde{F}(x, y, w)$ is another $(2d + n) \times (2d + n)$ matrix complex function of x, y, and w.

III. PARAMETRIZED SYMMETRY TRANSFORMATIONS

By virtue of solutions $F(t)$, $\tilde{F}(w)$ of linear systems (2.22), (2.23), (2.24), (2.30), and (2.31), we can explicitly construct parametrized symmetry transformations for the two-dimensional heterotic string theory.

We consider the following infinitesimal transformation $\delta = \delta(l)$ of potential H:

$$\delta H = \frac{1}{l}[F(l)TF(l)^{-1} - T]\Omega, \tag{3.1}$$

where l is a (finite) real parameter, $F(l)$ is a solution of (2.22), (2.23), and (2.24) with t being replaced by l, $T = T_a\alpha^a \in o(d, d + n)$ (the Lie algebra of the orthogonal group $O(d, d + n)$), T_a are generators of $o(d, d + n)$, and α^a are infinitesimal real constants. Thus we have the relation

$$T^\top\Omega + \Omega T = 0. \tag{3.2}$$

Now we prove that (3.1) is a hidden symmetry transformation of the motion equation (2.15) and conditions (2.9). First, from (2.24a), (3.1), and (3.2) and $T^+ = T^\dagger = T^\top$ in the real Lie algebra $o(d, d + n)$, we have

$$\begin{aligned}\delta H - \delta H^\dagger &= \frac{1}{l}[F(l)TF(l)^{-1} - T] \\ &\quad \times \Omega\frac{1}{l}\Omega[F(l)^{+-1}T^\top F(l)^+ - T^\top] \\ &\quad \times \frac{1}{l}F(l)[TF(l)^{-1}\Omega F(l)^{+-1} \\ &\quad + F(l)^{-1}\Omega F(l)^{+-1}T^\top]F(l)^+ \\ &= \frac{\lambda(l)}{l}F(l)(T\Omega + \Omega T^\top)F(l)^+ = 0. \end{aligned} \tag{3.3}$$

From (2.12) and (2.14), Eq. (3.3) implies that $\delta M^\top = \delta M$ and $\delta z = 0$.

In addition, Eqs. (2.12), (2.17), and (3.3) give $M = (1/4l)(A(l) - \overline{A(l)})\Omega$ and $\delta M = (1/2)(\delta H + \delta H^\top)$, then from (2.24b), (3.1), and (3.2) and we have

$$\delta M\Omega M + M\Omega\delta M = \frac{1}{8l^2}[(F(l)TF(l)^{-1} + \Omega F(l)^{\top -1}T^{\top}F(l)^{\top}\Omega)(A(l) - \overline{A(l)}) + (A(l) - \overline{A(l)})(F(l)TF(l)^{-1} - \Omega F(l)^{\top -1}T^{\top}F(l)^{\top}\Omega)]\Omega$$
$$= \frac{1}{4l^2}[A(l)F(l)TF(l)^{-1} - F(l)TF(l)^{-1}\overline{A(l)} + \overline{A(l)}\Omega F(l)^{\top -1}T^{\top}F(l)^{\top}\Omega - \Omega F(l)^{\top -1}T^{\top}F(l)^{\top}\Omega A(l)]\Omega$$
$$= \frac{-1}{4l^2}[\Omega F(l)^{\top -1}\Omega T F(l)^{-1} + \Omega F(l)^{\top -1}T^{\top}\Omega F(l)^{-1} - \lambda(l)^2 F(l)T\Omega F(l)^{\top}\Omega - \lambda(l)^2 F(l)\Omega T^{\top}F(l)^{\top}\Omega]\Omega = 0, \tag{3.4}$$

where the relations

$$A(l) + \overline{A(l)} = 2(1 - 2lz), \qquad A(l)\overline{A(l)} = \lambda(l)^2 \tag{3.5}$$

have been used. Equation (3.4) implies that, under the transformation (3.1), the condition (2.9b) is preserved and $\delta\rho = 0$.

Now we investigate the equation satisfied by δH. From (2.22) and (3.1), it follows that $d(\delta H) = (1/l) \times [\Gamma(l)\Omega, F(l)TF(l)^{-1}]\Omega$, this and (2.15) and (2.18) further followed by

$$2(z + \rho^*)d(\delta H) = (H + H^{\top})\Omega d(\delta H) - \frac{1}{l}[(H + H^{\top}) \times \Omega, F(l)TF(l)^{-1}]\Gamma(l). \tag{3.6}$$

On the other hand, from (2.17), (2.21), (2.24b), (3.1), and (3.2) we have

$$(\delta H + \delta H^{\top})\Omega dH = \frac{1}{l^2}[F(l)TF(l)^{-1}\Omega - \Omega F(l)^{\top -1}T^{\top}F(l)^{\top}]\Omega A(l)\Gamma(l) = -\frac{1}{l}[(H + H^{\top})\Omega, F(l)TF(l)^{-1}]\Gamma(l).$$

Substituting this into Eq. (3.6), we finally obtain

$$2(z + \rho^*)d(\delta H) = (H + H^{\top})\Omega d(\delta H) + (\delta H + \delta H^{\top})\Omega dH. \tag{3.7}$$

Equations (3.3), (3.4), and (3.7) show that $H + \delta H$ with δH given by (3.1) satisfies the same Eq. (2.15) and conditions (2.9a) and (2.9b) as H does, i.e. (3.1) is indeed a symmetry transformation for the motion equations of two-dimensional heterotic string theory.

Similarly, by using the solution $\tilde{F}(s)$ of (2.30) and (2.31), we can construct another parametrized infinitesimal symmetry transformation of the studied string theory as

$$\tilde{\delta}H = -s[\tilde{F}(s)T\tilde{F}(s)^{-1} - T]\Omega, \tag{3.8}$$

where s is a finite real parameter.

The set of symmetry transformations of the two-dimensional heterotic string theory can be further enlarged. In addition to (3.1) and (3.8), we propose two other infinitesimal transformations

$$\Delta H = -\sigma\dot{F}(l)F(l)^{-1}\Omega, \tag{3.9}$$

$$\tilde{\Delta}H = \epsilon s[s\dot{\tilde{F}}(s)\tilde{F}(s)^{-1} + \tfrac{1}{2}]\Omega, \tag{3.10}$$

where l, s both are finite real parameters and σ, ϵ are infinitesimal real constants.

From (2.24a) and (3.9),

$$\Delta H - \Delta H^{\dagger} = -\sigma[\dot{F}(l)F(l)^{-1}\Omega + \Omega F(l)^{+-1}\dot{F}(l)^{+}] = \sigma\lambda(l)^{-1}\frac{\partial}{\partial l}\lambda(l)\Omega = -\frac{2\sigma}{\lambda(l)^2}[z(1 - 2lz) - 2l\rho^2]\Omega, \tag{3.11}$$

this and (2.12) and (2.14) imply $(\Delta M)^{\top} = \Delta M$ and $\Delta z = \frac{\sigma}{\lambda(l)^2}[z(1 - 2lz) - 2l\rho^2]$.

Moreover, since $M = \frac{1}{2}(H + \bar{H})$ and $(\Delta M)^{\top} = \Delta M$ by (3.11), we have

$$\Delta M = \tfrac{1}{2}(\Delta H + \Delta\bar{H}) = \tfrac{1}{2}(\Delta H^{\dagger} + \Delta H^{\top}), \tag{3.12a}$$

$$(\Delta M\Omega M + M\Omega\Delta M)^{\top} = \Delta M\Omega M + M\Omega\Delta M. \tag{3.12b}$$

Thus from (2.24b), (3.5), and (3.12a), it follows that

$$\Delta M\Omega M + M\Omega\Delta M = \frac{1}{8l}[(\Delta H^{\dagger} + \Delta H^{\top})\Omega(A(l) - \overline{A(l)}) + (A(l) - \overline{A(l)})(\Delta H^{\dagger} + \Delta H^{\top})\Omega]\Omega$$
$$= \frac{1}{4l}[\Delta H^{\top}\Omega A(l) - \Delta H^{\dagger}\Omega\overline{A(l)} + A(l)\Delta H^{\dagger}\Omega - \overline{A(l)}\Delta H^{\top}\Omega]\Omega$$
$$= \frac{\sigma}{4l}\left[\Omega\frac{\partial}{\partial l}F(l)^{+-1}\Omega\overline{F(l)}^{-1}\Omega + \Omega\frac{\partial}{\partial l}F(l)^{\top -1}\Omega F(l)^{-1}\Omega - \frac{1}{\lambda(l)^2}\left(\overline{A(l)}\Omega\frac{\partial}{\partial l}F(l)^{\top -1}\Omega F(l)^{-1}\overline{A(l)}\Omega + A(l)\Omega\frac{\partial}{\partial l}F(l)^{+-1}\Omega\overline{F(l)}^{-1}A(l)\Omega\right)\right],$$

then from (2.24b), (3.5), and (3.12b) we obtain

$$\begin{aligned}\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J) &= \frac{1}{2}[\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J) + (\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J))^{\top}]\\ &= -\frac{\sigma}{8l}\Big[\Big(\frac{\partial}{\partial l}A(l) + \frac{\partial}{\partial l}\overline{A(l)}\Big) - \frac{1}{\lambda(l)^2}\Big(\overline{A(l)}\frac{\partial}{\partial l}A(l)\overline{A(l)} + A(l)\frac{\partial}{\partial l}\overline{A(l)}A(l)\Big)\Big]\Omega\\ &= -\frac{\sigma}{8l\lambda(l)^2}\Big[2\lambda(l)^2\frac{\partial}{\partial l}(A(l)+\overline{A(l)}) - \frac{\partial}{\partial l}(\lambda(l)^2)(A(l)+\overline{A(l)})\Big]\Omega = \frac{2\sigma}{\lambda(l)^2}\rho^2\Omega. \quad (3.13)\end{aligned}$$

This result shows that the transformation (3.9) preserves the condition (2.9b) provided $\Delta\rho = \frac{\sigma}{\lambda(l)^2}\rho$, and we can also verify, by direct calculations, that $^*d(\Delta\rho) = d(\Delta z)$ as desired.

Now we consider the equation satisfied by the transformed fields. From (2.15), (2.18), (2.19), (3.11), and (3.13), we have

$$\begin{aligned}2(\Delta z + \Delta\rho^*)dH &= 2\sigma(z+\rho^*)\Lambda(l)^{-1}dH\\ &= \frac{\sigma}{l}(H + H^{\top})\Omega\Gamma(l). \quad (3.14)\end{aligned}$$

Moreover, from (2.18), (2.19), (2.22), and (3.9) we obtain

$$d\Delta H = \sigma\dot{\Gamma}(l) - \sigma[\Gamma(l)\Omega, \dot{F}(l)F(l)^{-1}]\Omega. \quad (3.15)$$

Multiplying (3.15) from the left by $2(z+\rho^*)$ and using (2.15) and (3.15) again, it follows that

$$\begin{aligned}2(z+\rho^*)d\Delta H &= \sigma[(H+H^{\top})\Omega, \dot{F}(l)F(l)^{-1}]\Gamma(l)\\ &\quad + (H+H^{\top})\Omega d\Delta H. \quad (3.16)\end{aligned}$$

On the other hand, from (2.17), (2.21), (2.24b), and (3.9) we have

$$\begin{aligned}(\Delta H + \Delta H^{\top})\Omega dH &= -\sigma l^{-1}[\dot{F}(l)F(l)^{-1}\Omega\\ &\quad + \Omega F(l)^{\top -1}\dot{F}(l)^{\top}]\Omega A(l)\Gamma(l)\\ &= \sigma[(H+H^{\top})\Omega, \dot{F}(l)F(l)^{-1}]\Gamma(l)\\ &\quad + \sigma l^{-1}(H+H^{\top})\Omega\Gamma(l). \quad (3.17)\end{aligned}$$

Finally, (3.14), (3.16), and (3.17) give

$$\begin{aligned}&2(\Delta z + \Delta\rho^*)dH + 2(z+\rho^*)d\Delta H\\ &\quad = (\Delta H + \Delta H^{\top})\Omega dH + (H+H^{\top})\Omega d\Delta H. \quad (3.18)\end{aligned}$$

The above results show that (3.9) is indeed a symmetry transformation of Eq. (2.15) with conditions (2.9a) and (2.9b).

Similarly, we can prove that (3.10), which gives $\tilde{\Delta}z = \frac{\epsilon s}{\tilde{\lambda}(s)^2}[z(s-2z) - 2\rho^2]$ and $\tilde{\Delta}\rho = \frac{\epsilon s^2}{\tilde{\lambda}(s)^2}\rho$, is also a symmetry transformation of Eqs. (2.15), (2.9a), and (2.9b).

IV. INFINITE-DIMENSIONAL ALGEBRA STRUCTURES OF THE SYMMETRIES

From the structures of the transformations (3.1) and (3.8), we expand the right-hand sides of them in powers of l and s, respectively, as

$$\delta H = \sum_{k=0}^{\infty} l^k\delta^{(k)}H, \quad (4.1a)$$

$$\tilde{\delta}H = \sum_{m=1}^{\infty} s^m\tilde{\delta}^{(m)}H, \quad (4.1b)$$

where the analytic property of $F(l)$, $\tilde{F}(s)$ around $l=0$, $s=0$ is noted. Each of $\delta^{(k)}$ and $\tilde{\delta}^{(m)}$ satisfies the same equations and conditions as δ and $\tilde{\delta}$ do, thus we have, in fact, constructed infinite many infinitesimal hidden symmetry transformations of the considered theory. The algebraic structures of these transformations can be obtained as follows. Noticing the dependence of (3.1) and (3.8) on the parameters l, s and the infinitesimal constants α^a in T, we denote the corresponding transformations by $\delta_\alpha(l)$, $\tilde{\delta}_\alpha(s)$, respectively. Thus we have

$$\begin{aligned}[\delta_\alpha(l), \delta_\beta(l')]H &= \frac{1}{l}[\delta_\beta(l')F(l)F(l)^{-1}, F(l)T_\alpha F(l)^{-1}]\Omega\\ &\quad - \frac{1}{l'}[\delta_\alpha(l)F(l')F(l')^{-1}, F(l')T_\beta F(l')^{-1}]\Omega, \quad (4.2)\end{aligned}$$

$$\begin{aligned}[\delta_\alpha(l), \tilde{\delta}_\beta(s)]H &= \frac{1}{l}[\tilde{\delta}_\beta(s)F(l)F(l)^{-1}, F(l)T_\alpha F(l)^{-1}]\Omega\\ &\quad + s[\delta_\alpha(l)\tilde{F}(s)\tilde{F}(s)^{-1}, \tilde{F}(s)T_\beta\tilde{F}(s)^{-1}]\Omega, \quad (4.3)\end{aligned}$$

$$\begin{aligned}[\tilde{\delta}_\alpha(s), \tilde{\delta}_\beta(s')]H &= -s[\tilde{\delta}_\beta(s')\tilde{F}(s)\tilde{F}(s)^{-1}, \tilde{F}(s)T_\alpha\tilde{F}(s)^{-1}]\Omega\\ &\quad + s'[\tilde{\delta}_\alpha(s)\tilde{F}(s')\tilde{F}(s')^{-1},\\ &\quad \tilde{F}(s')T_\beta\tilde{F}(s')^{-1}]\Omega, \quad (4.4)\end{aligned}$$

where $T_\alpha = \alpha^a T_a$, $\delta(l')F(l) = F(l, H + \delta(l')H) - F(l, H)$, etc.

To obtain the above commutators explicitly, we need the variations of $F(l)$, $\tilde{F}(s)$ induced by $\delta(l')H$, $\tilde{\delta}(s')H$. It may be verified by tedious but straightforward calculations that we can take

$$\delta_\alpha(l')F(l) = \frac{l}{l-l'}[F(l')T_\alpha F(l')^{-1} - F(l)T_\alpha F(l)^{-1}]F(l), \quad (4.5)$$

$$\tilde{\delta}_{\alpha}(s)F(l) = \frac{ls}{1-ls}[\tilde{F}(s)T_{\alpha}\tilde{F}(s)^{-1} - F(l)T_{\alpha}F(l)^{-1}]F(l), \tag{4.6}$$

$$\delta_{\alpha}(l)\tilde{F}(s) = \frac{1}{1-ls}[F(l)T_{\alpha}F(l)^{-1} - \tilde{F}(s)T_{\alpha}\tilde{F}(s)^{-1}]\tilde{F}(s), \tag{4.7}$$

$$\tilde{\delta}_{\alpha}(s')\tilde{F}(s) = \frac{s'}{s-s'}[\tilde{F}(s')T_{\alpha}\tilde{F}(s')^{-1} - \tilde{F}(s)T_{\alpha}\tilde{F}(s)^{-1}]\tilde{F}(s), \tag{4.8}$$

such that $F(l) + \delta_{\alpha}(l')F(l)$, $F(l) + \tilde{\delta}_{\alpha}(s)F(l)$ satisfy the same Eq. (2.22) and conditions (2.23) and (2.24) as $F(l)$ does; while $\tilde{F}(s) + \delta_{\alpha}(l)\tilde{F}(s)$, $\tilde{F}(s) + \tilde{\delta}_{\alpha}(s')\tilde{F}(s)$ satisfy the same Eq. (2.30) and conditions (2.31) as $\tilde{F}(s)$ does.

Substituting (4.5), (4.6), (4.7), and (4.8) into (4.2), (4.3), and (4.4), using (3.1) and (3.8) again and writing $\delta_{\alpha}(l)H = \alpha^a\delta_a(l)H$, etc., we obtain

$$[\delta_{\alpha}(l), \delta_{\beta}(l')]H = \frac{\alpha^a\beta^b}{l-l'}C^c_{ab}(l\delta_c(l)H - l'\delta_c(l')H), \tag{4.9}$$

$$[\delta_{\alpha}(l), \tilde{\delta}_{\beta}(s)]H = \frac{\alpha^a\beta^b}{1-ls}C^c_{ab}(ls\delta_c(l)H + \tilde{\delta}_c(s)H), \tag{4.10}$$

$$[\tilde{\delta}_{\alpha}(s), \tilde{\delta}_{\beta}(s')]H = \frac{\alpha^a\beta^b}{s-s'}C^c_{ab}(s'\tilde{\delta}_c(s)H - s\tilde{\delta}_c(s')H), \tag{4.11}$$

where C^c_{ab}'s are structure constants of the Lie algebra $o(d, d+n)$. Writing (4.1a) and (4.1b) in the explicitly α related forms as

$$\delta_{\alpha}(l)H = \alpha^a\sum_{k=0}^{\infty} l^k\delta_a^{(k)}H, \tag{4.12a}$$

$$\tilde{\delta}_{\alpha}(s)H = \alpha^a\sum_{m=1}^{\infty} s^m\tilde{\delta}_a^{(m)}H, \tag{4.12b}$$

and then expanding both sides of (4.9), (4.10), and (4.11), we finally obtain

$$[\delta_a^{(k)}, \delta_b^{(m)}]H = C^c_{ab}\delta_c^{(k+m)}H, \qquad k, m = 0, \pm1, \pm2, \cdots, \tag{4.13}$$

where $\delta_a^{(-m)}H := \tilde{\delta}_a^{(m)}H$ for $m \geq 1$. Thus, the infinite set of symmetry transformations $\{\delta_a^{(k)}, k = 0, \pm1, \pm2, \cdots\}$ constitute an affine Kac-Moody $o(d, \widehat{d+n})$ algebra (without center charge).

Now we consider transformations (3.9) and (3.10). They can be expanded as

$$\Delta H = \sigma\sum_{k=0}^{\infty} l^k\Delta^{(k)}H, \tag{4.14a}$$

$$\tilde{\Delta}H = \epsilon\sum_{m=1}^{\infty} s^m\tilde{\Delta}^{(m)}H. \tag{4.14b}$$

Thus we obtain another infinite set of symmetry transformations $\{\Delta^{(k)}, \tilde{\Delta}^{(m)}, k = 0, 1, 2, \cdots, m = 1, 2, \cdots\}$ of the two-dimensional heterotic string theory. To calculate their commutators, we first denote (3.9) and (3.10) by $\Delta_{\sigma}(l)H$, $\tilde{\Delta}_{\epsilon}(s)H$, respectively, and then have

$$\begin{aligned}[\Delta_{\sigma}(l), \Delta_{\sigma'}(l')]H = &-\sigma\frac{\partial}{\partial l}(\Delta_{\sigma'}(l')F(l)F(l)^{-1})\Omega \\ &+ \sigma'\frac{\partial}{\partial l'}(\Delta_{\sigma}(l)F(l')F(l')^{-1})\Omega \\ &- \sigma[\Delta_{\sigma'}(l')F(l)F(l)^{-1}, \dot{F}(l)F(l)^{-1}]\Omega \\ &+ \sigma'[\Delta_{\sigma}(l)F(l')F(l')^{-1}, \dot{F}(l')F(l')^{-1}]\Omega,\end{aligned} \tag{4.15}$$

$$\begin{aligned}[\Delta_{\sigma}(l), \tilde{\Delta}_{\epsilon}(s)]H = &-\sigma\frac{\partial}{\partial l}(\tilde{\Delta}_{\epsilon}(s)F(l)F(l)^{-1})\Omega \\ &- \epsilon s^2\frac{\partial}{\partial s}(\Delta_{\sigma}(l)\tilde{F}(s)\tilde{F}(s)^{-1})\Omega \\ &- \sigma[\tilde{\Delta}_{\epsilon}(s)F(l)F(l)^{-1}, \dot{F}(l)F(l)^{-1}]\Omega \\ &- \epsilon s^2[\Delta_{\sigma}(l)\tilde{F}(s)\tilde{F}(s)^{-1}, \dot{\tilde{F}}(s)\tilde{F}(s)^{-1}]\Omega,\end{aligned} \tag{4.16}$$

$$\begin{aligned}[\tilde{\Delta}_{\epsilon}(s), \tilde{\Delta}_{\epsilon'}(s')]H = &\,\epsilon s^2\frac{\partial}{\partial s}(\tilde{\Delta}_{\epsilon'}(s')\tilde{F}(s)\tilde{F}(s)^{-1})\Omega \\ &- \epsilon' s'^2\frac{\partial}{\partial s'}(\tilde{\Delta}_{\epsilon}(s)\tilde{F}(s')\tilde{F}(s')^{-1})\Omega \\ &+ \epsilon s^2[\tilde{\Delta}_{\epsilon'}(s')\tilde{F}(s)\tilde{F}(s)^{-1}, \dot{\tilde{F}}(s)\tilde{F}(s)^{-1}]\Omega \\ &- \epsilon' s'^2[\tilde{\Delta}_{\epsilon}(s)\tilde{F}(s')\tilde{F}(s')^{-1}, \\ &\dot{\tilde{F}}(s')\tilde{F}(s')^{-1}]\Omega.\end{aligned} \tag{4.17}$$

As for $\Delta_{\sigma}(l')F(l)$, $\Delta_{\sigma}(l)\tilde{F}(s)$, etc., we propose

$$\Delta_{\sigma}(l')F(l) = \sigma\frac{l}{l-l'}[l\dot{F}(l)F(l)^{-1} - l'\dot{F}(l')F(l')^{-1}]F(l), \tag{4.18}$$

$$\tilde{\Delta}_{\epsilon}(s)F(l) = \epsilon\frac{ls}{ls-1}\left[l\dot{F}(l)F(l)^{-1} + s\dot{\tilde{F}}(s)\tilde{F}(s)^{-1} + \frac{1}{2}\right]F(l), \tag{4.19}$$

$$\Delta_{\sigma}(l)\tilde{F}(s) = \sigma\frac{1}{ls-1}\left[s\dot{\tilde{F}}(s)\tilde{F}(s)^{-1} + l\dot{F}(l)F(l)^{-1} + \frac{1}{2}\right]\tilde{F}(s), \tag{4.20}$$

$$\tilde{\Delta}_{\epsilon}(s')\tilde{F}(s) = \epsilon \frac{s'}{s - s'}[s\dot{\tilde{F}}(s)\tilde{F}(s)^{-1} - s'\dot{\tilde{F}}(s')\tilde{F}(s')^{-1}]\tilde{F}(s). \quad (4.21)$$

By some lengthy but straightforward calculations, it can be verified that (4.18) and (4.19) are symmetry transformations of Eq. (2.22) with conditions (2.23) and (2.24); while (4.20) and (4.21) are symmetry transformations of Eq. (2.30) with conditions (2.31).

Substituting (4.18), (4.19), (4.20), and (4.21) into (4.15), (4.16), and (4.17) and using (3.9) and (3.10) again, it follows that

$$[\Delta_{\sigma}(l), \Delta_{\sigma'}(l')]H = \sigma\frac{\partial}{\partial l}\left[\frac{l}{l-l'}(l\Delta_{\sigma'}(l)H - l'\Delta_{\sigma'}(l')H)\right] - \sigma'\frac{\partial}{\partial l'}\left[\frac{l'}{l'-l}(l'\Delta_{\sigma}(l')H - l\Delta_{\sigma}(l)H)\right], \quad (4.22)$$

$$[\Delta_{\sigma}(l), \tilde{\Delta}_{\epsilon}(s)]H = \sigma\frac{\partial}{\partial l}\left[\frac{ls}{ls-1}(l\Delta_{\epsilon}(l)H - s^{-1}\tilde{\Delta}_{\epsilon}(s)H)\right] + \epsilon s^2\frac{\partial}{\partial s}\left[\frac{1}{ls-1}(l\Delta_{\sigma}(l)H - s^{-1}\tilde{\Delta}_{\sigma}(s)H)\right], \quad (4.23)$$

$$[\tilde{\Delta}_{\epsilon}(s), \tilde{\Delta}_{\epsilon'}(s')]H = \epsilon s^2\frac{\partial}{\partial s}\left[\frac{s'}{s-s'}(s^{-1}\tilde{\Delta}_{\epsilon'}(s)H - s'^{-1}\tilde{\Delta}_{\epsilon'}(s')H)\right] - \epsilon' s'^2\frac{\partial}{\partial s'}\left[\frac{s}{s'-s}(s'^{-1}\tilde{\Delta}_{\epsilon}(s')H - s^{-1}\tilde{\Delta}_{\epsilon}(s)H)\right]. \quad (4.24)$$

By using (4.14a) and (4.14b) to expand both sides of (4.22), (4.23), and (4.24), we obtain

$$[\Delta^{(m)}, \Delta^{(k)}]H = (m-k)\Delta^{(m+k)}H, \qquad m, k = 0, \pm 1, \pm 2, \cdots, \quad (4.25)$$

where we have written $\Delta^{(-k)}H := \tilde{\Delta}^{(k)}H$ for $k \geq 1$. This shows that the infinite set of symmetry transformations $\{\Delta^{(k)}, k = 0, \pm 1, \pm 2, \cdots\}$ constitute the Virasoro algebra (without central charge).

Next we investigate the commutators between the members of $\{\delta^{(m)}\}$ and $\{\Delta^{(k)}\}$. For example, from (3.1), (3.9), (4.5), and (4.18) we have, by some calculations

$$[\Delta_{\sigma}(l), \delta_a(s)]H = \sigma\frac{\partial}{\partial l}\left[\frac{l}{l-s}(l\delta_a(l)H - s\delta_a(s)H)\right] - \sigma\frac{l}{l-s}\frac{\partial}{\partial l}(l\delta_a(l)H) + \sigma\frac{s}{l-s}\frac{\partial}{\partial s}(s\delta_a(s)H). \quad (4.26)$$

Similarly, we can give out the expressions of $[\Delta_{\sigma}(l), \tilde{\delta}_a(s)]H$, $[\tilde{\Delta}_{\sigma}(l), \delta_a(s)]H$, and $[\tilde{\Delta}_{\sigma}(l), \tilde{\delta}_a(s)]H$. Then by using (4.12a), (4.12b), (4.14a), and (4.14b) to expand both sides of these results, we finally obtain

$$[\Delta^{(m)}, \delta_a^{(k)}]H = -k\delta_a^{(m+k)}H, \qquad m, k = 0, \pm 1, \pm 2, \cdots. \quad (4.27)$$

The Eqs. (4.13), (4.25), and (4.27) show that the symmetry transformations (3.1), (3.8), (3.9), and (3.10) give a representation of a semidirect product of the affine $o(d, \widehat{d+n})$ and Virasoro algebras. These give an expression that the infinite-dimensional symmetry structures of the two-dimensional heterotic string theory contain not only the Kac-Moody $o(d, \widehat{d+n})$ algebra but also the Virasoro algebra. The results demonstrate that the string theory under consideration possesses more and richer symmetry structures than previously expected.

V. SUMMARY AND DISCUSSIONS

The symmetry structures of the two-dimensional heterotic string theory are further studied in this paper. A $(2d+n) \times (2d+n)$ matrix complex H-potential is introduced in (2.12) and the motion equations of the studied theory are written as a complex form (2.15). Moreover, we establish a pair of HE-type linear systems (2.22), (2.23), (2.24), (2.30), and (2.31). We would like to indicate that although Eqs. (2.22) and (2.30) are, in form, interrelated by $t \leftrightarrow w = 1/t$, the analytic properties of $F(t)$ and $\tilde{F}(w)$ as well as the conditions (2.23), (2.24), and (2.31) do not have this interrelation, therefore as whole linear systems they are different and give rise to different symmetries of the considered theory. Based on these linear systems, we explicitly construct symmetry transformations (3.1), (3.8), (3.9), and (3.10). These symmetries are verified to constitute infinite-dimensional Lie algebras, which is a semidirect product of the Kac-Moody $o(d, \widehat{d+n})$ and Virasoro algebras.

Finite symmetry transformations relating to the above infinitesimal ones and soliton solutions of the studied theory need more and further investigations and will be considered in some forthcoming works.

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