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Approximate Killing vectors on S^2

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We present a new method for computing the best approximation to a Killing vector on closed 2-surfaces that are topologically S^2. When solutions of Killing's equation do not exist, this method is shown to yield results superior to those produced by existing methods. In addition, this method appears to provide a new tool for studying the horizon geometry of distorted black holes.

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I. INTRODUCTION

An exact geometric sphere possesses a two parameter family of rotational Killing vectors while even a slightly distorted sphere may possess no Killing vectors whatsoever. Nevertheless, one could imagine defining perturbations of these initial vectors which would, in some well-defined "best" sense, represent the closest available approximation to vectors which almost satisfy Killing's equation on such a slightly distorted sphere. In this paper we introduce a definition which is best in a least squared sense and discuss some of its attributes.

In general relativity, rotational Killing vectors play an important role in providing a quasilocal definition for the spin of a rotating body. A system of astrophysical interest, such as a pair of orbiting black holes, possesses no global rotational Killing vectors. In this case, angular momentum can only be rigorously defined for the system as a whole in terms of asymptotic rotational Killing vectors.

A quantity of great importance in the evolution of black-hole binaries is the spin of the individual black holes. The spin of such black holes can only be determined by some approximate quasilocal definition (see Ref. [1] for a review). There exist many different quasilocal definitions for angular momentum, but they all take the form of an integral over a 2-surface with topology S^2, and all require a rotational Killing vector on this surface.

In numerical relativity, the quasilocal spin of a black hole is most often expressed as

$$S_{(\xi)} = \frac{1}{8\pi G}\oint_{\mathcal{S}} K_{ij}\xi^j d^2S^i, \tag{1}$$

where K_{ij} is the extrinsic curvature of a spatial hypersurface with metric γ_{ij}, d^2S^i is the area element of an S^2 surface of integration $\mathcal{S}$ taken to be a black hole's apparent horizon, and ξ^i is a Killing vector of the metric h_{ij} induced on $\mathcal{S}$ by γ_{ij}. This form was derived by Brown and York [2] and later within the Isolated Horizons framework (see Ref. [3] for a review). It gives the angular momentum of the rotation associated with the rotational Killing vector ξ^i. Unfortunately, the induced metric h_{ij} will not admit a solution of Killing's equation for the case of orbiting black-hole binaries. In this situation, one has no recourse but to find some reasonable approximation for the Killing vector required in Eq. (1).

In some cases, conformal Killing vectors have been used, and have yielded physically reasonable results [4]. Better still, a "Killing Transport" (KT) technique [5], which finds exact Killing vectors when they are present, has been recently adopted, and appears to give physically reasonable results (cf. Refs. [4,6]). Our definition is shown to be even better in a well-defined least squared sense and, for coalescing binary black holes, yields other interesting results worthy of further investigation.

II. EQUATIONS FOR THE BEST APPROXIMATE KILLING VECTOR

An arbitrary vector field ξ^i on S^2 can be decomposed into two scalars d and v

$$\xi^i \equiv D^i d + \epsilon^{ij}D_j v, \tag{2}$$

where D_i is the covariant derivative compatible with the metric h_{ij} induced on the surface $\mathcal{S}$ and ϵ_{ij} is the Levi-Civita tensor. Similarly, the general gradient of a 1-form can be expressed as

$$D_i\xi_j \equiv L\epsilon_{ij} + h_{ij}\Lambda + S_{ij}, \tag{3}$$

where L and Λ are scalars, and S_{ij} is symmetric and trace free. Equations (2) and (3) immediately imply that

$$\Lambda = \tfrac{1}{2}D^iD_i d, \tag{4}$$

$$L = -\tfrac{1}{2}D^iD_i v, \tag{5}$$

$$S_{ij} = D_iD_j d - \tfrac{1}{2}h_{ij}D^kD_k d + \tfrac{1}{2}(\epsilon_{ik}D_jD^k v + \epsilon_{jk}D_iD^k v). \tag{6}$$

For ξ^i to be Killing, it must satisfy Killing's equation $D_{(i}\xi_{j)} = 0$ where the parentheses denote symmetrization. This implies that Λ and S_{ij} must vanish if ξ^i is Killing. We

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may choose Λ to vanish, in which case Eq. (4) implies that d is harmonic. But, assuming a nonsingular metric, the only harmonic function on S^2 is a constant, which makes no contribution to ξ^i. Thus, we are left with

$$\xi^i \equiv \epsilon^{ij} D_j v, \tag{7}$$

$$D_i \xi_j \equiv L \epsilon_{ij} + S_{ij}. \tag{8}$$

Our goal is to find an approximate Killing vector that minimizes the non-Killing aspects of ξ^i. Clearly, having already set Λ to zero, a solution with S_{ij} as close to zero as possible is what we desire, so we proceed by finding a vector ξ^i that minimizes $S_{ij}S^{ij}$.

From Eqs. (7) and (8) it follows that

$$L = \tfrac{1}{2}\epsilon_{ij} D^i \xi^j, \tag{9}$$

and

$$S_{ij}S^{ij} = (D_i D_j v)(D^i D^j v) - \tfrac{1}{2}(D^k D_k v)^2. \tag{10}$$

Now, we wish to extremize $S_{ij}S^{ij}$ with respect to v. But, we must do this in a way that is independent of the normalization of ξ^i. Using $|\xi|^2 = (D_i v)(D^i v)$, we choose the following scalar function on $\mathcal{S}$

$$\mathcal{L} \equiv S_{ij}S^{ij} + \tfrac{1}{2}\,{}^2R\Theta(D_k v)(D^k v), \tag{11}$$

where ${}^2R$ is the Ricci scalar associated with h_{ij} and Θ is a *dimensionless* constant. The factor of ${}^2R$ is included in the constraint term of Eq. (11) on dimensional grounds and also gives extra weight to regions where the curvature is large, however other choices for the constraint term may also prove useful [7]. Varying with respect to v, $\delta\mathcal{L}/\delta v = 0$ yields a fourth-order scalar elliptic equation for v that can be rewritten as a pair of second-order scalar elliptic equations for v and L

$$D^i D_i L - (1-\Theta)[\tfrac{1}{2}(D^i\,{}^2R)D_i v - {}^2RL] = 0, \tag{12}$$

$$D^i D_i v + 2L = 0. \tag{13}$$

Equations (12) and (13) can be solved for L and v, with the Lagrange multiplier Θ fixed by the requirement that Eq. (12) be integrable on S^2. Given a solution for v, the approximate rotational Killing vector is given by Eq. (7) once it is normalized to have affine length 2π. We note that Eq. (12) is satisfied by a Killing vector ξ^i when $\Theta = 0$ (with L defined by Eq. (9) and $D_i v = (2/{}^2R)D_i L$).

It has been noted [8,9] that an approximate Killing vector should be divergence free $D_i\xi^i = 0$, in part because the angular momenta computed using such an approximate Killing vector possess a certain gauge invariance. Approximate Killing vectors constructed using the KT method are not guaranteed to be divergenceless, although at least in certain cases [4] this can be enforced *a posteriori*. These approximate Killing vectors also inherit an additional problem. A defining equation of the KT method [5], which can be written in the form of Eq. (9), is not satisfied by solutions of the KT equations unless there is a Killing vector. This is due to the path dependence of the solution scheme. Moreover, Eq. (9) *cannot* be enforced *a posteriori*.

Because our approximate Killing vector is defined from the solution of Eqs. (12) and (13) via Eq. (7), it is guaranteed to be divergenceless. Furthermore, because our solution is obtained via a global solution of elliptic equations, Eq. (9) is also guaranteed to be satisfied.

III. TESTS

We have implemented a code to solve Eqs. (12) and (13) for situations where the metric h_{ij} is conformal to a unit 2-sphere:

$$ds^2 = \psi^4 r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{14}$$

Details of the solution scheme will be presented in a future paper [10]. While this form for the metric may seem to be a strong simplification, a scheme based on this form is suitably general since any sufficiently smooth metric on S^2 is conformally equivalent to a unit 2-sphere.

To explore our method, we first consider the case where the conformal factor is given by an $\ell = 2$, $m = 0$ scalar spherical harmonic with its axis of symmetry rotated to a direction given by (θ', ϕ')

$$\psi(\theta, \phi) = A + B\sum_{m=-2}^{2} Y_{2m}(\theta, \phi)Y^*_{2m}(\theta', \phi'), \tag{15}$$

and A and B are real constants chosen to guarantee that $\psi > 0$ everywhere. The form of Eq. (15) guarantees that the metric possesses a rotational Killing vector and in all the cases we attempted, solving Eqs. (12) and (13) returned a solution where the axis of symmetry was correctly rotated by (θ', ϕ') and for which $\Theta = 0$. As mentioned above, the solutions are divergenceless and satisfy Eq. (9) to the level of roundoff error. Furthermore, we find $S_{ij}S^{ij} = 0$ also to the level of roundoff error, as expected for a solution that yields a true Killing vector.

Interestingly, Eqs. (12) and (13) in general have *multiple solutions*. In fact for the example given by Eq. (15), there exist an infinitely degenerate set of solutions where the axis of approximate symmetry lies anywhere in the rotated equatorial plane. For these solutions, $\Theta \neq 0$ and $S_{ij}S^{ij} \neq 0$ since these solutions are not true Killing vectors. Again, the solutions are divergenceless and satisfy Eq. (9) to the level of roundoff error.

We have also tested the system of equations against the numerically generated initial data for corotating and nonspinning equal-mass black-hole binaries as described in Ref. [4]. We use data for different orbital separations, parametrized by the dimensionless orbital angular velocity $M\Omega_0$, where M is the total irreducible mass of the binary. For the case of corotating black holes, the spin of each black hole is aligned with the direction of the orbital angular momentum. In all cases tested, we have found that the measured spins based on an approximate Killing vector obtained using Eqs. (12) and (13) are *nearly identical* to those based on the KT method, with differences growing only to a few parts in 10^7. While the resulting spins are nearly identical, our solutions are measurably different from the results of the KT method, and we find that our system of equations produces solutions for which $\langle S_{ij}S^{ij}\rangle \equiv (4\pi)^{-1}\oint S_{ij}S^{ij}d\Omega$ is always smaller than that produced by the KT method. Note that in both cases, the approximate Killing vectors have been properly normalized to have affine length 2π. Figure 1 shows $\langle S_{ij}S^{ij}\rangle$ as computed by our method and the KT method, and the analogous quantity $\langle 2\Lambda^2\rangle$ obtained by using conformal Killing vectors. On a separate scale, Fig. 1 also shows the value of Θ obtained by our method for the same data.

As with the first example given by Eq. (15), our equations yield multiple solutions for the numerically generated black-hole data. The solutions shown in Fig. 1 have their axis of approximate symmetry aligned with the orbital angular momentum (z-axis) as expected. In all cases, we find *two additional solutions*. Both have their axis of approximate symmetry in the orbital plane and we refer to them as Solutions 1 and 2. We find that for all orbital separations, the directions of these two symmetry axes are orthogonal to the level of numerical error in the code (truncation error). For large separation (small $M\Omega_0$), Solution 1 has its axis of approximate symmetry pointed roughly in the direction of motion of the black hole (y-axis) at an azimuthal angle of ϕ_1, and Solution 2 has its axis pointed roughly toward the companion black hole (x-axis) at an angle of ϕ_2. The directions ϕ_1 and ϕ_2 (mod π) for these two solutions are displayed in Fig. 2. Here, zero azimuthal angle is in the direction of the positive x-axis. For small separations, the roles are reversed and we find that Solution 1 points roughly toward the x-axis and Solution 2 toward the y-axis.

The regime between large and small orbital separations, where the two solutions swap orientations, is quite interesting. Over most of the range of $M\Omega_0$ considered, ϕ_1 and ϕ_2 change gradually. However, over a narrow range of $M\Omega_0$, the angles change rapidly but smoothly, rotating by an angle of approximately $\pi/2$. Interestingly, the value of $\langle S_{ij}S^{ij}\rangle$ for Solution 1 is smaller than that for Solution 2 for all separations. At the point where $\phi_1 = \phi_2 - \pi/2 = \pi/4$, curves of $\langle S_{ij}S^{ij}\rangle$ for the two solutions "appear" to cross. However, a careful examination shows this to be an "avoided" crossing. We shall examine this behavior in more detail in a future paper [10].

Finally, if we measure the spin of the black holes using the approximate Killing vectors associate with Solutions 1 and 2, all cases yield zero to truncation error. So, for the case of corotating equal-mass black-hole binaries, we find three "orthogonal" solutions and only one of them yields a nonvanishing spin.

We find similar results for the case of nonspinning black-hole-binary initial data. There is a solution with an axis of approximate symmetry aligned with the orbital angular momentum and two additional solutions with their axes in the orbital plane and similarly orthogonal to each other. As discussed in Ref. [4], the nonspinning black-hole data we use are defined by setting the spin measured by the KT method to zero. The corresponding spins measured by

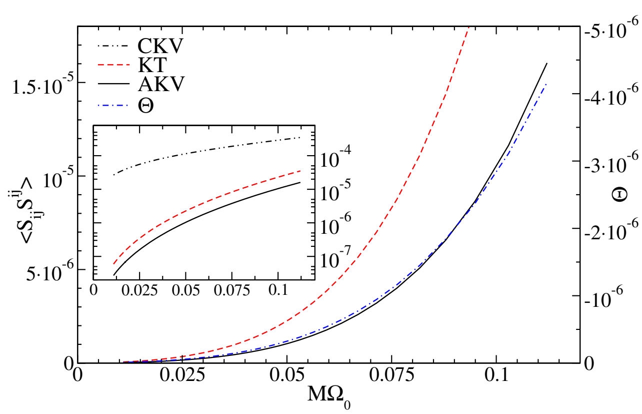

FIG. 1 (color online). The value of $\langle S_{ij}S^{ij}\rangle$ for the new method is displayed as a solid (black) line and its value when using the KT method is shown as a dashed (red) line. The analogous quantity $\langle 2\Lambda^2\rangle$ when using conformal Killing vectors is shown as a dot-dot-dashed (black) line visible in the inset where a logarithmic scaling is used. The value of the Lagrange multiplier Θ obtained using the new method is also displayed as a dot-dashed (blue) line and its scale is shown on the right. All data is from one corotating black hole in an equal-mass binary.

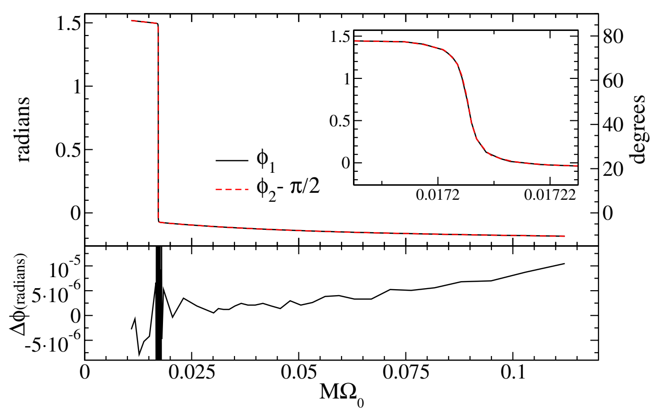

FIG. 2 (color online). The top portion of the figure displays the directions of the additional approximate symmetry axes with $\phi = 0$ in the positive x direction. The axis direction of the Solution 1 is displayed as a solid (black) curve and the direction for Solution 2, with $\pi/2$ subtracted, is displayed as a dashed (red) line. The lines are visually coincident. The lower portion of the figure displays $\Delta\phi \equiv \phi_1 - \phi_2 + \pi/2$ showing the degree to which the two axes are approximately orthogonal. All data is from one corotating black hole in an equal-mass binary.

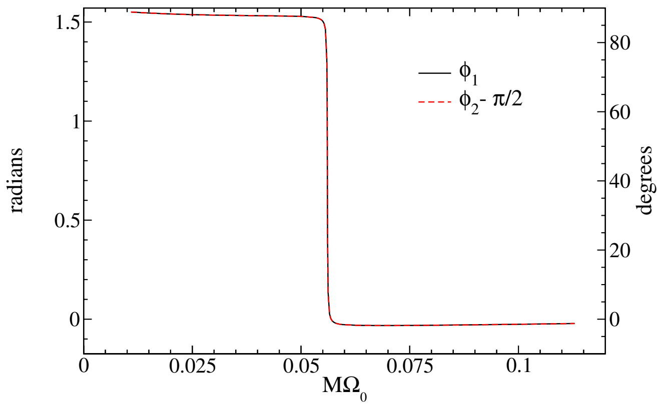

FIG. 3 (color online). The directions of the additional approximate symmetry axes from one nonspinning black hole in an equal-mass binary. See description for Fig. 2.

our method again differ at most by a few parts in 10^7, and in all cases we find the value of $\langle S_{ij}S^{ij}\rangle$ for the new method to be smaller than the value from the KT solution. The behavior of the additional solutions is qualitatively the same as seen for the case of corotation and is displayed in Fig. 3. While the behaviors are generally similar, the directions of the approximate symmetry axes are somewhat different and the rapid change in the direction of the solutions occurs at much smaller separation.

IV. DISCUSSION

We have mentioned two previous methods defined in the literature for use in computing the spin of rotating black holes that lack axial symmetry. Both have been considered useful in the past, and yet both have shortcomings. When a Killing vector does not exist, the conformal Killing approach returns a vector for which $D_i\xi^i \neq 0$. By contrast, the Killing Transport method constructs a ξ^i that can often be made divergenceless. It also constructs the scalar L (see Eq. (3)), but generally this does not satisfy Eq. (9). Our new method not only ensures both that ξ^i is divergenceless and that Eq. (9) is satisfied, but it also is best in the sense that $\langle S_{ij}S^{ij}\rangle$ is minimal. Since a Killing vector cannot be produced where one does not exist, the usefulness of our results for an approximate Killing vector will depend on the extent to which physical questions (such as concern black-hole spins) can be given meaningful answers. In particular, it may have immediate application in giving a more refined definition for binaries containing black holes without individual spin.

One sense in which our results already appear meaningful relates to the vectors found to reside in our xy-plane, for which the corresponding spins are computed to be zero to the level of truncation error. Any other outcome for these would have been somewhat unpalatable. Their actual orientation for binary black holes at large separation can be easily interpreted in terms of boosted frames. Closer in, their combined dramatic rotation at some critical separation warrants further investigation, as does the apparent occurrence of avoided crossings for $\langle S_{ij}S^{ij}\rangle$ and for Θ at the critical separation.

Another sense in which our results appear meaningful relates to the solution associated with our z-direction. For sufficiently large black-hole separation, our result leads to spins which are in good agreement with those of the KT method. This is not unreasonable, even though we find small differences between the two solutions for ξ^i over the surface of the apparent horizon. However, for highly distorted black holes—such as near merger or with near maximal rotation, or for unaligned spins—we can imagine that our results could prove to be more robust. More extensive comparisons than we have been able to carry out here will be necessary before this expectation might be practically substantiated.

To a high degree of precision, it appears that the axes of the approximate symmetries we find form an orthogonal basis in all the cases we have examined. This is of considerable interest, because this basis is directly related to intrinsic properties of the apparent horizons we have studied. Thus, in addition to the reasonableness of our results associated with both the rotation axis and the orbital plane, our method appears to give a new tool for studying the horizon geometry of distorted black holes in general, and the volatile dynamics of black holes during collision, in particular. Further investigation of the usefulness of this new tool will be forthcoming.

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[1] L. B. Szabados, Living Rev. Relativity **7** (2004), http://www.livingreviews.org/lrr-2004-4.

[2] J. D. Brown and J. W. York, Jr., Phys. Rev. D **47**, 1407 (1993).

[3] A. Ashtekar and B. Krishnan, Living Rev. Relativity **7** (2004), http://www.livingreviews.org/lrr-2004-10.

[4] M. Caudill, G. B. Cook, J. D. Grigsby, and H. P. Pfeiffer, Phys. Rev. D **74**, 064011 (2006).

[5] O. Dreyer, B. Krishnan, D. Shoemaker, and E. Schnetter, Phys. Rev. D **67**, 024018 (2003).

[6] M. Campanelli, C. O. Lousto, Y. Zlochower, B. Krishnan, and D. Merritt, Phys. Rev. D **75**, 064030 (2007).

[7] R. A. Matzner, J. Math. Phys. (N.Y.) **9**, 1657 (1968).

[8] E. Schnetter, B. Krishnan, and F. Beyer, Phys. Rev. D **74**, 024028 (2006).

[9] S. A. Hayward, Phys. Rev. D **74**, 104013 (2006).

[10] G. B. Cook and B. F. Whiting (unpublished).