Quantum stability of a $w < -1$ phase of cosmic acceleration

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We consider a massless, minimally coupled scalar with a quartic self-interaction which is released in Bunch-Davies vacuum in the locally de Sitter background of an inflating universe. It was shown, in this system, that quantum effects can induce a temporary phase of superacceleration, causing a violation of the weak energy condition on cosmological scales. In this paper, we investigate the system's stability by studying the behavior of linearized perturbations in the quantum-corrected effective field equation at one- and two-loop order. We show that the amplitude of the quantum-corrected mode function is reduced in time, starting from its initial classical (Bunch-Davies) value. This implies that the linear perturbations do not grow; hence, the model is stable. The decrease in the amplitude is in agreement with the system developing a positive (growing) mass squared due to quantum processes. The induced mass, however, remains perturbatively small and does not go tachyonic. This ensures the stability.

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I. INTRODUCTION

Present cosmological observations [1] do not exclude the possibility of an evolving dark energy equation of state $w \equiv p/\rho$, whose current value is less than minus one [2], i.e., a phase of superacceleration. Although the data are consistent with $w = -1$, which can be explained by a simple cosmological constant, the possibility of $w < -1$ has been an area of great interest in recent years [3].

Superacceleration is difficult to explain with *classical* models on account of the problem with stability [4]. One can achieve models exhibiting $w < -1$, by postulating scalar fields, for example. Such models, however, decay irrespective of how this is achieved. The observed persistence of the universe, therefore, can only be consistent with a relatively brief self-limiting phase of superacceleration. One way to get such a self-limiting phase, without violating classical stability, is via quantum effects [5–11]. The energy-time uncertainty principle requires virtual particles to emerge from the vacuum and then disappear back into it. The inflationary expansion of spacetime, however, causes the virtual particles to persist longer than the flat spacetime [12]. In fact, any sufficiently long wavelength virtual particle-antiparticle pairs, which are *massless* on the Hubble scale, are pulled apart by the Hubble flow before they find time to annihilate each other. Hence, they become real and may persist forever, recalling the analogy with the Hawking radiation. The rate at which the virtual particles emerge from the vacuum, on the other hand, is suppressed by the inverse of the scale factor for conformally invariant particles. Thus, quantum effects are enhanced during inflation for particles that are effectively massless (with respect to the Hubble parameter H) and classically conformally noninvariant. Gravitons and massless minimally coupled (MMC) scalars are unique in possessing zero mass without having classical conformal invariance. One is lead, naturally, to a self-limiting quantum effect in a classically stable theory, such as the MMC scalar with a quartic self-interaction in the locally de Sitter background of an inflating universe. The Lagrangian density that describes this system is

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{\lambda}{4!}\sqrt{-g}\varphi^4 + \text{counterterms}. \tag{1}$$

The dynamical variable in the model is the scalar field $\varphi(x)$. The metric $g_{\mu\nu}$ is a nondynamical background which is taken to be a D-dimensional locally de Sitter geometry. The invariant element can be expressed conveniently in either comoving or conformal coordinates

$$ds^2 = -dt^2 + e^{2Ht}d\vec{x}\cdot d\vec{x} = a^2(\eta)[-d\eta^2 + d\vec{x}\cdot d\vec{x}], \tag{2}$$

respectively. The conformal factor and the transformation which relate the two coordinate systems are

$$a(\eta) = -\frac{1}{H\eta} = e^{Ht}. \tag{3}$$

The Hubble constant H is related to the cosmological constant $\Lambda = (D-1)H^2$. It is the cosmological constant that drives inflation in the model. The scalar is a spectator to Λ-driven (de Sitter) inflation. We adopt the following notations: $x^\mu = (x^0, \vec{x})$, $x^0 \equiv \eta$, $\partial_\mu = (\partial_0, \vec{\nabla})$.

We release the state in Bunch-Davies vacuum at $t = 0$, corresponding to conformal time $\eta = \eta_i \equiv -H^{-1}$. Hence, the scale factor is normalized to $a = 1$ when the state is released so that $a > 1$ throughout the evolution. Note that the infinite future corresponds to $\eta \to 0^-$, so the possible variation of causally related conformal coordinates in either space or time is at most $\Delta x = \Delta\eta = H^{-1}$.

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Applying the Schwinger-Keldysh formalism [13,14] and using dimensional regularization, the fully renormalized vacuum expectation value (VEV) of the stress-energy tensor $\langle\Omega|T_{\mu\nu}(x)|\Omega\rangle$ is calculated [8,9] in this system. The energy density $\rho = \langle\Omega|T_{00}|\Omega\rangle/a^2(\eta)$ and pressure $p\delta_{ij} = \langle\Omega|T_{ij}|\Omega\rangle/a^2(\eta)$ are obtained as

$$\rho_{\rm ren} = \frac{\Lambda}{8\pi G} + \frac{\lambda H^4}{2^6\pi^4}\bigg\{\frac{1}{2}\ln^2(a) + \frac{2}{9}a^{-3} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{n+2}{(n+1)^2}a^{-(n+1)}\bigg\} + O(\lambda^2), \tag{4}$$

$$p_{\rm ren} = -\frac{\Lambda}{8\pi G} - \frac{\lambda H^4}{2^6\pi^4}\bigg\{\frac{1}{2}\ln^2(a) + \frac{1}{3}\ln(a) + \frac{1}{6}\sum_{n=1}^{\infty}\frac{n^2-4}{(n+1)^2}a^{-(n+1)}\bigg\} + O(\lambda^2). \tag{5}$$

Notice that $\rho_{\rm ren}$ and $p_{\rm ren}$ obey [8,9] the covariant conservation law $T_{\mu\nu}(x)^{;\nu} = 0$, i.e., $\dot{\rho}_{\rm ren} = -3H(\rho_{\rm ren} + p_{\rm ren})$, where the dot denotes the derivative with respect to the comoving time t. Their sum, however, violates the weak energy condition (WEC) $\rho + p \geq 0$ on cosmological scales

$$\rho_{\rm ren} + p_{\rm ren} = \frac{\lambda H^4}{2^6\pi^4}\bigg\{-\frac{1}{3}\ln(a) + \frac{2}{9}a^{-3} - \frac{1}{6}\sum_{n=1}^{\infty}\frac{n+2}{n+1}a^{-(n+1)}\bigg\} + O(\lambda^2). \tag{6}$$

Although the value for $w + 1$ is unobservably small in this model, the calculation shows that quantum effects can induce a self-limiting phase of superacceleration, in which a classically stable theory violates the WEC on cosmological scales in the average of $\rho + p$ not just in fluctuations about an average that obeys the condition $\rho + p = 0$. This is because inflationary particle production causes the scalar to undergo a random walk such that its average distance from the minimum of the potential $\frac{\lambda}{4!}\varphi^4$ increases. In our model, $\langle\Omega|\varphi^2(x)|\Omega\rangle = (\text{UV divergence}) + H^2\ln(a)/4\pi^2 + O(\lambda)$ [15]; recall that $\ln(a) = Ht$. [See the calculations in Sec. IV for the $O(\lambda)$ and $O(\lambda^2)$ corrections.] Hence, after the ultraviolet divergence is removed, the VEV of φ^2 gets pushed up its potential by inflationary particle production. This increases the vacuum energy which leads to the violation of the WEC by virtue of the covariant conservation $\dot{\rho} = -3H(\rho + p)$; since $\dot{\rho} > 0$ due to inflationary particle production, $\rho + p$ has to be less than zero.

The process, however, must be self-limiting because (i) as the scalar rises up its potential, the classical restoring force $-\lambda\varphi^3/6$ pushes it back down, and (ii) the curvature $\lambda\varphi^2/2$ associated with being away from the minimum of the potential *acts like* a positive "mass squared" to reduce the inflationary particle production responsible for pushing the scalar away from the configuration $\varphi = 0$, where the potential is minimum. [In quantum field theory (QFT), the mass squared is calculated via self-energy diagrams as is rigorously done in Ref. [10]. The VEV of the curvature of the potential provides a heuristic picture to understand the effect.] Since the classical restoring force (i) gets bigger as the field rolls up its potential and the mass generation (ii) cuts off particle production, the field cannot continue rolling up its potential. It must eventually come to a halt. Indeed, Starobinsky and Yokoyama showed [7] that $\langle\Omega|\varphi^2(x)|\Omega\rangle$ asymptotes to the constant $3H^2\Gamma(3/4)/\pi\Gamma(1/4)\sqrt{\lambda}$ in this model, which *proves* that the field strength does not grow forever. The curvature of the potential, that acts like a mass squared, should asymptote to $\lambda/2$ times this expectation value. They also estimated the time scale for the process as $T \approx 18.7/H\sqrt{\lambda}$. Thus, by choosing $\lambda \ll 1$, it is possible to have a long duration for the effect. We assume $\lambda \ll 1$ in this paper.

We study the stability of the system in this paper. To decide whether the system is stable [15] or not, one needs to check (i) if the VEV $\langle\Omega|\varphi^2(x)|\Omega\rangle$ continues to grow without a bound and (ii) if the small, position-dependent perturbations grow. If neither happens, the system is stable; otherwise, it is unstable. The above arguments show that $\langle\Omega|\varphi^2(x)|\Omega\rangle$ cannot continue to grow forever in the interacting theory (it asymptotes to a constant). Checking criterion (ii) is the main object of this paper. To do that, one solves the quantum-corrected effective field equation at linearized order

$$\Box\,\varphi(x) - \int d^4x' M^2(x;x')\varphi(x') = 0 \tag{7}$$

and obtains the quantum-corrected mode function. Although the scalar is classically massless in our model, quantum processes generate a nonzero self-mass-squared $M^2(x;x')$. Potential instabilities would come from the field developing a negative mass squared. In that case, the amplitude of the mode function would be an increasing function of time, indicating growth of perturbations and, hence, the instability. The fully renormalized scalar self-mass-squared $M^2(x;x')$ is calculated rigorously in Ref. [10] at one- and two-loop order, using the Schwinger-Keldysh formalism. $M^2(x;x')$ is indeed *positive* at one loop. However, one must go to two-loop order to see corrections of the derivative terms. To interpret the two-loop result, and hence to check the stability of the system, one needs to investigate how the self-mass-squared $M^2(x;x')$ modifies the effective field equations and its solution, i.e., the quantum-corrected mode function. If the amplitude of the solution is a decreasing function of time, one can conclude that perturbations do not grow; therefore, the model is stable.

The outline is as follows. In Sec. II, we define the effective mode equation, summarize the Schwinger-Keldysh formalism, and discuss our limitations in solving the effective mode equation. In Sec. III, we solve the effective mode equation in the late time limit and obtain the mode function in the leading logarithm approximation. Late time, for us, means $\ln(a) \gg 1$. In Sec. IV, we alternatively compute the same mode function using Starobinsky's stochastic inflation technique and compare it with the result obtained in Sec. III. Our conclusions are summarized in Sec. V.

II. EFFECTIVE MODE EQUATION FOR THE MMC SCALAR

In this section, we describe the operator formalism and effective field equation correspondence. Then we review the Schwinger-Keldysh formalism that one must use to calculate expectation values. We use the one- and two-loop results [10] for the scalar self-mass-squared $M^2(x; x')$, obtained by applying the Schwinger-Keldysh formalism in our model, to write the effective (quantum-corrected) mode equation and discuss how we "solve" it.

A. Relation to fundamental operators

The relation between the fundamental Heisenberg operator of the scalar field $\varphi(x)$ and the $\mathbb{C}$-number plane wave mode solution $\Phi(x; \vec{k})$ of the linearized effective field equation can be given [16,17] as

$$\Phi(x; \vec{k}) = \langle\Psi_f|[\varphi(x), \alpha^\dagger(\vec{k})]|\Psi_i\rangle. \tag{8}$$

Here $|\Psi_i\rangle$ and $|\Psi_f\rangle$ are the states, and $\alpha^\dagger(\vec{k})$ is the free creation operator. In flat space scattering problems, $|\Psi_i\rangle$ and $|\Psi_f\rangle$ correspond to the states whose wave functionals are free vacuum in the asymptotic past and future, respectively. The universe, however, begins at a finite time and evolves to some unknown state in the asymptotic future. Therefore, in cosmology, we release the universe from a prepared state at a given time and then let it evolve. We seek to know expectation values in the presence of this state. This corresponds to the choice $|\Psi_f\rangle = |\Psi_i\rangle$. For computational convenience, we assume that both of the states are free vacuum at $\eta = \eta_i$. Had the choices of flat space scattering theory been used, acausal effective field equations would have been obtained. The matrix elements of Hermitian operators would also be complex in that case.

To define the free creation and annihilation operators, recall that the full Lagrangian density $\mathcal{L}$ of the MMC scalar is

$$\mathcal{L} = -\frac{(1+\delta Z)}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{(\lambda+\delta\lambda)}{4!}\sqrt{-g}\varphi^4 - \frac{\delta m^2}{2}\sqrt{-g}\varphi^2. \tag{9}$$

The field strength (δZ), coupling constant ($\delta\lambda$), and mass (δm^2) counterterms are needed to remove divergences at one- and two-loop order in the scalar self-mass-squared. It turns out that δZ and $\delta\lambda$ are of order λ^2, whereas δm^2 has contributions of order λ and λ^2 [10].

Let us now integrate the invariant field equation of the MMC scalar

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) - \frac{\sqrt{-g}}{1+\delta Z}\left[\frac{(\lambda+\delta\lambda)}{6}\varphi^3 + \delta m^2\varphi\right] = 0. \tag{10}$$

The result is

$$\varphi(x) = \varphi_0(x) + \int_{\eta_i}^0 d\eta' \int d^{D-1}x' G(x; x') I[\varphi(x')], \tag{11}$$

where $\varphi_0(x)$ is the free field. We define the interaction term as

$$I[\varphi] \equiv \frac{\sqrt{-g}}{1+\delta Z}\left[\frac{(\lambda+\delta\lambda)}{6}\varphi^3 + \delta m^2\varphi\right]. \tag{12}$$

The Green's function $G(x; x')$ is any solution of the equation

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu G(x; x')) = \delta^D(x - x'). \tag{13}$$

Although the Green's functions would obey Feynman boundary conditions for flat space scattering problems, it is more natural to use retarded boundary conditions in cosmology. The fundamental field operator $\varphi(x)$, on the other hand, is unique. It does not depend on the choices of the boundary conditions for the Green's functions or on η_i. What changes with those choices is the free scalar $\varphi_0(x)$. Because $\varphi_0(x)$ obeys the linearized equations of motion and agrees with the full fields at $\eta = \eta_i$, it can be expanded in terms of free creation and annihilation operators $\alpha(\vec{k})$ and $\alpha^\dagger(\vec{k})$ as

$$\varphi_0(x) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}}\{u(\eta, k)e^{i\vec{k}\cdot\vec{x}}\alpha(\vec{k}) + u^*(\eta, k)e^{-i\vec{k}\cdot\vec{x}}\alpha^\dagger(\vec{k})\}, \tag{14}$$

where the Bunch-Davies mode function [18]

$$u(\eta, k) = \frac{H}{\sqrt{2k^3}}(1 + ik\eta)e^{-ik\eta}. \tag{15}$$

Although the creation and annihilation operators change as different Green's functions are used in Eq. (11), their nonzero commutation relation remains fixed

$$[\alpha(\vec{k}), \alpha^\dagger(\vec{k}')] = (2\pi)^{D-1}\delta^{D-1}(\vec{k} - \vec{k}'). \tag{16}$$

By iterating Eq. (11), one can expand the full field $\varphi(x)$ in terms of the free field $\varphi_0(x)$ as

$$\varphi(x) = \varphi_0(x) + \int_{\eta_i}^0 d\eta' \int d^{D-1}x' G_{\rm ret}(x; x') I[\varphi(x')] \tag{17}$$

$$= \varphi_0(x) + \int_{\eta_i}^0 d\eta' \int d^{D-1}x' G_{\rm ret}(x; x') \times I[\varphi_0(x')] + \cdots. \tag{18}$$

Hence, choosing $|\Psi_f\rangle = |\Psi_i\rangle$ as free vacuum at η_i, one can see [16,17] that the quantum-corrected plane wave mode solution (8) yields

$$\Phi(x;\vec{k}) = \langle\Omega|[\varphi(x), \alpha^\dagger(\vec{k})]|\Omega\rangle = u(\eta, k)e^{i\vec{k}\cdot\vec{x}} + O(\lambda). \tag{19}$$

The $O(\lambda)$ and $O(\lambda^2)$ corrections in Eq. (19) are obtained in Sec. III, by "solving" the quantum-corrected effective field equation at one- and two-loop order. In Sec. IV, we obtain the same corrections by calculating the commutator (8) stochastically. The results yielded by the two approaches are in perfect agreement.

B. Schwinger-Keldysh formalism

Because of the fact that "in" ($t \to -\infty$) vacuum is not equal to the "out" ($t \to \infty$) vacuum in the de Sitter background, we need to calculate expectation values rather than in-out matrix elements. This is done by applying the Schwinger-Keldysh formalism [13,14]. The end points of propagators acquire a $\pm$ polarity, in this formalism. Hence, every propagator $i\Delta(x;x')$ of the in-out formalism generalizes to four Schwinger-Keldysh propagators: $i\Delta_{++}(x;x')$, $i\Delta_{+-}(x;x')$, $i\Delta_{-+}(x;x')$, and $i\Delta_{--}(x;x')$. Each propagator can be obtained from the Feynman propagator by replacing the de Sitter conformal coordinate interval

$$\begin{aligned}\Delta x^2(x,x') &= \Delta x^2_{++}(x;x')\\ &\equiv \|\vec{x}-\vec{x}'\|^2 - (|\eta-\eta'| - i\delta)^2\end{aligned} \tag{20}$$

with the appropriate coordinate interval

$$\begin{aligned}\Delta x^2_{+-}(x;x') &\equiv \|\vec{x}-\vec{x}'\|^2 - (\eta-\eta'+i\delta)^2\\ &= (\Delta x^2_{-+}(x;x'))^*,\end{aligned} \tag{21}$$

$$\Delta x^2_{--}(x;x') = (\Delta x^2_{++}(x;x'))^*. \tag{22}$$

Vertices are either all $+$ or all $-$. A $+$ vertex is the usual one of the in-out formalism, whereas the $-$ vertex is its conjugate.

Because each external line can be either $+$ or $-$ in the Schwinger-Keldysh formalism, each N-point 1PI function of the in-out formalism corresponds to 2^N Schwinger-Keldysh N-point 1PI functions. The Schwinger-Keldysh effective action is the generating functional of these 1PI functions, so it depends upon two background fields $\varphi_+(x)$ and $\varphi_-(x)$. For example, there are four Schwinger-Keldysh 2-point 1PI functions $M^2_{\pm\pm}(x;x')$. The $++$ one is the same as the in-out self-mass-squared, and the others are related as the propagators

$$\begin{aligned}-iM^2_{--}(x;x') &= (-iM^2_{++}(x;x'))^*,\\ -iM^2_{-+}(x;x') &= (-iM^2_{+-}(x;x'))^*.\end{aligned} \tag{23}$$

The various self-mass-squared terms enter [14] the effective action as follows:

$$\begin{aligned}\Gamma[\varphi_+,\varphi_-] = S[\varphi_+] - S[\varphi_-] - \frac{1}{2}\int d^Dx\int d^Dx'\\ \times\{\varphi_+(x)M^2_{++}(x;x')\varphi_+(x')\\ + \varphi_+(x)M^2_{+-}(x;x')\varphi_-(x')\\ + \varphi_-(x)M^2_{-+}(x;x')\varphi_+(x')\\ + \varphi_-(x)M^2_{--}(x;x')\varphi_-(x')\} + O(\varphi^3_\pm),\end{aligned} \tag{24}$$

where $S[\varphi]$ is the classical scalar action. The effective field equations of the Schwinger-Keldysh formalism are obtained by varying with respect to either polarity and then setting the two polarities equal [14]. Up to order $O(\varphi^2)$, we have

$$\begin{aligned}\frac{\delta\Gamma[\varphi_\pm]}{\delta\varphi_+(x)}\bigg|_{\varphi_\pm=\varphi} &= \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi(x))\\ &\quad - \int_{\eta_i}^0 d\eta'\int d^3x'\{M^2_{++}(x;x')\\ &\quad + M^2_{+-}(x;x')\}\varphi(x').\end{aligned} \tag{25}$$

Note that we have taken the regularization parameter D to its unregulated value of $D = 4$, in view of the fact that the self-mass-squared is assumed to be fully renormalized. It is this *linearized* effective field equation which $\Phi(x;\vec{k})$ [Eq. (8)] obeys [16,17]

$$\begin{aligned}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi(x;\vec{k})) - \int_{\eta_i}^0 d\eta'\int d^3x'\{M^2_{++}(x;x')\\ + M^2_{+-}(x;x')\}\Phi(x';\vec{k}) = 0.\end{aligned} \tag{26}$$

Thus, the two renormalized 1PI 2-point functions we need are $M^2_{++}(x;x')$ and $M^2_{+-}(x;x')$. At one-loop order, we have [10]

$$M^2_{1++}(x;x') = \frac{\lambda H^2}{8\pi^2}a^4\ln(a)\delta^4(x-x') + O(\lambda^2). \tag{27}$$

The $+-$ case vanishes at this order because there is no mixed interaction. Fully renormalized two-loop results for the $++$ and $+-$ cases are [10]

$$M^2_{2++} = \frac{i\lambda^2}{2^9\pi^6}\bigg\{\frac{aa'}{24}\partial^4\bigg[\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}}\bigg] - H^2(aa')^2\partial^2\bigg[\ln\bigg(\frac{He^{3/4}}{2\mu}\bigg)\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}} + \frac{\ln^2(\mu^2\Delta x^2_{++})}{4\Delta x^2_{++}}\bigg] - H^4(aa')^3\frac{\ln^2(\frac{\sqrt{e}}{4}H^2\Delta x^2_{++})}{\Delta x^2_{++}} + \frac{H^6}{6}(aa')^4\ln^3\bigg(\frac{\sqrt{e}}{4}H^2\Delta x^2_{++}\bigg)\bigg\} + \frac{\lambda^2}{2^9 3\pi^4}a^2\{-\ln(a)\partial^2 + (2\ln(a)+1)Ha\partial_0\} \times \delta^4(x-x') + \frac{\lambda^2H^2}{2^7\pi^4}\bigg\{-\frac{4}{9}\ln^3(a) - \frac{23}{18}\ln^2(a) + \bigg[\frac{13}{3} + 3\ln\bigg(\frac{H}{2\mu}\bigg) - \frac{2}{9}\pi^2\bigg]\ln(a)\bigg\}a^4\delta^4(x-x') + \frac{\lambda^2H^2}{2^7\pi^4}\bigg\{\frac{a^{-3}}{81} - \sum_{n=1}^{\infty}\frac{n+5}{(n+1)^3}a^{-(n+1)} + 4\sum_{n=1}^{\infty}\frac{a^{-(n+2)}}{(n+2)^3} + 4\sum_{n=1}^{\infty}\frac{a^{-(n+3)}}{n(n+3)^3}\bigg\}a^4\delta^4(x-x'), \tag{28}$$

$$M^2_{2+-} = -\frac{i\lambda^2}{2^9\pi^6}\bigg\{\frac{aa'}{24}\partial^4\bigg[\frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}}\bigg] - H^2(aa')^2\partial^2\bigg[\ln\bigg(\frac{He^{3/4}}{2\mu}\bigg)\frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}} + \frac{\ln^2(\mu^2\Delta x^2_{+-})}{4\Delta x^2_{+-}}\bigg] - H^4(aa')^3 \times \frac{\ln^2(\frac{\sqrt{e}}{4}H^2\Delta x^2_{+-})}{\Delta x^2_{+-}} + \frac{H^6}{6}(aa')^4\ln^3\bigg(\frac{\sqrt{e}}{4}H^2\Delta x^2_{+-}\bigg)\bigg\}. \tag{29}$$

The $++$ and $+-$ terms in (26) exactly cancel for $\eta' > \eta$ and also, in the limit $\delta \to 0$, for x'^μ outside the light cone of x^μ. This is how the Schwinger-Keldysh formalism gives causal effective field equations. In the next section, we discuss what we mean by solving the quantum-corrected effective mode equation (26). The one- and two-loop corrected mode solution is obtained in the late time limit, i.e., for $\ln(a) \gg 1$, in Sec. III.

C. Solving the quantum-corrected effective mode equation

Here we discuss the limitations that one has in solving the effective mode equation (26). The full scalar self-mass-squared can be expressed, as a series, in powers of the loop counting parameter λ

$$M^2_{++}(x;x') + M^2_{+-}(x;x') = \sum_{\ell=1}^{\infty}\lambda^\ell\mathcal{M}^2_\ell(x;x'). \tag{30}$$

The first limitation is that we have only the $\ell = 1$ and $\ell = 2$ terms

$$M^2_{1++}(x;x') = \lambda\mathcal{M}^2_1(x;x'), \tag{31}$$

$$M^2_{2++}(x;x') + M^2_{2+-}(x;x') = \lambda^2\mathcal{M}^2_2(x;x'), \tag{32}$$

which are given by Eqs. (27)–(29), respectively. So we can only solve the effective mode equation perturbatively. We first substitute a series solution of the form

$$\Phi(x;\vec{k}) \equiv u(\eta,k)e^{i\vec{k}\cdot\vec{x}} + \sum_{\ell=1}^{\infty}\lambda^\ell\Phi_\ell(\eta,k)e^{i\vec{k}\cdot\vec{x}} \tag{33}$$

into Eq. (26) and then solve the equation order by order in powers of λ and λ^2. The zeroth order ($\ell = 0$) solution of Φ_ℓ is the well-known Bunch-Davies mode function $u(\eta, k)$ [Eq. (15)] times the exponential $e^{i\vec{k}\cdot\vec{x}}$.

The second limitation is due to the lower bound "η_i" on the temporal integration in Eq. (26). We release the universe in free vacuum at time $\eta = \eta_i$. Little is known about the wave functionals of interacting QFTs in curved space, but free vacuum can hardly be realistic. In fact, all of the finite energy states of interacting flat space QFTs have important corrections. Similar corrections are expected in curved space, too. Although it is possible to correct the free state functionals perturbatively as in nonrelativistic quantum mechanics, the usual procedure in flat space QFT is to release the system in free vacuum at asymptotic past and let the infinite time evolution resolve the difference between free vacuum and true vacuum into shifts of the mass, field strength, and background field [19]. In cosmology, however, one cannot typically apply this procedure, for the reasons noted in Sec. II A. One can still correct the state wave functionals perturbatively, though. Corrections to the initial state would appear as new interaction vertices on the initial value surface. They are expected to have a large effect on the expectation values of operators near the initial value which would decay in the late time limit. For example, it is the exponentially falling portions of the renormalized stress-energy tensor (4) and (5)

$$\rho_{\text{falling}} = \frac{\lambda H^4}{2^6\pi^4}\bigg\{\frac{2}{9}a^{-3} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{n+2}{(n+1)^2}a^{-(n+1)}\bigg\} + O(\lambda^2), \tag{34}$$

$$p_{\text{falling}} = \frac{\lambda H^4}{2^6\pi^4}\bigg\{-\frac{1}{6}\sum_{n=1}^{\infty}\frac{n^2-4}{(n+1)^2}a^{-(n+1)}\bigg\} + O(\lambda^2) \tag{35}$$

that it is conjectured [9] can be absorbed into an order λ correction of the initial ($a = 1$) free Bunch-Davies vacuum state. The fact that they fall off as one evolves away from the initial value surface suggests that they can be absorbed into a kind of local interaction there, leaving only the infrared logarithms

$$\rho_{\rm conj} = \frac{\Lambda}{8\pi G} + \frac{\lambda H^4}{2^6\pi^4}\left\{\frac{1}{2}\ln^2(a)\right\} + O(\lambda^2), \quad (36)$$

$$p_{\rm conj} = -\frac{\Lambda}{8\pi G} - \frac{\lambda H^4}{2^6\pi^4}\left\{\frac{1}{2}\ln^2(a) + \frac{1}{3}\ln(a)\right\} + O(\lambda^2). \quad (37)$$

Notice that they are separately conserved, i.e., $\dot{\rho}_{\rm conj} = -3H(\rho_{\rm conj} + p_{\rm conj})$ and $\dot{\rho}_{\rm falling} = -3H(\rho_{\rm falling} + p_{\rm falling})$. This is exactly what would be the case if they could be canceled by a new interaction vertex. Note that Eqs. (34) and (35) diverge on the initial value surface at $a = 1$, which indicates that free vacuum is very far away from any physically accessible state. Thus, although Eq. (26) determines the quantum corrections to the mode function (19) for free vacuum, that mode function has little physical relevance, because free vacuum is inaccessible. To find physically relevant mode functions, which are also valid for initial times, the corrections to the state wave functional must be included. Unfortunately, we have neither order λ nor order λ^2 corrections to the state wave functional. It therefore makes no sense to solve Eq. (26) for all times. The effects of the state corrections, however, must fall off at late times $[\ln(a) \gg 1]$ as in Eqs. (34) and (35). Because of time evolution, initially free vacuum and true vacuum become indistinguishable, as in flat space QFT [19]. Hence, we may obtain valid information from Eq. (26) by solving it in the late time limit. That is the subject of the next section.

III. EFFECTIVE MODE FUNCTION FOR THE MMC SCALAR

The linearized effective field equation that the MMC scalar mode solution $\Phi(x, \vec{k})$ obeys is given in Eq. (26). Using Eqs. (30), (31), and (33), one obtains the integro-differential equation for the one-loop correction $\Phi_1(\eta, \vec{k})$ to the classical mode function $u(\eta, k)$

$$\begin{aligned} a^2[\partial_0^2 + 2Ha\partial_0 + k^2]\Phi_1(\eta, k) &= -\int_{\eta_i}^0 d\eta' \int d^3x' \mathcal{M}_1^2(x; x') u(\eta', k) e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \\ &= -\int_{\eta_i}^0 d\eta' \int d^3x' \frac{H^2}{8\pi^2} a^4 \ln(a) \\ &\quad \times \delta^4(x - x') u(\eta', k) e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \\ &= -\frac{H^2}{8\pi^2} u(\eta, k) a^4 \ln(a). \end{aligned} \quad (38)$$

As is discussed in Sec. II C, the only sensible and physically interesting regime in which we can solve the effective mode equation is the late time limit $\ln(a) \gg 1$. The zeroth order mode function $u(\eta, k)$ can be replaced by its limit $u(0, k) = H/\sqrt{2k^3}$ in this regime. Solving Eq. (38) in the late time limit, we find

$$\Phi_1(\eta, k) \sim -\frac{1}{2^4 3\pi^2} u(0, k)\left\{\ln^2(a) - \frac{2}{3}\ln(a)\right\}, \quad (39)$$

in leading logarithm orders.

The order λ^2 correction $\Phi_2(\eta, k)$, on the other hand, has contributions due to both one- and two-loop self-mass-squared terms. It obeys

$$\begin{aligned} a^2[\partial_0^2 + 2Ha\partial_0 + k^2]\Phi_2(\eta, k) &= -\int_{\eta_i}^0 d\eta' \int d^3x' \{\mathcal{M}_1^2(x; x')\Phi_1(\eta', k) \\ &\quad + \mathcal{M}_2^2(x; x') u(\eta', k)\} e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}. \end{aligned} \quad (40)$$

The first integral is evaluated, in leading logarithm order, by inserting Eqs. (27) and (39) into Eq. (40). We find

$$\begin{aligned} &-\int_{\eta_i}^0 d\eta' \int d^3x' \mathcal{M}_1^2(x; x')\Phi_1(\eta', k) \\ &\sim \frac{H^2}{2^7 3\pi^4} u(0, k) a^4\left[\ln^3(a) - \frac{2}{3}\ln^2(a)\right]. \end{aligned} \quad (41)$$

The second integral is evaluated, in the late time limit, in Appendix A. Expanding in terms of powers of infrared logarithms, we find

$$\begin{aligned} &-\int_{\eta_i}^0 d\eta' \int d^3x' \mathcal{M}_2^2(x; x') u(\eta', k) e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \\ &\to -u(0, k)\int_{\eta_i}^0 d\eta' \int d^3x' \mathcal{M}_2^2(x; x') \\ &\sim \frac{H^2}{2^4 3^2\pi^4} u(0, k) a^4\left[\ln^3(a) + \frac{23}{16}\ln^2(a)\right. \\ &\quad \left. + \left(\frac{27}{8}\ln\left(\frac{2\mu}{H}\right) - \frac{189}{32} + \frac{\pi^2}{2}\right)\ln(a)\right]. \end{aligned} \quad (42)$$

Using Eqs. (41) and (42) in Eq. (40) yields

$$\begin{aligned} &a^2[\partial_0^2 + 2Ha\partial_0 + k^2]\Phi_2(\eta, k) \\ &\to \frac{H^2}{2^7 3^2\pi^4} u(0, k) a^4\left[11\ln^3(a) + \frac{19}{2}\ln^2(a)\right. \\ &\quad \left. + \left(27\ln\left(\frac{2\mu}{H}\right) - \frac{189}{4} + 4\pi^2\right)\ln(a)\right]. \end{aligned} \quad (43)$$

In leading orders, the solution for $\Phi_2(\eta, k)$ is

$$\begin{aligned} \Phi_2(\eta, k) &\sim \frac{1}{2^8 3^3\pi^4} u(0, k)\left\{\frac{11}{2}\ln^4(a) - \ln^3(a)\right. \\ &\quad + \left[27\ln\left(\frac{2\mu}{H}\right) - \frac{185}{4} + 4\pi^2\right] \\ &\quad \left. \times \left[\ln^2(a) - \frac{2}{3}\ln(a)\right]\right\}. \end{aligned} \quad (44)$$

Thus, keeping the leading logarithm terms at each order of perturbation (i.e., in Φ_1 and Φ_2), we find that the quantum-corrected mode solution (33) asymptotes to

$$\Phi(x, \vec{k}) \sim u(0, k)e^{i\vec{k}\cdot\vec{x}}\Big\{1 - \frac{1}{2^4 3\pi^2}\lambda \ln^2(a) + \frac{11}{2^9 3^3 \pi^4}\lambda^2 \ln^4(a)\Big\} + O(\lambda^3). \quad (45)$$

One can immediately see from Eq. (45) that perturbation theory breaks down when $\ln(a(t))$ is of order $1/\sqrt{\lambda}$. This, however, does not invalidate the reliability of our late time $[\ln(a) \gg 1]$ solution, because by choosing $\lambda \ll 1$, as we assume in this paper, one can have a long period of time during which $1 \ll \ln(a) \ll 1/\sqrt{\lambda}$.

Equation (45) also shows that, at $t = 0$, the mode solution $\Phi(x, \vec{k})$ is equal to the well-known (Bunch-Davies) classical result $u(0, k)e^{i\vec{k}\cdot\vec{x}}$. As time goes on, it decreases proportional to the factor $1 - \lambda H^2 t^2/48\pi^2 + O(\lambda^2)$ (the stochastic calculation of Sec. IV yields the same result). Thus, the amplitude of the quantum-corrected mode function (hence, of the field) is reduced (consistent with the model developing a *positive* mass squared, as one- and two-loop self-mass-squared terms and the VEV of the curvature of the potential imply; see the discussion in the next section). This means that linear perturbations do not grow in this system; therefore, it is stable.

IV. STOCHASTIC ANALYSIS

Starobinsky developed a stochastic inflation technique [6,7] which gives the leading infrared logarithms at each order in perturbation theory. Recently, his technique was proven to all orders and extended to various models [20]. In this section, we introduce the stochastic technique briefly and use it to calculate the quantum-corrected plane wave mode solution (8) and the VEV of the curvature (associated with the field being away from the minimum) of the potential which acts like a mass squared in the classical action.

The equation of motion for the scalar field with quartic self-interaction in $D = 3 + 1$-dimensional de Sitter background is

$$\ddot{\varphi}(t, \vec{x}) + 3H\dot{\varphi}(t, \vec{x}) - \frac{\nabla^2}{a^2}\varphi(t, \vec{x}) + \frac{\lambda}{6}\varphi^3(t, \vec{x}) = 0. \quad (46)$$

The solution of Eq. (46) can be obtained by iterating

$$\varphi(t, \vec{x}) = \varphi_0(t, \vec{x}) - \frac{\lambda}{6}\int_0^t dt' a^3(t') \times \int d^3x' G_{\rm ret}(t, \vec{x}; t', \vec{x}')\varphi^3(t', \vec{x}'), \quad (47)$$

where the retarded Green's function [20] is

$$G_{\rm ret} \equiv \frac{H^2}{4\pi}\Theta(t - t')\Big[\frac{\delta(H\|\vec{x} - \vec{x}'\| + a^{-1}(t) - a^{-1}(t'))}{a(t)a(t')H\|\vec{x} - \vec{x}'\|} + \Theta(H\|\vec{x} - \vec{x}'\| + a^{-1}(t) - a^{-1}(t'))\Big]. \quad (48)$$

As in Eq. (14), the free field $\varphi_0(t, \vec{x})$ can be expanded in terms of mode function $u(t, k)$ and annihilation and creation operators $\alpha(\vec{k})$ and $\alpha^\dagger(\vec{k})$, satisfying the canonical commutation relation (16) in $D = 3 + 1$ dimensions. Starobinsky's stochastic technique cuts out the ultraviolet modes $k > Ha$ of the field and applies the following rules to the equation of motion: (i) retain only the term with the smallest number of derivatives of the field, (ii) replace the field variable by a stochastic variable, and (iii) subtract the stochastic source term f for each time derivative of the field. Applying these rules to Eq. (46) yields

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{\nabla^2}{a^2}\varphi + \frac{\lambda}{6}\varphi^3 = 0 \to 3H\dot{\varphi} + \frac{\lambda}{6}\varphi^3 = 0 \to 3H(\dot{\phi} - f_\phi) + \frac{\lambda}{6}\phi^3 = 0. \quad (49)$$

(The scalar field φ became a stochastic field ϕ.) The source term f_ϕ is the time derivative of the *infrared truncated* free field (14)

$$\phi_0(t, \vec{x}) \equiv \int \frac{d^3k}{(2\pi)^3}\theta(Ha(t) - k)\frac{H}{\sqrt{2k^3}}\{e^{i\vec{k}\cdot\vec{x}}\alpha(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}}\alpha^\dagger(\vec{k})\}. \quad (50)$$

Here the leading infrared limit of the Bunch-Davies mode function (15) $u(t, k) \sim H/\sqrt{2k^3}$ is used. Hence,

$$f_\phi(t, \vec{x}) \equiv \dot{\phi}_0(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3}\delta(k - Ha(t))\frac{H^2}{\sqrt{2k}}\{e^{i\vec{k}\cdot\vec{x}}\alpha(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}}\alpha^\dagger(\vec{k})\}. \quad (51)$$

In Eq. (49), we obtained a Langevin-like equation which can be recast as

$$\dot{\phi}(t, \vec{x}) = f_\phi(t, \vec{x}) - \frac{\lambda}{18H}\phi^3(t, \vec{x}). \quad (52)$$

In this section, we use this equation to stochastically calculate (i) the VEV of the curvature associated with being away from the minimum of the potential, i.e., $\lambda\langle\Omega|\phi^2(x)|\Omega\rangle/2$, which acts like a field-dependent mass squared, and (ii) the quantum-corrected mode function (8), i.e., $\langle\Omega|[\phi(x), \alpha^\dagger(\vec{k})]|\Omega\rangle$. As a check, we calculate (i) also using perturbative QFT at one- and two-loop order and show that the two realizations agree perfectly in leading logarithm order. The quantum-corrected mode function was already obtained in Sec. III by applying QFT. Comparing Eq. (45) of Sec. III with the stochastic result for (ii) will show that the agreement is again perfect in leading logarithm order.

In stochastic calculations (i) and (ii), we express $\phi(x)$ in terms of the infrared truncated free field ϕ_0 perturbatively, by first integrating Eq. (52) and then iterating the result up to the desired power of λ:

$$\phi(t,\vec{x}) = \phi_0(t,\vec{x}) - \frac{\lambda}{18H}\int_0^t dt'\phi^3(t',\vec{x}) = \phi_0(t,\vec{x}) - \frac{\lambda}{18H}\int_0^t dt'\phi_0^3(t',\vec{x}) + \frac{\lambda^2}{2^2 3^3 H^2} \times \int_0^t dt'\phi_0^2(t',\vec{x})\int_0^{t'} dt''\phi_0^3(t'',\vec{x}) + O(\lambda^3). \tag{53}$$

In the noninteracting (free) theory, $\phi(t,\vec{x}) = \phi_0(t,\vec{x})$. Hence, using Eq. (50), the VEV of the scalar field strength squared is obtained trivially in this ($\lambda = 0$) limit

$$\langle\Omega|\phi_0^2(x)|\Omega\rangle = \frac{H^2}{4\pi^2}\ln(a). \tag{54}$$

This stochastic result is the same as the results of Ref. [15] applying QFT.

Now we start calculating (i) the VEV $\lambda\langle\Omega|\phi^2(x)|\Omega\rangle/2$ in the interacting ($\lambda \neq 0$) theory. Using Eq. (53), we find

$$\frac{\lambda}{2}\langle\Omega|\phi^2(x)|\Omega\rangle = \frac{\lambda}{2}\Big[\langle\Omega|\phi_0^2(x)|\Omega\rangle - \frac{\lambda}{9H}\int_0^t dt'\langle\Omega|\phi_0(t,\vec{x})\phi_0^3(t',\vec{x})|\Omega\rangle\Big] + O(\lambda^3) = \frac{\lambda}{2}\Big[\langle\Omega|\phi_0^2(x)|\Omega\rangle - \frac{\lambda}{3H}\int_0^t dt'\langle\Omega|\phi_0(t,\vec{x})\phi_0(t',\vec{x})|\Omega\rangle\langle\Omega|\phi_0^2(t',\vec{x})|\Omega\rangle\Big] + O(\lambda^3). \tag{55}$$

Inserting Eq. (50) into Eq. (55) yields the stochastic result

$$\frac{\lambda}{2}\langle\Omega|\phi^2(x)|\Omega\rangle = \frac{\lambda}{2}\Big[\frac{H^2}{4\pi^2}\ln(a) - \frac{\lambda}{3H}\int_0^t dt'\frac{H^4}{16\pi^4}\ln^2(a') + O(\lambda^2)\Big] = \frac{H^2}{2^3\pi^2}\lambda\ln(a)\Big[1 - \frac{1}{2^2 3^2\pi^2}\lambda\ln^2(a)\Big] + O(\lambda^3). \tag{56}$$

Next, we want to calculate the same VEV $\lambda\langle\Omega|\phi^2|\Omega\rangle/2$ using QFT. Figure 1 depicts the one-loop contribution. Hence, at one-loop order, the VEV is given in terms of the coincident limit of the scalar propagator and the mass counterterm δm^2:

$$\frac{\lambda}{2}\langle\Omega|\phi^2|\Omega\rangle = \frac{\lambda}{2}i\Delta(x;x) + \delta m^2 + O(\lambda^2). \tag{57}$$

The scalar propagator in D-dimensional locally de Sitter background is [8,9]

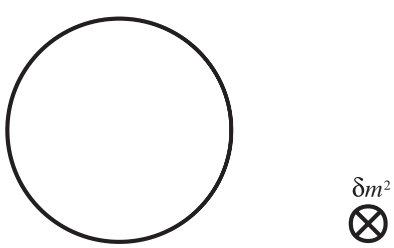

FIG. 1. Generic one-loop diagram with the mass counterterm.

$$i\Delta(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\Big\{-\sum_{n=0}^{\infty}\frac{1}{n-\frac{D}{2}+1}\frac{\Gamma(n+\frac{D}{2})}{\Gamma(n+1)} \times \Big(\frac{y}{4}\Big)^{n-(D/2)+1} - \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\Big(\pi\frac{D}{2}\Big) + \sum_{n=1}^{\infty}\frac{1}{n}\frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})}\Big(\frac{y}{4}\Big)^n + \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\ln(aa')\Big\}. \tag{58}$$

Here the modified de Sitter length function $y(x;x')$ is given in terms of the de Sitter conformal coordinate interval Δx^2 [Eq. (20)]

$$y(x;x') = H^2aa'\Delta x^2 = H^2aa'[\|\vec{x}-\vec{x}'\|^2 - (|\eta-\eta'| - i\delta)^2]. \tag{59}$$

To facilitate dimensional regularization, we express the dimension of spacetime in terms of its deviation from four: $D = 4 - \epsilon$. Therefore, the coincident limit of the scalar propagator

$$i\Delta(x;x) = \lim_{x'\to x} i\Delta(x;x') = \frac{H^{2-\epsilon}}{(4\pi)^{2-(\epsilon/2)}}\frac{\Gamma(3-\epsilon)}{\Gamma(2-\frac{\epsilon}{2})}\Big\{2\ln(a) + \pi\cot\Big(\frac{\pi\epsilon}{2}\Big)\Big\}. \tag{60}$$

Because of the finite, time-dependent term in Eq. (60), we cannot make the one-loop VEV (57) vanish for all time. Our renormalization condition is that it should be zero at $t = 0$, which implies

$$\delta m^2 = -\frac{\lambda H^{2-\epsilon}}{2^{5-\epsilon}\pi^{2-(\epsilon/2)}}\frac{\Gamma(3-\epsilon)}{\Gamma(2-\frac{\epsilon}{2})}\pi\cot\Big(\frac{\pi\epsilon}{2}\Big) + O(\lambda^2). \tag{61}$$

Therefore, Eq. (57) yields

$$\frac{\lambda}{2}\langle\Omega|\phi^2|\Omega\rangle_{\text{1-loop}} = \frac{\lambda H^2}{8\pi^2}\ln(a). \tag{62}$$

The two-loop diagram that contributes to the VEV is known as the snowman diagram depicted on the left of Fig. 2. The right-hand side diagram depicts the one-loop mass counterterm which naturally combines with it (δm^2 denotes the mass counterterm vertex). In Schwinger-Keldysh formalism (Sec. II B) the internal vertices are summed over both + and − polarities. A simple application of Feynman rules gives

$$\frac{\lambda}{2}\langle\Omega|\phi^2|\Omega\rangle_{\text{2-loop}} = \frac{\lambda}{2}\int d^Dx' a'^D\{[i\Delta_{++}(x;x')]^2 - [i\Delta_{+-}(x;x')]^2\}\bigg\{\frac{(-i\lambda)}{2}i\Delta(x';x') - i\delta m^2\bigg\}. \tag{63}$$

Both ++ and +− propagators are the same function (58) of the appropriate version of the modified de Sitter length function $y(x;x')$. By definition (20), $y_{++}(x;x') \equiv H^2aa'\Delta x^2_{++} = y(x;x')$, given in Eq. (59). On the other hand, $y_{+-}(x;x') \equiv H^2aa'\Delta x^2_{+-}$, where the coordinate interval Δx^2_{+-} is given in Eq. (21). The coincident propagator and the mass counterterm are calculated in Eqs. (60) and (61), respectively. Because both diagrams in Fig. 2 have the same lower loop, they possess the common factor given in the first curly bracket of Eq. (63). The first term in the second curly bracket comes from the left-hand side diagram, whereas the second term comes from the right-hand side diagram.

The integral in Eq. (63) is calculated explicitly in Ref. [10]. The result can be read off directly from its Eq. (61). After renormalizing the overlapping divergence $-(\lambda^2H^2/2^7\pi^4)(\frac{2\pi}{H\mu})^\epsilon\frac{\ln(a)}{\epsilon}$ of the snowman diagram by the two-loop mass counterterm, one obtains

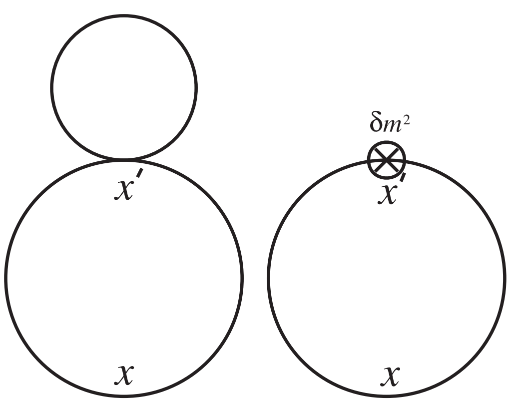

FIG. 2. Generic snowman diagram with the mass counterterm.

$$\frac{\lambda}{2}\langle\Omega|\phi^2|\Omega\rangle_{\text{2-loop}} = \frac{\lambda^2H^2}{2^7\pi^4}\bigg\{-\frac{4}{9}\ln^3(a) + \frac{13}{18}\ln^2(a) + \bigg[\ln\bigg(\frac{H}{2\mu}\bigg) + \frac{8}{3} - \gamma - \frac{2}{9}\pi^2\bigg]\ln(a) - \frac{238}{81} + \frac{13}{54}\pi^2 + \frac{4}{3}\zeta(3) + \frac{a^{-3}}{81} - \sum_{n=1}^{\infty}\frac{n+5}{(n+1)^3}a^{-(n+1)} + 4\sum_{n=1}^{\infty}\frac{a^{-(n+2)}}{(n+2)^3} + 4\sum_{n=1}^{\infty}\frac{a^{-(n+3)}}{n(n+3)^3}\bigg\} + O(\lambda^3), \tag{64}$$

where Euler's constant $\gamma \simeq 0.577$. (The constants and exponentially decaying terms may be subsumed into the definition of the vacuum state.) Combining one- and two-loop results, Eqs. (62) and (64), we find

$$\frac{\lambda}{2}\langle\Omega|\phi^2|\Omega\rangle = \frac{H^2}{2^3\pi^2}\lambda\ln(a)\bigg[1 - \frac{1}{2^23^2\pi^2}\lambda\ln^2(a)\bigg] + O(\lambda^3), \tag{65}$$

in leading logarithm order. This QFT result is exactly the same as stochastic result Eq. (56).

Until the breakdown of the perturbation theory—that occurs around $\lambda\ln^2(a) \sim 1$, as was pointed out earlier—expectation value (56) remains positive. This means that the curvature associated with the scalar being away from the minimum of the potential assumes a growing positive expectation value which *acts like* a positive "mass squared" during the process. This agrees with the decreasing mode function obtained in Eq. (45) by solving the one- and two-loop corrected effective field equation, in the context of QFT.

Next, in calculation (ii), we recompute the very same mode solution $\Phi(x;\vec{k})$ obtained in Eq. (45) using QFT. This time, however, we calculate $\Phi(x;\vec{k}) = \langle\Omega|[\phi(x),\alpha^\dagger(\vec{k})]|\Omega\rangle$ [Eq. (8)] by applying the stochastic technique to show that the two realizations yield exactly the same result, in leading logarithm order. The commutator

$$[\phi(x),\alpha^\dagger(\vec{k})] = u(0,k)e^{i\vec{k}\cdot\vec{x}}\bigg\{1 - \frac{\lambda}{6H}\int_0^t dt'\phi_0^2(t',\vec{x}) + \frac{\lambda^2}{2^23^2H^2}\int_0^t dt'\phi_0^2(t',\vec{x})\int_0^{t'}dt''\phi_0^2(t'',\vec{x}) + \frac{\lambda^2}{23^3H^2}\int_0^t dt'\phi_0(t',\vec{x})\int_0^{t'}dt''\phi_0^3(t'',\vec{x})\bigg\} + O(\lambda^3), \tag{66}$$

which implies

$$\Phi(x;\vec{k}) = \langle\Omega|[\phi(x), \alpha^\dagger(\vec{k})]|\Omega\rangle$$
$$= u(0,k)e^{i\vec{k}\cdot\vec{x}}\Big\{1 - \frac{\lambda}{6H}\int_0^t dt'\langle\Omega|\phi_0^2(t',\vec{x})|\Omega\rangle + \frac{\lambda^2}{2^2 3^2 H^2}\Big[\int_0^t dt'\int_0^{t'} dt''\langle\Omega|\phi_0^2(t',\vec{x})|\Omega\rangle\langle\Omega|\phi_0^2(t'',\vec{x})|\Omega\rangle$$
$$+ 2\int_0^t dt'\int_0^{t'} dt''(\langle\Omega|\phi_0(t',\vec{x})\phi_0(t'',\vec{x})|\Omega\rangle)^2\Big] + \frac{\lambda^2}{2 3^2 H^2}\int_0^t dt'\int_0^{t'} dt''\langle\Omega|\phi_0(t',\vec{x})\phi_0(t'',\vec{x})|\Omega\rangle$$
$$\times\langle\Omega|\phi_0^2(t'',\vec{x})|\Omega\rangle\Big\} + O(\lambda^3). \tag{67}$$

Using Eq. (50) in Eq. (67) yields

$$\Phi(x;\vec{k}) = u(0,k)e^{i\vec{k}\cdot\vec{x}}\Big\{1 - \frac{\lambda}{6H}\int_0^t dt'\frac{H^2}{4\pi^2}\ln(a') + \frac{\lambda^2}{2^2 3^2 H^2}\int_0^t dt'\frac{H^2}{4\pi^2}\ln(a')\int_0^{t'} dt''\frac{H^2}{4\pi^2}\ln(a'') + \frac{\lambda^2}{3^2 H^2}$$
$$\times\int_0^t dt'\int_0^{t'} dt''\frac{H^4}{16\pi^4}\ln^2(a'')\Big\} + O(\lambda^3)$$
$$= u(0,k)e^{i\vec{k}\cdot\vec{x}}\Big\{1 - \frac{1}{2^4 3\pi^2}\lambda\ln^2(a) + \frac{11}{2^9 3^3\pi^4}\lambda^2\ln^4(a)\Big\} + O(\lambda^3). \tag{68}$$

This result is in perfect agreement with Eq. (45), which is obtained by lengthy and highly nontrivial calculation using quantum field theory.

V. CONCLUSIONS

Massless, minimally coupled $\frac{\lambda}{4!}\varphi^4$ on a locally de Sitter background can induce enhanced quantum effects causing superaccelerated phase of cosmic expansion, a possibility not excluded by present observations. In this paper, we have studied the stability of this system for $\lambda \ll 1$. In Sec. II, we have obtained the quantum-corrected effective field equations at linearized order, using the fully renormalized Schwinger-Keldysh self-mass-squared terms at one- and two-loop orders. In Sec. III, we have solved the effective field equations in the late time limit, i.e., for $\ln(a) \gg 1$, and obtained the scalar mode function in leading powers of infrared logarithms at each order of perturbation. In Sec. IV, we have used Starobinsky's stochastic inflation technique to compute the mode function in the leading logarithm approximation and compared it with the quantum field theory result of Sec. III.

There are three main conclusions that we draw: (i) perturbation theory breaks down for $\ln(a(t)) \sim 1/\sqrt{\lambda}$. This, however, does not invalidate the reliability of our late time solutions since one can have a long period of time during which $1 \ll \ln(a) \ll 1/\sqrt{\lambda}$ for $\lambda \ll 1$. (ii) The quantum-corrected mode function decreases in time—consistent with the field developing a positive (nontachyonic) mass squared—starting from its initial classical (Bunch-Davies) value. This means that linear perturbations do not grow in this model. Thus, the model is stable. (iii) The results of quantum field theory and Starobinsky's stochastic technique are in perfect agreement.

The effect can be understood physically as follows. Heisenberg's energy-time uncertainty principle implies that *virtual* particle-antiparticle pairs continually emerge from the vacuum and then disappear back into it. However, massless particles which are also conformally noninvariant have a certain amplitude for appearing with a wavelength greater than the inverse of the Hubble parameter. In an *inflating* universe, these virtual pairs are pulled apart by the Hubble flow before they find time to annihilate each other. Hence, they become real, recalling the analogy with the Hawking radiation. Therefore, one gains particles out of nothing which also means that the scalar field strength *grows*. In fact, $\langle\Omega|\varphi^2|\Omega\rangle = (H^2/4\pi^2)Ht + O(\lambda)$ in our model. Thus, inflationary particle production causes the scalar to undergo a random walk such that its average distance from the minimum of the quartic potential increases. This makes the vacuum energy larger; hence, $\dot{\rho} > 0$. Now recall the covariant stress-energy conservation law $\dot{\rho} = -3H(\rho + p)$. Because inflationary particle production causes $\dot{\rho} > 0$, we must have $\rho + p < 0$ to satisfy this law. Hence, the weak energy condition is violated, and the equation of state parameter $w < -1$. Will this effect be terminated? If the growing of the field, which generates the effect, *stops*, then the effect terminates. There are two causes, in the interacting theory, which would yield the growing of the field come to a halt eventually. The first cause is the *classical restoring force* $-\frac{\lambda}{6}\varphi^3$. This force pushes the field back down to the configuration where the potential is minimum, i.e., to $\varphi = 0$, as the scalar rises up its potential. The second cause is the curvature associated with the field being away from the minimum of the potential, i.e., $\frac{\lambda}{2}\varphi^2$, which acts like a "mass squared." Because $\frac{\lambda}{2}\langle\Omega|\varphi^2|\Omega\rangle$ is a growing positive real number, the scalar develops a growing positive mass. That cuts off particle production, since inflationary particle production requires effective masslessness (with respect to the Hubble parameter). Because of these two causes, the field cannot continue to roll up its position. It must eventually come to a halt. In fact, Starobinsky and Yokoyama showed [7] that $\langle\Omega|\varphi^2|\Omega\rangle$ does asymptote to a positive *constant*.

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APPENDIX A: INTEGRATING THE TWO-LOOP TERMS

In this appendix, we calculate the integral

$$-\int_{\eta_i}^0 d\eta' \int d^3x' \mathcal{M}_2^2(x;x')u(\eta',k)e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}, \tag{A1}$$

which is a part of Eq. (40). $\mathcal{M}_2^2(x;x')$ is defined in Eq. (32). In the late time regime of physical interest, we replace $u(\eta',k)$ [Eq. (15)] with its constant limit $u(0,k)$. Moreover, as can be seen below, when $++$ and $+-$ terms are added, factors of the Heaviside function $\Theta(\Delta\eta - \Delta r)$ arise. They require $\|\vec{x}-\vec{x}'\| \leq \eta - \eta'$, ensuring casuality in the Schwinger-Keldysh formalism. In the late time limit, this means that the spatial plane wave factor $e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$ can also be dropped in Eq. (A1), since $\eta - \eta' \to 0$ in that regime. We, therefore, break the integral (A1) into a sum of six terms of the general form $\sum_{n=1}^6 I_n(\eta)$, with

$$I_n(\eta) \equiv -u(0,k)\int_{\eta_i}^0 d\eta' \int d^3x' \mathcal{M}_{2,n}^2(x;x'). \tag{A2}$$

Here we define the integrands

$$\mathcal{M}_{2,1}^2(x;x') = \frac{i}{2^{12}3\pi^6} aa'\partial^4\left[\frac{\ln(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\right], \tag{A3}$$

$$\mathcal{M}_{2,2}^2(x;x') = -\frac{iH^2}{2^9\pi^6}(aa')^2\partial^2\left[\ln\left(\frac{He^{3/4}}{2\mu}\right)\left(\frac{\ln(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\right)\right], \tag{A4}$$

$$\mathcal{M}_{2,3}^2(x;x') = -\frac{iH^2}{2^{11}\pi^6}(aa')^2\partial^2\left[\frac{\ln^2(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln^2(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\right], \tag{A5}$$

$$\mathcal{M}_{2,4}^2(x;x') = -\frac{iH^4}{2^9\pi^6}(aa')^3\left[\frac{\ln^2(\frac{\sqrt{e}}{4}H^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln^2(\frac{\sqrt{e}}{4}H^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\right], \tag{A6}$$

$$\mathcal{M}_{2,5}^2(x;x') = \frac{iH^6}{2^{10}3\pi^6}(aa')^4\left[\ln^3\left(\frac{\sqrt{e}}{4}H^2\Delta x_{++}^2\right) - \ln^3\left(\frac{\sqrt{e}}{4}H^2\Delta x_{+-}^2\right)\right], \tag{A7}$$

$$\begin{aligned}\mathcal{M}_{2,6}^2(x;x') = &-\frac{1}{2^9 3\pi^4}a^2\{\ln(a)\partial^2 - (2\ln(a)+1)Ha\partial_0\}\delta^4(x-x') - \frac{H^2}{2^7 3^2\pi^4}a^4\Big\{4\ln^3(a) + \frac{23}{2}\ln^2(a) \\ &- \left[39 + 27\ln\left(\frac{H}{2\mu}\right) - 2\pi^2\right]\ln(a)\Big\}\delta^4(x-x') + \frac{H^2}{2^7\pi^4}a^4\Big\{\frac{a^{-3}}{81} - \sum_{n=1}^\infty \frac{n+5}{(n+1)^3}a^{-(n+1)} + 4\sum_{n=1}^\infty\frac{a^{-(n+2)}}{(n+2)^3} \\ &+ 4\sum_{n=1}^\infty \frac{a^{-(n+3)}}{n(n+3)^3}\Big\}\delta^4(x-x').\end{aligned} \tag{A8}$$

We evaluate $I_1(\eta)$ and $I_3(\eta)$ explicitly to illustrate the relevant calculation techniques and give only the final answers for the remaining four that can be obtained similarly. The first integral is

$$I_1(\eta) \equiv -u(0,k)\int_{\eta_i}^\eta d\eta' \int d^3x' \mathcal{M}_{2,1}^2(x;x'). \tag{A9}$$

It is useful to break up the logarithms in $\mathcal{M}_{2,1}^2$ as

$$\ln(\mu^2\Delta x^2) = \ln\left(\frac{H^2\Delta x^2}{4}\right) + 2\ln\left(\frac{2\mu}{H}\right). \tag{A10}$$

We then partially integrate the inverse powers of Δx^2, using the identities

$$\frac{\ln(H^2\Delta x^2)}{\Delta x^2} = \frac{\partial^2}{8}\ln^2(H^2\Delta x^2) - \frac{1}{\Delta x^2}, \tag{A11}$$

$$\frac{1}{\Delta x^2} = \frac{\partial^2}{4} \ln(H^2 \Delta x^2). \tag{A12}$$

Because the integral is over x'^{μ}, the derivatives with respect to x^{μ} can be taken outside the integral. The remaining integrand possesses only logarithmic singularities. There is no distinction between the $++$ and $+-$ terms at this stage. We define the temporal and spatial intervals $\Delta\eta \equiv \eta - \eta'$ and $\Delta r \equiv \|\vec{x} - \vec{x}'\|$, respectively. The $++$ and $+-$ terms cancel for $\Delta\eta < 0$, so we can restrict the integration to $\Delta\eta > 0$. Then the logarithms can be expanded as

$$\ln\left(\frac{H^2}{4}\Delta x^2_{+\pm}\right) = \ln\left(\frac{H^2}{4}(\Delta\eta^2 - \Delta r^2)\right) \pm i\pi\Theta(\Delta\eta^2 - \Delta r^2). \tag{A13}$$

Making the change of variables $\vec{r} = \vec{x} - \vec{x}'$ and performing the angular integrals yield

$$I_1(\eta) = -\frac{1}{2^{11}3\pi^4} u(0,k) a \partial_0^6 \int_{\eta_i}^{\eta} d\eta' a' \int_0^{\Delta\eta} dr r^2 \times \left[\ln\left(\frac{H^2}{4}(\Delta\eta^2 - \Delta r^2)\right) + 2\ln\left(\frac{2\mu}{H}\right) - 1\right],$$

where the initial time $\eta_i \equiv -H^{-1}$. Next, we make the change of variables $r \equiv \Delta\eta z$ and perform the integration over z, using

$$\int_0^1 dz z^2 \ln\left(\frac{1-z^2}{4}\right) = -\frac{8}{9}. \tag{A14}$$

The result is

$$I_1(\eta) = -\frac{1}{2^{10}3^2\pi^4} u(0,k) a \partial_0^3 \int_{\eta_i}^{\eta} d\eta' a' \partial_0^3 \times \left[\Delta\eta^3\left(\ln(H\Delta\eta) + \ln\left(\frac{2\mu}{H}\right) - \frac{11}{6}\right)\right]. \tag{A15}$$

Note that, owing to the factor $\Delta\eta^3$, three of the external derivatives were brought inside the integral. This cubic derivative gives $6\ln(2\mu\Delta\eta)$, when it acts upon the terms inside the square bracket. At this stage, one makes the change of variables $\eta' = -(Ha')^{-1}$ and looks up the relevant integral [Eq. (B1)] from Appendix B. Acting on the remaining derivatives using $\partial_0 = Ha^2\frac{\partial}{\partial a}$, one obtains

$$I_1(\eta) = \frac{H^2}{2^8 3\pi^4} u(0,k) a^4 \left\{\ln(a) - \ln\left(\frac{2\mu}{H}\right) + \frac{3}{2} + \sum_{n=1}^{\infty} \frac{(n-1)(n-2)}{2n} a^{-n}\right\}. \tag{A16}$$

Evaluation of $I_2(\eta)$, defined by Eqs. (A2) and (A4), is similar to that of $I_1(\eta)$. Using Eqs. (A10)–(A14) and (B2) yields

$$I_2(\eta) = \frac{H^2}{2^6\pi^4} \ln\left(\frac{He^{3/4}}{2\mu}\right) u(0,k) a^4 \left\{\ln(a) - \ln\left(\frac{2\mu}{H}\right) + 1 - \ln\left(1 - \frac{1}{a}\right) - a^{-1}\right\}. \tag{A17}$$

Next, we explicitly evaluate

$$I_3(\eta) \equiv -u(0,k) \int_{\eta_i}^{\eta} d\eta' \int d^3x' \mathcal{M}^2_{2,3}(x;x'), \tag{A18}$$

where $\mathcal{M}^2_{2,3}$ is given in Eq. (A5). We break up the logarithm squared in $\mathcal{M}^2_{2,3}$ as

$$\ln^2(\mu^2\Delta x^2) = \ln^2\left(\frac{H^2\Delta x^2}{4}\right) + 4\ln\left(\frac{2\mu}{H}\right)\ln\left(\frac{H^2\Delta x^2}{4}\right) + 4\ln^2\left(\frac{2\mu}{H}\right). \tag{A19}$$

Then we use the identity

$$\frac{\ln^2(H^2\Delta x^2)}{\Delta x^2} = \frac{\partial^2}{12}\ln^3(H^2\Delta x^2) - \frac{\partial^2}{4}\ln^2(H^2\Delta x^2) + \frac{2}{\Delta x^2} \tag{A20}$$

and Eqs. (A12) and (A13). To evaluate the radial integral, we make the change of variables $r \equiv \Delta\eta z$ and use Eq. (A14) and the integral

$$\int_0^1 dz z^2 \ln^2\left(\frac{1-z^2}{4}\right) = \frac{104}{27} - \frac{\pi^2}{9}. \tag{A21}$$

We find

$$I_3(\eta) = -\frac{H^2}{2^8 3\pi^4} u(0,k) a^2 \partial_0 \int_{\eta_i}^{\eta} d\eta' a'^2 \partial_0^3 [\Delta\eta^3(\ln^2(H\Delta\eta) + A\ln(H\Delta\eta) + B)], \tag{A22}$$

where $A \equiv 2\ln(2\mu/H) - 11/3$ and $B \equiv \ln^2(2\mu/H) - (11/3)\ln(2\mu/H) + 85/18 - \pi^2/6$. The cubic derivative in the integrand yields $6\ln^2(2\mu\Delta\eta) - \pi^2$, when it acts upon the terms inside the square bracket. Making the change of variables $\eta' = -(Ha')^{-1}$ and using Eqs. (B2)–(B6) and $\partial_0 = Ha^2\frac{\partial}{\partial a}$, one obtains

$$I_3(\eta) = \frac{H^2}{2^6\pi^4} u(0,k) a^4 \left\{-\frac{1}{2}\ln^2(a) + \left[\ln\left(\frac{2\mu}{H}\right) - 1\right]\ln(a) - \left[\frac{1}{2}\ln\left(\frac{2\mu}{H}\right) - 1 + a^{-1}\right]\ln\left(\frac{2\mu}{H}\right) - \frac{\pi^2}{12} - \left[\ln\left(\frac{2\mu}{H}\right) + a^{-1}\right]\ln\left(1 - \frac{1}{a}\right) - \frac{1}{2}\ln^2\left(1 - \frac{1}{a}\right) - \sum_{n=1}^{\infty}\frac{(n-1)}{n^2} a^{-n}\right\}. \tag{A23}$$

Evaluation of $I_4(\eta)$, defined by Eqs. (A2) and (A6), is similar to that of $I_3(\eta)$. Using Eqs. (A12)–(A14), (A19)–(A21), (B2), (B3), (B6), and (B7) yields

$$I_4(\eta) = \frac{H^2}{2^6\pi^4}u(0,k)a^4\Big\{\ln^2(a) - \frac{1}{2}\ln(a) + \frac{1}{16} + \frac{\pi^2}{6} + \Big[\frac{1}{2} + a^{-1} - \frac{3}{2}a^{-2}\Big]\ln\Big(1-\frac{1}{a}\Big) + [1 - 2a^{-1} + a^{-2}]\ln^2\Big(1-\frac{1}{a}\Big) - \Big[\frac{13}{8} - \frac{\pi^2}{3}\Big]a^{-1} + \Big[\frac{25}{16} - \frac{\pi^2}{6}\Big]a^{-2} - 2\sum_{n=1}^{\infty}\frac{a^{-n}}{n^2}\Big\}. \tag{A24}$$

To evaluate $I_5(\eta)$, we expand $\ln^3(\sqrt{e}H^2\Delta x^2/4) = [\ln(H^2\Delta x^2/4) + (1/2)]^3$ and use Eqs. (A13), (A14), (A21), and (B5)–(B8). The result is

$$I_5(\eta) = \frac{H^2}{2^5 3^2\pi^4}u(0,k)a^4\Big\{\ln^3(a) - \frac{9}{4}\ln^2(a) + \Big[\frac{15}{16} + \frac{\pi^2}{2}\Big]\ln(a) - \frac{2035}{288} + \frac{\pi^2}{6} + \Big[\frac{71}{12} - \frac{25}{2}a^{-1} + \frac{35}{4}a^{-2} - \frac{13}{6}a^{-3}\Big] \times \ln\Big(1-\frac{1}{a}\Big) - \Big[\frac{11}{2} - 9a^{-1} + \frac{9}{2}a^{-2} - a^{-3}\Big]\ln^2\Big(1-\frac{1}{a}\Big) + \Big[\frac{671}{48} - \frac{3\pi^2}{2}\Big]a^{-1} - \Big[\frac{883}{96} - \frac{3\pi^2}{4}\Big]a^{-2} + \Big[\frac{329}{144} - \frac{\pi^2}{6}\Big]a^{-3} - 6\sum_{n=1}^{\infty}\frac{a^{-n}}{n^2}\Big[\psi(n) + \gamma - \frac{3}{4} - \frac{1}{n}\Big]\Big\}, \tag{A25}$$

where the Digamma function $\psi(n) \equiv -\gamma + \sum_{k=1}^{n-1}k^{-1}$ and Euler's gamma $\gamma \simeq 0.577$. It is straightforward to show that the remaining integral

$$I_6(\eta) = \frac{H^2}{2^5 3^2\pi^4}u(0,k)a^4\Big\{\ln^3(a) + \frac{23}{8}\ln^2(a) + \Big[\frac{27}{4}\ln\Big(\frac{2\mu}{H}\Big) - \frac{39}{4} + \frac{\pi^2}{2}\Big]\ln(a) - \frac{1}{36}a^{-3} + \frac{9}{4}\sum_{n=1}^{\infty}\frac{n+5}{(n+1)^3}a^{-(n+1)} - 9\sum_{n=1}^{\infty}\frac{a^{-(n+2)}}{(n+2)^3} - 9\sum_{n=1}^{\infty}\frac{a^{-(n+3)}}{n(n+3)^3}\Big\}. \tag{A26}$$

Summing the six terms gives the total two-loop contribution in Eq. (40)

$$-\int_{\eta_i}^{\eta}d\eta'\int d^3x'\mathcal{M}_2^2(x;x')u(\eta',k)e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \rightarrow \sum_{n=1}^{6}I_n(\eta) = \frac{H^2}{2^4 3^2\pi^4}u(0,k)a^4\Big\{\ln^3(a) + \frac{23}{16}\ln^2(a) + \Big[\frac{27}{8}\ln\Big(\frac{2\mu}{H}\Big) - \frac{189}{32} + \frac{\pi^2}{2}\Big] \times \ln(a) + \Big[\frac{9}{8}\ln\Big(\frac{2\mu}{H}\Big) - \frac{15}{8}\Big]\ln\Big(\frac{2\mu}{H}\Big) - \frac{205}{144} + \frac{13}{48}\pi^2 + \Big[\frac{115}{48} - \frac{25}{4}a^{-1} + a^{-2} - \frac{13}{12}a^{-3}\Big]\ln\Big(1-\frac{1}{a}\Big) - \Big[\frac{13}{8} - \frac{a^{-3}}{2}\Big] \times \ln^2\Big(1-\frac{1}{a}\Big) + \frac{79}{48}a^{-1} - \frac{13}{12}a^{-2} + \Big[\frac{325}{288} - \frac{\pi^2}{12}\Big]a^{-3} + \frac{9}{8}\sum_{n=1}^{\infty}\frac{n+5}{(n+1)^3}a^{-(n+1)} - \frac{9}{2}\sum_{n=1}^{\infty}\frac{a^{-(n+2)}}{(n+2)^3} - \frac{9}{2}\sum_{n=1}^{\infty}\frac{a^{-(n+3)}}{n(n+3)^3} + \frac{3}{8}\sum_{n=1}^{\infty}\Big[\frac{n}{4} - \frac{3}{4} - \frac{11}{2n} - \frac{8}{n^2}\Big(\psi(n) + \gamma - \frac{1}{n}\Big)\Big]a^{-n}\Big\}. \tag{A27}$$

APPENDIX B: USEFUL INTEGRAL IDENTITIES

In Appendix A, to calculate the temporal integrations over η', we make the change of variables $\eta' = -(Ha')^{-1}$ and use the following integral identities:

$$\int_1^a \frac{da'}{a'}\ln\Big(\frac{1}{a'} - \frac{1}{a}\Big) = -\frac{1}{2}\ln^2(a) - \sum_{n=1}^{\infty}\frac{(1-a^{-n})}{n^2}, \tag{B1}$$

$$\int_1^a da'\ln\Big(\frac{1}{a'} - \frac{1}{a}\Big) = -a\ln(a) + (a-1)\ln\Big(1-\frac{1}{a}\Big), \tag{B2}$$

$$\int_1^a da'a'\ln\Big(\frac{1}{a'} - \frac{1}{a}\Big) = -\frac{a^2}{2}\ln(a) - \frac{1}{2}(a^2 - a) + \frac{1}{2}(a^2-1)\ln\Big(1-\frac{1}{a}\Big), \tag{B3}$$

$$\int_1^a da' a'^2 \ln\Big(\frac{1}{a'} - \frac{1}{a}\Big) = -\frac{a^3}{3}\ln(a) - \frac{a^3}{2} + \frac{a^2}{3} + \frac{a}{6} + \frac{1}{3}(a^3 - 1)\ln\Big(1 - \frac{1}{a}\Big), \tag{B4}$$

$$\int_1^a \frac{da'}{a'}\ln^2\Big(\frac{1}{a'} - \frac{1}{a}\Big) = \frac{1}{3}\ln^3(a) + \frac{\pi^2}{3}\ln(a) + 2\sum_{n=1}^{\infty}\frac{(1 - a^{-n})}{n^2} \times \Big[\psi(n) + \gamma - \frac{1}{n}\Big], \tag{B5}$$

where $\psi(n)$ is the Digamma function and γ is the Euler's constant, as defined in Appendix A,

$$\int_1^a da' \ln^2\Big(\frac{1}{a'} - \frac{1}{a}\Big) = a\ln^2(a) + 2a\sum_{n=1}^{\infty}\frac{(1 - a^{-n})}{n^2} + (a - 1)\ln^2\Big(1 - \frac{1}{a}\Big), \tag{B6}$$

$$\int_1^a da' a' \ln^2\Big(\frac{1}{a'} - \frac{1}{a}\Big) = \frac{a^2}{2}\ln^2(a) + a^2\ln(a) + a^2\sum_{n=1}^{\infty}\frac{(1 - a^{-n})}{n^2} - a(a - 1)\ln\Big(1 - \frac{1}{a}\Big) + \frac{1}{2}(a^2 - 1)\ln^2\Big(1 - \frac{1}{a}\Big), \tag{B7}$$

$$\int_1^a da' a'^2 \ln^2\Big(\frac{1}{a'} - \frac{1}{a}\Big) = \frac{a^3}{3}\ln^2(a) + a^3\ln(a) + \frac{a^2}{3}(a - 1) + \frac{2}{3}a^3\sum_{n=1}^{\infty}\frac{(1 - a^{-n})}{n^2} - \frac{a}{3}(a - 1)(3a + 1)\ln\Big(1 - \frac{1}{a}\Big) + \frac{1}{3}(a^3 - 1)\ln^2\Big(1 - \frac{1}{a}\Big). \tag{B8}$$

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