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Frame dragging and superenergy

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We show that the vorticity appearing in stationary vacuum spacetimes is always related to the existence of a flow of superenergy on the plane orthogonal to the vorticity vector. This result, together with the previously established link between vorticity and superenergy in radiative (Bondi-Sachs) spacetimes, strengthens further the case for this latter quantity as the cause of frame dragging.

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I. INTRODUCTION

The appearance of vorticity in the congruence of world-lines of observers in the gravitational field of a massive rotating ball (Lense-Thirring effect), led Shiff [1] to propose the use of gyroscopes to measure such an effect. Since then this idea has been developed extensively (see [2–5] and references cited therein).

However, the appearance of vorticity is not always related (at least explicitly) to rotating sources. Indeed, this is the case of the field of a magnetic dipole plus an electric charge [6], and the case of gravitational radiation [7–10].

In the former case the vorticity is accounted for by the existence of a flow of electromagnetic energy on the plane orthogonal to the vorticity vector. As a matter of fact it appears that in all stationary electrovacuum solutions [11], at least part of the vorticity has that origin.

In the case of gravitational radiation (Bondi-Sachs) we have recently shown [12] that the appearing vorticity is related to the existence of a flow of superenergy on the plane orthogonal to the vorticity vector.

Here we prove that such a link between a flow of superenergy and vorticity, is also present in stationary vacuum spacetimes.

In order to motivate our study, we shall start by considering simple particular cases, before dealing with the general stationary situation. Thus, we shall first present the Lense-Thirring case (Kerr up to the first order in m/r and a/r), then we will treat the Kerr case, and finally we establish the aforementioned link for the general stationary vacuum spacetime.

Doing so we provide a "universal" mechanism (i.e. one which applies to all known situations) for the occurrence of frame dragging. At the same time this result brings out the relevance of the Bel-Robinson tensor in the study of self-gravitating systems.

II. THE LENSE-THIRRING METRIC

As is well known the Lense-Thirring metric [13]

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 + \frac{2m}{r}\right)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + \frac{4J\sin^2\theta}{r} d\phi dt \tag{1}$$

describes the gravitational field outside a spinning sphere of constant density, and is valid up to first order in m/r and J/r^2, with m and J denoting the mass and the angular momentum, respectively.

It is also well known that, up to that order, it is the Kerr metric, with the identification

$$ma = -J, \tag{2}$$

where a is the Kerr parameter [14].

Now, the congruence of the worldline of observers at rest in the frame of (1) is

$$u^\alpha = \left(\frac{1}{\sqrt{1 - \frac{2m}{r}}}, 0, 0, 0\right), \tag{3}$$

and the vorticity vector, defined as usual by

$$\omega^\alpha = \tfrac{1}{2}\eta^{\alpha\eta\iota\lambda} u_\eta u_{\iota,\lambda}, \tag{4}$$

yields, up to order a/r and m/r

$$\omega^r = \frac{2ma\cos\theta}{r^3}, \tag{5}$$

$$\omega^\theta = \frac{ma\sin\theta}{r^4}, \tag{6}$$

or, for the absolute value of the vector ω^α

$$\Omega = (\omega^\alpha \omega_\alpha)^{1/2} = \frac{ma}{r^3}\sqrt{1 + 3\cos^2\theta}. \tag{7}$$

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At the equator ($\theta = \frac{\pi}{2}$)

$$\Omega = \frac{ma}{r^3} \tag{8}$$

which is a very well-known result.

The leading term of the super-Poynting gravitational vector at the equator, calculated from (1), is (see next section)

$$P^{\Phi} \approx 9\frac{m^2}{r^2}\frac{a}{r}\frac{1}{r^5}, \tag{9}$$

implying that $P^{\phi} = 0 \Leftrightarrow a = 0 \Leftrightarrow \omega^{\alpha} = 0$.

We shall now prove our conjecture in the exact case (Kerr metric).

III. THE KERR METRIC

In Boyer-Linquist coordinates, the Kerr metric takes the form

$$\begin{aligned} ds^2 = {} & \left(-1 + \frac{2mr}{r^2 + a^2\cos^2\theta}\right)dt^2 - \left(\frac{4mar\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)dtd\phi \\ & + \left(\frac{r^2 + a^2\cos^2\theta}{r^2 - 2mr + a^2}\right)dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 \\ & + \left(r^2\sin^2\theta + a^2\sin^2\theta + \frac{2mra^2\sin^4\theta}{r^2 + a^2\cos^2\theta}\right)d\phi^2. \end{aligned} \tag{10}$$

The congruence of worldline of observers at rest in (10) is defined by the vector field

$$u^{\alpha} = \left(\frac{1}{\sqrt{1 - \frac{2mr}{r^2 + a^2\cos^2\theta}}}, 0, 0, 0\right). \tag{11}$$

There are two nonvanishing components of the vorticity vector for such a congruence, which are

$$\begin{aligned} \omega^r = {} & 2mra\cos\theta(r^2 - 2mr + a^2)(r^2 + a^2\cos^2\theta)^{-2} \\ & \times (r^2 - 2mr + a^2\cos^2\theta)^{-1} \end{aligned} \tag{12}$$

and

$$\begin{aligned} \omega^{\theta} = {} & ma\sin\theta(r^2 - a^2\cos^2\theta)(r^2 + a^2\cos^2\theta)^{-2} \\ & \times (r^2 - 2mr + a^2\cos^2\theta)^{-1} \end{aligned} \tag{13}$$

which of course coincide with (5) and (6) up to first order in m/r and a/r.

Next, the super-Poynting vector based on the Bel-Robinson [15] tensor, as defined in Maartens and Basset [16], is

$$P_{\alpha} = \eta_{\alpha\beta\gamma\delta}E^{\beta}_{\rho}H^{\gamma\rho}u^{\delta}, \tag{14}$$

where $E_{\mu\nu}$ and $H_{\mu\nu}$ are the electric and magnetic parts of Weyl tensor, respectively, formed from Weyl tensor $C_{\alpha\beta\gamma\delta}$ and its dual $\tilde{C}_{\alpha\beta\gamma\delta}$ by contraction with the four velocity vector, given by

$$E_{\alpha\beta} = C_{\alpha\gamma\beta\delta}u^{\gamma}u^{\delta}, \tag{15}$$

$$H_{\alpha\beta} = \tilde{C}_{\alpha\gamma\beta\delta}u^{\gamma}u^{\delta} = \tfrac{1}{2}\eta_{\alpha\gamma\epsilon\delta}C^{\epsilon\delta}{}_{\beta\rho}u^{\gamma}u^{\rho}. \tag{16}$$

Then, a direct calculation of P^{μ} using the package GR-Tensor running on Maple yields, for the Kerr spacetime:

$$P^{\mu} = (P^t, P^{\phi}, 0, 0), \tag{17}$$

where

$$\begin{aligned} P^t = {} & -18m^3ra^2\sin^2\theta(r^2 - 2mr + a^2\sin^2\theta + a^2) \\ & \times (r^2 + a^2\cos^2\theta)^{-4}(r^2 - 2mr + a^2\cos^2\theta)^{-2} \\ & \times \left(\frac{r^2 - 2mr + a^2\cos^2\theta}{r^2 + a^2\cos^2\theta}\right)^{-1/2} \end{aligned} \tag{18}$$

and

$$\begin{aligned} P^{\phi} = {} & 9m^2a(r^2 - 2mr + 2a^2 - a^2\cos^2\theta) \\ & \times \left(\frac{r^2 - 2mr + a^2\cos^2\theta}{r^2 + a^2\cos^2\theta}\right)^{-1/2}(r^2 + a^2\cos^2\theta)^{-4} \\ & \times (r^2 - 2mr + a^2\cos^2\theta)^{-1}. \end{aligned} \tag{19}$$

From the above it can be seen that $P^{\phi} = 0 \Leftrightarrow a = 0 \Leftrightarrow \omega^{\alpha} = 0$. In other words, there is always an azimuthal flow of superenergy, as long as $a \neq 0$, and vice versa, the vanishing of such a flow, implies $a = 0$. Also observe that the leading term in the power series of m/r and a/r in (17) is of order $(m/r)^2$.

Let us now consider the general stationary and axisymmetric vacuum case.

IV. THE GENERAL CASE

The line element for a general stationary and axisymmetric vacuum spacetime may be written as [17]

$$\begin{aligned} ds^2 = {} & -fdt^2 + 2f\omega dtd\phi + f^{-1}e^{2\gamma}(d\rho^2 + dz^2) \\ & + (f^{-1}\rho^2 - f\omega^2)d\phi^2, \end{aligned} \tag{20}$$

where $x^0 = t$; $x^1 = \rho$; $x^2 = z$; and $x^3 = \phi$ and metric functions depend only on ρ and z which must satisfy the vacuum field equations:

$$\gamma_{\rho} = \frac{1}{4\rho f^2}[\rho^2(f_{\rho}^2 - f_z^2) - f^4(\omega_{\rho}^2 - \omega_z^2)], \tag{21}$$

$$\gamma_z = \frac{1}{2\rho f^2}[\rho^2 f_{\rho}f_z - f^4\omega_{\rho}\omega_z], \tag{22}$$

$$f_{\rho\rho} = -f_{zz} - \frac{f_{\rho}}{\rho} - \frac{f^3}{\rho^2}(\omega_{\rho}^2 + \omega_z^2) + \frac{1}{f}(f_{\rho}^2 + f_z^2), \tag{23}$$

$$\omega_{\rho\rho} = -\omega_{zz} + \frac{\omega_{\rho}}{\rho} - \frac{2}{f}(f_{\rho}\omega_{\rho} + f_z\omega_z). \tag{24}$$

The four velocity vector for an observer at rest in the frame of (20) is

$$u^\alpha = (f^{-1/2}, 0, 0, 0). \tag{25}$$

The super-Poynting vector can now be calculated for the general class of spacetimes represented by the above metric (20) (i.e., without making any assumption about the matter content of the spacetime), and one gets (using again GR-Tensor)

$$P^\mu = (P^t, P^\phi, 0, 0) \quad \text{with} \quad P^t = \omega P^\phi, \qquad \text{hence} \quad P_\mu = \left(0, \frac{\rho^2}{f}P^\phi, 0, 0\right). \tag{26}$$

Thus, the relevant quantity is P^ϕ which is given by (again in the general case, i.e., without taking into account the field equations):

$$P^\phi = f^{3/2}e^{-4\gamma}\rho^{-5}\{A11\}. \tag{27}$$

Substituting now the vacuum field equations (21)–(24) in the above expression one gets

$$P^\phi = -\tfrac{1}{32}f^{-3/2}e^{-4\gamma}\rho^{-5}\{A12\}, \tag{28}$$

where $A11$ and $A12$ are given in the appendix.

Now, it has been shown in [11] that for the general metric (20) the following relations hold

$$H_{\alpha\beta} = 0 \Leftrightarrow \omega^\alpha = 0 \Leftrightarrow \omega = 0 \tag{29}$$

and of course we know that

$$H_{\alpha\beta} = 0 \Rightarrow P^\mu = 0. \tag{30}$$

So, what we want to show here is that $P^\mu = 0$ implies necessarily that $\omega = 0$ the solution becomes then static, not just stationary, and therefore the so-called "dragging of inertial frames" effect disappears as the vorticity vanishes.

In other words we want to establish the relation

$$P^\mu = 0 \Leftrightarrow H_{\alpha\beta} = 0 \Leftrightarrow \omega^\alpha = 0 \Leftrightarrow \omega = 0. \tag{31}$$

Now, as the above equation is far too complicated to be treated in full, we shall start instead by analyzing what happens in the neighborhood of the symmetry axis $\rho = 0$ and far away from any matter source along it, that is, for $z \to \infty$. In so doing, the following two assumptions will be made:

(1) The spacetime is regular at the axis.
(2) The spacetime is asymptotically flat in spacelike directions.

The super-Poynting vector in a neighborhood of the axis

The geometry of axisymmetric spacetimes in the vicinity of the axis was studied in [18] (see also [17,19]). It then follows that $g_{t\phi}$ must tend to zero when $\rho \to 0$ at least as ρ^2, $g_{\phi\phi}$ tends to zero as ρ^2, and g_{tt}, $g_{\rho\rho}$, and g_{zz} cannot vanish on the symmetry axis. All these imply (but are not equivalent to) the so-called "elementary flatness condition" on the axis, namely,

$$\frac{X^a X_a}{4X} \to 1, \qquad X \equiv g_{\phi\phi}. \tag{32}$$

Let us then put

$$\begin{aligned} \omega(\rho, z) &= \rho^{2+k}A(z) + O(\rho^3),\\ f(\rho, z) &= m(z) + \rho n(z) + \rho^2 S(z) + O(\rho^3), \end{aligned} \tag{33}$$

where $k \geq 0$ is a constant and $m(z) \neq 0$ necessarily; further, the elementary flatness condition mentioned above implies that $\gamma \to 0$ as ρ tends to zero, and expanding P^ϕ in a power series around $\rho = 0$ one gets (in the case $k = 0$)

$$\begin{aligned} P^\phi \propto {}& -\frac{8Am^2n^2}{\rho^2} - \frac{8m^2}{\rho}\{-4A^3nm^4 + 3m^2n_zA_z + m^2nA_{z,z} + 4mA_znm_z - 4mAnS - 2mAm_{z,z}n + 6mAn_zm_z\\ &+ 3Am_z{}^2n - An^3\} + 2m\{-16A^5m^8 + (8A^2A_{z,z} + 12A_z{}^2A)m^6 + (-16A^3m_{z,z} + 56A_zA^2m_z - 32A^3S)m^5\\ &+ (-72A^3n^2 + 24A^3m_z{}^2)m^4 + (8SA_{z,z} + 24A_zS_z + 8m_{z,z}A_{z,z})m^3 + [10n^2A_{z,z} + (76A_zn_z - 8An_{z,z})n + 24An_z{}^2\\ &+ 20A_zm_{z,z}m_z - 16AS^2 - 16Am_{z,z}S - 2A_{z,z}m_z{}^2 + 32A_zSm_z + 48AS_zm_z]m^2 + [(-12AS - 20Am_{z,z} + 28A_zm_z)n^2\\ &+ 88An_znm_z - 2A_zm_z{}^3 + 24Am_z{}^2S - 4Am_{z,z}m_z{}^2]m + 7Am_z{}^4 + 10Am_z{}^2n^2 - An^4\} + O(\rho). \end{aligned} \tag{34}$$

Setting $P^\phi = 0$ all over the spacetime implies that the above must also vanish in a neighborhood of $\rho = 0$, which in turn implies that the coefficients of ρ^{-2}, ρ^{-1}, and ρ^0 must be zero. Since $m \neq 0$, it must be either $A = 0$ (which is what we aim at showing), or else $n = 0$.

Let us assume that $n = 0$, the coefficients of the terms in ρ^{-2} and ρ^{-1} then vanish identically, whereas the last term is much reduced.

Further, our second requirement above (asymptotic flatness), implies that far away from the source and in a small neighborhood of the axis, $m(z) = 1$, m_z, $m_{zz} = 0$, and also $S(z) = 0$ (for otherwise $f \neq 1$ at infinity in spatial directions), therefore one is left with

$$2AA_{zz} + 3A_z^2 - 4A^4 = 0 \tag{35}$$

which must hold for z large and in a neighborhood of the axis. Further, its solution must be bounded, since otherwise ω would increase without limit thus violating again the condition of asymptotic flatness.

This is an autonomous equation, a first integral of which can be readily found to be

$$A_z = \pm \frac{1}{\sqrt{7}} \frac{1}{A^2} \sqrt{A(4A^7 + C)}, \tag{36}$$

C being a constant of integration. It is then immediate to show, using numerical simulations, that the function A diverges for large values of z, and therefore it must vanish, which is what we wanted to show.

So far, we have only analyzed the case $k = 0$, however it is a simple matter to check that field equation (24) together with (33) rule out all values of $k > 0$.

In order to complete our proof we have to show that the result above (the vanishing of vorticity, implied by the vanishing of super-Poynting, within an infinite cylinder around the axis of symmetry), can be analytically extended to the whole spacetime.

In other words we have to prove that it is not possible to smoothly match a static axially symmetric, asymptotically flat vacuum spacetime to a stationary (nonstatic) axially symmetric vacuum spacetime which is also asymptotically flat, across an infinite cylinder (say Σ) around the axis of symmetry.

Such a matching is not possible [20], but it can also be checked at once from the continuity of the first fundamental forms on the cylinder. Indeed, this last requirement implies that ω should vanish also at the outer part of Σ, thereby indicating that the static condition can be analytically extended.

V. CONCLUSIONS

We have seen that Bonnor's original idea to associate the frame dragging in some electrovac solutions with the existence of a flow of electromagnetic energy (as described by the Poynting vector), can be successfully extended to the same effect in vacuum stationary spacetimes, by replacing the flow of electromagnetic energy by a flow of superenergy, as described by the super-Poynting vector defined from the Bel-Robinson tensor.

Because of the lack of a covariant definition of gravitational energy, superenergy appears to be the best candidate for playing such a role. On the other hand, the fact that it is also associated to frame dragging in radiative spacetimes, reinforces further our conjecture, that it is responsible for such an effect in any general relativistic scenario.

Before concluding, the following remark is in order: All along this work, the vorticity is calculated for a congruence with a distinct physical meaning, namely, the congruence of worldlines of observers at rest with respect to the source, i.e. observers at rest in the frame of (1), (10), and (20), respectively. This is particularly clear in the case of the Lense-Thirring metric (1).

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APPENDIX

$$\begin{aligned}
A11 = {} & [-2\rho\omega_\rho(\omega_\rho^2 + \omega_z^2)\gamma_\rho - 2\rho\omega_z(\omega_\rho^2 + \omega_z^2)\gamma_z - \omega_z^2\omega_{\rho\rho}\rho + \omega_z^2\omega_{zz}\rho + 4\omega_z\omega_\rho\omega_{\rho z}\rho - \omega_\rho^2\omega_{zz}\rho - \omega_\rho^3 \\
& + \omega_\rho^2\omega_{\rho\rho}\rho]f^4 + 3\rho(\omega_\rho^2 + \omega_z^2)(\omega_z f_z + \omega_\rho f_\rho)f^3 - 2\rho(-2\rho\gamma_z\omega_{\rho z} + 2\gamma_z^2\omega_\rho\rho + 2\rho\gamma_\rho^2\omega_\rho + \gamma_z\omega_z + \gamma_\rho\omega_\rho \\
& + \rho\gamma_\rho\omega_{zz} - \rho\gamma_\rho\omega_{\rho\rho})f^2 + [4\rho^3(f_z\omega_z + f_\rho\omega_\rho)\gamma_\rho^2 + 2\rho^2(\rho f_{zz}\omega_\rho - 2\rho f_{\rho z}\omega_z - 2f_z\omega_z + 4f_\rho\omega_\rho - 2\rho f_z\omega_{\rho z} \\
& - \rho f_{\rho\rho}\omega_\rho - \rho f_\rho\omega_{\rho\rho} + \rho f_\rho\omega_{zz})\gamma_\rho + 4\rho^3(f_z\omega_z + f_\rho\omega_\rho)\gamma_z^2 + 2\rho^2(4f_\rho\omega_z + \rho f_z\omega_{\rho\rho} - 2\rho f_{\rho z}\omega_\rho + \rho f_{\rho\rho}\omega_z \\
& - 2\rho f_\rho\omega_{\rho z} - \rho f_{zz}\omega_z - \rho f_z\omega_{zz} + 2\omega_\rho f_z)\gamma_z + 4\rho^3 f_{\rho z}\omega_{\rho z} - \rho^3 f_{zz}\omega_{\rho\rho} - \rho^3 f_{\rho\rho}\omega_{zz} + \rho^2 f_{zz}\omega_\rho - 2\rho^2 f_{\rho z}\omega_z \\
& - \rho^2 f_{\rho\rho}\omega_\rho + \rho^3 f_{zz}\omega_{zz} + \rho^3 f_{\rho\rho}\omega_{\rho\rho}]f - 6\rho^3(f_\rho^2 + f_z^2)\omega_\rho\gamma_\rho - 6\rho^3(f_\rho^2 + f_z^2)\omega_z\gamma_z \\
& + 3\rho^3(f_{\rho\rho}f_\rho\omega_\rho + f_{zz}f_z\omega_z + 2f_{\rho z}f_z\omega_\rho - f_{\rho\rho}f_z\omega_z + 2f_{\rho z}f_\rho\omega_z - f_{zz}f_\rho\omega_\rho),
\end{aligned} \tag{A1}$$

$$
\begin{aligned}
A12 = {} & \omega_\rho(-7\omega_z^4 - 6\omega_\rho^2\omega_z^2 + \omega_\rho^4)f^9 + [-\rho\omega_\rho f_\rho(\omega_z^4 + \omega_\rho^4 + 2\omega_\rho^2\omega_z^2) - \rho f_z\omega_z(\omega_\rho^4 + 2\omega_z^2\omega_\rho^2 + \omega_z^4)]f^8 \\
& + [-4\rho\omega_{zz}(\omega_\rho^2 + 3\omega_z^2) + 4\omega_\rho(-2\rho\omega_z\omega_{\rho z} + \omega_z^2)]f^7 + [4\rho\omega_z(-8\omega_z^2 f_z + \rho\omega_z^2 f_{\rho z} - 3\rho\omega_\rho\omega_z f_{zz} - 3\rho\omega_\rho^2 f_{\rho z} \\
& - \rho f_\rho\omega_z\omega_{zz} - 5\omega_\rho^2 f_z - 2\rho f_\rho\omega_\rho\omega_{\rho z} - 2\omega_z\omega_\rho f_\rho - 2\rho f_z\omega_\rho\omega_{zz} + \rho f_z\omega_z\omega_{\rho z}) + 4\rho\omega_\rho(-\rho\omega_\rho f_z\omega_{\rho z} \\
& + \rho\omega_\rho f_\rho\omega_{zz} + \omega_\rho^2 f_\rho + \rho\omega_\rho^2 f_{zz})]f^6 + [-6\rho^2\omega_\rho^3(f_z^2 + f_\rho^2) - 2\rho^2 f_z\omega_\rho\omega_z(2\omega_\rho f_\rho + 5\omega_z f_z) + 2\rho^2 f_\rho\omega_z^2(2\omega_z f_z \\
& - \omega_\rho f_\rho)]f^5 + [8\rho^2(f_{\rho z}\omega_z - f_\rho\omega_{zz}) - 16\rho^3(f_{\rho z}\omega_{\rho z} + f_{zz}\omega_{zz}) + 10\rho^3 f_z\omega_z f_\rho\omega_\rho(f_z\omega_z + f_\rho\omega_\rho) \\
& - 2\rho^3 f_z(f_\rho f_z\omega_\rho^3 + f_\rho^2\omega_z^3) + 2\rho^3 f_\rho^3(\omega_\rho^3 - \omega_\rho\omega_z^2) + 2\rho^3 f_z^3(\omega_z^3 - \omega_z\omega_\rho^2)]f^4 + [-24\rho^3 f_{\rho z}(\omega_\rho f_z + \omega_z f_\rho) \\
& + 4\rho^2 f_\rho(f_\rho\omega_\rho - 4f_z\omega_z) + 4\rho^3(3f_\rho^2\omega_{zz} + 2f_{zz}f_\rho\omega_\rho - 2\omega_{\rho z}f_\rho f_z + f_z^2\omega_{zz} - 10f_{zz}\omega_z f_z)]f^3 \\
& + [4\rho^4 f_{\rho z}(f_\rho^2\omega_z - f_z^2\omega_z + 2\omega_\rho f_\rho f_z) + 4\rho^4 f_{zz}(f_z^2\omega_\rho - f_\rho^2\omega_\rho + 2\omega_z f_\rho f_z) + 4\rho^4\omega_{\rho z}(-f_z^3 + 3f_z f_\rho^2) \\
& + 4\rho^4\omega_{zz}(-f_\rho^3 + 3f_\rho f_z^2) + 4\rho^3(4f_\rho^2 f_z\omega_z - 3f_z^2 f_\rho\omega_\rho - 2f_\rho^3\omega_\rho + 3f_z^3\omega_z)]f^2 + [\rho^4\omega_\rho(14f_z^2 f_\rho^2 - 7f_z^4 + 5f_\rho^4) \\
& + 4\rho^4\omega_z(f_\rho^3 f_z + 5f_z^3 f_\rho)]f - \rho^5\omega_z(2f_z^3 f_\rho^2 + f_z^5 + f_z f_\rho^4) - \rho^5\omega_\rho(2f_\rho^3 f_z^2 + f_\rho^5 + f_\rho f_z^4). \qquad \text{(A2)}
\end{aligned}
$$

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