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Onset of inflation in loop quantum cosmology

Cristiano Germani
SISSA and INFN, via Beirut 4, 34014 Trieste, Italy

William Nelson and Mairi Sakellariadou
Department of Physics, King's College London, London WC2R 2LS, United Kingdom
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Using a Liouville measure, similar to the one proposed recently by Gibbons and Turok, we investigate the probability that single-field inflation with a polynomial potential can last long enough to solve the shortcomings of the standard hot big bang model, within the semiclassical regime of loop quantum cosmology. We conclude that, for such a class of inflationary models and for natural values of the loop quantum cosmology parameters, a successful inflationary scenario is highly improbable.

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I. INTRODUCTION

Cosmological inflation [1] is, at present, the most promising candidate to solve the shortcomings of the standard hot big bang model, although other mechanisms have been proposed [2–4]. Inflation essentially consists of a phase of accelerated expansion which took place at a very high energy scale. One of the appealing features of inflation is that it is deeply rooted in the basic principles of general relativity and field theory. In addition, when the principles of quantum mechanics are taken into account, inflation provides a simple explanation for the origin of the large scale structures and the associated temperature anisotropies in the cosmic microwave background radiation.

Despite its success, inflation is still a paradigm in search of a model and its strength is based on the assumption that its onset is generically independent of the initial conditions. However, even when the issue of the onset of inflation was addressed [5,6], no robust conclusions could be drawn as a quantum theory of gravity was missing. Recently, it has been argued [7] that the probability of having N (or more) e-foldings of inflation within single-field, slow-roll inflationary models is suppressed by an order of $\exp(-3N)$. However, in finding this result the authors have used a classical theory even at energy scales for which the quantum effects can no longer be neglected. Moreover, as we shall discuss, the analysis of Ref. [7] is not always valid. In what follows, we estimate the probability to obtain a sufficiently long inflationary era, in the context where not only general relativistic but also quantum effects are taken into account. More precisely, we study the probability of having a sufficiently long inflationary era in the context of loop quantum cosmology.

We organize the rest of the paper as follows. In Sec. II, we briefly discuss some elements of loop quantum cosmology. In Sec. III, we discuss inflation within the loop quantum cosmology framework. In Sec. IV, we study the probability of having successful inflation within this context. We round up with our conclusions in Sec. V.

II. LOOP QUANTUM COSMOLOGY

Loop quantum gravity [8,9] is, at present, the most developed approach to a background independent and nonperturbative quantization of general relativity, which can deal with the extreme conditions realized at classical singularities. The full theory is still not completely understood, and in a number of cases not even the continuum limit of space-time can be explicitly found. Nevertheless, by introducing symmetries, one may resolve the theory at a nonperturbative level. More precisely, applying loop quantum gravity to homogeneous and isotropic cosmologies, the theory becomes analytically tractable and loop quantum cosmology can be studied [10]. It is worth noting that such mini-superspace models may not encompass all features of the full inhomogeneous theory; however, it is reasonable to expect that they have, at least qualitatively, the correct behavior [11].

In loop quantum cosmology, the evolution of the universe is divided into three distinct phases [12], depending on the value of the scales probed by the universe. At first, very close to the Planck scale, the concept of space-time has no meaning, full quantum gravity is the correct framework, and the universe is in a discrete quantum phase. Applying loop quantum cosmology during this phase, one gets a finite bounded spectrum for eigenvalues of inverse powers of the three-volume density, which we shall call "the geometrical density." As the volume of the universe increases with time, the universe enters a semiclassical phase.

For length scales above $L_{\rm Pl} \equiv \sqrt{\gamma} l_{\rm Pl}$ ($\gamma \approx 0.2735$ is the Barbero-Immirzi parameter and $l_{\rm Pl}$ denotes the Planck length, with $l_{\rm Pl}^2 = 8\pi G$[1]), the space-time can be approximated by a continuous manifold and the equations of motion take a continuous form, which differs from the classical behavior due to the nonperturbative quantization effects. This intermediate phase is characterized by a sec-

[1] Units $\hbar = c = 1$ are used in this paper.

ond length scale $L_\star$, with $L_\star \equiv \sqrt{(\gamma j \mu_0)/3} l_{\rm Pl}$, which determines the size below which the geometrical density is significantly different from its classical form. For length scales below $L_\star$, quantum corrections can no longer be neglected. The half-integer j labels the ambiguity in choosing the representation in which the matter part of the Hamiltonian constraint for a scalar field is quantized. The length parameter μ_0, related to the underlying discrete structure, is the scale of the finite fiducial cell that spatial integration is restricted to, so as to remove the divergences that occur in noncompact topologies [13]. As it was shown in Ref. [11], one can use an arbitrary value of μ_0. However, one should keep in mind that the same value should be adopted for both the Hamiltonian constraint and the inverse volume operator. In what follows we set, for simplicity, $\mu_0 = 1$; as $\mu_0 \sim \mathcal{O}(1)$, different values of μ_0 do not sensibly modify our conclusions. For $j > 3$, the two scales $L_{\rm Pl}$ and $L_\star$ overlap and space-time can be considered as continuous. The intermediate phase is the most important one regarding the phenomenological consequences of loop quantum cosmology as it may lead to distinct signatures [14]. At later times, and therefore larger scales, the universe enters the full classical phase and standard cosmology becomes valid. The main feature of loop quantum cosmology is the resolution of the cosmological singularity. Indeed, one can show that, upon quantization, the operator associated with the inverse of the three-volume never diverges.

The metric of a Friedmann-Lemaître-Robertson-Walker (FLRW) space-time reads

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \\ &= a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j], \end{aligned} \tag{1}$$

where $t(\eta)$ is the cosmological (conformal) time (with $dt = a d\eta$), a the scale factor, and δ_{ij} the Krönecker symbol. The geometric density is therefore a^{-3}. Promoting a^{-3} and its inverse (a^3) at the level of operators, we have that, upon quantization,[2]

$$\langle \hat{a}^3 \rangle = a^3, \qquad \langle \hat{a}^{-3} \rangle = d_{j,l}(a), \tag{2}$$

where the modified density, $d_{j,l}(a)$, in the continuum limit of loop quantum cosmology is given by the following approximated expression:

$$d_{j,l}(a) = D_l(q) a^{-3} \quad \text{with} \quad q = a^2/a_\star^2. \tag{3}$$

The parameter l determines the behavior of the effective geometrical density on small scales with respect to $L_\star$ and $l \in [0, 1]$. However, some values of l are preferred and only a discrete sequence is used, $l_k = 1 - (2k)^{-1}$ with $k \in N$. The function $D_l(q)$ can be written as [12]

[2] We define $\hat{A}$ to be the operator associated with the function A.

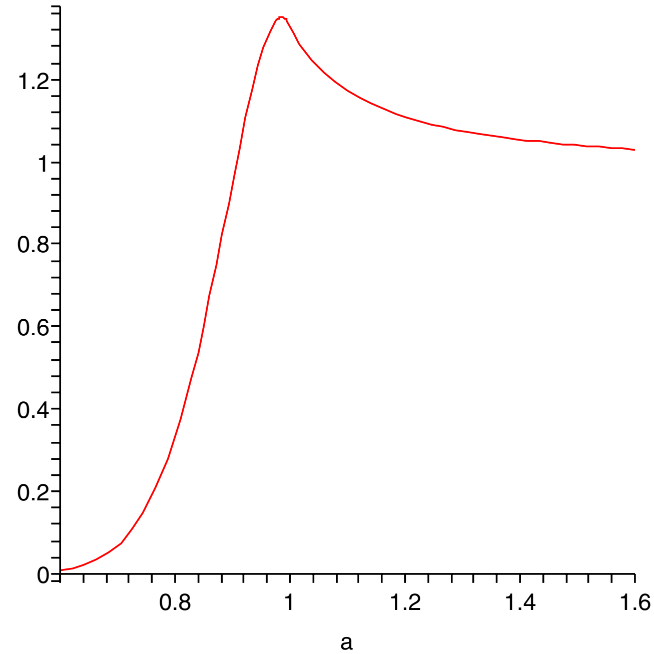

FIG. 1 (color online). $D(q)$ is plotted as a function of a, with $a_\star = 1$ and $l = 0.75$. For $a \gg a_\star$, $D \to 1$, giving us the classical result.

$$D_l(q) = \left\{ \frac{3}{2l} q^{1-l} \left[\frac{(q+1)^{l+2} - |q-1|^{l+2}}{l+2} - \frac{q}{1+l} \times ((q+1)^{l+1} - \mathrm{sgn}(q-1)|q-1|^{l+1}) \right] \right\}^{3/(2-2l)}. \tag{4}$$

For all allowed values of the parameters l, j, the effective geometrical density behaves as $d_{j,l}(a) \to a^{-3}$, which is the classical behavior, as $a \gg a_\star$. Around $a_\star$ the effective geometrical density $d_{j,l}(a)$ becomes maximal (see Fig. 1). For $a \ll a_\star$ the density approaches zero,

$$d_{j,l}(a) \sim \left(\frac{3}{1+l}\right)^{3/(2-2l)} \left(\frac{a}{a_\star}\right)^{3(2-l)/(1-l)} a^{-3}, \tag{5}$$

resolving the singularity present in the classical theory as $a \to 0$ (see Fig. 1). The parameters j and l can only be weakly restricted by considerations of the discrete structure of the theory. These parameters can, in principle, be fixed by knowing the full loop quantum gravity; we will not consider them here as ambiguities.[3] However, as we shall

[3] These two parameters capture the typical properties of functions such as $D(q)$, which is the position of the peak (j) and the power law increase at small q (l). As we mentioned previously, other ambiguities in quantitative details exist, which at the current stage are not relevant for an effective analysis. Precise functions would follow by relating isotropic models to the inhomogeneous situation [15,16]. We will briefly discuss this issue later.

see, there is a more dangerous ambiguity coming from the fact that $\langle \hat{a}^3 \rangle \neq 1/\langle \hat{a}^{-3} \rangle$.

Dynamics and ambiguities

Let us first briefly, and in a rather schematic way, discuss the issue of quantization ambiguities. Dynamics are controlled by the Hamiltonian constraint, which classically gives the Friedmann equation. The Hamiltonian of the whole system, gravity plus matter, reads [12]

$$\mathcal{H} \equiv -3\dot{a}^2 a + 8\pi G \mathcal{H}_{\rm m}, \tag{6}$$

where the first term in the right-hand side is the gravity part and $\mathcal{H}_{\rm m}$ is the matter Hamiltonian. The equations of motion are satisfied requiring quantum mechanically that $\hat{\mathcal{H}}|\Psi\rangle = 0$, where $|\Psi\rangle$ is the *wave function* of the universe and $\hat{\mathcal{H}}$ is the promotion to an operator of the classical Hamiltonian. Semiclassically, this implies $\langle\Psi|\hat{\mathcal{H}}|\Psi\rangle = \langle\mathcal{H}\rangle = 0$. However, as we shall see, we have ambiguities on deciding which composite operator of the geometrical density (such as the Hamiltonian) is the correct one to be used.

In principle, instead of the Hamiltonian, one may consider the classically equivalent operator

$$\hat{\mathcal{Q}} = -3\hat{a}\hat{\dot{a}}^2 + 8\pi G \mathcal{H}_{\rm m}\hat{a}^n\hat{a}^{-n}. \tag{7}$$

Requiring that classical symmetries, such as diffeomorphism invariance, are not broken at the quantum level, one obtains that $\hat{\mathcal{Q}}$ should, as well as the Hamiltonian, define diffeomorphism invariance, implying

$$\hat{\mathcal{Q}}|\Phi\rangle = 0; \tag{8}$$

$|\Phi\rangle$ is a new state associated with $\hat{\mathcal{Q}}$ [12]. In this case however, the semiclassical equations $\langle\hat{\mathcal{Q}}\rangle = 0$ differ from $\langle\hat{\mathcal{H}}\rangle = 0$ during the quantum regime. Although the difference is at the quantum level, the trajectories defined by $\langle\hat{\mathcal{Q}}\rangle = 0$ and $\langle\hat{\mathcal{H}}\rangle = 0$ coincide at the classical limit (as $a \to \infty$). Therefore, as the two evolutions are indistinguishable at the classical level, it is impossible to decide whether or not the state $|\Psi\rangle$ is more fundamental than the $|\Phi\rangle$ one. This is a dangerous ambiguity in loop quantum cosmology as it appears whenever, at the quantum level, $\langle \hat{a}^3 \rangle \neq 1/\langle \hat{a}^{-3} \rangle$.

An extra ambiguity appears in the choice of the representation in which the gravity part of the total Hamiltonian is quantized. This ambiguity is defined by a similar parameter used to label the representation for the quantization of the matter Hamiltonian, we will call this parameter $j_{\rm G}$ [13]. As in the matter part, one can in fact define a new scale $L_{\rm G} = \sqrt{\gamma j_{\rm G}\mu_0/3}\, l_{\rm Pl}$ above which quantum corrections are negligible. The ambiguity arises as, in principle, the representations, in which the matter and the gravitational Hamiltonians are quantized, can be different. However, as we shall discuss in the following, our results are not sensible on this ambiguity.

We address the genericity of inflation in this setup, and we investigate whether one can constrain the parameter space by requiring a sufficiently long inflationary era to be as likely as possible during the continuum limit of loop quantum cosmology.

III. INFLATION WITHIN LOOP QUANTUM COSMOLOGY

During the inflationary era the FLRW scale factor $a(t)$ underwent an accelerated expansion [17], $\mathrm{d}^2a/\mathrm{d}t^2 > 0$. Equivalently, during inflation the universe was dominated by a fluid with negative pressure which is usually identified as a scalar field. The scalar field action is

$$S_\phi = \int \mathrm{d}^4x \mathcal{L}_\phi = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}[(\partial\phi)^2 - 2V(\phi)], \tag{9}$$

where the metric is taken to be of the form given in Eq. (1). Inflation is successful in solving the problems which plague the standard big bang model, provided the slow-roll conditions

$$\left(\frac{\partial V/\partial\phi}{V}\right)^2 \ll 1 \quad \text{and} \quad \left|\frac{\partial^2 V/\partial\phi^2}{V}\right| \ll 1 \tag{10}$$

are satisfied for a period of about 60 e-folds; i.e. the final value of the scale factor, $a_{\rm s}$, must be $a_{\rm s} \approx \exp(60)a_{\rm i}$, where $a_{\rm i}$ stands for the value of the scale factor at the beginning of inflation. We define the number of e-foldings N to be given from $N = \ln(a_{\rm s}/a_{\rm i})$.

Consider a single-field inflationary model, with an inflaton field ϕ having a potential $V(\phi)$. The Hamiltonian for ϕ obtained from the action S_ϕ, Eq. (9), reads

$$\mathcal{H}_\phi = \tfrac{1}{2}a^{-3}P_\phi^2 + a^3V(\phi), \tag{11}$$

where the momentum P_ϕ is defined as $P_\phi = -\partial\mathcal{L}_\phi/\partial\dot{\phi}$; an over-dot defines a derivative with respect to the cosmic time. We then promote the scalar field Hamiltonian to an operator; thus the full Hamiltonian for inflation within loop quantum cosmology reads

$$\hat{\mathcal{H}} \equiv -3a\dot{a}^2 + 8\pi G[\tfrac{1}{2}a^{-3}P_\phi^2 + a^3V(\phi)]. \tag{12}$$

As introduced before, we can define the new, classically equivalent operator as

$$\hat{\mathcal{Q}} \equiv -3a\dot{a}^2 + 8\pi G[\tfrac{1}{2}a^{-3(n+1)}a^{3n}P_\phi^2 + a^{-3m}a^{3(m+1)}V(\phi)], \tag{13}$$

where m, n are positive constants. Upon quantization, $\langle\hat{\mathcal{Q}}\rangle = 0$, we obtain, for $V(\phi) \ll l_{\rm Pl}^{-4}$, in the slow-roll region, the semiclassical equation [10,18]

$$H^2 = \frac{8\pi G S(q_G)}{3}\left[\frac{1}{2}D_l^{-(n+1)}\dot{\phi}^2 + D_l^m V(\phi)\right], \quad (14)$$

where the Hubble parameter H is defined as $H \equiv \dot{a}/a$. The function $S(q_G)$ in Eq. (14),

$$S(q_G) = \frac{4}{\sqrt{q G}}\left\{\frac{1}{10}[(q_G+1)^{5/2} + \mathrm{sgn}(q_G - 1)|q_G - 1|^{5/2}] - \frac{1}{35}[(q_G+1)^{7/2} - |q_G - 1|^{7/2}]\right\}, \quad (15)$$

accounts for the quantization of the gravity part of the Hamiltonian using a $j_G \neq 1/2$ representation [18]; the case $j_G = 1/2$ represents the irreducible representation. We have defined as before $q_G = a^2/a_G^2$. For $q_G > 1$, the function $S(q_G)$ is $S(q_G) \approx 1$, while for small volume, $S(q_G) \approx (6/5)\sqrt{q_G}$ (see Fig. 2). As discussed previously, it is not necessary to use the same j_G representation to quantize both the matter and gravity parts of the Hamiltonian constraint. To simplify the following calculations we set $j_G = 1/2$ for the quantization of the gravity part, which implies $S(q_G) = 1$. We discuss the effects of a more general choice of $S(q_G)$ in Sec. IV B 3.

Finally, for $V(\Phi) \sim l_{\mathrm{Pl}}^{-4}$ an extra term producing a bouncing in Eq. (14) appears [19]. However, our conclusions apply only if $V(\phi) \ll l_{\mathrm{Pl}}^{-4}$.

Using Eq. (14) one can write an effective Lagrangian, which leads to the following conservation equation for the scalar field ϕ:

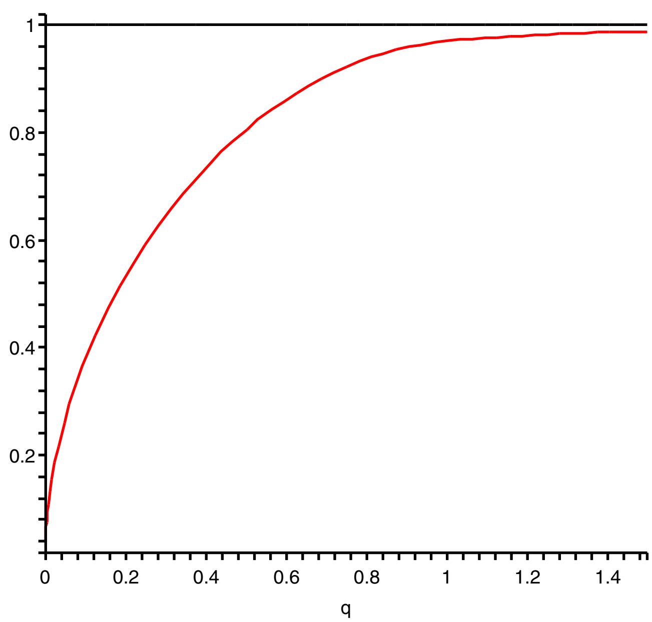

FIG. 2 (color online). $S(q_G)$ is plotted as a function of a, with $a_G = 1$. Notice that $S(q_G) < 1$ for all a, and that for $a \gg a_G$, $S \to 1$, giving us the $j = 1/2$ result.

$$\ddot{\phi} + \left[3H - (1+n)\frac{\dot{D}_l}{D_l}\right]\dot{\phi} + D_l^{m+n+1}V'(\phi) = 0, \quad (16)$$

where $V' \equiv \partial V/\partial \phi$. We see in Eqs. (14) and (16) that there is an ambiguity in choosing the parameters m and n.

IV. THE PROBABILITY TO GET SUCCESSFUL INFLATION IN LOOP QUANTUM COSMOLOGY

It has been recently shown [7] that it is possible to define a canonical measure in cosmology. More precisely, it has been shown [7] that the volume of phase space of possible orbits, for certain inflationary models, is finite if a coarse graining cutoff is introduced. The authors of Ref. [7] argued that two cosmologies cannot be experimentally distinguished if they differ by a small amount of spatial curvature, and this removed the divergence present in the phase space of the relativistic trajectories found in Ref. [20]. Considering a finite phase-space volume of possible orbits, they could calculate the fraction of the whole phase space occupied by inflationary trajectories. In this way they could define the probability of having inflationary initial conditions in the framework of classical general relativity. However, the authors of Ref. [7] allowed the possible trajectories to reach the Planck scale, where the classical general relativistic Hamiltonian should not be used.

Here, we modify the proposal of Ref. [7], defining a quantum gravity cutoff instead of an observational one, namely, we calculate the volume of the phase space of solutions only in the continuum limit of loop quantum cosmology, i.e. in the space where $H^{-1} \gg \sqrt{\gamma} l_{\mathrm{Pl}}$ (which we later refer to as the "quantum gravity cutoff"). We show that the volume is again finite for the same inflationary models implicitly used in Ref. [7].

A. Measure: definition

The canonical cosmological measure of Ref. [21] is given as follows: As with any phase space we have a symplectic form

$$\Omega = \sum_{i=1}^{k} \mathrm{d}P_i \wedge \mathrm{d}Q^i, \quad (17)$$

where Q_i and P_i are the dynamical degrees of freedom and their conjugate momenta, respectively. The kth power of Ω gives the volume element of the space. The Hamiltonian constraint restricts the space of trajectories to lie on a $(2k-1)$-dimensional subspace M of the full phase space, referred to as the multiverse. It can be shown [21] that M also contains a closed symplectic form $\omega = \sum_{i=1}^{k-1} \mathrm{d}P_i \wedge \mathrm{d}Q^i$, which is related to Ω via

$$\Omega = \omega + \mathrm{d}\mathcal{H} \wedge \mathrm{d}t \Rightarrow \omega = \Omega|_{\mathcal{H}=0}. \quad (18)$$

In particular, this construction can be easily extended to our case by replacing $\mathcal{H}$ with the effective Hamiltonian $\langle \hat{\mathcal{Q}} \rangle$ of the system.

In FLRW universes containing a scalar field there are only two canonical variables (a, ϕ), so we set $k = 2$. Given this symplectic form it is possible to define a divergenceless field

$$B_a \equiv \tfrac{1}{2}\epsilon_{abc}\omega_{bc}; \tag{19}$$

ϵ_{abc} is totally antisymmetric with $\epsilon_{123} = 1$ and $a, b, c = 1, 2, 3$. Each orbit in the phase space is associated with a *line of force* of $\mathbf{B}$, i.e. $\mathbf{B}$ defines the flow of trajectories across surfaces in the phase space. We thus define a measure as

$$\mathcal{N} = \int \mathbf{B}\cdot \mathrm{d}\mathbf{S}, \tag{20}$$

where $\mathbf{S}$ is an open surface where, to ensure that there is no over-counting, the orbits cross only once. We schematically show this method in Fig. 3. Since $\mathbf{B}$ is divergenceless we can define $\mathrm{d}\mathbf{A} = \mathbf{B}$ and, in the case of a nondisconnected surface $\mathbf{S}$, using Stoke's theorem

$$\mathcal{N} = \oint \mathbf{A}\cdot \mathrm{d}\boldsymbol{l}, \tag{21}$$

where $\boldsymbol{l} = \partial\mathbf{S}$ is the boundary of $\mathbf{S}$. The quantity $\mathcal{N}$ is the canonical measure of all trajectories crossing topologically equivalent surfaces which are bounded by $\partial\mathbf{S}$.

The reader should keep in mind that, since Eq. (17) defines a flat metric on the phase space, each trajectory has the same weight; hence we have a measure on the *number* of trajectories, and not the volume they occupy. This is indeed a crucial point for the estimation of the probability to have successful inflation, since such a solution is an attractor, meaning that the volume occupied by

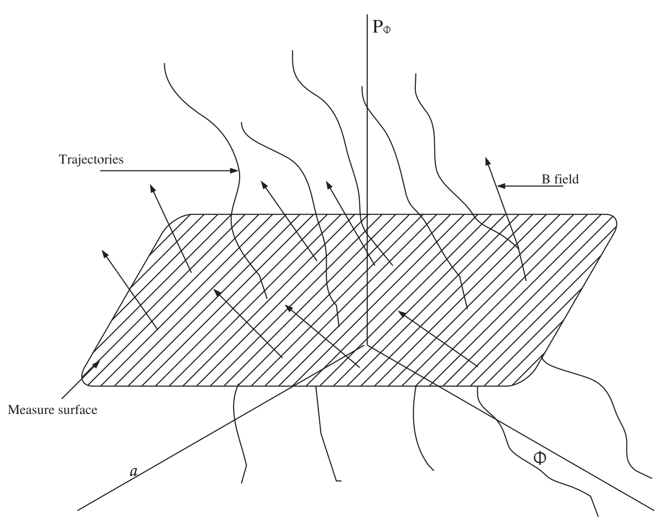

FIG. 3. The probability measure is defined by integrating the $\mathbf{B}$ field over a constant surface in the 3-dimensional phase space produced by using the Hamiltonian constraint to eliminate one of the dynamical variables.

inflationary trajectories decreases, as the attractor solution is approached. In conclusion, we are confident in our estimated probability of successful inflation, since we are just counting the *number* of trajectories and not the phase-space volume they enclose. The measure we use is certainly not the most general measure of the gravitational phase-space volume, but just the simplest (a uniform distribution) one. The reader should then keep in mind that different distributions for the phase-space trajectories may be adopted (see for example [22]).

B. Estimation of the probability

An estimation of the probability of a set of trajectories $\mathcal{C}$ in the phase space of all possible trajectories of a Hamiltonian system is given by the ratio of the measure of $\mathcal{C}$ to the measure of all possible trajectories and not the phase-space volume they enclose. For this ratio to be well defined we require that the trajectories do not cross, as this would lead to a time dependent measure of $\mathcal{C}$. The time reversibility of the system ensures that such crossing does not take place within a finite time. To define the ratio, we also require the measure of the entire phase space to be finite. It is well known that in the classical case this is not true [20], unless one introduces a coarse graining cutoff [7]. In quantum loop cosmology this classical divergence is removed since we are restricted to scales above $L_{\rm Pl}$. However, as we will see in the next section, there is the possibility of a further divergence associated with the form of the potential. To ensure that this divergence is not present we must restrict our attention to a specific class of potentials, a limitation that is also present in the classical theory of [7].

1. The volume of the phase space

We now turn our attention to the calculation of the total volume using Eqs. (14) and (16). The momentum associated with the scale factor and the scalar field ϕ is

$$P_a = -6a^2H, \qquad P_\phi = a^3D_l^{-(n+1)}\dot{\phi}, \tag{22}$$

or, in terms of $q = a^2/a_\star^2$ and using Eq. (14),

$$\begin{aligned} P_q &= -6a_\star^2 qH, \\ P_\phi &= a_\star^3 q^{3/2}D_l^{-(n+1)}\dot{\phi} \\ &= D_l^{-(n+1)/2}a_\star^3 q^{3/2}\sqrt{\frac{3H^2}{4\pi G} - 2D_l^m V}, \end{aligned} \tag{23}$$

leading to the constraint

$$H^2 > \tfrac{1}{3}D_l^m V. \tag{24}$$

We now calculate the symplectic form

$$\omega = \mathrm{d}P_\phi \wedge \mathrm{d}\phi + \mathrm{d}P_q \wedge \mathrm{d}q, \tag{25}$$

which gives

$$B_\phi = -6a_\star^2 q, \tag{26}$$

$$B_H = \frac{1}{2}a_\star^3 q^{1/2}\sqrt{\frac{3H^2}{4\pi G} - 2D_l^m V} D_l^{-(n+1)/2}\left[3 - (n+1)\frac{a_s}{D_l}\times\left(\frac{\partial D_l}{\partial q}\right)\right] - \frac{m a_\star^3 q^{3/2}}{\sqrt{\frac{3H^2}{4\pi G} - 2D_l^m V}} D_l^{(2m-n-3)/2} V\left(\frac{\partial D_l}{\partial q}\right), \tag{27}$$

$$B_q = -\frac{3Ha_\star^3 q^{3/2} D_l^{-(n+1)/2}}{4\pi G\sqrt{\frac{3H^2}{4\pi G} - 2D_l^m V}}. \tag{28}$$

An associated vector potential, $\mathbf{A} = P_i \mathrm{d}Q^i$, reads

$$\mathbf{A} = \left[a_\star^3 q^{3/2}\sqrt{\frac{3H^2}{4\pi G} - 2D_l^m V} D_l^{-(n+1)/2}, 0, -6a_\star^2 qH\right]. \tag{29}$$

To calculate the measure, as described in the previous section, we need to define a surface that is cut only once by the trajectories in the multiverse. It is convenient to use the surface defined by $q = q_s$, where q_s is a constant. We consider expanding universes; thus from $\langle\hat{\mathcal{Q}}\rangle = 0$ we get that $\mathrm{d}a/\mathrm{d}t$, and hence $\mathrm{d}q/\mathrm{d}t$ is monotonic and positive if and only if $V > 0$, which is guaranteed by the dominant energy condition.[4] Integrating $\mathbf{B}$ over a constant $q = q_s$ surface, we obtain

$$\mathcal{N} = -\frac{3}{4\pi G}\iint \frac{Ha_\star^3 q_s^{3/2} D_s^{-(n+1)/2}}{\sqrt{\frac{3H^2}{4\pi G} - 2D_s^m V}}\mathrm{d}H\mathrm{d}\phi, \tag{30}$$

where (from now on we drop the l label) we use the notation $f(a_s) = f_s$ so that $D_s \equiv D(q_s)$.

We are now able to integrate Eq. (30). At this point we introduce the quantum gravity cutoff $H^{-1} \gg \sqrt{\gamma} l_{\rm Pl}$. Considering the physical limit[5] $H \gg \sqrt{\frac{8}{3}\pi G D^m V}$, we also obtain in turn that $V \ll l_{\rm Pl}^{-4}$ which avoids, as anticipated, the quantum gravity bouncing region. We now may perform the integral over H to obtain

$$\mathcal{N} = -a_\star^3 q_s^{3/2} D_s^{-(n+1)/2}\sum_k \int_{\phi_i^k}^{\phi_f^k}\sqrt{\frac{3}{4\pi\gamma l_{\rm Pl}^4} - 2D_s^m V(\phi)}\mathrm{d}\phi, \tag{31}$$

where $[\phi_i^k, \phi_f^k]$ represent the allowed (possibly disconnected) ranges for ϕ such that $\mathcal{N}$ is real. We note that the integral in Eq. (31) is by no means always convergent, as it was assumed in Ref. [7], in the $D \to 1$ limit for potentials with only one minima. Its convergence indeed depends on the choice of $V(\phi)$. However, a large class of potentials makes it convergent, for example, potentials with only one minima but diverging in the large ϕ limit. These potentials are phenomenologically very important. First, the requirement of having a minimum makes it possible to have only one specific vacuum for the scalar field ϕ, from which ordinary matter may be produced. Moreover, in this class of potentials belongs the case of a massive scalar field with $V(\phi) = \frac{1}{2}\mu^2\phi^2$, where μ is a constant mass, which seems to be the favorite model [23] to match the WMAP data. In Sec. IV B 3 we will calculate explicitly the probability to have successful inflation for single-field polynomial inflation with a potential of the form $V(\phi) = \frac{\mu^4}{2\alpha!}(\frac{\phi}{\mu})^{2\alpha}$; the integer constant α is $\alpha \geq 1$ and the self-coupling constant μ has dimensions of mass (the $\alpha = 1$ case reduces to the scalar field mass). For the class of potentials for which the integral in Eq. (31) converges, the range of allowed ϕ is given by the roots of the equation

$$\frac{3}{4\pi\gamma l_{\rm Pl}^4} - 2D_s^m V(\phi) = 0, \tag{32}$$

implying

$$\mathcal{N} = \frac{a_\star^3 q_s^{3/2} D_s^{-(n+1)/2}}{l_{Pl}^3}\left[\frac{3}{4\pi\gamma}\right]^{(\alpha+1)/2\alpha}\left[\frac{2\alpha!}{2D_s^m}\right]^{1/2\alpha} \times\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1}{2\alpha})}{\alpha\Gamma(\frac{3\alpha+1}{2\alpha})}(l_{\rm Pl}\mu)^{(\alpha-2)/\alpha}. \tag{33}$$

This volume is clearly finite, as D is a finite function of q that is bounded below by the cutoff $L_{\rm Pl}$.

2. The probability

We now need to calculate the volume of the phase space that contains inflationary trajectories. In other words, following exactly the same calculations as before, we will estimate the measure

$$\mathcal{M} = -\frac{3}{4\pi G}\iint \frac{Ha_\star^3 q_s^{3/2} D_s^{-(n+1)/2}}{\sqrt{\frac{3H^2}{4\pi G} - 2D_s^m V}}\mathrm{d}H\mathrm{d}\phi\Bigg|_{\rm inflation}. \tag{34}$$

Using Stoke's theorem for the path $H = H_s = $ const, we get

$$\mathcal{M} = -\oint |P_\phi|\mathrm{d}\phi\Bigg|_{\rm inflation}. \tag{35}$$

Note that the above integral is positive as inflation runs from higher to lower values of the scalar field ϕ. From Eq. (14) and using $F^2 = H^2 D^{-m}$, we obtain

$$F^2 = 8\pi G[\tfrac{1}{6}D^{-(n+m+1)}\dot{\phi}^2 + \tfrac{1}{3}V], \tag{36}$$

[4] The dominant energy condition reads $\rho \geq |p_i|$. In the case of a scalar field ϕ, one has $\rho \sim$ K.E. $+ V(\phi)$ and $p_i \sim$ K.E. $- V(\phi)$, where K.E. denotes the kinetic term. Thus, even with the corrections to the $1/a^3$ factor in the kinetic energy terms, the dominant energy condition ensures that $V(\phi) > 0$.

[5] For an expanding universe $H > 0$.

or

$$\dot{\phi}^2 = \frac{1}{4\pi G} 3D^{n+m+1}F^2 - 2D^{n+m+1}V. \tag{37}$$

Using Eq. (16), after lengthy but straightforward calculations, we obtain

$$\dot{\phi} = -\frac{D^{n+(m/2)+1}(\frac{dF}{d\phi})}{4\pi G[1-(n-m+1)\frac{q}{3D}(\frac{\partial D}{\partial q})]}, \tag{38}$$

for

$$1-(n-m+1)\frac{q}{3D}\left(\frac{\partial D}{\partial q}\right) \neq 0.$$

Equation (38) implies inflation should happen only in the $(1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}) > 0$ region for an expanding universe ($H > 0$) and for graceful exit from inflation ($\dot{\phi} < 0$). One can easily see this by considering that during slow roll $dF/d\phi \sim V'/\sqrt{V} > 0$, for expanding cosmologies. Substituting Eq. (38) into Eq. (35), we get

$$\begin{aligned}\mathcal{M} &= -\frac{a_s^3 D_s^{m/2}}{4\pi G|1-(n-m+1)\frac{q_s}{3D_s}(\frac{\partial D}{\partial q}|_s)|} \\ &\quad\times \oint \mathrm{d}\phi \frac{\mathrm{d}F}{\mathrm{d}\phi}\bigg|_{\text{inflation}} \\ &= -\frac{a_s^3}{4\pi G|1-(n-m+1)\frac{q_s}{3D_s}(\frac{\partial D}{\partial q}|_s)|}\delta H|_{H_s,q_s},\end{aligned} \tag{39}$$

where $\delta H|_{H_s,q_s}$ measures the space of inflationary trajectories cutting the $H = H_s$ surface at $q = q_s$. To calculate $\delta H|_{H_s,q_s}$, we substitute Eq. (38) into Eq. (36) and evaluate on $q = q_s$. Introducing the new positive definite variable

$$\epsilon = \frac{D^{(n+1)/2}}{|1-(n-m+1)\frac{q}{3D}(\frac{\partial D}{\partial q})|}F \equiv \frac{F}{A}, \tag{40}$$

we obtain

$$A_s^2\epsilon_s^2 = \frac{1}{12\pi G}\left(\frac{\mathrm{d}\epsilon_s}{\mathrm{d}\phi}\right)^2 + \frac{8\pi G}{3}V, \tag{41}$$

where ϵ_s and A_s are ϵ and A evaluated on $a = a_s$. The slow-roll condition in loop quantum cosmology is equivalent to $D^{-n-m-1}\dot{\phi}^2 \ll 2V$, which implies

$$\frac{1}{3}\frac{(\partial V/\partial\phi)^2}{V^2} \ll 8\pi G A^2. \tag{42}$$

Using the variable ϵ defined above, the slow-roll condition can equivalently be written as $(\mathrm{d}\epsilon_s/\mathrm{d}\phi)^2 \ll (96\pi G/3)V$.

Let us now consider the perfect slow-roll solution (ϵ_{sr}) such that

$$A_s^2\epsilon_{sr}^2 = \frac{8\pi G}{3}V. \tag{43}$$

The volume of all inflationary trajectories will be an expansion on small values of ϵ. We will therefore study Eq. (41) using $\epsilon_s \longrightarrow \epsilon_{sr} + \delta\epsilon$, where $\delta\epsilon$ is a small perturbation in ϵ_{sr}. This gives

$$\frac{\mathrm{d}\delta\epsilon}{\mathrm{d}\phi} = \frac{3A_s^2\epsilon_s\delta\epsilon}{\sqrt{\frac{3A_s^2\epsilon_s^2}{4\pi G} - 2V}}. \tag{44}$$

If we now define the function

$$N_s = \int_a^{a_s}\frac{\mathrm{d}\tilde{a}}{\tilde{a}}, \tag{45}$$

we have

$$\frac{\mathrm{d}N_s}{\mathrm{d}\phi} = \frac{H_s}{\dot{\phi}_s} = \frac{\epsilon_s A_s D_s^{-(n+1)/2}}{\sqrt{\frac{3A_s^2\epsilon_s^2}{4\pi G} - 2V}}, \tag{46}$$

so that

$$\frac{\mathrm{d}\delta\epsilon}{\mathrm{d}N_s} = 3\left|1-(n-m+1)\frac{q_s}{3D_s}\left(\frac{\partial D}{\partial q}\bigg|_s\right)\right|\delta\epsilon. \tag{47}$$

We finally have

$$\delta\epsilon = C\exp\left(3\left|1-(n-m+1)\frac{q_s}{3D_s}\left(\frac{\partial D}{\partial q}\bigg|_s\right)\right|N_s\right), \tag{48}$$

where C is a constant. Taking a_s to be the scale at the end of inflation and measuring N_s from a_s to the beginning of inflation, we get

$$\delta\epsilon_i = \delta\epsilon_f\exp\left(3\left|1-(n-m+1)\frac{q_s}{3D_s}\left(\frac{\partial D}{\partial q}\bigg|_s\right)\right|N\right), \tag{49}$$

where $\delta\epsilon_f$ and $\delta\epsilon_i$ are the perturbation evaluated at the end and beginning of inflation, respectively, and N stands here for the total number of e-foldings during the slow roll.

By iterating Eq. (41) it is easy to see that

$$\epsilon_s \approx \frac{1}{A_s}\sqrt{\frac{8\pi G V(\phi)}{3}}\left[1 + \frac{1}{96A_s^2\pi G}\left(\frac{1}{V(\phi)}\frac{\partial V(\phi)}{\partial\phi}\right)^2 + \ldots\right]. \tag{50}$$

Using this expansion we can write

$$\delta\epsilon_i \approx \frac{1}{12A_s^3}\sqrt{\frac{V(\phi_i)}{24\pi G}}\left(\frac{1}{V(\phi_i)}\frac{\partial V(\phi)}{\partial\phi}\bigg|_i\right)^2. \tag{51}$$

From Eq. (49) we have

$$\delta\epsilon_f = \delta\epsilon_i\exp\left(-3|1-(n-m+1)\frac{q_s}{3D_s}\left(\frac{\partial D}{\partial q}\bigg|_s\right)\right|N\bigg). \tag{52}$$

Since we perturbed ϵ on the constant $a = a_s$ surface, we have that

$$\delta\epsilon_{\rm f} = \frac{D_{\rm s}^{(n-m+1)/2}}{|1-(n-m+1)\frac{q_{\rm s}}{3D_{\rm s}}(\frac{\partial D}{\partial q}|_{\rm s})|}\delta H|_{H_{\rm s},a_{\rm s}}. \quad (53)$$

Putting these together and substituting into Eq. (39) gives the measure of the trajectories that inflate,

$$\mathcal{M} = \frac{a_\star^3 q_{\rm s}^{3/2}}{4\pi G} D_{\rm s}^{-(n-m+1)/2}\delta\epsilon_{\rm i} \times \exp\left(-3\left|1-(n-m+1)\frac{q_{\rm s}}{3D_{\rm s}}\left(\frac{\partial D}{\partial q}\Big|_{\rm s}\right)\right|N\right). \quad (54)$$

Thus, the probability of getting N or more e-foldings is

$$\mathcal{P}(N) = \frac{1}{\mathcal{N}}\frac{a_\star^3 q_{\rm s}^{3/2} D_{\rm s}^{-(n-m+1)/2}}{4\pi G}\delta\epsilon_{\rm i} \times \exp\left(-3\left|1-(n-m+1)\frac{q_{\rm s}}{3D}\left(\frac{\partial D}{\partial q}\Big|_{\rm s}\right)\right|N\right). \quad (55)$$

3. Probability for polynomial potentials

As we already discussed, our measure is valid only for a subclass of possible inflationary potentials. In particular, we now discuss polynomial potentials

$$V(\phi) = \frac{\mu^4}{2\alpha!}\left(\frac{\phi}{\mu}\right)^{2\alpha}. \quad (56)$$

From the slow-roll conditions and Eq. (50) we obtain

$$\frac{H}{\dot\phi} \approx -8\pi G\left(\frac{1}{V(\phi)}\frac{\partial V(\phi)}{\partial\phi}\right)^{-1}\frac{[1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}]}{D^{n+1}}. \quad (57)$$

Integrating the identity $(1/a) = (H/\dot\phi)(\mathrm{d}\phi/\mathrm{d}a)$, we obtain that for polynomial potentials

$$-\int_{a_{\rm i}}^{a_{\rm s}}\frac{D^{n+1}}{[1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}]}\frac{\mathrm{d}a}{a} \approx \frac{4\pi G}{\alpha}(\phi_{\rm f}^2-\phi_{\rm i}^2), \quad (58)$$

where, as before, we took $a_{\rm s}$ as the scale factor at the end of inflation and $a_{\rm i}$ the scale factor at the begin of inflation. Equation (58) implies that slow roll (for which $[1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}] > 0$) ends when $\phi_{\rm f} < \phi_{\rm i}$. In particular, using the standard definition for the end of inflation, $\phi_{\rm f} \ll \phi_{\rm i}$, we obtain the following good approximation:

$$\phi_{\rm i} \approx \left(\frac{\alpha}{4\pi G}\int_{a_{\rm i}}^{a_{\rm s}}\frac{D^{n+1}}{[1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}]}\frac{\mathrm{d}a}{a}\right)^{1/2}. \quad (59)$$

Note that for $\alpha = 1$ and $D = 1$ we recover the usual result $\phi_{\rm i} \approx \sqrt{N/(4\pi G)}$.

We are now able to evaluate $\delta\epsilon_{\rm i}$. Using Eq. (51) we get

$$\delta\epsilon_{\rm i} \approx \frac{l_{\rm Pl}^{-1}}{\sqrt{2\alpha!}}\frac{\alpha^2}{3A_{\rm s}^3\sqrt{24\pi}}\left(\frac{\mu}{\phi_{\rm i}}\right)^{2-\alpha}. \quad (60)$$

Finally, from Eqs. (33), (55), and (60), the probability $\mathcal{P}(N)$ of having N e-folds of slow-roll inflation reads

$$\mathcal{P}(N) \approx \beta^2\left(\frac{\mu}{\phi_{\rm i}}\right)^{2-\alpha}(l_{\rm Pl}\mu)^{(2-\alpha)/\alpha} \times \exp\left(-3\left|1-(n-m+1)\frac{q_{\rm s}}{3D}\left(\frac{\partial D}{\partial q}\Big|_{\rm s}\right)\right|N\right), \quad (61)$$

where

$$\beta^2 = \frac{\alpha^3}{144}\left[\frac{2}{3\pi(2\alpha!)}\right]^{(\alpha+1)/2\alpha}2^{(\alpha+2)/2\alpha}\pi^{(\alpha-1)/\alpha}\gamma^{(\alpha+1)/2\alpha} \times \frac{\Gamma(\frac{3\alpha+1}{2\alpha})}{\Gamma(\frac{1}{2\alpha})}D_s^{m[(\alpha+1)/2\alpha]}A_s^{-3}. \quad (62)$$

The above probability changes qualitatively for renormalizable ($\alpha \le 2$) and nonrenormalizable ($\alpha > 2$) potentials. We will concentrate on renormalizable potentials [24]; thus $\alpha = 1, 2$.

The above calculation can be repeated using $S(q_{\rm G}) \neq 1$ to give

$$\mathcal{P}(N) \approx \beta^2\left(\frac{\mu}{\phi_{\rm i}}\right)^{2-\alpha}(l_{\rm Pl}\mu)^{(2-\alpha)/\alpha}\left[S\left(\frac{a^2}{a_{\rm G}^2}\right)\right]^{(\alpha+4)/4\alpha} \times \exp\left(-3\left|1-(n-m+1)\frac{q_{\rm s}}{3D}\left(\frac{\partial D}{\partial q}\Big|_{\rm s}\right)\right|N\right), \quad (63)$$

where now

$$\phi_{\rm i} \approx \left(\frac{\alpha}{4\pi G}\int_{a_{\rm i}}^{a_{\rm s}}\frac{D^{n+1}}{S(\frac{a^2}{a_G^2})[1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}]}\frac{\mathrm{d}a}{a}\right)^{1/2}. \quad (64)$$

Since $S(q_{\rm G}) < 1$ it is clear that choosing $j_{\rm G} \neq 1/2$ slightly reduces the probability of inflation. However, for $a_{\rm i} > a_{\rm G}$, $S(q_{\rm G})$ is well approximated by 1; thus the conclusions for the $j_{\rm G} = 1/2$ case remain qualitatively unchanged. We shall therefore consider in the following only the $j_{\rm G} = 1/2$ fundamental representation case.

C. Estimation of the probability

Let us first note that, since we have assumed conditions favoring the onset of inflation (i.e. FLRW universes), finding a high probability in this context only gives a necessary condition for inflation to be likely. In order to have enough inflation we require

$$e^{60} \approx \frac{a_{\rm s}}{a_{\rm i}} < \frac{a_{\rm s}}{a_{\rm Pl}}; \quad (65)$$

$a_{\rm Pl}$ is the minimal scale which can be probed in our approach. There are the following possibilities, either $a_{\rm s} \leq a_\star$ or $a_{\rm s} > a_\star$. In loop quantum gravity j gives the scale for which semiclassical effects can be observed. Obviously j cannot be too large, otherwise we will probe quantum gravity at everyday scales. In fact, particle physics experiments restrict $j < 10^{20}$ [10]. With this bound for j it is easy to show that the only possibility for which the necessary condition, Eq. (65), is satisfied is in the large $a_{\rm s}$ limit, i.e. $a_{\rm s} \gg a_\star$. With this inequality we can expand any function evaluated at $a_{\rm s}$ in the large q limit.

Let us discuss the magnitude of the probability, Eq. (61), for renormalizable potentials:

(1) The probability, Eq. (60), is suppressed by a factor

$$\exp\left(-3\left[1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}\right]\Big|_{a_{\rm s}} N\right).$$

In order to make the probability high enough, one could naively think that, by just finding appropriate values for n, m, $a_{\rm s}$ which make the exponent of $\mathcal{O}(1)$, one can overcome the negative result of Ref. [7]. However,

$$\beta^2 \propto \left[\left(1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}\right)\Big|_{a_{\rm s}}\right]^3$$

acts against this reduction. In fact, increasing the value of the exponential would actually make the probability of having a successful slow-roll inflation closer to zero. Therefore, the higher estimation of the probability may be found only when

$$\left(1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}\right)\Big|_{a_{\rm s}} \sim \mathcal{O}(1).$$

(2) The second term to take care of is the factor $l_{\rm Pl}\mu$. However, natural conditions for inflation [24] require the scalar field mass to be much lower than the Planck mass, i.e. $l_{\rm Pl}\mu \ll 1$. For $\alpha = 1$ the probability is therefore suppressed by the factor $(l_{\rm Pl}\mu)^2$.

(3) The most interesting term is the factor $\mu/\phi_{\rm i}$. We have already discussed that $|1-(n-m+1)\frac{q}{3D}\times\frac{\partial D}{\partial q}|$ has to be far from zero at the end of inflation. However, in principle it can be close to zero at the beginning of or during inflation, in compatibility with Eq. (42). In this case the probability is again suppressed as the integral defining $\phi_{\rm i}$ contains $|1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}|^{-1}$. In order to increment the probability $a_{\rm i}$ then has to be far from the zero of $|1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}|$. To have $\phi_{\rm i}$ as small as possible we therefore need to have D, in the range $[a_{\rm i}, a_{\rm s}]$ as small as possible. In particular, in order to improve the classical result $\phi_{\rm i} \sim \sqrt{N}$, one needs

$$D^{n+1}\left|1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}\right|^{-1} < 1.$$

In the case of $n-m+1 \lesssim -\epsilon_-$, where $-1.5 < \epsilon_- < -4$ (depending on the value of l taken), there are two zeros of $|1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}|$ (see Fig. 4), $a_{\rm c_1} \approx a_\star$ and $a_{\rm c_2} > a_\star$, so $a_\star < a_{\rm i} < a_{\rm s}$. In this region $D > 1$ (see Fig. 1) and

$$\left|1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}\right| < 1,$$

hence $\phi_{\rm i} > \sqrt{N/4\pi G}$.

In the case of $n-m+1 > \epsilon_+$, $0 < \epsilon_+ < 1$ (depending on the value of l taken) there is only one zero of $|1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}|$ that is always close to $a_\star$ (see Fig. 5). In this case the function $D^{n+1}|1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}|^{-1}$ can be less or greater than 1, depending on the choice of n and a; however in the region we are concerned with it is always greater than $(a-a_{\rm i})/(a_{\rm s}-a_{\rm i})$ (see Fig. 6).

This gives

$$\phi_{\rm i} > \left[\frac{\alpha}{4\pi G}\left(1-\frac{a_{\rm i}N}{a_{\rm s}-a_{\rm i}}\right)\right]^{1/2} \approx \left[\frac{\alpha}{4\pi G}\right]^{1/2}, \qquad (66)$$

where we use the fact that $a_{\rm s} \gg a_{\rm i}$.

In the case $\epsilon_- < n-m+1 < 0$ we do not have any

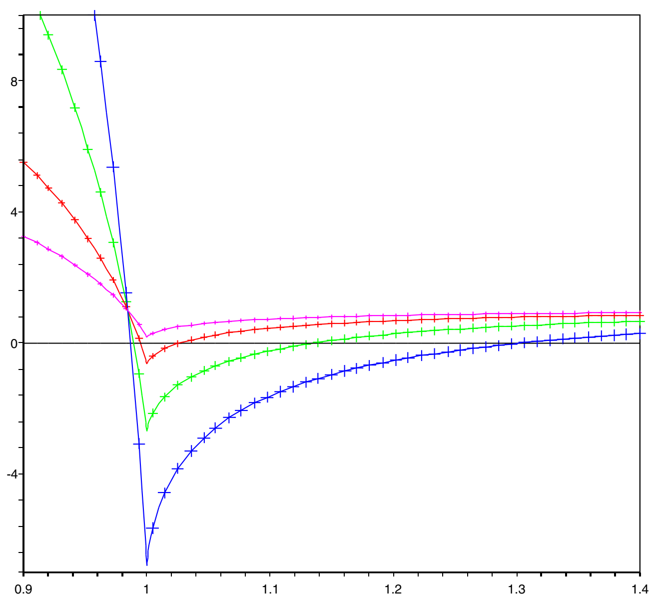

FIG. 4 (color online). The function $[1-(n-m+1)\frac{q}{3D}(\frac{\partial D}{\partial q})]$ is plotted as a function of a, with $a_\star = 1$ and $l = 0.75$, for $n-m+1 = -1, -4, -9, -19$ (small to large crosses, respectively). This is the term that determines the suppression of the probability of having N e-foldings of inflation. Notice that for $n-m+1 < -2$ we have two zeros, although this number is weakly dependent on l.

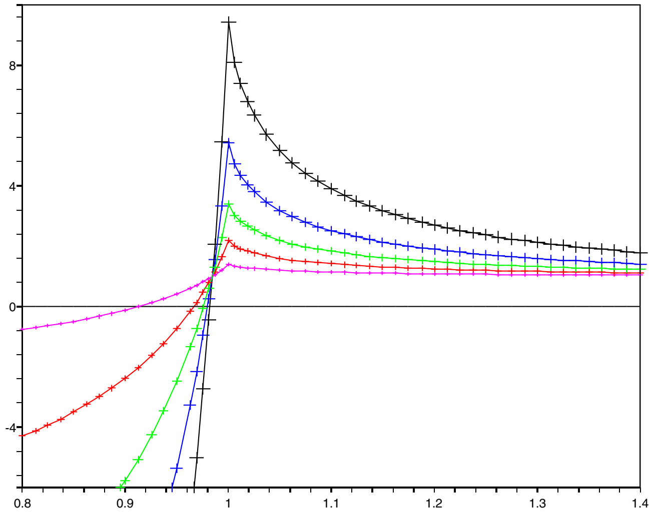

FIG. 5 (color online). The function $[1-(n-m+1)\frac{q}{3D}(\frac{\partial D}{\partial q})]$ is plotted as a function of a, with $a_\star = 1$ and $l = 0.75$, for $n-m+1 = 1, 3, 6, 11, 21$ (small to large crosses, respectively). This is the term that determines the suppression of the probability of having N e-foldings of inflation. Notice that for $n-m+1>0$ we have only one zero that is always close to $a_\star$.

zeros of $|1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}|$, and $a_{\rm i}$ is only restricted to be above the Planck scale. In this case we can consider the following estimation:

$$\phi_{\rm i} \approx \left(\frac{\alpha}{4\pi G}\int_{a_{\rm i}}^{a_{\rm s}} \frac{D^{n+1}}{|1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}|}\frac{{\rm d}a}{a}\right)^{1/2} \tag{67}$$

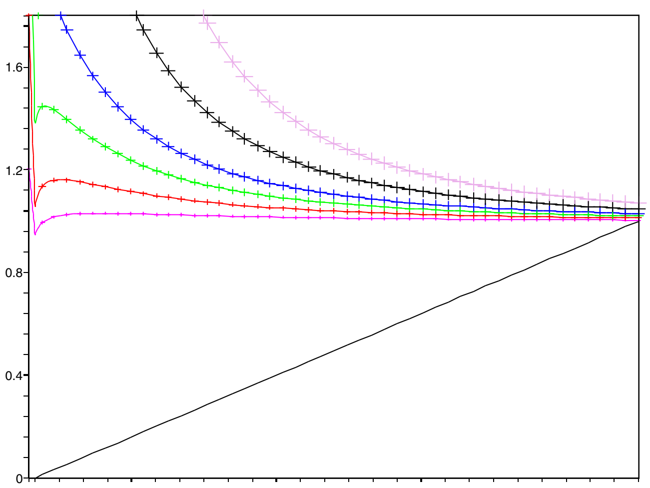

FIG. 6 (color online). The function $D^{n+1}[1-(n-m+1)\times \frac{q}{3D}(\frac{\partial D}{\partial q})]^{-1}$ is plotted as a function of a, with $a_\star = 1$ and $l = 0.75$, for $n-m+1 = 1, 3, 5, 7, 11, 16$ (small to large crosses, respectively). Also plotted is an example of $(a-a_{\rm i})/(a_s - a_i)$ for $a_i = a_\star = 1$ and $a_s = 2$, which allows us to calculate a lower bound on $\phi_{\rm i}$, which appears as one of the coefficients of the probability.

$$> \left(\frac{\alpha}{4\pi G}\int_{a_\star}^{a_{\rm s}} \frac{D^{n+1}}{|1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}|}\frac{{\rm d}a}{a}\right)^{1/2}. \tag{68}$$

By the arguments above we have

$$\phi_{\rm i} > \left(\frac{\alpha}{4\pi G}\int_{a_\star}^{a_{\rm s}} \frac{{\rm d}a}{a}\right)^{1/2}. \tag{69}$$

But as described above, particle physics experiments restrict $a_\star < 10^{10}\sqrt{\gamma}/\sqrt{3}$, which implies

$$\phi_{\rm i} > \left[\frac{\alpha}{4\pi G}(N - \ln(10^{10}/\sqrt{3}))\right]^{1/2}$$
$$> \left[\frac{\alpha}{4\pi G}(N - 22.5)\right]^{1/2}. \tag{70}$$

Finally, in the case $0 < n-m+1 < \epsilon_+$ we still do not have any zeros; however, the function $\frac{D^{n+1}}{|1-(n-m+1)\frac{q}{3D}\frac{\partial D}{\partial q}|}$ approaches 1 from above, so

$$\phi_{\rm i} > \sqrt{\frac{N}{4\pi G}}. \tag{71}$$

We have therefore shown that the probability of having slow-roll inflation is not significantly improved by the factor $\mu/\phi_{\rm i}$, for values of $N > 22.5$. It has been shown [25] that quantum loop cosmology can lead to a period of *superinflation* during which the scalar field is driven up its potential. Since this period does not satisfy the slow-roll conditions it is not accounted for by our analysis. However, perturbation theory is unstable in this *superinflationary* epoch [14], and hence, to produce the observed CMB anisotropies, we still require approximately 60 e-foldings of standard slow-roll inflation.

(4) In the probability, Eq. (61), the factor $\beta^2 \propto D_s^{[m(\alpha+1)-\alpha(n+1)]/2\alpha}$ [see the definition of β in (62) combined with the definition of A in (40)] can in principle be big for large values of m such that $m > \frac{\alpha}{\alpha+1}(n+1)$. This is due to the fact that the function D approaches 1 from above. We can estimate the magnitude of this factor by expanding D for large q's. We have (note that $l < 1$)

$$D_s \simeq 1 + 3\frac{2-l}{20}q_s^{-2}. \tag{72}$$

We have already discussed that, in all cases, $a_{\rm i} > \sqrt{\gamma}$ so, for $N = 60$, we have $q_{\rm s} > 10^{32}$. In this case, if we want the factor $D_s^{[m(\alpha+1)-\alpha(n+1)]/2\alpha}$ to at least overcome the exponential suppression $\exp(-180)$, we have the necessary condition

$$\frac{m(\alpha + 1) - \alpha(n + 1)}{2\alpha} \gtrsim 10^{110}, \qquad (73)$$

which is possible only for very "unnatural" parameters of the loop quantum cosmology [12].

V. CONCLUSION AND DISCUSSION

Cosmological inflation still remains the most appealing candidate in solving the puzzles of the standard hot big bang model. However, inflation must prove itself generic. This is an old question which has already been faced in the past [5,6]. Recently, this issue came back with the study of Ref. [7], where it was concluded that successful inflation is unlikely. More precisely, it was argued that the probability of having successful inflation is exponentially suppressed by the number of e-folds. Clearly, since such a conclusion leads necessarily to very severe implications, one should be sure of its validity and generality. The study of Ref. [7] has, in our opinion, a *weak* point. Classical physics has been applied all the way to very early times, and therefore very small scales as compared to the Planck length, a regime where quantum corrections can no longer be neglected. The estimation of the probability of having successful inflation should, in our opinion, be done taking quantum corrections into account. This is indeed the aim of our work. More precisely, in this paper we have addressed the question of how likely the onset of inflation is during the continuum phase of loop quantum cosmology.

Modifying the canonical measure introduced in Ref. [21], so that it is applicable in the context of loop quantum cosmology, we have found that it is not probable to get sufficiently long single-field inflation, for the phenomenologically favorite inflationary models, unless we accept extreme values for the ambiguity parameters m, n. Since, during the semiclassical era of loop quantum cosmology, the field ϕ can depart from the minimum of its potential [12], one may think that this could improve the classical probability of inflation [26]. This, however, is not what we obtain from our analysis, and the reason is that the same mechanism which forces ϕ away from its minimum will also increase $\dot{\phi}$, which would tend to reduce the probability of the onset of slow roll. In conclusion, our results show that, overall, quantum loop cosmology does not significantly improve on the classical probability, unless one accepts extreme values of the ambiguity parameters.

Our results hold for single-field inflation with potentials, which makes the volume of the phase space of possible trajectories finite. For example, inflationary models with potentials of the form $V(\phi) \sim \phi^{2\alpha}$ (with α an integer number) are within the class of models we have studied here. Our result implies limitations in the form of inflationary models within loop quantum cosmology. Since eventually the form of the inflationary model will be dictated from a fundamental theory, this *freedom* in modeling the inflationary potential will be alleviated.

From the analysis presented here, it is clear that, for the classes of models studied, the values $m = n = 0$ do not lead to a successful inflationary model. This implies important consequences. In the literature on loop quantum cosmology, the ambiguity parameters m, n have usually both been set equal to zero. Clearly, in this context successful cosmological inflation cannot take place in the semiclassical regime. Actually, one expects to constrain the ambiguity parameters by investigating the observational consequences to which inverse volume operators lead. For example, to study cosmological perturbations in loop quantum cosmology, one should perturb both the gravitational and the matter parts about the homogeneous background. This has only recently been accomplished. In Ref. [25] inhomogeneous cosmological perturbation equations have been derived without neglecting corrections in the gravitational part, thus treating both gravitational and matter terms on equal footing. This is indeed the appropriate framework to study cosmological perturbations in the context of loop quantum cosmology [27]. However, also in this study, m, n have been set equal to zero, which as we have shown here does not lead to successful inflation.

We have also analyzed the probability of having successful inflation for arbitrary values of the ambiguity parameters m, n. Our study has shown that successful inflation can be realized only for extreme values of the parameters—a result which goes against the *spirit* of inflation.

Our findings do not imply that inflation itself is improbable. What we have shown here is that, at least in the case of the semiclassical regime of loop quantum cosmology and therefore of general relativity, inflation is not as general as it is usually assumed. Thus, one has to address inflation in full quantum gravity, or in a string theory context.

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