

Lattice-refining loop quantum cosmology and inflation

William Nelson and Mairi Sakellariadou

King's College London, Department of Physics, Strand WC2R 2LS, London, U.K.

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We study the importance of lattice refinement in achieving a successful inflationary era. We solve, in the continuum limit, the second order difference equation governing the quantum evolution in loop quantum cosmology, assuming both a fixed and a dynamically varying lattice in a suitable refinement model. We thus impose a constraint on the potential of a scalar field, so that the continuum approximation is not broken. Considering that such a scalar field could play the role of the inflaton, we obtain a second constraint on the inflationary potential so that there is consistency with the cosmic microwave background data on large angular scales. For a $m^2\phi^2/2$ inflationary model, we combine the two constraints on the inflaton potential to impose an upper limit on m , which is severely fine-tuned in the case of a fixed lattice. We thus conclude that lattice refinement is necessary to achieve a natural inflationary model.

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I. INTRODUCTION

Loop quantum gravity (LQG) provides a method of quantizing gravity in a background independent, nonperturbative way. Quantum gravity is essential when curvature becomes large, as, for example, in the early stages of the evolution of the universe. Applying LQG in a cosmological context leads to loop quantum cosmology (LQC) [1], which is a symmetry reduction of the infinite dimensional phase space of the full theory, allowing us to study certain aspects of the theory analytically. In particular, it has been shown that classical big bang [2] and black hole singularities [3] are removed in LQC, in a well-defined manner. The discreteness of spatial geometry, a key element of the full theory, leads to successes in LQC which do not hold in the Wheeler-DeWitt (WDW) quantum cosmology.

In LQC, the quantum evolution is governed by a second order difference equation, rather than the second order differential equation of the WDW theory. This is at the heart of LQC's success at removing singularities [2,3]. Of equal importance is the continuum limit of these equations, which can lead to a behavior which is qualitatively different from the classical cosmological evolution [4]. As the universe becomes large and enters the semiclassical regime, the WDW differential equation becomes a very good approximation to the difference equation of LQC.

By deriving the effective equations of motion in the continuum limit, a semiclassical approach can be used to investigate possible phenomenological signatures of the theory. In particular, it was hoped that LQC would help overcome the extreme fine-tuning necessary to achieve successful inflation in general relativity [5]. However, it has been shown [6] that semiclassical corrections are insufficient to alleviate this difficulty. There remains the possibility that effects due to the underlying discrete nature of LQC may come to inflation's aid. Here, we will explore this possibility and show that the discrete nature of the

early universe naturally introduces constraints on a subsequent inflationary era.

It has been suggested, by heuristic considerations, that for LQC to be considered physical the scale at which the discrete structure of space-time becomes significant must shrink as the universe expands [7]. This lattice refinement has been put on a more rigorous footing in Ref. [8]. In particular, Ref. [8] has analyzed implications of lattice refinement for the semiclassical behavior of the dynamical difference equations. Studying the effect of different choices for the lattice spacings on the solutions of the Hamiltonian constraint, the authors of Ref. [8] deduced the stability properties of the various refinement models. Combining the stability analysis with a confirmation of the appropriate classical limit, the suitable refinement models were constrained [8].

In what follows, in addition to discussing the effects of LQC on an inflationary era, we will show that lattice refinement is necessary to support a massive scalar field (in our context the inflaton), and we will generalize to matter components with an arbitrary scale factor dependence. In particular, we will argue that lattice refinement renders a successful inflationary era more natural.

In Section II, we briefly discuss elements of LQC which we will later use. We are interested in isotropic models described in terms of one dynamical parameter. In Section III, we quantize the theory and derive the Hamiltonian constraint. We then take its continuum limit, assuming that the regulating length (which is proportional to the length of the holonomy) is constant. We also discuss quantization ambiguities. In Section III C, we investigate solutions of the full Hamiltonian constraint for large scales, which is indeed the WDW equation for a massive scalar field. In Section IV, we discuss lattice refinement and we deduce the constraints imposed on the inflationary potential. In Section V, we briefly discuss nonflat geometries. We round up our conclusions in Section VI.

II. ELEMENTS OF LOOP QUANTUM COSMOLOGY

We deal with isotropic models described in terms of the scalar field $a(t)$, thus there is only one kinematical degree of freedom. In connection variables, it is parametrized by the triad component \tilde{p} conjugate to the connection component \tilde{c} . Their relation to the metric variable a is [9]

$$|\dot{\tilde{p}}| = a^2 \quad \text{and} \quad \dot{\tilde{c}} = k + \gamma \dot{a}, \quad (1)$$

where over-dot denotes derivatives with respect to cosmological time, γ is the Barbero-Immirzi parameter, representing a quantum ambiguity parameter of the theory, and k depends on the intrinsic spatial curvature ($k = 0, \pm 1$). We will concentrate on flat universes, thus $k = 0$. The triad has an orientation, determined by the sign of \tilde{p} , and since the scale factor has no orientation, we used an absolute value in the above equation. The triad \tilde{p} incorporates information about the three-geometry through the scale factor a , while the connection \tilde{c} has information about the curvature, which in our case (flat and isotropic geometry) is contained in the extrinsic curvature proportional to \dot{a} . The canonical variables \tilde{c} , \tilde{p} are related through

$$\{\tilde{c}, \tilde{p}\} = \frac{\kappa\gamma}{3} V_0, \quad (2)$$

where $\kappa \equiv 8\pi G$ and V_0 is the volume of the elementary cell \mathcal{V} adapted to the fiducial triad. Defining

$$p = V_0^{2/3} \tilde{p} \quad \text{and} \quad c = V_0^{1/3} \tilde{c}, \quad (3)$$

with the triad component p determining the physical volume of the fiducial cell, and the connection component c determining the rate of change of the physical edge length of the fiducial cell, one obtains

$$\{c, p\} = \frac{\kappa\gamma}{3}, \quad (4)$$

independent of the volume of the fiducial cell.

To proceed in the quantum theory, one follows the procedure used in the full LQG theory. Thus, in LQC one takes $e^{i\mu_0 c/2}$, with μ_0 an arbitrary real number and p as the elementary classical variables, which have well-defined operator analogues [9].

Using the Dirac bra-ket notation and setting $e^{i\mu_0 c/2} = \langle c|\mu\rangle$, the eigenstates of \hat{p} are the basis vectors $|\mu\rangle$:

$$\hat{p}|\mu\rangle = \frac{\kappa\gamma\hbar|\mu|}{6} |\mu\rangle. \quad (5)$$

Using the volume operator $\hat{V} = |\hat{p}|^{3/2}$, representing the volume of the elementary cell \mathcal{V} , with eigenvalues $V_\mu = (\kappa\gamma\hbar|\mu|/6)^{3/2}$, one gets [4]

$$\hat{V}|\mu\rangle = \left(\frac{\kappa\gamma\hbar|\mu|}{6}\right)^{3/2} |\mu\rangle. \quad (6)$$

Notice that since we will be concerned with the large scale

behavior of the LQC equations, we will typically neglect the sign ambiguity that arises because the triads can have two different orientations. We thus avoid numerous factors of $\text{sgn}(\mu)$ appearing in subsequent equations.

To define the inverse volume operator, one has to trace over SU(2) valued holonomies. Since there is a freedom in choosing the irreducible representation to perform the trace, an ambiguity, labeled by J , arises. More precisely, the half-integer J stands for the spin of the representation. At this point, it is important to note that the same ambiguity arises in the gravitational part of the Hamiltonian constraint, since one has again to trace over SU(2) valued holonomies. Usually one quantizes the gravitational part of the Hamiltonian constraint using the fundamental $J = 1/2$ representation, and the ambiguity is only investigated for the matter part. This issue is further discussed in Section III B.

Let us use the $J = 1/2$ irreducible representation of SU(2). The inverse volume operator is diagonal in the $|\mu\rangle$ basis and is given by [9]

$$\widehat{V}^{-1}|\mu\rangle = \left| \frac{6}{\kappa\gamma\hbar\mu_0} \left(V_{\mu+\mu_0}^{1/3} - V_{\mu-\mu_0}^{1/3} \right) \right|^3 |\mu\rangle, \quad (7)$$

where μ_0 is proportional to the length of the holonomy. At this point, we would like to stress that the regulating length μ_0 is the crucial parameter in the quantum corrections. In Eq. (7) the eigenvalues are bounded and approach zero near the classical singularity, in contrast to the classical case where the eigenvalues diverge at the singularity $\mu = 0$. The eigenvalues reach their maximum at a characteristic scale equal to μ_0 , while at larger μ they approach the classical values and below μ_0 they are suppressed compared to their classical analogues [7]. The inverse volume is cut off for small volumes and, as it has been shown [10], the generalization to higher J will push the cutoff region to larger volumes.

III. HAMILTONIAN CONSTRAINT

The dynamics are determined completely by the Hamiltonian operator, the gravitational part of which, in the fundamental representation, is [7]

$$\hat{\mathcal{C}}_g = \frac{2i}{\kappa^2\hbar\gamma^3\mu_0^3} \text{tr} \sum_{ijk} \epsilon^{ijk} (\hat{h}_i \hat{h}_j \hat{h}_i^{-1} \hat{h}_j^{-1} \hat{h}_k [\hat{h}_k^{-1}, \hat{V}]) (\hat{p}), \quad (8)$$

where

$$\hat{h}_i = \cos\left(\frac{\mu_0 c}{2}\right) - i\sigma_i \sin\left(\frac{\mu_0 c}{2}\right), \quad (9)$$

denote the holonomies, with σ_i the Pauli spin matrices; the trace is taken over the SU(2) indices. We quantize this by setting

$$\hat{p}|\mu\rangle = p|\mu\rangle \quad \text{and} \quad \widehat{e^{\mu_0 c/2}}|\mu\rangle = e^{\mu_0(d/d\mu)}|\mu\rangle = |\mu + \mu_0\rangle. \quad (10)$$

The action of the self-adjoint Hamiltonian constraint operator, $\hat{\mathcal{H}}_g = (\hat{C}_g + \hat{C}_g^\dagger)/2$, on the basis states, $|\mu\rangle$, is

$$\begin{aligned} \hat{\mathcal{H}}_g |\mu\rangle = & \frac{3}{4\kappa^2 \gamma^3 \hbar \mu_0^3} \{ [R(\mu) + R(\mu + 4\mu_0)] |\mu + 4\mu_0\rangle \\ & - 4R(\mu) |\mu\rangle \\ & + [R(\mu) + R(\mu - 4\mu_0)] |\mu - 4\mu_0\rangle \}, \end{aligned} \quad (11)$$

where

$$R(\mu) = (\kappa\gamma\hbar/6)^{3/2} |\mu + \mu_0|^{3/2} - |\mu - \mu_0|^{3/2}. \quad (12)$$

We impose the constraint equation on the physical wave-functions $|\Psi\rangle$, which are those annihilated by the constraint operator, i.e., they have to fulfill the Wheeler-DeWitt equation, since the classical expression must vanish. Such states can be expanded using the basis states as $|\Psi\rangle = \sum_\mu \Psi_\mu(\phi) |\mu\rangle$, with summation over values of μ and where the dependence of the coefficients on ϕ represents the matter degrees of freedom. Since the states $|\mu\rangle$ are eigenstates of the triad operator, the coefficients $\Psi_\mu(\phi)$ represent the state in the triad representation. Thus, quantizing the Friedmann equation along the lines of the constraint in the full LQG theory, one gets the following difference equation [11]

$$\begin{aligned} & [V_{\mu+5\mu_0} - V_{\mu+3\mu_0}] + [V_{\mu+\mu_0} - V_{\mu-\mu_0}] \Psi_{\mu+4\mu_0}(\phi) \\ & - 4[V_{\mu+\mu_0} V_{\mu-\mu_0}] \Psi_\mu(\phi) \\ & + [V_{\mu-3\mu_0} - V_{\mu-5\mu_0}] + [V_{\mu+\mu_0} - V_{\mu-\mu_0}] \Psi_{\mu-4\mu_0}(\phi) \\ & = -\frac{4\kappa^2 \gamma^3 \hbar \mu_0^3}{3} \mathcal{H}_\phi(\mu) \Psi_\mu(\phi), \end{aligned} \quad (13)$$

where the matter Hamiltonian $\hat{\mathcal{H}}_\phi$ is assumed to act diagonally on the basis states with eigenvalue $\mathcal{H}_\phi(\mu)$. Equation (13) is indeed the quantum evolution (in internal time μ) equation. There is no continuous variable (the scale factor in classical cosmology), but a label μ with discrete steps. The wave-function $\Psi_\mu(\phi)$ depending on internal time μ and matter fields ϕ determines the dependence of matter fields on the evolution of the universe.

A. Continuum limit

We consider the continuum (i.e., $\mu \gg \mu_0$) limit of the Hamiltonian constraint operator acting on the physical states, i.e., $\hat{\mathcal{H}}_g |\Psi\rangle$. Expanding around the small regulating length, μ_0 , we get

$$\begin{aligned} V_{\mu+\alpha\mu_0} - V_{\mu+\beta\mu_0} = & \frac{3}{2} \left(\frac{\kappa\gamma\hbar}{6} \right)^{3/2} \mu_0 \mu^{1/2} \left[\alpha - \beta + \frac{\mu_0}{4\mu} \right. \\ & \times (\alpha^2 - \beta^2) - \frac{\mu_0^2}{24\mu^2} (\alpha^3 - \beta^3) \\ & \left. + \mathcal{O}\left(\frac{\mu_0^3}{\mu^3}\right) + \text{higher orders} \right]. \end{aligned} \quad (14)$$

From Eqs. (13) and (14) we obtain, in the small μ_0 limit, that

$$\begin{aligned} & \frac{9}{2\kappa^2 \gamma^3 \hbar} \left(\frac{\kappa\gamma\hbar}{6} \right)^{3/2} \mu_0^{-2} \mu^{1/2} \left[\Psi_{\mu+4\mu_0}(\phi) - 2\Psi_\mu(\phi) \right. \\ & \quad \left. + \Psi_{\mu-4\mu_0}(\phi) + \frac{\mu_0}{\mu} \{ \Psi_{\mu+4\mu_0}(\phi) - \Psi_{\mu-4\mu_0}(\phi) \} \right. \\ & \quad \left. - \frac{\mu_0^2}{\mu^2} \{ \Psi_{\mu+4\mu_0}(\phi) + \Psi_{\mu-4\mu_0}(\phi) \} \right. \\ & \quad \left. - \frac{\mu_0^2}{24\mu^2} \{ \Psi_{\mu+4\mu_0}(\phi) - 2\Psi_\mu(\phi) + \Psi_{\mu-4\mu_0}(\phi) \} \right. \\ & \quad \left. + \mathcal{O}\left(\frac{\mu_0^3}{\mu^3}\right) \right] = -\mathcal{H}_\phi(\mu) \Psi_\mu(\phi). \end{aligned} \quad (15)$$

The above second order difference equation distinguishes the components of the wave-functions in different lattices of spacing $4\mu_0$. Assuming that Ψ does not vary much on scales of the order of $4\mu_0$ (known as *preclassicality* [12]), one can smoothly interpolate between the points on the discrete function $\Psi_\mu(\phi)$ and approximate them by the continuous function $\Psi(\mu, \phi)$. Under this assumption, the difference equation is very well approximated by a differential equation for a continuous wave-function. As we will discuss later, it is this assumption that can break down at large scales, leading to deviations from the classical behavior.

Let us consider the parameter μ_0 entering in the regularization of the Hamiltonian constraint to be a constant real number (the case of lattice refinement will be addressed in Section IV). In other words, let us assume a fixed spatial lattice throughout the whole evolution of the universe. Expanding the wave-function $\Psi(\mu \pm 4\mu_0, \phi)$ in Taylor series we obtain

$$\begin{aligned} \Psi(\mu \pm 4\mu_0, \phi) = & \Psi(\mu, \phi) \pm \frac{\partial \Psi(\mu, \phi)}{\partial \mu} (4\mu_0) \\ & + \frac{1}{2} \frac{\partial^2 \Psi(\mu, \phi)}{\partial \mu^2} (16\mu_0^2) \\ & + \mathcal{O}\left(\mu_0^3 \frac{\partial^3 \Psi(\mu, \phi)}{\partial \mu^3}\right) \\ & + \mathcal{O}\left(\mu_0^4 \frac{\partial^4 \Psi(\mu, \phi)}{\partial \mu^4}\right) + \dots \end{aligned} \quad (16)$$

Applying Eq. (16) in the difference equation Eq. (15) we get

$$\begin{aligned} & \frac{36}{\kappa^2 \gamma^3 \hbar} \left(\frac{\kappa\gamma\hbar}{6} \right)^{3/2} \left[\sqrt{\mu} \frac{\partial^2 \Psi(\mu, \phi)}{\partial \mu^2} + \frac{\partial^2}{\partial \mu^2} (\sqrt{\mu} \Psi(\mu, \phi)) \right] \\ & \quad + \mathcal{O}(\mu_0) + \mathcal{O}(\mu_0^2) + \dots = -\mathcal{H}_\phi(\mu) \sqrt{\mu} \Psi(\mu, \phi), \end{aligned} \quad (17)$$

where \dots denote higher order corrections in μ_0 . Classically, the matter part of the Hamiltonian constraint

for a massive scalar field is just

$$\mathcal{H}_\phi = \kappa \left[\frac{P_\phi^2}{2a^3} + a^3 V(\phi) \right], \quad (18)$$

with momentum P_ϕ and potential $V(\phi)$. To quantize the above Hamiltonian constraint we use

$$\begin{aligned} \hat{P}_\phi \Psi(p, \phi) &= -i\hbar \frac{\partial \Psi(p, \phi)}{\partial \phi} \quad \text{and} \\ \hat{\phi} \Psi(p, \phi) &= \phi \Psi(p, \phi), \end{aligned} \quad (19)$$

while the expressions for the volume and inverse volume operators, \widehat{V} and \widehat{V}^{-1} , are given by Eqs. (6) and (7), respectively. Taking then the continuum limit, one obtains

$$\begin{aligned} \hat{\mathcal{H}}_\phi \Psi(\mu, \phi) &= -3 \left(\frac{6\hbar}{\kappa\gamma^3} \right)^{1/2} \mu^{-3/2} \frac{\partial^2 \Psi(\mu, \phi)}{\partial \phi^2} \\ &+ \left(\frac{\kappa\gamma\hbar}{6} \right)^{3/2} \mu^{3/2} V(\phi) \Psi(\mu, \phi) \\ &+ \mathcal{O}(\mu_0) + \dots \end{aligned} \quad (20)$$

From Eqs. (17) and (20), the full Hamiltonian constraint, $(\hat{\mathcal{H}}_g + \hat{\mathcal{H}}_\phi)|\Psi\rangle = 0$, reads

$$\begin{aligned} \sqrt{\mu} \frac{\partial^2 \Psi(\mu, \phi)}{\partial \mu^2} + \frac{\partial^2}{\partial \mu^2} (\sqrt{\mu} \Psi(\mu, \phi)) \\ - 3\mu^{-3/2} \frac{\partial^2 \Psi(\mu, \phi)}{\partial \phi^2} + \frac{\kappa^2 \gamma^3 \hbar}{36} V(\phi) \mu^{3/2} \Psi(\mu, \phi) \\ + \mathcal{O}(\mu_0) + \dots = 0, \end{aligned} \quad (21)$$

or, equivalently, in terms of p with

$$p = \kappa\gamma\hbar\mu/6, \quad (22)$$

one obtains

$$\begin{aligned} \sqrt{p} \frac{\partial^2 \Psi(p, \phi)}{\partial p^2} + \frac{\partial^2}{\partial p^2} (\sqrt{p} \Psi(p, \phi)) - 3p^{-3/2} \frac{\partial^2 \Psi(p, \phi)}{\partial \phi^2} \\ + \frac{6}{\kappa\hbar^2} p^{3/2} V(\phi) \Psi(p, \phi) + \mathcal{O}(\mu_0) + \dots = 0, \end{aligned} \quad (23)$$

which is just a particular factor ordering of the Wheeler-DeWitt equation for a massive scalar field.

B. Ambiguities

Thus far we have neglected any quantization ambiguities, which can have significant consequences for the predictions of LQC [6,10]. As was discussed in Section II, there is a fundamental ambiguity arising from our choice of representation for the holonomies. It is possible to account for this ambiguity in the eigenvalues of the inverse volume operator [10], however constructing the gravitational part of the Hamiltonian constraint in non-fundamental ($J \neq 1/2$) representations results in a higher order difference equation [7]. For $J = 1$ the difference

equation has fundamental step-size equal to $2\mu_0$, and it is of order $8J = 8$ [11]. Being of higher order, spurious solutions may arise [11] suggesting that representations of higher order than the fundamental representation may not lead to the correct quantum theory. While it may appear more natural to choose the same representation for both the matter and gravitational parts of the Hamiltonian constraint, there is no *a priori* reason this should be the case. Here we set $J = 1/2$ in the gravitational part of the constraint since it is the most tractable, while leaving it unspecified in the matter part, to remain as general as possible.

The J ambiguity arises only in the inverse volume eigenvalues, which are given in general by [7]

$$\begin{aligned} \widehat{V}^{-1}|\mu\rangle &= \left| \frac{9}{\kappa\gamma\hbar\mu_0 L J (J+1) (2J+1)} \right. \\ &\times \left. \sum_{m=-J}^J m V_{\mu+2m\mu_0}^{2L/3} \right|^{3/(2(1-L))} |\mu\rangle, \end{aligned} \quad (24)$$

where $0 < L < 1$ is a second quantization ambiguity coming from the classical Poisson bracket [7]

$$\{c, |p|^L\} = \frac{1}{3} \gamma \kappa L \text{sgn}(p) |p|^{L-1}. \quad (25)$$

Taking the $\mu \gg \mu_0$ limit, Eq. (24) implies

$$\begin{aligned} \widehat{V}^{-1}|\mu\rangle &= \left[1 + \frac{(L-2)(3J^2 + 3J - 1)\mu_0^2}{5\mu^2} \right. \\ &\left. + \mathcal{O}\left(\frac{\mu_0^4}{\mu^4}\right) + \dots \right] |p|^{-3/2} |\mu\rangle. \end{aligned} \quad (26)$$

There are also ambiguities arising from the classical matter Hamiltonian we choose to quantize [6,13]. Classically,

$$\mathcal{H}_\phi = \kappa \left[\frac{P_\phi^2}{2a^{3(n+1)}} a^{3n} + \frac{1}{a^{3m}} a^{3(m+1)} V(\phi) \right] \quad (27)$$

is identical to Eq. (18), however they will differ in the quantum regime. Equation (26) implies

$$\begin{aligned} (\widehat{V} \widehat{V}^{-1})^n |\mu\rangle &= \left[1 + \frac{n(L-2)(3J^2 + 3J - 1)\mu_0^2}{5\mu^2} \right. \\ &\left. + \mathcal{O}\left(\frac{\mu_0^4}{\mu^4}\right) + \dots \right] |\mu\rangle. \end{aligned} \quad (28)$$

Equations (26) and (28) imply that all quantum ambiguities are of order $\mathcal{O}(\mu_0^2/\mu^2)$; thus they do not affect the results of Section III A.

C. Wheeler-DeWitt solutions

From Eq. (23) we see that for large scales the equation we want to solve is the standard Wheeler-DeWitt equation,

$$\sqrt{p} \frac{\partial^2 \Psi(p, \phi)}{\partial p^2} + \frac{\partial^2}{\partial p^2} (\sqrt{p} \Psi(p, \phi)) + \alpha p^{3/2} V(\phi) \Psi(p, \phi) = 0, \quad (29)$$

where

$$\alpha = \frac{6}{\kappa \hbar^2}. \quad (30)$$

At this point we approximate the dynamics of ϕ to be such that

$$V(\phi) = p^{\delta-3/2} V_\phi, \quad (31)$$

with V_ϕ a constant. For an inflaton field during slow-roll inflation, $V(\phi) \approx \text{const.}$, so we expect $\delta \approx 3/2$ in this era (which should be during the continuum epoch to be compatible with observations). By separation of variables $\Psi(p, \phi) = Y(p)\Phi(\phi)$, we get

$$\sqrt{p} \frac{d^2 Y(p)}{dp^2} + \frac{d^2}{dp^2} (\sqrt{p} Y(p)) + \alpha p^\delta V_\phi Y(p) \approx 0. \quad (32)$$

Equation (32) has solutions

$$Y(p) = C_1 p^{1/4} J_{\sqrt{3}/(2\delta+3)} \left(\frac{2\sqrt{2\alpha}\sqrt{V_\phi}}{2\delta+3} p^{(2\delta+3)/4} \right) + C_2 p^{1/4} Y_{\sqrt{3}/(2\delta+3)} \left(\frac{2\sqrt{2\alpha}\sqrt{V_\phi}}{2\delta+3} p^{(2\delta+3)/4} \right), \quad (33)$$

where J and Y are Bessel functions of the first and second kind, respectively, and C_1 and C_2 are integration constants. It is clear that for $\delta > 1/2$, the period of the solution decreases with increasing μ . Without lattice refinement (to be discussed in the next section) the discrete nature of the underlying lattice would eventually be unable to support the oscillations, and the assumption of preclassicality will break down, implying that the discrete nature of space-time becomes significant on very large scales.

We can see this rigorously by using the Bessel function expansions

$$J_\beta(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\beta\pi}{2} - \frac{\pi}{4}\right),$$

$$Y_\beta(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\beta\pi}{2} - \frac{\pi}{4}\right),$$

for large x , which give

$$Y(p) \approx \sqrt{\frac{2\delta+3}{\pi\sqrt{2\alpha}\sqrt{V_\phi}}} p^{-(2\delta+1)/8} \left[C_1 \cos\left(x - \frac{\sqrt{3}\pi}{2(2\delta+3)} - \frac{\pi}{4}\right) + C_2 \sin\left(x - \frac{\sqrt{3}\pi}{2(2\delta+3)} - \frac{\pi}{4}\right) \right], \quad (34)$$

where $x = 2\sqrt{2\alpha}\sqrt{V_\phi}(2\delta+3)^{-1} p^{(2\delta+3)/4}$. The zeros of $Y(p)$ are at

$$\tan\left(x - \frac{\pi}{2(\delta+2)} - \frac{\pi}{4}\right) = -\frac{C_1}{C_2}, \quad (35)$$

which gives the scale of the n th zero to be

$$p_n = \left[\frac{(2\delta+3)\pi}{2\sqrt{2\alpha}\sqrt{V_\phi}} \right]^{4/(2\delta+3)} [C+n]^{4/(2\delta+3)}, \quad (36)$$

with

$$C = \tan^{-1}\left(-\frac{C_1}{C_2}\right) \frac{1}{\pi} + \frac{\sqrt{3}}{2(2\delta+3)} \pm \frac{1}{2} - \frac{1}{4} \quad (37)$$

a constant. Note that we take the n th zero of $\tan(x)$ to be at $x = (2n \pm 1)\pi/2$.

We are interested in the separation between two successive zeros in the limit of large n (large p), which is given by

$$\Delta p \equiv \lim_{n \rightarrow \infty} (p_{n+1} - p_n) = \frac{2\pi}{\sqrt{2\alpha}\sqrt{V_\phi}} p^{(1-2\delta)/4} + \mathcal{O}(p^{-(2\delta+1)/2}). \quad (38)$$

For μ_0 constant, it is clear that for $\alpha \neq 0$ ($\alpha = 0$ corresponds to the case where the wave-function is linear and hence has an infinite period), there will be a scale at which Δp is of the order of $2\kappa\gamma\hbar\mu_0/3$ (i.e., when $\Delta\mu$ is of the order of $4\mu_0$), at which point the assumption that the wave-function is smooth on scales of the order of $4\mu_0$ breaks down and the semiclassical description is no longer valid. If we want the end of inflation to be describable using classical general relativity then it must end before this scale is reached. During slow-roll inflation the scalar field potential is approximately constant so we can set $\delta \approx 3/2$.

For $\Delta p > 2\kappa\gamma\hbar\mu_0/3$, Eqs. (22), (30), and (38), lead to the following constraint for the scale, a_f , at the end of inflation:

$$a_f < \frac{3\pi}{2\gamma\mu_0\sqrt{3\kappa V_\phi}}, \quad (39)$$

where we have also used that $a = |p|^{1/2} = (\kappa\gamma\hbar|\mu|/6)^{1/2}$. We can then calculate the maximum number of e-foldings, $N_{\text{cl}} = \ln(a_f/a_i)$, where a_i is the initial scale, which can occur within the classical era, given by

$$a_i \gg (2\kappa\gamma\hbar\mu_0/3)^{1/2}. \quad (40)$$

Thus, from Eqs. (39) and (40) the number of e-foldings N_{cl} must satisfy the constraint

$$N_{\text{cl}} \ll \ln\left(\frac{3\pi}{2\kappa\sqrt{2V_\phi}\hbar\gamma^3\mu_0^3}\right). \quad (41)$$

Setting [4] $\mu_0 = 3\sqrt{3}/2$ and $\gamma \approx 0.24$, the above constraint, Eq. (41), on the number of e-foldings in units of $\hbar = 1$ reads

$$N_{\text{cl}} \ll \ln\left(\frac{0.27}{\sqrt{V_\phi} l_{\text{pl}}^2}\right), \quad (42)$$

with $l_{\text{pl}} = \sqrt{8\pi G}$. Thus, to have N_{cl} e-foldings of inflation with negligible quantum gravity effects, the potential must satisfy

$$V_\phi \ll 0.07 e^{-2N_{\text{cl}}} l_{\text{pl}}^{-4}. \quad (43)$$

If we, conservatively, say that only half of inflation takes place in the classical era, the above constraint leads to

$$V_\phi \ll 10^{-28} l_{\text{pl}}^{-4}. \quad (44)$$

Clearly, the above constraint becomes even tighter if we increase the number of e-foldings during the classical era. Considering lattice refinement we will see that this strong constraint on the value of the inflaton potential (during slow-roll) becomes considerably less fine-tuned.

IV. LATTICE REFINEMENT

In the derivation of the Hamiltonian constraint we have assumed that the parameter μ_0 is a constant real number. This parameter is related to the edge length used in holonomies. Since holonomies refer to embedded edges and loops, a constant μ_0 corresponds to a fixed spatial lattice, so that only edges of a given coordinate length are used, and consequently the loop size remains the same as the total volume increases. In what follows we relax the assumption of a constant μ_0 and we investigate the consequences, and thus the necessity, of a parameter $\tilde{\mu}$ being dependent on the phase space variable p , or equivalently, on μ .

Allowing the length scale of the holonomies to vary dynamically, the form of the difference equation, Eq. (13), changes. Since the parameter μ_0 determines the step-size of the difference equation, assuming the lattice size is growing, the step-size of the difference equation is not constant in the original triad variables. The exact form of the difference equation depends on the lattice refinement used. Here we will take the particular case of

$$\mu_0 \rightarrow \tilde{\mu}(\mu) = \mu_0 \mu^{-1/2}. \quad (45)$$

This type of lattice refinement is suggested by certain intuitive heuristic approaches such as noting that the *minimum area* used to regulate the holonomies should be a *physical area* [14], or that the discrete step-size of the difference equation should always be of the order of the Planck volume [7]. This choice also results in a significant simplification of the difference equation, compared to more general lattice refinement schemes.

The basic operators are given by replacing μ_0 with $\tilde{\mu}$. Upon quantization we have [14]

$$\widehat{e^{i\tilde{\mu}c/2}}|_{\mu} = e^{-i\tilde{\mu}(d/d\mu)}|\mu\rangle, \quad (46)$$

which is no longer a simple shift operator since $\tilde{\mu}$ is a

function of μ . If we change our basis to

$$\nu = \mu_0 \int \frac{d\mu}{\tilde{\mu}} = \frac{2}{3} \mu^{3/2}, \quad (47)$$

we find

$$e^{-i\tilde{\mu}(d/d\mu)}|\nu\rangle = e^{-i\mu_0(d/d\nu)}|\nu\rangle = |\nu + \mu_0\rangle. \quad (48)$$

Thus, the basis $|\nu\rangle$ is a much more natural choice than $|\mu\rangle$. The action of the volume operator on these basis states is

$$\hat{V}|\nu\rangle = \frac{3\nu}{2} \left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2} |\nu\rangle, \quad (49)$$

and the self-adjoint Hamiltonian constraint operator acts as [7]

$$\begin{aligned} \hat{\mathcal{H}}_{\text{g}}|\nu\rangle &= \frac{9|\nu|}{16\mu_0^3} \left(\frac{\hbar}{6\kappa\gamma^3}\right)^{1/2} \\ &\times \left[\frac{1}{2} \{U(\nu) + U(\nu + 4\mu_0)\} |\nu + 4\mu_0\rangle \right. \\ &\quad - 2U(\nu)|\nu\rangle \\ &\quad \left. + \frac{1}{2} \{U(\nu) + U(\nu - 4\mu_0)\} |\nu - 4\mu_0\rangle \right], \quad (50) \end{aligned}$$

where

$$U(\nu) = |\nu + \mu_0| - |\nu - \mu_0|. \quad (51)$$

Notice that since μ is dimensionless, so is ν .

Following the same approach as previously, we expand $|\Psi\rangle = \sum_\nu \Psi_\nu(\phi)|\nu\rangle$ and the Hamiltonian constraint becomes

$$\begin{aligned} &\frac{1}{2} |\nu + 4\mu_0| [U(\nu + 4\mu_0) + U(\nu)] \Psi_{\nu+4\mu_0}(\phi) \\ &\quad + 2|\nu| U(\nu) \Psi_\nu(\nu) \\ &\quad + \frac{1}{2} |\nu - 4\mu_0| [U(\nu - 4\mu_0) + U(\nu)] \Psi_{\nu-4\mu_0}(\phi) \\ &= -\frac{16\mu_0^3}{9} \left(\frac{6\kappa\gamma^3}{\hbar}\right)^{1/2} \mathcal{H}_\phi(\nu) \Psi_\nu(\phi). \quad (52) \end{aligned}$$

Now we take the continuum ($\nu \gg \mu_0$) limit of Eq. (52) to get

$$\begin{aligned} &\frac{\partial^2}{\partial \nu^2} [\nu \Psi(\nu, \phi)] + \frac{16}{9} \left(\frac{6\kappa\gamma^3}{\hbar}\right)^{1/2} \mathcal{H}_\phi(\nu) \Psi(\nu, \phi) \\ &\quad + \mathcal{O}(\mu_0) + \dots = 0. \quad (53) \end{aligned}$$

In terms of μ Eq. (53) reads

$$\begin{aligned} &\mu^{-1/2} \frac{\partial}{\partial \mu} \left[\mu^{-1/2} \frac{\partial}{\partial \mu} (\mu^{3/2} \Psi(\mu, \phi)) \right] \\ &\quad + \frac{8}{3} \left(\frac{6\kappa\gamma^3}{\hbar}\right)^{1/2} \mathcal{H}_\phi(\mu) \Psi(\mu, \phi) + \mathcal{O}(\mu_0) + \dots = 0, \quad (54) \end{aligned}$$

which is just a different factor ordering of the Wheeler-

DeWitt equation, Eq. (21). Substituting in for \mathcal{H}_ϕ from Eq. (20), and for p from Eq. (22), we get

$$p^{-1/2} \frac{\partial}{\partial p} \left[p^{-1/2} \frac{\partial}{\partial p} (p^{3/2} \Psi(p, \phi)) \right] - 48 p^{-3/2} \frac{\partial^2 \Psi(p, \phi)}{\partial \phi^2} + \frac{96}{\kappa \hbar^2} V(\phi) p^{3/2} \Psi(p, \phi) + \mathcal{O}(\mu_0) + \dots = 0. \quad (55)$$

Thus, in the large scale limit the equation we have to solve reads

$$p^{-1/2} \frac{\partial}{\partial p} \left[p^{-1/2} \frac{\partial}{\partial p} (p^{3/2} \Psi(p, \phi)) \right] + \beta V(\phi) p^{3/2} \Psi(p, \phi) = 0, \quad (56)$$

where

$$\beta = \frac{96}{\kappa \hbar^2}. \quad (57)$$

This equation is analogous to Eq. (23) and thus we are following the same procedure as in Section III C. We first separate the wave-function $\Psi(p, \phi)$ into $\Psi(p, \phi) = Y(p)\Phi(\phi)$ and we then approximate the dynamics of the inflaton field by setting $V(\phi) = V_\phi p^{\delta-3/2}$ to get

$$p^{-1/2} \frac{d}{dp} \left[p^{-1/2} \frac{d}{dp} (p^{3/2} Y(p)) \right] + \beta V_\phi p^\delta Y(p) = 0. \quad (58)$$

The solutions of Eq. (58) are

$$Y(p) = C_1 p^{-3/4} J_{3/(2\delta+3)} \left(\frac{4\sqrt{\beta V_\phi}}{2\delta+3} p^{(2\delta+3)/4} \right) + C_2 p^{-3/4} Y_{3/(2\delta+3)} \left(\frac{4\sqrt{\beta V_\phi}}{2\delta+3} p^{(2\delta+3)/4} \right). \quad (59)$$

Expanding Eq. (59), as we did in Section III C, we obtain

$$Y(p) \approx p^{-(9+2\delta)/8} \sqrt{\frac{2\delta+3}{2\sqrt{\beta V_\phi} \pi}} \left[C_1 \cos \left(x - \frac{3\pi}{2(2\delta+3)} - \frac{\pi}{4} \right) + C_2 \sin \left(x - \frac{3\pi}{2(2\delta+3)} - \frac{\pi}{4} \right) \right], \quad (60)$$

where

$$x = 4\sqrt{\beta V_\phi} (2\delta+3)^{-1} p^{(2\delta+3)/4}, \quad (61)$$

with β defined in Eq. (57). The scale of the n th zero is

$$p_n = \left[\frac{(2\delta+3)\pi}{4\sqrt{\beta V_\phi}} \right]^{4/(2\delta+3)} [C+n]^{4/(2\delta+3)}, \quad (62)$$

where

$$C = \tan^{-1} \left(\frac{-C_1}{C_2} \right) \frac{1}{\pi} + \frac{3}{2(2\delta+3)} \pm \frac{1}{2} - \frac{1}{4}. \quad (63)$$

Calculating the separation between two successive zeros, we obtain

$$\Delta p \equiv \lim_{p \rightarrow \infty} (p_{n+1} - p_n) = \frac{\pi}{\sqrt{\beta V_\phi}} p^{(1-2\delta)/4}. \quad (64)$$

For the continuum limit to be valid, the wave-function must vary slowly on scales of the order of $\mu_c = 4\tilde{\mu}$. Thus, using $p = \kappa\gamma\hbar\mu/6$, we arrive at the constraint

$$\Delta p > 4\mu_0 \left(\frac{\kappa\gamma\hbar}{6} \right)^{3/2} p^{-1/2}. \quad (65)$$

From Eqs. (64) and (65) we obtain the following constraint on V_ϕ :

$$V_\phi < \frac{27\pi^2}{192\mu_0^2\gamma^3\kappa^2\hbar} p^{(3-2\delta)/2}. \quad (66)$$

For slow-roll inflation, $V(\phi)$ must be approximately constant, thus $\delta \approx 3/2$. As previously, we use $\mu_0 = 3\sqrt{3}/2$ and $\gamma \approx 0.24$ to find the constraint on the inflationary potential in units of $\hbar = 1$,

$$V(\phi) \lesssim 2.35 \times 10^{-2} l_{\text{pl}}^{-4}, \quad (67)$$

where $l_{\text{pl}}^2 = G\hbar$. Clearly, this is a milder constraint than the one obtained for fixed lattices, Eq. (43).

V. OPEN/CLOSED ($k = \pm 1$) GEOMETRIES

In the previous sections we looked at the $k = 0$ case only. The reason for this is that there are several conceptual difficulties associated with the quantization procedure in the other cases, particularly for $k = -1$ [15,16]. These difficulties arise because vectors no longer commute and hence the four holonomies previously used to regulate the Hamiltonian no longer form a closed loop. There are methods of dealing with this (see Ref. [7] and references therein) that lead to discrete evolution equations of the same form as Eqs. (13) and (52), but with an extra term involving the curvature.

The correct quantization should be able to reproduce the Wheeler-DeWitt equation including a curvature term in a suitable limit,

$$\frac{\partial^2 S(\mu, \phi)}{\partial \mu^2} + kS(\mu, \phi) + \tilde{\alpha}\mu^\delta S(\mu, \phi) = 0, \quad (68)$$

where $S(\mu, \phi) = \sqrt{\mu}\Psi(\mu, \phi)$. This no longer has general solutions, however, during inflation when $\delta \approx 1$, analytic solutions do exist:

$$T(\mu) = C_1 \text{Airy}_A(-(k + \alpha\mu)\alpha^{-2/3}) + C_2 \text{Airy}_B(-(k + \alpha\mu)\alpha^{-2/3}), \quad (69)$$

where $T(\mu)$ is the μ dependent part of $S(\mu, \phi)$ (i.e., $S(\mu, \phi) = T(\mu)\Phi(\phi)$). Using the Airy functions expansions,

$$\lim_{x \rightarrow \infty} \text{Airy}_A(-|x|) = \frac{1}{\sqrt{\pi x^{1/4}}} \sin\left(\frac{2}{3}x^{2/3} + \frac{\pi}{4}\right),$$

$$\lim_{x \rightarrow \infty} \text{Airy}_B(-|x|) = \frac{1}{\sqrt{\pi x^{1/4}}} \cos\left(\frac{2}{3}x^{2/3} + \frac{\pi}{4}\right),$$

we obtain an explicit form for $T(\mu)$, which has zeros at the scales

$$\mu_n = \alpha^{-1/3} \left(\frac{3\pi}{2}\right)^{2/3} (C + n)^{2/3} - \frac{k}{\alpha}, \quad (70)$$

where now

$$C = \tan^{-1}(-C_2/C_1) - 3/4 \quad (71)$$

is again constant. Clearly then, the presence of a curvature will not affect the large scale behavior of the scale between subsequent zeros and so we expect the previous section to hold for any quantization schemes that have a continuum limit of the form of Eq. (68). As we have seen, changing the factor ordering of the Wheeler-DeWitt equation often means that analytical solutions are not available; in particular, adding a $k\sqrt{p}Y(p)$ term to Eq. (56) results in an equation that cannot be solved exactly. However, the success of this factor ordering (which arises from a non self-adjoint Hamiltonian constraint) suggests that the results for other cases may be similar.

VI. DISCUSSION/CONCLUSIONS

We have studied phenomenological aspects of LQC. In particular, we have investigated the effect of lattice refinement, which as we will now argue, makes the conditions for successful inflation more natural.

Let us consider that the scalar field ϕ plays the role of the inflaton. We will then combine the condition imposed earlier on the potential $V(\phi)$ with the constraint so that the perturbations induced by ϕ are consistent with the cosmic microwave background (CMB) measurements on large angular scales. As a concrete example we will use a simple inflationary model, namely $V(\phi) = m^2 \phi^2/2$.

The contribution of the inflaton field ϕ with potential $V(\phi)$ to the fractional over-density in Fourier space, at the epoch of horizon crossing, $k = aH$, is given by

$$\delta_H^2(k) = \frac{1}{75\pi^2 M_{\text{Pl}}^6} \left. \frac{V^3(\phi)}{[V'(\phi)]^2} \right|_{k=aH}, \quad (72)$$

where $V'(\phi) \equiv dV/d\phi$ and the Planck mass is $M_{\text{Pl}} = (8\pi G)^{-1/2} \simeq 2.4 \times 10^{18}$ GeV. From the COBE-DMR measurements, $\delta_H \simeq 1.91 \times 10^{-5}$ [17], which implies from Eq. (72) that

$$\frac{[V(\phi)]^{3/2}}{V'(\phi)} \simeq 5.2 \times 10^{-4} M_{\text{Pl}}^3. \quad (73)$$

To ensure the continuum approximation is not broken at large scales, we found in previous sections (Sections III and IV) that the potential must be $V(\phi) \lesssim x$, where x

depends on whether we have static or dynamically varying holonomy length scales. In the case of a fixed lattice and assuming N_{cl} e-foldings of inflation within the classical era, we found $x^{\text{fixed lattice}} \sim 7 \times 10^{-2} e^{-2N_{\text{cl}}} l_{\text{pl}}^{-4}$. While considering lattice refinement we got $x^{\text{lattice refinement}}(\phi) \lesssim 10^{-2} l_{\text{pl}}^{-4}$.

We now consider the simple inflationary case $V(\phi) = m^2 \phi^2/2$, favored by the latest WMAP measurements. In this case, Eq. (73) implies

$$m\phi^2 \simeq 1.5 \times 10^{-3} M_{\text{Pl}}^3, \quad (74)$$

and the constraint on the potential so that the continuum approximation is valid on large scales reads

$$m^2 \phi^2 \lesssim 2x. \quad (75)$$

Equations (74) and (75) imply

$$\Rightarrow m \lesssim 10^3 x / M_{\text{Pl}}^3. \quad (76)$$

For the fixed and varying lattices, the above constraint reads

$$m \lesssim 70(e^{-2N_{\text{cl}}}) M_{\text{Pl}} \quad (77a)$$

and

$$m \lesssim 10 M_{\text{Pl}}, \quad (77b)$$

respectively. Clearly, for any significant proportion of inflation taking place in the classical era, Eq. (77a) is a very strong, and therefore fine-tuned, constraint on the mass of the inflaton field. The condition on the inflaton mass becomes however natural, Eq. (77b), once lattice refinement is taken into account.

Equations (77) can be compared to restrictions on the inflaton mass coming from the WMAP data [18] for the same inflationary model as the one we consider here, namely,

$$-5.24 < \log\left(\sqrt{\kappa} \frac{m}{M_{\text{Pl}}}\right) < -5.18. \quad (78)$$

The fixed lattice case, Eq. (77a), is consistent with Eq. (78) only if $N_{\text{cl}} < 8.96$, which would suggest that loop quantum gravity effects should be observable in the spectrum of the CMB temperature anisotropies. In the lattice refinement case, however, no such incompatibility between theoretical and observational constraint exists.

In conclusion, lattice refinement is required to naturally obtain a successful inflationary model. In contrast to this the fixed lattice case requires the number of e-foldings of inflation that can be considered classical to be severely fine-tuned to match the observational data, leading to the conclusion that most of the inflationary era requires loop quantum gravity corrections to be included. This is at odds with the startling agreement between classical CMB calculations and observation.

Finally, it is important to note that while $\tilde{\mu} = \mu_0 \mu^{-1/2}$ has been suggested as a natural choice for the scaling behavior of the lattice [7,14], there is no rigorous justifi-

cation for this choice. A different choice would result in a different Hamiltonian constraint equation, Eq. (52), and all the subsequent analysis would change. Dynamic lattice refinement represents a large freedom in the formulation of LQC that, at present, is fixed only by phenomenological arguments. However, deriving the form of this refinement from the exact LQG equations remains, at present, not possible.

Last, but not least, we would like to bring to the attention of the reader that an alternative approach to the one we followed here is to perform a stability analysis of the dynamical difference equations, as suggested in Ref. [8]. More precisely, to investigate the scale at which the lattice

is unable to support the continuum oscillations, one could perform a von Neumann analysis of the difference equations on the grid [19]. Unfortunately, applying this method to a refining lattice case is more complicated since plane waves could no longer be used.

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