Derivative expansion of the heat kernel in curved space

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The heat kernel in curved space-time is computed to fourth order in a strict expansion in the number of covariant derivatives. The computation is made for arbitrary non-Abelian gauge and scalar fields and for the Riemann connection in the coordinate sector. The expressions obtained hold for arbitrary tensor representations of the matter field. Complete results are presented for the diagonal matrix elements and for the trace of the heat kernel operator. In addition, Chan's formula is extended to curved space-time. As a byproduct, the bosonic effective action is also obtained to fourth order.

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I. INTRODUCTION

The heat kernel operator, the exponential of the Klein-Gordon operator, is a useful tool in quantum field theory since it is ultraviolet finite, one valued, and gauge covariant and allows to obtain the propagator and the effective action [1–4]. Quite different applications of the heat kernel (such as spectral densities, index theorems, ζ-function, quantum anomalies, chiral gauge theories, effective theories of QCD, Casimir effect, black hole entropies, membranes, etc.) are illustrated in [5–14]. The exact evaluation of the heat kernel is not possible in general (see however, [15–19] for particular cases). Nevertheless the series expansion in powers of the proper time is available with computable coefficients, the so-called Hadamard-Minakshisundaram-DeWitt-Seeley (HMDS) or heat kernel coefficients [4,20]. These coefficients have been computed with different techniques in several setups and to dimension ten [21–27]. For reviews on spectral geometry and field theory on curved space see [28–36]. For an alternative approach based on covariant perturbation theory see [37]. Extensions to finite temperature can be found in [38,39]. Expansions around non c-number mass terms are given in [40–42]. The extension to noncommutative quantum field theory has been presented in [43].

In [44] we investigated a different expansion for the diagonal matrix elements of the heat kernel operator where the terms are classified by their number of covariant derivatives. This can be regarded as a resummation of the HMDS coefficients to all orders in the (non-Abelian) mass term, which needs not be small. There, explicit results were presented for boundaryless compact flat manifolds to four derivatives for the diagonal matrix elements and to six derivatives for the trace of the heat kernel operator, for Klein-Gordon operators with arbitrary non-Abelian gauge and scalar external fields. In the present work we extend those results to the case of curved manifolds endowed with the Levi-Civita connection in the coordinate sector.

The scope and ideas involved as well as some definitions are given in Sec. II. In particular, we avoid the standard approach of reducing tensor fields (e.g., the photon field) to scalars by introduction of a tetrad [34]. That approach allows to use the heat kernel formulas derived for coordinate scalars but at the price of complicating the internal space (due to the new tetrad index) with its corresponding modified connection. Instead, we provide formulas which are equally simple regardless of the tensor representation of the fields (on which the Klein-Gordon operator is acting) using only the original gauge connection in the sector of internal indices and the Levi-Civita connection in the coordinate sector.

In Sec. III we discuss the shortcomings of Chan's approach in the curved case and develop our own approach based on the use of covariant symbols [45,46]. The covariant symbols are in fact multiplicative operators (with respect to x) which define a faithful representation of the algebra of pseudodifferential operators. This technique is similar to that of symbols for pseudodifferential operators [20,47–49] and shares with it the feature of providing diagonal matrix elements of generic operators, not necessarily related to the heat kernel, since it is not based on recurrence relations. However there is an important difference between both techniques: in the standard method of symbols covariance is not manifest prior to momentum integration whereas with covariant symbols covariance (under both gauge and coordinate transformations) is manifest at every step of the calculation. In that section results are provided for the diagonal matrix elements to four derivatives. These results are presented in compact form using the technique of labeled operators, also used in [44]. In this notation the mass term carries a label indicating its position in the expression; in this way it becomes effectively a c-number and momentum integrals can be carried out explicitly.

Explicit formulas for the derivative expansion of the trace of the heat kernel operator are presented in Sec. IV. This is of interest in the computation of the effective action. It is noted that many, in fact most, of the new terms introduced by the curvature can be eliminated by a suitable redefinition of the mass term in the Klein-Gordon operator.

In Sec. V we obtain the expressions for the diagonal matrix elements and trace of the heat kernel in the form first derived by Chan [50] for the effective action to four derivatives and extended in [51] to six derivatives. This is done following a rather indirect path since Chan's derivation is not used. Instead a Chan's form is proposed and the coefficients are adjusted to reproduce the results previously derived using covariant symbols. The question of whether Chan's elegant method can be used in curved manifolds is left open. In any case, as expected from the experience in the flat case, Chan's form is truly much more compact. Drawbacks are that the expressions are less explicit because one parametric integral is left undone, and for the same reason there is a by parts integration ambiguity in the formulas.

Section VI is devoted to computing the bosonic effective action in an explicit way, with the help of labeled operators. The necessary momentum integrals can be done with dimensional regularization. They are given in [52] using the $\overline{\mathrm{MS}}$ scheme.

Somewhat more explicit details on the calculation with covariant symbols are given in Appendix A. The trace cyclic property and integration by parts identities are derived in Appendix B.

II. DERIVATIVE EXPANSION OF THE HEAT KERNEL

We assume a compact boundaryless Riemannian manifold of dimension d and Euclidean metric $g_{\mu\nu}(x)$ to represent the space-time upon Wick rotation. Further, for the world sector indices[1] we take the Levi-Civita connection (i.e., torsionless and metric preserving). The Klein-Gordon operator will act on wave functions $\psi(x)$ (matter fields) defined on the space-time manifold. The wave functions are vectors with respect to some representation of a certain gauge group. Without loss of generality we will assume that ψ carries a single gauge (or internal) index. ψ is also allowed to carry world indices, that is, we do not assume ψ to be a world scalar.[2] Following for instance [34], we use a single covariant derivative ∇_μ which acts on all indices with the appropriate connection, $\nabla = \partial + \Gamma + \omega$. $\Gamma^\lambda_{\mu\nu}$ is the connection on world indices and $\omega_\mu(x)$ the connection on gauge indices, a matrix in internal space. Our convention for the Riemann tensor is such that, if ψ is a world vector,

$$[\nabla_\mu, \nabla_\nu]\psi_\lambda = \Omega_{\mu\nu}\psi_\lambda + R_{\mu\nu\lambda\sigma}\psi_\sigma, \qquad (2.1)$$

where

$$\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu] \qquad (2.2)$$

is the gauge field strength tensor. In addition, for the Ricci tensor and scalar curvature

$$\mathcal{R}_{\mu\nu} = R_{\lambda\mu\lambda\nu}, \qquad \boldsymbol{R} = \mathcal{R}_{\mu\mu}. \qquad (2.3)$$

In the previous formulas we use the same notation for contravariant and covariant indices. We will follow this convention throughout unless an ambiguity arises.

The Klein-Gordon operator is of the form

$$K = g^{\mu\nu}\nabla_\mu\nabla_\nu + X. \qquad (2.4)$$

$X(x)$ is a scalar field with respect to world indices and a matrix with respect to the internal space. As is well known the heat kernel operator $e^{\tau K}$ is ultraviolet finite for $\mathrm{Re}(\tau) > 0$, and its matrix elements admit an asymptotic expansion in powers of τ. For the diagonal matrix elements

$$\langle x|e^{\tau K}|x\rangle \asymp \frac{1}{(4\pi\tau)^{d/2}}\sum_{n=0}^\infty \tau^n a_n(x). \qquad (2.5)$$

The coefficients a_n are still operators with respect to gauge and world indices since the brackets $\langle x|$, $|x\rangle$ refer only to x space. To lowest orders, the well-known result is

$$\begin{aligned} a_0 &= 1, \qquad a_1 = X + \tfrac{1}{6}\boldsymbol{R}, \\ a_2 &= \tfrac{1}{2}X^2 + \tfrac{1}{6}X_{\mu\mu} + \tfrac{1}{12}Z^2_{\mu\nu} + \tfrac{1}{6}\boldsymbol{R}X + \tfrac{1}{30}\boldsymbol{R}_{\mu\mu} \\ &\quad + \tfrac{1}{72}\boldsymbol{R}^2 - \tfrac{1}{180}\mathcal{R}^2_{\mu\nu} + \tfrac{1}{180}R^2_{\mu\nu\alpha\beta}. \end{aligned} \qquad (2.6)$$

In these formulas (and hereafter) we use the following notational convention: the covariant derivative of an object is represented by adding a world index to it and further covariant derivatives add further indices to the left. So for instance,

$$X_\mu = [\nabla_\mu, X], \quad X_{\mu\nu} = [\nabla_\mu,[\nabla_\nu, X]], \quad \mathcal{R}_{\alpha\mu\nu} = \nabla_\alpha\mathcal{R}_{\mu\nu}. \qquad (2.7)$$

We have also introduced the quantity

$$Z_{\mu\nu} := [\nabla_\mu, \nabla_\nu]. \qquad (2.8)$$

Let us emphasize that $Z_{\mu\nu}$ is different from $\Omega_{\mu\nu}$. The operators $Z_{\mu\nu}$ and $\Omega_{\mu\nu}$ are both multiplicative with respect to x space and matrices in internal space, however, unlike $\Omega_{\mu\nu}$, $Z_{\mu\nu}$ acts also on world indices (cf. (2.1)). This implies, for instance, that $Z_{\mu\nu}$ and $\mathcal{R}_{\alpha\beta}$ do not commute:

$$[Z_{\mu\nu}, \mathcal{R}_{\alpha\beta}] = R_{\mu\nu\alpha\sigma}\mathcal{R}_{\sigma\beta} + R_{\mu\nu\beta\sigma}\mathcal{R}_{\alpha\sigma}. \qquad (2.9)$$

Likewise, $[Z_{\mu\nu}, X_\lambda] = [\Omega_{\mu\nu}, X_\lambda] + R_{\mu\nu\lambda\sigma}X_\sigma$, etc. The derivative of $Z_{\mu\nu}$ will also be needed,[3]

[1] In this work we will use the label *world* interchangeably with *coordinate* or *space-time* in expressions like "world tensor," "world index," etc., to refer to properties tied to indices $\mu, \nu, \ldots$, associated to natural bases, $\partial/\partial x^\mu$, of the tangent space of the Riemannian manifold.

[2] Of course one can choose to transform such world indices into internal or gauge indices using a tetrad field. This is the standard approach [34]. Our formulas hold whether this choice is made or not.

[3] This is an exception to the index convention. The second term in the definition of $Z_{\alpha\mu\nu}$ is required to make it a multiplicative operator with respect to x. E.g.

$$[Z_{\alpha\mu\nu}, X_\lambda] = [\Omega_{\alpha\mu\nu}, X_\lambda] + R_{\alpha\mu\nu\lambda\sigma}X_\sigma.$$

Similar terms appear in higher derivatives, $Z_{\alpha\beta\mu\nu}$, etc., [46].

$$Z_{\alpha\mu\nu} := [\nabla_\alpha, Z_{\mu\nu}] - \tfrac{1}{2}\{\nabla_\lambda, R_{\lambda\alpha\mu\nu}\}. \tag{2.10}$$

In our formulas we will use $Z_{\mu\nu}$ rather than $\Omega_{\mu\nu}$. Nevertheless, if desired one can always move the Z's to the right using their commutation properties and apply them to the wave function ψ to produce Ω's and R's. (Of course, the result so obtained will depend on the concrete tensor representation of the wave function.) For instance, the term $Z_{\mu\nu}^2/12$ in a_2 is equivalent to more standard $\Omega_{\mu\nu}^2/12$ when the wave function happens to be a world scalar. The advantage of using $Z_{\mu_1\cdots\mu_n}$ is that the expressions take the same form regardless of the world-tensor representation of ψ and yet the original connections Γ and ω are used. (See footnote 2.) Fuller details can be found in [46].

By dimensional counting it is clear that the standard heat kernel expansion (2.5) is an expansion with coefficients ordered by their mass dimension: if $g_{\mu\nu}$ carries dimension 0, ∇_μ carries dimension 1, and X carries dimension 2, the coefficient a_n has dimension $2n$. In what follows we will set $\tau = 1$ since dimensional counting allows to restore the parameter τ at any time if needed.

In this work we consider a different classification of terms in the heat kernel, namely, operators are classified by the number of covariant derivatives they carry, so ∇_μ counts as order 1 while X and $g_{\mu\nu}$ count as order zero. Thus, e.g., XX_μ is of first order, $R_{\mu\nu\alpha\beta}$ and $Z_{\mu\nu}$ of second order, $Z_{\alpha\mu\nu}$ of third order, and so on. In this expansion

$$\langle x|e^K|x\rangle \asymp \frac{1}{(4\pi)^{d/2}}\sum_{n=0}^{\infty} A_n(x), \tag{2.11}$$

where the coefficient A_n collects all operators with $2n$ derivatives and any number of X. In turn, the A_n can be reexpanded using operators classified by their mass dimension,

$$\begin{aligned} A_0 &= 1 + X + \tfrac{1}{2}X^2 + \cdots, \\ A_1 &= \tfrac{1}{6}\boldsymbol{R} + \tfrac{1}{6}X_{\mu\mu} + \tfrac{1}{6}\boldsymbol{R}X + \cdots, \\ A_2 &= \tfrac{1}{12}Z_{\mu\nu}^2 + \tfrac{1}{30}\boldsymbol{R}_{\mu\mu} + \tfrac{1}{72}\boldsymbol{R}^2 - \tfrac{1}{180}\mathcal{R}_{\mu\nu}^2 \\ &\quad + \tfrac{1}{180}R_{\mu\nu\alpha\beta}^2 + \cdots. \end{aligned} \tag{2.12}$$

(Of course, when τ is not set to unity these coefficients are also functions of τ.)

In [44] we presented complete formulas (i.e., valid to all orders in X) for A_0, A_1, and A_2, as well as B_0, B_1, B_2, and B_3 (defined below), for the flat case. Presently we extend those results (except B_3) to curved manifolds.

The key tool to write the result to all orders in X in closed form is the use of labeled operators. For instance, all terms in A_1 of the type $X^nX_{\mu\mu}X^m$, with $n, m = 0, 1, \ldots$, can be collected as

$$I_{2,2}X_{\mu\mu} = \left(\frac{e^{X_1}+e^{X_2}}{(X_1-X_2)^2} - 2\frac{e^{X_1}-e^{X_2}}{(X_1-X_2)^3}\right)X_{\mu\mu}. \tag{2.13}$$

The label 1 in X_1 indicates that the corresponding X should be placed just before (i.e. to the left of) the fixed operator $X_{\mu\mu}$, likewise, X_2 indicates that X is to be put just after (i.e., to the right of) the fixed operator. Upon a series expansion in powers of X_1 and X_2

$$\begin{aligned} I_{2,2}X_{\mu\mu} &= (\tfrac{1}{6} + \tfrac{1}{12}X_1 + \tfrac{1}{12}X_2 + \tfrac{1}{40}X_1^2 + \tfrac{1}{40}X_2^2 \\ &\quad + \tfrac{1}{30}X_1X_2 + \cdots)X_{\mu\mu} \\ &= \tfrac{1}{6}X_{\mu\mu} + \tfrac{1}{12}XX_{\mu\mu} + \tfrac{1}{12}X_{\mu\mu}X + \tfrac{1}{40}X^2X_{\mu\mu} \\ &\quad + \tfrac{1}{40}X_{\mu\mu}X^2 + \tfrac{1}{30}XX_{\mu\mu}X + \cdots. \end{aligned} \tag{2.14}$$

Likewise, $X_1^2X_2X_3X_\mu^2$ would stand for $X^2X_\mu XX_\mu X$, etc. The labels always refer to the position of X's with respect to "fixed operators" such as $X_{\alpha\mu\nu}$, $Z_{\mu\nu}$, etc. Because X commutes with $R_{\mu\nu\alpha\beta}$ we will not need to include the Riemann tensor among the set of fixed operators. So for instance, $I_{2,2}\boldsymbol{R}X_{\mu\mu}$ will be used to mean $\boldsymbol{R}$ multiplied by $I_{2,2}X_{\mu\mu}$.

The important point is that the labeled operators can be treated as c-numbers, e.g., $X_1X_2 = X_2X_1$; the true position of the labeled operator is given by its label. This is similar to what happens within normal or chronological orders. Labeled operators were introduced in [53,54].

The function $I_{2,2}$ belongs to the family of functions $I_{r_1,\ldots,r_n}$, with arguments $X_1, \ldots, X_n$. They are defined as

$$I_{r_1,r_2,\ldots,r_n} := \int_\Gamma \frac{dz}{2\pi i} e^z N_1^{r_1}N_2^{r_2}\cdots N_n^{r_n}, \tag{2.15}$$

where

$$N = (z - X)^{-1} \tag{2.16}$$

and $N_i = (z - X_i)^{-1}$, and Γ is a positively oriented closed simple path on the complex plane enclosing the eigenvalues of X (at the given point x). These functions enjoy a number of properties described in [44]. In particular they are entire functions of the X_i and can be computed by recurrence relations starting from $I_1 = e^{X_1}$. They are not linearly independent; integration by parts implies the relation

$$I_{r_1,r_2,\ldots,r_n} = \sum_{j=1}^{n} r_j I_{r_1,r_2,\ldots,r_j+1,\ldots,r_n}. \tag{2.17}$$

The trace of the heat kernel operator,

$$\mathrm{Tr}\,(e^K) = \int d^dx\sqrt{g}\,\mathrm{tr}\langle x|e^K|x\rangle, \tag{2.18}$$

is also of great interest in applications, such as the computation of the effective action (see Sec. VI). The trace tr refers to gauge and world indices. The trace of the heat kernel admits an asymptotic expansion with terms classified by their mass dimension:

$$\mathrm{Tr}\,(e^K) \asymp \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \int d^d x \sqrt{g}\, \mathrm{tr}(b_n(x)), \tag{2.19}$$

with

$$\begin{aligned} &b_0 = 1, \qquad b_1 = X + \tfrac{1}{6}\boldsymbol{R}, \\ &b_2 = \tfrac{1}{2}X^2 + \tfrac{1}{12}Z_{\mu\nu}^2 + \tfrac{1}{6}\boldsymbol{R}X + \tfrac{1}{72}\boldsymbol{R}^2 - \tfrac{1}{180}\mathcal{R}_{\mu\nu}^2 + \tfrac{1}{180}R_{\mu\nu\alpha\beta}^2. \end{aligned} \tag{2.20}$$

Likewise, the terms can be classified by their number of derivatives

$$\mathrm{Tr}\,(e^K) \asymp \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \int d^d x \sqrt{g}\, \mathrm{tr}(B_n(x)). \tag{2.21}$$

Because of the trace cyclic property and integration by parts, there is an ambiguity in the definition of $B_n(x)$. This allows to bring $B_n(x)$ to a simpler form starting from $A_n(x)$. In turn, functional derivation allows one to obtain A_n from B_n

$$A_n(x) = \frac{\delta}{\delta X(x)} \int d^d x \sqrt{g}\, \mathrm{tr}(B_n(x)). \tag{2.22}$$

Before closing this section, let us comment on the nature of the derivative expansion in the present context. [We reinstate τ for this discussion.] As is well known, the standard expansion in powers of τ is an asymptotic one, reliable for small τ only. Intuitively, this is because the coefficients $a_n(x)$ are local, i.e., they depend on a finite number of derivatives of the external fields (namely, the metric, the gauge connection, and the scalar field X). Consequently, the expansion at a given point x_0 is not sensitive to modifications of these external fields taking place outside a fixed neighborhood of x_0. Such modifications affect the exact matrix element at x_0 (and hence its τ dependence) but this is not seen by the asymptotic expansion. By the same token, the derivative expansion is expected to be asymptotic too, since the coefficients $A_n(x;\tau)$ are also local. To make mathematical sense of the asymptotic series it is necessary to write it as a power series expansion. The derivative expansion can be viewed as a power series as follows: for a given point x_0 consider a deformation of the external fields such that within a fixed neighborhood of x_0 the deformation is just a dilatation by a parameter λ

$$\begin{aligned} X(x) &\mapsto X(x_\lambda), & g_{\mu\nu}(x) &\mapsto g_{\mu\nu}(x_\lambda), \\ \omega_\mu(x) &\mapsto \lambda\omega_\mu(x_\lambda), & x_\lambda^\mu &= x_0^\mu + \lambda(x^\mu - x_0^\mu). \end{aligned} \tag{2.23}$$

The deformation is smoothly continued outside of the neighborhood. This produces a family of Klein-Gordon operators K_λ.[4] By construction the parameter λ counts the number of derivatives, that is,

$$A_n(x_0;\tau,\lambda) = \lambda^{2n} A_n(x_0;\tau). \tag{2.24}$$

Therefore the derivative expansion can be read off from an expansion of $\langle x_0|e^{\tau K_\lambda}|x_0\rangle$. This construction suggests that, although the series in λ is asymptotic, the coefficients $A_n(x;\tau)$ themselves are well-defined quantities, as is also the case for the coefficients $a_n(x)$. This point is to be settled by a rigorous mathematical approach. Note that this definition of the coefficients (from an expansion in λ) is not identical to the alternative definition

$$A_n(x;\tau) = \sum_{m\geq n} a_{m,n}(x)\tau^{m-n} \tag{2.25}$$

(using $a_{m,n}(x)$ to denote the terms of $a_m(x)$ with $2n$ derivatives). We expect both definitions to coincide. This expectation relies on (i) the fact that the functions $I_{r_1,r_2,\ldots,r_n}$ are entire functions of X_i and so of τ, thus their expansion in τ is absolutely convergent, and (ii) the obvious reason for the series in τ to be asymptotic does not apply here since $A_n(x;\tau)$ is itself local. In summary, we expect the derivative expansion to be valid for finite (not small) τ, provided λ is sufficiently small.

[4] The dilatation depends not only on x_0 but also on the coordinate system used. Equation (2.24) is coordinate independent.

III. DIAGONAL COEFFICIENTS

In [44] the calculation of A_n in flat space-time was based on that of B_n. In turn B_n was adapted from the result of Chan [50] for the effective action to four derivatives and its extension in [51] to six derivatives. Unfortunately, it is not obvious how the elegant approach of [50] is to be extended to the curved case. As we may recall, in that approach a symbol method ($D_\mu \to D_\mu + p_\mu$) is applied, to $\mathrm{Tr}(\log(K))$. The expression is then formally expanded in the number of derivatives and brought to a canonical form by using integration by parts (with respect to p_μ) as well as formal cyclic property. This canonical form is not manifestly gauge invariant but it allows to unambiguously identify the originating gauge invariant expression from which it comes, because no two different gauge invariant expressions may have the same canonical form. The trouble is that a similar statement does not hold in the curved case. For instance,

$$\mathrm{Tr}\,(Z_{\mu\nu}[p_\mu, p_\nu N^2]) = 0 \tag{3.1}$$

(with $N = (p^2 - X)^{-1}$), since p_μ and X commute. However, under formal cyclic property the same expression would be equivalent to

$$\begin{aligned} \mathrm{Tr}\,(p_\nu N^2[Z_{\mu\nu}, p_\mu]) &= \mathrm{Tr}(p_\nu N^2 R_{\mu\nu\mu\sigma} p_\sigma) \\ &= \mathrm{Tr}(p_\mu p_\nu \mathcal{R}_{\mu\nu} N^2). \end{aligned} \tag{3.2}$$

This is equivalent to $-\mathrm{Tr}(p^2 \boldsymbol{R} N^2)/d$ and does not vanish.

Since Chan's approach is not available, we will compute A_n from scratch and then obtain B_n from it. The starting point is the representation

$$e^K = \int_\Gamma \frac{dz}{2\pi i} \frac{e^z}{z - \nabla^2 - X}, \tag{3.3}$$

where we can apply the method of covariant symbols [45,46]. This is the second key ingredient of our approach. This gives

$$\langle x|e^K|x\rangle = \int_\Gamma \frac{dz}{2\pi i} \frac{1}{\sqrt{g}} \int \frac{d^d p}{(2\pi)^d} \frac{e^z}{z - \bar{\nabla}^2 - \bar{X}}. \tag{3.4}$$

The covariant symbols are designed for computing diagonal matrix elements of general operators, not necessarily the heat kernel. The covariant symbols were introduced by Pletnev and Banin in [45] for the gauge connection and extended to curved space-time in [46] where they have been computed to four derivatives. Explicitly, to two derivatives

$$\begin{aligned} \bar{X} &= X - X_\alpha \partial^\alpha + \frac{1}{2!} X_{\alpha\beta}\partial_\alpha\partial_\beta + \cdots, \\ \bar{\nabla}^2_\mu &= p_\mu^2 + \frac{1}{6}\boldsymbol{R} + Z_{\alpha\beta} p_\alpha \partial_\beta - \frac{1}{3}\mathcal{R}_{\alpha\beta} p_\alpha \partial_\beta \\ &\quad + \frac{1}{3} R_{\alpha\mu\beta\nu} p_\alpha p_\beta \partial_\mu \partial_\nu + \cdots, \end{aligned} \tag{3.5}$$

where $\partial_\mu = \partial/\partial p_\mu$ and the dots refer to terms with three derivatives or more. Note that, unlike ordinary symbols, these operators are multiplicative and covariant already before momentum integration. The calculation is straightforward along the lines of the examples presented in [46]. Details are provided in Appendix A. One obtains

$$A_0 = I_1 = e^X \tag{3.6}$$

to zeroth order and

$$A_1 = (I_{1,2} - 2I_{1,3})X_{\mu\mu} + (2I_{1,1,2} - 4I_{1,1,3} - 2I_{1,2,2})X_\mu^2 + \tfrac{1}{3} I_3 \boldsymbol{R} \tag{3.7}$$

to second order. Using the identities (2.17), this can be rewritten as

$$A_1 = I_{2,2} X_{\mu\mu} + 2I_{2,1,2} X_\mu^2 + \tfrac{1}{6} I_1 \boldsymbol{R}. \tag{3.8}$$

Let us emphasize that the functions $f(X_1, X_2)$ multiplying $X_{\mu\mu}$ and $f(X_1, X_2, X_3)$ multiplying X_μ^2, are well defined and unambiguous. The only ambiguity enters in how they are written in terms of the overcomplete basis $I_{r_1,\ldots,r_n}$.

The two terms without $\boldsymbol{R}$ are identical to those of the flat case in [44], only with the "minimal coupling" replacement $D = \partial + \omega \to \nabla = \partial + \Gamma + \omega$. These are then minimal terms required by covariance (under general coordinate transformations). The term with $\boldsymbol{R}$ is nonminimal; covariant but not required by covariance. In general we will obtain coefficients of the form

$$A_n = A_n^m + A_n^R. \tag{3.9}$$

The terms in A_n^R are those which contain explicitly the Riemann tensor and so vanish in the flat space case. On the other hand, the minimal terms A_n^m can be reconstructed by minimal coupling from the flat space expressions. Let us warn, however, that this separation is not an unambiguous one: in general A_n^m will depend of the concrete A_n of flat space used to apply the minimal coupling. This is because reordering of world indices in the flat space expression may introduce Riemann tensors in the curved case.[5]

The calculation of A_2 gives

$$\begin{aligned} A_2^m &= 2I_{2,1,2}Z_{\mu\nu}Z_{\mu\nu} + (2I_{2,2,2} - 4I_{3,0,3} + 4I_{2,1,3})Z_{\mu\mu\nu}X_\nu + (2I_{2,2,2} - 4I_{3,0,3} + 4I_{3,1,2})X_\mu Z_{\nu\mu\nu} + (4I_{2,2,1,2} - 16I_{3,0,1,3} \\ &\quad - 8I_{3,0,2,2} + 8I_{3,1,1,2})X_\mu Z_{\mu\nu} X_\nu + (16I_{3,0,1,3} - 8I_{2,1,1,3} - 4I_{2,1,2,2} + 8I_{3,0,2,2})Z_{\mu\nu}X_\mu X_\nu + (16I_{3,1,0,3} - 8I_{3,1,1,2} \\ &\quad - 4I_{2,2,1,2} + 8I_{2,2,0,3})X_\mu X_\nu Z_{\mu\nu} + 2I_{3,3}X_{\mu\mu\nu\nu} + (2I_{2,3,2} + 4I_{3,1,3} + 2I_{2,2,3} + 2I_{3,2,2})X_{\mu\mu}X_{\nu\nu} + 8I_{3,1,3}X_{\mu\nu}X_{\mu\nu} \\ &\quad + (8I_{3,1,3} + 4I_{3,2,2})X_{\mu\nu\nu}X_\mu + (8I_{3,1,3} + 4I_{2,2,3})X_\mu X_{\mu\nu\nu} + (4I_{2,2,2,2} + 16I_{3,1,1,3} + 8I_{2,2,1,3} + 8I_{3,1,2,2})X_\mu X_{\mu\nu}X_\nu \\ &\quad + (2I_{2,2,2,2} + 8I_{3,1,1,3} + 4I_{2,2,1,3} + 4I_{3,1,2,2})X_\mu X_{\nu\nu}X_\mu + (2I_{2,2,2,2} + 8I_{3,1,1,3} + 4I_{2,2,1,3} + 4I_{2,3,1,2} + 4I_{3,1,2,2} \\ &\quad + 4I_{3,2,1,2})X_{\mu\mu}X_\nu X_\nu + (2I_{2,2,2,2} + 8I_{3,1,1,3} + 4I_{2,2,1,3} + 4I_{2,1,3,2} + 4I_{3,1,2,2} + 4I_{2,1,2,3})X_\mu X_\mu X_{\nu\nu} + (16I_{3,1,1,3} \\ &\quad + 8I_{3,1,2,2})X_{\mu\nu}X_\mu X_\nu + (16I_{3,1,1,3} + 8I_{2,2,1,3})X_\mu X_\nu X_{\nu\mu} + (4I_{2,2,1,2,2} + 16I_{3,1,1,1,3} + 8I_{2,1,2,1,3} + 8I_{3,1,2,1,2} \\ &\quad + 4I_{2,2,2,1,2} + 4I_{2,1,2,2,2} + 8I_{2,1,3,1,2} + 8I_{2,2,1,1,3} + 8I_{3,1,1,2,2})X_\mu X_\mu X_\nu X_\nu + (4I_{2,2,1,2,2} + 16I_{3,1,1,1,3} + 8I_{2,2,1,1,3} \\ &\quad + 8I_{3,1,1,2,2})X_\mu X_\nu X_\mu X_\nu + (4I_{2,2,1,2,2} + 16I_{3,1,1,1,3} + 8I_{2,2,1,1,3} + 8I_{3,1,1,2,2})X_\mu X_\nu X_\nu X_\mu, \end{aligned} \tag{3.10}$$

[5]For instance,

$$Z_{\alpha\beta\mu\nu} = Z_{\beta\alpha\mu\nu} + [Z_{\alpha\beta}, Z_{\mu\nu}] - \tfrac{1}{2}\{Z_{\alpha\lambda}, R_{\lambda\beta\mu\nu}\} + \tfrac{1}{2}\{Z_{\beta\lambda}, R_{\lambda\alpha\mu\nu}\}.$$

$$\begin{aligned} A_2^R &= \tfrac{1}{30} I_1 \boldsymbol{R}_{\mu\mu} + \tfrac{1}{72} I_1 \boldsymbol{R}^2 + \tfrac{1}{6} I_{2,2} \boldsymbol{R} X_{\mu\mu} + \tfrac{1}{4} I_{2,2} \boldsymbol{R}_\mu X_\mu \\ &\quad + \tfrac{4}{3} I_{3,3} \mathcal{R}_{\mu\nu} X_{\mu\nu} + \tfrac{1}{3} I_{2,1,2} \boldsymbol{R} X_\mu^2 + (\tfrac{2}{3} I_{2,2,2} + \tfrac{4}{3} I_{2,3,2} \\ &\quad - \tfrac{16}{3} I_{3,1,3}) \mathcal{R}_{\mu\nu} X_\mu X_\nu - \tfrac{1}{180} I_1 \mathcal{R}_{\mu\nu}^2 + \tfrac{1}{180} I_1 R_{\mu\nu\alpha\beta}^2. \end{aligned} \tag{3.11}$$

A_2^m is formally identical to the expression in (4.10) of [44]. (There mirror symmetry of A_n was exploited to write A_2 in a shorter, less explicit form.) As said before, in the formula for A_2^R the Riemann tensor is not to be considered as one of the fixed operators.

The coefficients A_1 and A_2, for curved space and non-Abelian gauge group, with contributions to all orders in X are computed here for the first time. Upon expansion in powers of X they reproduce the corresponding terms with up to four derivatives of the standard expansion a_n. In particular the coefficients quoted in Eqs. (4.26–4.29) of [34] are correctly reproduced. (The comparison with [34] is achieved by restricting our results to the case of a world scalar wave function and using the trace cyclic property, but not integration by parts.) Results to all orders in X and up to four derivatives for curved space are also available from [55] for the so-called minimal case, i.e., $X(x)$ a c-number and $\omega_\mu = 0$, and ψ a world scalar. Our formulas also check Eq. (22) of [55].

IV. COEFFICIENTS FOR THE TRACE

Application of integration by parts and the trace cyclic property allows to obtain the simpler coefficients B_n. To four derivatives they take the form

$$\begin{aligned} B_0 &= I_1, \qquad B_1 = -\tfrac{1}{2} I_{2,2} X_\mu X_\mu + \tfrac{1}{6} I_1 \boldsymbol{R}, \\ B_2 &= (-I_{2,2,2,2} + 4 I_{3,1,3,1}) X_\mu X_\mu X_\nu X_\nu \\ &\quad + \tfrac{1}{2} I_{2,2,2,2} X_\mu X_\nu X_\mu X_\nu + 4 I_{3,1,3} X_\mu X_\mu X_{\nu\nu} \\ &\quad + I_{3,3} X_{\mu\mu} X_{\nu\nu} + 2 I_{2,2,2} X_\mu X_\nu Z_{\mu\nu} + \tfrac{1}{2} I_{2,2} Z_{\mu\nu} Z_{\mu\nu} \\ &\quad - \tfrac{1}{12} I_{2,2} \boldsymbol{R} X_\mu X_\mu - \tfrac{2}{3} I_{3,3} \mathcal{R}_{\mu\nu} X_\mu X_\nu + \tfrac{1}{30} I_1 \boldsymbol{R}_{\mu\mu} \\ &\quad + \tfrac{1}{72} I_1 \boldsymbol{R}^2 - \tfrac{1}{180} I_1 \mathcal{R}_{\mu\nu}^2 + \tfrac{1}{180} I_1 R_{\mu\nu\alpha\beta}^2. \end{aligned} \tag{4.1}$$

[For short we have written $I_{2,2}$ for $I_{2,2,0}$, etc.] The simplest way to obtain this result is starting from its Chan's form, to be discussed in the next section. The validity of integration by parts for general world representations of the wave function space is shown in Appendix B. The cyclic property for multiplicative operators is also subtle when $Z_{\mu\nu}$ (or more generally $Z_{\mu_1\cdots\mu_n}$) is present because this operator acts on world indices. For instance, under trace,

$$X_\mu Z_{\mu\nu} X_\nu \equiv X_\nu X_\mu Z_{\mu\nu} - \mathcal{R}_{\mu\nu} X_\mu X_\nu. \tag{4.2}$$

The last term would not be present under the standard trace cyclic property of matrices. The rationale of the extra term is that in $X_\mu Z_{\mu\nu} X_\nu$, $Z_{\mu\nu}$ acts on the index ν of X_ν (as well as on the subsequent world indices in the wave function) while in $X_\nu X_\mu Z_{\mu\nu}$ it does not; the missing contribution is added by the extra term. This equation is derived in Appendix B.[6]

It can be verified that A_n and B_n are indeed equivalent inside $\int d^d x \sqrt{g}\, \mathrm{tr}()$ and that (2.22) is fulfilled.

To this order one can see that most of the nonminimal terms can be generated by a modified "minimal coupling" prescription. Namely, $\partial + \omega \to \partial + \omega + \Gamma$ and $X \to X'$, with the new scalar field

$$X' = X + \tfrac{1}{6} \boldsymbol{R} + \tfrac{1}{180} (\boldsymbol{R}_{\mu\mu} - \mathcal{R}_{\mu\nu}^2 + R_{\mu\nu\alpha\beta}^2) + \mathcal{O}(\nabla^6). \tag{4.3}$$

The trace of the heat kernel can then be expanded as

$$\mathrm{Tr}\,(e^K) \asymp \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \int d^d x \sqrt{g}\, \mathrm{tr}(B_n'(x)), \tag{4.4}$$

with

$$\begin{aligned} B_0' &= I_1', \qquad B_1' = -\tfrac{1}{2} I_{2,2}' X_\mu' X_\mu', \\ B_2' &= (-I_{2,2,2,2}' + 4 I_{3,1,3,1}') X_\mu' X_\mu' X_\nu' X_\nu' \\ &\quad + \tfrac{1}{2} I_{2,2,2,2}' X_\mu' X_\nu' X_\mu' X_\nu' + 4 I_{3,1,3}' X_\mu' X_\mu' X_{\nu\nu}' \\ &\quad + I_{3,3}' X_{\mu\mu}' X_{\nu\nu}' + 2 I_{2,2,2}' X_\mu' X_\nu' Z_{\mu\nu} \\ &\quad + \tfrac{1}{2} I_{2,2}' Z_{\mu\nu} Z_{\mu\nu} - \tfrac{2}{3} I_{3,3}' \mathcal{R}_{\mu\nu} X_\mu' X_\nu'. \end{aligned} \tag{4.5}$$

($I'_{r_1,\ldots,r_n}$ being defined as in (2.15) but using X' instead of X.) There are corresponding coefficients A_n'. The relations $\int d^d x \sqrt{g}\, \mathrm{tr}(A_n) = \int d^d x \sqrt{g}\, \mathrm{tr}(B_n)$ and (2.22) hold also for the primed coefficients. The redefinition $X' = X + \frac{1}{6}\boldsymbol{R}$ is quite standard in the literature [15] to eliminate some of the terms. Note that beyond that the primed expansion is no longer a strict expansion in the number of covariant derivatives (and X' will depend on τ when τ is restored).

V. CHAN'S FORM OF THE COEFFICIENTS

Let us call the coefficients A_n and B_n just derived the coefficients in X form, to distinguish them from their Chan's or N form, to be discussed in this section.

Let us briefly summarize Chan's method in flat space [50,51]. In this method the results are obtained in terms of derivatives of $N = (z - X)^{-1}$, instead of derivatives of X, that is, $N_\mu = [\nabla_\mu N]$, $N_{\mu\nu}$, etc. Because z appears inside $N_{\mu_1\cdots\mu_n}$ the integral over z cannot be carried out explicitly and is left undone. Integration by parts (with respect to z) allows to reorder terms so that in each term of B_n the quantity N (derivated or not) appears exactly $2n$ times. A virtue of this approach is that for each B_n there is only a limited number of available covariant structures constructed with $2n$ N's and $2n$ ∇'s, thus the expressions so obtained are quite compact. To pass a result given in N

[6]Of course, relations like (4.2) imply that starting from the same B_n of flat space but written in different ways will in general yield different results for the minimal coupling part of B_n in curved space.

form to X form is, of course, straightforward using the relation $N_\mu = NX_\mu N$ and its derivatives. As pointed out before, the result in X form is free from ambiguities.

Because the extension of Chan's method to curved space is not known, the existence of a Chan's form for the heat kernel coefficients in the curved case is not obvious. In principle, undoing the z integrals in the $I_{r_1,\ldots,r_n}$ and using identities of the type

$$\begin{aligned} X_\mu &= N^{-1}N_\mu N^{-1}, \\ X_{\mu\nu} &= N^{-1}N_{\mu\nu}N^{-1} - N^{-1}N_\mu N^{-1}N_\nu N^{-1} \\ &\quad - N^{-1}N_\nu N^{-1}N_\mu N^{-1}, \end{aligned} \tag{5.1}$$

would allow to bring the coefficients computed in X form to an almost Chan's form, except that, in general, negative powers of N will be present. Moreover the precise result will be subject to ambiguities due to integration by parts on z and no algorithm is available to bring it to a compact form. This method works for B_0 and B_1.

The procedure that we have followed to obtain an N form for the coefficient B_2 is as follows. In the flat case the N form is known for B_2 and hence for A_2 (by functional variation with respect to X), therefore we apply minimal coupling there. This already reproduces most of the terms of the known full A_2 in X form, (3.10) and (3.11). For the few remaining terms of A_2 it is relatively easy to bring them to a rather compact N form by hand. This remainder is the functional variation of the nonminimal remainder in B_2, B_2^R, not yet determined. To obtain B_2^R we simply write down the most general terms having four derivatives, at least one Riemann tensor, and no more than four N's and with arbitrary numerical coefficients. These coefficients are then chosen to reproduce the nonminimal remainder of A_2. This procedure gives[7]

$$\begin{aligned} B_0 &= \langle N\rangle_z, \qquad B_1 = \langle -\tfrac{1}{2}N_\mu^2 + \tfrac{1}{6}\boldsymbol{R}N\rangle_z, \\ B_2 &= \langle -N_\mu^2 N_\nu^2 + \tfrac{1}{2}(N_\mu N_\nu)^2 + (NN_{\mu\mu})^2 \\ &\quad + 2NN_\mu N_\nu NZ_{\mu\nu} + \tfrac{1}{2}(NZ_{\mu\nu}N)^2 - \tfrac{1}{12}\boldsymbol{R}N_\mu^2 \\ &\quad - \tfrac{2}{3}\mathcal{R}_{\mu\nu}NN_\mu NN_\nu + \tfrac{1}{30}\boldsymbol{R}_{\mu\mu}N + \tfrac{1}{72}\boldsymbol{R}^2 N \\ &\quad - \tfrac{1}{180}\mathcal{R}_{\mu\nu}^2 N + \tfrac{1}{180}R_{\mu\nu\alpha\beta}^2 N\rangle_z, \end{aligned} \tag{5.2}$$

where we use the shorthand notation

$$\langle\rangle_z := \int_\Gamma \frac{dz}{2\pi i} e^z(). \tag{5.3}$$

B_2 in X form, (4.1), has been obtained from this N form. Equations (5.2) are the extension of Chan's formulas to curved space-time. Once again, to this order, the only nonminimal term surviving is $-\frac{2}{3}\mathcal{R}_{\mu\nu}NN_\mu NN_\nu$ if X' is used throughout.

Using (2.22), A_n in N form is easily obtained from B_n. This gives[8]

$$\begin{aligned} A_0 &= \langle N\rangle_z, \qquad A_1 = \langle NN_{\mu\mu}N + \tfrac{1}{6}\boldsymbol{R}N\rangle_z, \\ A_2 &= \langle 2N^2N_{\mu\mu\nu\nu}N^2 + 4NN_\mu N_{\mu\nu}N_\nu N + 2NN_\mu N_{\nu\nu}N_\mu N + 2NN_\mu^2 N_{\nu\nu}N + 2NN_{\mu\mu}N_\nu^2 N + 4NN_\mu N_{\mu\nu\nu}N^2 \\ &\quad + 4N^2N_{\mu\nu\nu}N_\mu N + 2N^2N_{\mu\mu}^2 N + 2NN_{\mu\mu}NN_{\nu\nu}N + 2NN_{\mu\mu}^2N^2 + 4N^2Z_{\mu\nu}N_\mu N_\nu N + 4NN_\mu N_\nu Z_{\mu\nu}N^2 \\ &\quad + 4NN_\mu NZ_{\mu\nu}N_\nu N + 2N^2Z_{\mu\mu\nu}NN_\nu N + 2NN_\mu NZ_{\nu\mu\nu}N^2 + 2N^2Z_{\mu\nu}NZ_{\mu\nu}N^2 + \tfrac{1}{6}\boldsymbol{R}NN_{\mu\mu}N + \tfrac{1}{4}\boldsymbol{R}_\mu NN_\mu N \\ &\quad + \tfrac{1}{30}\boldsymbol{R}_{\mu\mu}N + \tfrac{4}{3}\mathcal{R}_{\mu\nu}N^2N_{\mu\nu}N^2 + \tfrac{2}{3}\mathcal{R}_{\mu\nu}NN_\mu N_\nu N + \tfrac{4}{3}\mathcal{R}_{\mu\nu}NN_\mu NN_\nu N + \tfrac{1}{72}\boldsymbol{R}^2N - \tfrac{1}{180}\mathcal{R}_{\mu\nu}^2 N + \tfrac{1}{180}R_{\mu\nu\alpha\beta}^2 N\rangle_z. \end{aligned} \tag{5.4}$$

As noted, the existence of a Chan's form for the coefficients was not completely obvious *a priori* in the curved case. The fact that this Chan's form exists suggests that perhaps Chan's method could find a suitable extension in the case of curved space.

VI. THE EFFECTIVE ACTION

After functional integration, the effective action of a complex bosonic field is given by $-\mathrm{Tr}(\log K)$. This can be related to the heat kernel by

$$-\mathrm{Tr}\log K = \int_0^\infty \frac{d\tau}{\tau}\,\mathrm{Tr}(e^{\tau K}). \tag{6.1}$$

Upon restoring τ in the expressions, a contribution $I_{r_1,\ldots,r_n}\mathcal{O}$ in B_n picks up a factor $\tau^{\gamma+1-\rho-d/2}$, where 2γ is the mass dimension of the operator $\mathcal{O}$ and $\rho = \sum_{i=1}^n r_i$, (e.g., $\tau^{4+1-4-d/2}$ for $I_{2,2}\boldsymbol{R}X_\mu X_\mu$ in B_2). After carrying out the integral over τ, the integral over z in $I_{r_1,\ldots,r_n}$ can be traded by a momentum integral. This gives the replacement rule for going from the heat kernel to the effective action

$$\frac{1}{(4\pi)^{d/2}} I_{r_1,\ldots,r_n}\mathcal{O} \to I^{\rho-\gamma}_{r_1,\ldots,r_n}\mathcal{O}, \tag{6.2}$$

where we have defined

$$I^k_{r_1,\ldots,r_n} := \frac{\Gamma(d/2)}{\Gamma(k+d/2)}\int \frac{d^dq}{(2\pi)^d}(q^2)^k N_1^{r_1}\cdots N_n^{r_n}, \tag{6.3}$$

with $N = (q^2 - X)^{-1}$. (Note that $I^k_{r_1,\ldots,r_n}$ depends also on d.) The contributions to the effective action may be ultra-

[7] Note that the sign of the fourth term of B_2 is incorrect in [50].

[8] Let us note that the minimal parts of A_2 in X form and in N form are different:

$$A^m_{2,N} = A^m_{2,X} - 8I_{3,1,3}\mathcal{R}_{\mu\nu}X_\mu X_\nu.$$

violet and infrared divergent. Dimensional regularization applies here. These integrals have been computed in [52] using minimal subtraction.

The replacement rule gives for the effective action expanded in derivatives

$$-\operatorname{tr}\log K = \int d^d x \sqrt{g} \sum_{n=0}^{\infty} \operatorname{tr} W_n \tag{6.4}$$

with

$$\begin{aligned}
W_0 &= I_1^1, \\
W_1 &= -\tfrac{1}{2} I_{2,2}^1 X_\mu X_\mu + \tfrac{1}{6} I_1^0 \boldsymbol{R}, \\
W_2 &= (-I_{2,2,2,2}^2 + 4 I_{3,1,3,1}^2) X_\mu X_\mu X_\nu X_\nu \\
&\quad + \tfrac{1}{2} I_{2,2,2,2}^2 X_\mu X_\nu X_\mu X_\nu + 4 I_{3,1,3}^2 X_\mu X_\mu X_{\nu\nu} \\
&\quad + I_{3,3}^2 X_{\mu\mu} X_{\nu\nu} + 2 I_{2,2,2}^2 X_\mu X_\nu Z_{\mu\nu} + \tfrac{1}{2} I_{2,2}^2 Z_{\mu\nu} Z_{\mu\nu} \\
&\quad - \tfrac{1}{12} I_{2,2}^0 \boldsymbol{R} X_\mu X_\mu - \tfrac{2}{3} I_{3,3}^2 \mathcal{R}_{\mu\nu} X_\mu X_\nu + \tfrac{1}{30} I_2^0 \boldsymbol{R}_{\mu\mu} \\
&\quad + \tfrac{1}{72} I_2^0 \boldsymbol{R}^2 - \tfrac{1}{180} I_2^0 \mathcal{R}_{\mu\nu}^2 + \tfrac{1}{180} I_2^0 R_{\mu\nu\alpha\beta}^2.
\end{aligned} \tag{6.5}$$

VII. SUMMARY

We have derived, for the first time, expressions valid to all orders in X and to four covariant derivatives for the diagonal matrix elements and also for the trace of the heat kernel operator in a Riemannian curved manifold. The expressions presented check previously available results for the so-called minimal case [55] and, when reexpanded in powers of X, they reproduce the known HMDS coefficients to four derivatives. We also extend Chan's formula, originally derived for the effective action, to include curvature. As in the flat case, the expressions in Chan's form are remarkably simple also in the curved case. This simplicity suggests a direct calculation of the energy-momentum tensor taking a variation of the effective action with respect to the metric. Such a calculation has not been addressed here. The method of covariant symbols allows to consider more general coordinate connections, including torsion. This would be of interest in the derivation of the Lorentz group generators since the coordinate connection couples to them. A virtue of our formulas is that they are equally simple for scalar wave functions and for tensor ones, without redefining the gauge connection to include the parallel transport of the new tetrad fields indices. This is achieved through a consistent use of the operators $Z_{\mu_1\cdots\mu_n}$ which are defined so that they are multiplicative. Remarkably the formulas obtained are independent of the tensor representation of the wave function even for the coefficients $B_n(x)$. This is because the general formulas for the trace cyclic property and integration by parts can also be written in a tensor representation independent way (see Appendix B).

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APPENDIX A: COVARIANT SYMBOLS

Let us give some details on the calculation of A_n using covariant symbols. Fuller details on the use of covariant symbols with derivative expansions can be found in [56] for flat space-time and [46] for curved space-time. Using the expansions (3.5) in (3.4), reexpanding the result and keeping terms with at most two covariant derivatives, gives

$$\begin{aligned}
\langle x|e^K|x\rangle &= \int_\Gamma \frac{dz}{2\pi i} \frac{1}{\sqrt{g}} \int \frac{d^d p}{(2\pi)^d} e^z \Big[N - N X_\alpha \partial^\alpha N \\
&\quad + N X_\alpha \partial^\alpha N X_\beta \partial^\beta N + N\Big(\frac{1}{2!} X_{\alpha\beta} \partial_\alpha \partial_\beta \\
&\quad + \frac{1}{6}\boldsymbol{R} + Z_{\alpha\beta} p_\alpha \partial_\beta - \frac{1}{3}\mathcal{R}_{\alpha\beta} p_\alpha \partial_\beta \\
&\quad + \frac{1}{3} R_{\alpha\mu\beta\nu} p_\alpha p_\beta \partial_\mu \partial_\nu \Big) N + \mathcal{O}(\nabla^4) \Big],
\end{aligned} \tag{A1}$$

with $N = (z - p_\mu^2 - X)^{-1}$. (Let us warn that we are using a purely imaginary p_μ, to avoid the proliferation of i's in the formulas.) The derivatives with respect to p_μ are then carried out (using $\partial_\mu N = 2 p_\mu N^2$). After that, the shift $z \to z + p_\mu^2$ allows to isolate the p_μ dependence in integrals of the type $\int d^d p e^{p_\mu^2} p_{\mu_1} \cdots p_{\mu_n}$ which are easily evaluated. These steps produce

$$\begin{aligned}
\langle x|e^K|x\rangle &= \frac{1}{(4\pi)^{d/2}} \int_\Gamma \frac{dz}{2\pi i} e^z \Big[N + N X_{\mu\mu} N^2 \\
&\quad - 2 N X_{\mu\mu} N^3 + 2 N X_\mu N X_\mu N^2 \\
&\quad - 4 N X_\mu N X_\mu N^3 - 2 N X_\mu N^2 X_\mu N^2 \\
&\quad + \frac{1}{3} N^3 \boldsymbol{R} + \mathcal{O}(\nabla^4) \Big]
\end{aligned} \tag{A2}$$

with $N = (z - X)^{-1}$. This expression immediately translates into those in (3.6) and (3.7).

APPENDIX B: TRACE AND INTEGRATION BY PARTS

In this work (and, in particular, in this appendix), all formulas hold for arbitrary tensor representations of the wave functions $\psi(x)$ on which the Klein-Gordon operator acts. The space of world tensors of rank r is spanned by $e_{a_1}^{\mu_1} \cdots e_{a_r}^{\mu_r}$ where $e_a^\mu(x)$ is a local basis of the space-time tangent space.

The trace tr in (2.18) refers to gauge indices and to world indices, $\operatorname{tr} = \operatorname{tr}_{\text{gauge}} \operatorname{tr}_{\text{world}}$. We need to consider only multiplicative operators $\hat{\mathcal{O}}$ (that is, $\hat{\mathcal{O}}\psi$ at x does not depend on ψ at $x' \neq x$). All operators in A_n and B_n are multiplicative. In this case, the trace on world indices in the representation of tensors of rank r will be

$$\mathrm{tr}_{\,\mathrm{world}}(\hat{\mathcal{O}}) = e^{a_1}_{\mu_1} \cdots e^{a_r}_{\mu_r}(\hat{\mathcal{O}} e^{\mu_1}_{a_1} \cdots e^{\mu_r}_{a_r}), \qquad \text{(B1)}$$

where e^a_μ is the dual basis, $e^a_\mu e^\mu_b = \delta_{ab}$. Note that $\mathrm{tr}_{\mathrm{world}}(\hat{\mathcal{O}})$ may only be nonvanishing when $\hat{\mathcal{O}}$ is itself a world scalar, that is, it maps tensors of rank r to tensors of rank r. As an illustration, in the space of tensors of rank r, an easy calculation yields

$$\mathrm{tr}\,(\tfrac{1}{12}Z^2_{\mu\nu}) = d^r \mathrm{tr}_{\mathrm{gauge}}(\tfrac{1}{12}\Omega^2_{\mu\nu}) - r d^{r-1} n_g \tfrac{1}{12} R^2_{\mu\nu\alpha\beta}, \qquad \text{(B2)}$$

n_g being the dimension of the gauge representation.

The use of the trace cyclic property and integration by parts requires some care. These properties work as usual when the operators involved are (i) multiplicative and (ii) *they do not act on world indices*. (For a scalar operator $\hat{\mathcal{O}}$, this means that the component μ of $\hat{\mathcal{O}}$ applied to $e_a(x)$ does not depend on $e^\nu_a(x)$, for $\nu \neq \mu$.) Instances of such operators are X, X_μ, and $\Omega_{\mu\nu}$. In the notation of [46], these are the operators in the class $\mathcal{C}(\underline{\nabla}, \underline{Z})$. On the other hand, modifications occur in the trace cyclic property and integration by parts when multiplicative operators acting on world indices are involved (class $\mathcal{C}(\underline{\nabla})$). An instance of this is $Z_{\mu\nu}$, since $(Z_{\mu\nu}e_a)^\lambda = R_{\mu\nu\lambda\sigma}e^\sigma_a + \Omega_{\mu\nu}e^\lambda_a$. (However, $[\nabla_\mu, A]$ and $[Z_{\mu\nu}, A] \in \mathcal{C}(\underline{\nabla}, \underline{Z})$ provided $A \in \mathcal{C}(\underline{\nabla}, \underline{Z})$.)

To write down the correct relations, let us define $Z^R_{\mu_1\cdots\mu_n}$ as the curvature parts of $Z_{\mu_1\cdots\mu_n}$, that is, obtained by dropping ω_μ in the covariant derivative. In particular,

$$Z_{\mu\nu} = Z^R_{\mu\nu} + \Omega_{\mu\nu}. \qquad \text{(B3)}$$

Furthermore, let us introduce the shorthand notation

$$\langle\hat{\mathcal{O}}\rangle := \int d^d x \sqrt{g}\, \mathrm{tr}(\hat{\mathcal{O}}). \qquad \text{(B.4)}$$

The two following useful properties are easily established

$$\begin{aligned} &\langle A Z^R_{\mu\nu}\rangle = 0, \qquad A \in \mathcal{C}(\underline{\nabla}, \underline{Z}) \\ &\langle A Z^R_{\alpha\mu\nu}\rangle = \langle -\tfrac{1}{2}R_{\sigma\sigma\alpha\mu\nu}A\rangle, \qquad A \in \mathcal{C}(\underline{\nabla}, \underline{Z}) \end{aligned} \qquad \text{(B5)}$$

which hold for arbitrary multiplicative operators A not acting on world indices.

Using these properties, one can prove the following relations for arbitrary operators $A, B, \ldots,$ in $\mathcal{C}(\underline{\nabla}, \underline{Z})$:

$$\langle[\nabla_\mu, A]B\rangle = \langle -A[\nabla_\mu, B]\rangle, \qquad \text{(B6)}$$

$$\begin{aligned} \langle[\nabla_\alpha, A]BZ_{\mu\nu}C\rangle = \langle -A(&B_\alpha Z_{\mu\nu}C + BZ_{\alpha\mu\nu}C + BZ_{\mu\nu}C_\alpha \\ &+ R_{\sigma\alpha\mu\nu}BC_\sigma + \tfrac{1}{2}R_{\sigma\sigma\alpha\mu\nu}BC)\rangle, \end{aligned} \qquad \text{(B7)}$$

$$\langle ABZ_{\mu\nu}C\rangle = \langle BZ_{\mu\nu}CA - BC[Z^R_{\mu\nu}, A]\rangle, \qquad \text{(B8)}$$

$$\langle ABZ_{\alpha\mu\nu}C\rangle = \langle BZ_{\alpha\mu\nu}CA - BC[Z^R_{\alpha\mu\nu}, A]\rangle, \qquad \text{(B9)}$$

$$\begin{aligned} \langle ABZ_{\mu\nu}CZ_{\alpha\beta}D\rangle = \langle &BZ_{\mu\nu}CZ_{\alpha\beta}DA - BZ_{\mu\nu}CD[Z^R_{\alpha\beta}, A] \\ &- BCZ_{\alpha\beta}D[Z^R_{\mu\nu}, A] \\ &+ BCD[Z^R_{\alpha\beta}, [Z^R_{\mu\nu}, A]]\rangle, \end{aligned} \qquad \text{(B10)}$$

$$\begin{aligned} \langle Z_{\mu\nu}AZ_{\alpha\beta}B\rangle = \langle &AZ_{\alpha\beta}BZ_{\mu\nu} - R_{\alpha\beta\mu\sigma}ABZ_{\sigma\nu} \\ &- R_{\alpha\beta\nu\sigma}ABZ_{\mu\sigma}\rangle. \end{aligned} \qquad \text{(B11)}$$

The first two identities refer to integration by parts while the other allow to apply the trace cyclic property. The relation (4.2) is a consequence of (B8).

[1] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Dover, New York, 1952).
[2] V. Fock, Phys. Z. Sowjetunion **12**, 404 (1937).
[3] J. S. Schwinger, Phys. Rev. **82**, 664 (1951).
[4] B. S. DeWitt, Phys. Rep. **19**, 295 (1975).
[5] P. B. Gilkey, J. Diff. Geom. **10**, 601 (1975).
[6] M. Atiyah, R. Bott, and V. K. Patodi, Inventiones Mathematicae **19**, 279 (1973).
[7] S. W. Hawking, Commun. Math. Phys. **55**, 133 (1977).
[8] K. Fujikawa, Phys. Rev. D **21**, 2848 (1980).
[9] R. D. Ball, Phys. Rep. **182**, 1 (1989).
[10] J. Bijnens, Phys. Rep. **265**, 370 (1996).
[11] E. Megías, E. Ruiz Arriola, and L. L. Salcedo, Phys. Rev. D **72**, 014001 (2005).
[12] M. Bordag, U. Mohideen, and V. M. Mostepanenko, Phys. Rep. **353**, 1 (2001).
[13] J. Callan, G. Curtis, and F. Wilczek, Phys. Lett. B **333**, 55 (1994).
[14] K. Belani, P. Kaura, and A. Misra, J. High Energy Phys. 10 (2006) 023.
[15] A. A. Bytsenko, G. Cognola, L. Vanzo, and S. Zerbini, Phys. Rep. **266**, 1 (1996).
[16] R. Camporesi, Phys. Rep. **196**, 1 (1990).
[17] I. G. Avramidi, J. Math. Phys. (N.Y.) **37**, 374 (1996).
[18] I. G. Avramidi, J. Math. Phys. (N.Y.) **36**, 5055 (1995).
[19] I. G. Avramidi, Rev. Math. Phys. **11**, 947 (1999).
[20] R. T. Seeley, Proc. Sympos. Pure Math. **10**, 288 (1967).
[21] A. A. Bel'kov, A. V. Lanyov, and A. Schaale, Comput. Phys. Commun. **95**, 123 (1996).
[22] A. E. M. van de Ven, Classical Quantum Gravity **15**, 2311 (1998).
[23] I. G. Moss and W. Naylor, Classical Quantum Gravity **16**,

2611 (1999).
[24] D. Fliegner, P. Haberl, M. G. Schmidt, and C. Schubert, Ann. Phys. (N.Y.) **264**, 51 (1998).
[25] I. G. Avramidi, Nucl. Phys. **B355**, 712 (1991).
[26] V. P. Gusynin, Phys. Lett. B **225**, 233 (1989).
[27] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994).
[28] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).
[29] P. B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem* (Publish or Perish, Wilmington, DE, 1984).
[30] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[31] S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, Cambridge, England, 1989).
[32] R. M. Wald, *Quantum Field Theory in Curved Space-time and Black Hole Thermodynamics* (University of Chicago, Chicago, 1994).
[33] I. G. Avramidi, *Heat Kernel and Quantum Gravity*, Lecture Notes in Physics Vol. m64 (Springer, New York, 2000), p. 1.
[34] D. V. Vassilevich, Phys. Rep. **388**, 279 (2003).
[35] K. Kirsten, *Spectral Functions in Mathematics and Physics* (Chapman and Hall/CRC, Boca Raton, FL, 2002).
[36] G. Esposito, G. Miele, and B. Preziosi, Nucl. Phys. B, Proc. Suppl. **104**, v (2002).
[37] A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. **B282**, 163 (1987).
[38] E. Megías, E. Ruiz Arriola, and L. L. Salcedo, Phys. Lett. B **563**, 173 (2003).
[39] E. Megías, E. Ruiz Arriola, and L. L. Salcedo, Phys. Rev. D **69**, 116003 (2004).
[40] A. A. Osipov and B. Hiller, Phys. Lett. B **515**, 458 (2001).
[41] A. A. Osipov and B. Hiller, Phys. Rev. D **64**, 087701 (2001).
[42] L. L. Salcedo, Eur. Phys. J. direct C **3**, 14 (2001).
[43] D. V. Vassilevich, Lett. Math. Phys. **67**, 185 (2004).
[44] L. L. Salcedo, Eur. Phys. J. C **37**, 511 (2004).
[45] N. G. Pletnev and A. T. Banin, Phys. Rev. D **60**, 105017 (1999).
[46] L. L. Salcedo, Eur. Phys. J. C **49**, 831 (2007).
[47] T. Eguchi, P. B. Gilkey, and A. J. Hanson, Phys. Rep. **66**, 213 (1980).
[48] R. I. Nepomechie, Phys. Rev. D **31**, 3291 (1985).
[49] L. L. Salcedo and E. Ruiz Arriola, Ann. Phys. (N.Y.) **250**, 1 (1996).
[50] L.-H. Chan, Phys. Rev. Lett. **57**, 1199 (1986).
[51] J. Caro and L. L. Salcedo, Phys. Lett. B **309**, 359 (1993).
[52] L. L. Salcedo, Eur. Phys. J. C **20**, 147 (2001).
[53] L. L. Salcedo, Nucl. Phys. **B549**, 98 (1999).
[54] C. García-Recio and L. L. Salcedo, Phys. Rev. D **63**, 045016 (2001).
[55] V. P. Gusynin and V. A. Kushnir, Classical Quantum Gravity **8**, 279 (1991).
[56] L. L. Salcedo, Eur. Phys. J. C **20**, 161 (2001).