Connecting giant magnons to the pp-wave: An interpolating limit of $AdS_5 \times S^5$

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We consider a particular large-radius limit of the world sheet S matrix for strings propagating on $AdS_5 \times S^5$. This limiting theory interpolates smoothly between the so-called plane-wave and giant-magnon regimes of the theory. The sigma model in this region simplifies; it stands as a toy model of the full theory, and may be easier to solve directly. The S matrix of the limiting theory is nontrivial, and receives contributions to all orders in the α' expansion. We analyze a guess for the full world sheet S matrix that was formulated recently by Beisert, Hernandez, and Lopez, and Beisert, Eden, and Staudacher, and take the corresponding limit. After doing a Borel resummation we find that the proposed S matrix reproduces the expected results in the giant-magnon region. In addition, we rely on general considerations to draw some basic conclusions about the analytic structure of the S matrix.

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I. INTRODUCTION

Recently there has been a lot of activity regarding the world sheet S matrix for type IIB string theory on $AdS_5 \times S^5$, and the corresponding S matrix[1] for planar $\mathcal{N} = 4$ super-Yang Mills theory. This is an object that arises when one considers operators or states with very large charge J under an $SO(2)$ subgroup of $SO(6)$. In the limit $J \to \infty$ with $\Delta - J$ finite, where Δ is the conformal dimension, we are led to consider a finite set of impurities propagating along an infinite spin chain [1]. The Hamiltonian of this spin chain is formulated to reproduce the action of the gauge theory dilatation generator on single-trace operators in $\mathcal{N} = 4$ super-Yang Mills theory. On the string theory side, one can go to light-cone gauge and obtain a rather complicated looking two-dimensional theory on an infinite line [2,3]. Both systems have elementary excitations that have been dubbed magnons. These theories are conjectured to be integrable, so that the full S matrix on either side of the correspondence is determined entirely in terms of $2 \to 2$ scattering processes. It was shown by Beisert that the *matrix* structure of the basic $2 \to 2S$ matrix is fixed by symmetries [4], so what remains is to determine the phase factor. Recently there has been an interesting guess for this phase proposed by Beisert, Eden, and Staudacher (BES) [5], based on previous work by Beisert, Hernandez, and Lopez (BHL) [6]. We will refer to this guess as BES/BHL.[2]

In this article we study an interesting strong-coupling limit of the world sheet S matrix, wherein the sigma model describing the system simplifies, but the S matrix itself remains nontrivial. We find this limit interesting, in that it still captures some important aspects of the full problem. As we take $\lambda \equiv g_{YM}^2 N \to \infty$, we can scale the magnon momentum p ($p \sim p + 2\pi$) in different ways [6]. The simplest limit is taken by scaling p such that $p\sqrt{\lambda} =$ fixed. This produces the plane-wave limit of [8], where excitations are free and the S matrix is unity. Another simple limit is to keep p fixed. In this case the elementary excitations can be viewed as nontopological solitons of a weakly-coupled two-dimensional theory. Here, the basic magnon excitations of the theory turn into large solitons, or "giant magnons" [9]. The giant-magnon S matrix can be computed using semiclassical methods, and one obtains an answer that scales as $\log S \sim \sqrt{\lambda} f(p_1, p_2)$, where the function f has a branch cut at $p_1 = p_2$. In the exact theory we expect that this branch cut is replaced by a sequence of poles or zeros with a spacing of order $1/\sqrt{\lambda}$. In both of these limits the leading-order answer for the S matrix is described by a weakly-coupled theory that can be solved easily.

The third interesting limit corresponds to scaling p in such a way that one probes the region in between the previous two regimes. Namely, we keep $p\lambda^{1/4}$ fixed. In this regime the S matrix is nontrivial and receives contributions to all orders in α'. The S matrix develops a singularity for complex momenta that approaches the real axis for large λ. After going to suitable rescaled variables, however, the singularity remains a finite distance from the axis. This singularity is intimately related to the structure Janik's crossing-symmetry equation [4,10], which remains nontrivial even after taking this limit. Furthermore, the full sigma model describing strings in $AdS_5 \times S^5$ simplifies significantly in this limit, and leads (after gauge fixing) to a rather simple-looking (albeit non Lorentz-invariant) theory in $1 + 1$ dimensions. The magnons in this theory have rescaled momenta that lie between zero and infinity. For momenta close to zero, the S matrix becomes the identity and we recover the plane-wave results as well as the leading finite-J deviation away from the plane-wave limit [11]. For large rescaled momenta, however, the elementary excitations turn into solitons, and we recover results that are similar to those obtained for the

[1]This S matrix should not be confused with the notion of a *spacetime* S matrix.

[2]The paper [6] contains a couple of guesses. The correct one, based on the results in Ref. [7], seems to be the one in Ref. [5].

giant-magnon regime in Ref. [9]. In particular, we find a semiclassical S matrix with a branch cut, which should be replaced by a string of poles in the exact answer.

The limit characterized by $p \sim \lambda^{-1/4}$ was studied in Ref. [12], as it arises when one considers the flat-space limit. Our discussion is not directly relevant to the computation of energies of string states in flat space, because the part of the S matrix we consider drops out from that computation. We are interested in the particular part of the S matrix we consider here because it stands as a toy version of the full S matrix, and displays several interesting features of the full problem. Moreover, it is described by a self-contained Lagrangian that might prove easier to solve directly than the full $AdS_5 \times S^5$ theory (though we were not able to solve it).

We can also consider the corresponding limit of the recent BES/BHL [5,6] proposal for the S matrix. After a Borel resummation, the phase is given by a simple-looking integral expression that allows us to explore some of its analytic properties. In particular, we show that we recover the expected structure in the giant-magnon region. Namely, we see that the branch cut in the semiclassical answer disappears and is replaced by a sequence of double poles. We also check that the crossing-symmetry equation is obeyed after choosing an appropriate contour.

The paper is organized as follows. In Sec. II we discuss the kinematics of this "near-flat-space" limit. In Sec. III we discuss the world sheet theory that is obtained in this limit. We first consider an analogous limit for the $O(N)$ sigma model, and then we turn to the full $AdS_5 \times S_5$ theory. For each case we find a pair of theories describing the system before or after imposing the Virasoro constraints. In Sec. III we consider the BES/BHL S matrix in this limit and study some of its properties. We also include several Appendices where we discuss various related topics.

II. THE "NEAR-FLAT-SPACE" LIMIT

As mentioned in the introduction, we are interested in a limit in which we scale the magnon momentum p such that $p \sim \lambda^{-1/4}$. This particular scaling choice can be motivated by introducing the kinematic variables used in Refs. [13,14]:

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}, \qquad e^{ip} = \frac{x^+}{x^-}, \qquad g^2 \equiv \frac{\lambda}{16\pi^2} = \frac{g_{\rm YM}^2 N}{16\pi^2}, \tag{2.1}$$

where $g_{\rm YM}$ is the conventional Yang Mills coupling. Note that we have replaced p by a pair of variables $x^{\pm}$ obeying a constraint equation. The matrix structure of the S matrix is dramatically simplified when expressed in terms of these variables [4]. Furthermore, crossing symmetry acts in a rather simple fashion: $x^{\pm} \to 1/x^{\pm}$.

We now want to focus on the regime where $x^+ \sim x^- \sim 1$, as it connects the two possible ways of approximately solving the equation in Eq. (2.1) at strong coupling, namely, $x^+ \sim (x^-)^{\pm 1}$. We define rescaled variables $w^{\pm}$ via

$$x^{\pm} = e^{w^{\pm}/\sqrt{g}}, \tag{2.2}$$

and take $g \to \infty$, keeping $w^{\pm}$ fixed. In this limit we can see that the constraint (2.1) linking x^+ and x^- becomes

$$(w^+)^2 - (w^-)^2 = i, \qquad k \equiv p\sqrt{g} = -i(w^+ - w^-), \tag{2.3}$$

where we have defined a rescaled momentum k that is kept fixed as we take the limit. These equations can be solved to express $w^{\pm}$ in terms of k:

$$w^{\pm} = \frac{1}{2}\left(\frac{1}{k} \pm ik\right). \tag{2.4}$$

It will also be useful to introduce a new variable u, defined as

$$u \equiv \frac{1}{2}((w^+)^2 + (w^-)^2) = (w^{\pm})^2 \mp \frac{i}{2} = \frac{1}{4}\left(\frac{1}{k^2} - k^2\right), \tag{2.5}$$

where the momentum k is positive. Small k ($k \ll 1$) corresponds to the large-momentum regime of the plane-wave limit, while large k ($k \gg 1$) probes the small-momentum regime of the giant-magnon region: see Fig. 1. In these two regimes we have a simple, weakly-coupled description of the dynamics. For $k \sim 1$ we are forced to consider an interacting theory.

It will also be useful to consider the following expression for the energy:

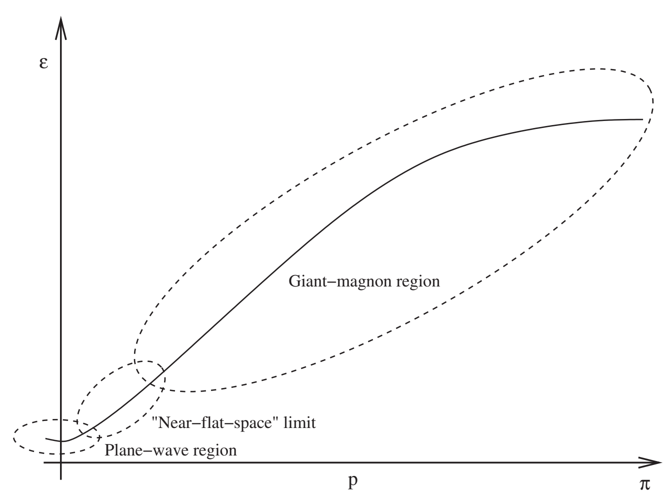

FIG. 1. The dispersion relation $\epsilon(p)$ at large λ. The plane-wave region corresponds to the region near $p \sim 0$, where the dispersion relation looks like that of a relativistic massive particle. In the giant-magnon region, the dispersion relation is essentially $\epsilon \sim \sqrt{\lambda}\sin p/2$. We are interested in the interpolating region, which overlaps with limits of both of the previous regimes.

$$\epsilon = \frac{g}{i}\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right) = \sqrt{1 + 16g^2\sin^2\frac{p}{2}}. \quad (2.6)$$

To leading order in the strong-coupling expansion ($g \gg 0$), we find that the energy is $\epsilon \sim 2\sqrt{g}k$, representing particles that move close to the speed of light to the right. The two approximate solutions under consideration correspond to the plane-wave and giant-magnon regimes. Of course, there is a similar region where $x^+ \sim x^- \sim -1$, where we get particles that move very fast to the left. This region is related by world sheet parity to the one discussed here.

It will be useful to define a rescaled energy by picking out the subleading k dependence as

$$\hat{\epsilon} = \lim_{g\to\infty}[2\sqrt{g}(\epsilon - 2\sqrt{g}k)] = \frac{1}{2k} - \frac{k^3}{6}. \quad (2.7)$$

Now that the dispersion relation is not exactly relativistic, we can see that excitations will travel with different velocity, and one can define an S matrix that encodes scattering between two of these right-moving excitations. In other words, the velocity is

$$v = \frac{d\epsilon}{dk} = 2\sqrt{g} - \frac{1}{4\sqrt{g}}\left(\frac{1}{k^2} + k^2\right). \quad (2.8)$$

Physically, the rescaling of the energy in Eq. (2.7) means that it will take a long time for the magnons to separate, and small deviations of the metric from flat space lead to large effects, which, in turn, lead to a nontrivial S matrix.

Let us pause here to consider the relation between this limit and the flat-space limit considered in Ref. [12]. When one is interested in reproducing the energies of strings in flat space, one should rescale $J \sim \lambda^{1/4}$. Strictly speaking, this is not the infinite-J limit, and one cannot be sure that the asymptotic S matrix formulas apply. Nevertheless, one can proceed and find that the order of magnitude of the momentum, $p \sim n/J \sim n/\lambda^{1/4}$, indeed scales as discussed above. In this case, one can keep the leading dependence of the energy $\epsilon \sim 2\sqrt{g}k$, but consider both left- and right-movers to have zero total momentum on the world sheet. Upon writing the Bethe equations, one needs only the S matrix between left- and right-movers, but the S matrix for the left-movers drops out of the analysis. The S matrix between left- and right-movers is rather simple, and one can obtain the flat-space results [12] in a straightforward manner. In summary, the discussion presented here will not be relevant for finding the *energies* of states in flat space (which was already done in Ref. [12]). Rather, we will be primarily concerned with understanding the full structure of the S matrix and the connection between the two simple strong-coupling regimes discussed above (i.e., those of the plane wave and giant magnon).

We can therefore view our limit as a toy version of the full problem, where we have retained some of the interesting structure of the complete theory, but we have lost one parameter, namely λ. In fact, one can check that in this limit the structure of the S matrix [4] remains nontrivial and, consequently, the crossing symmetry Eq. (2.10) is also nontrivial and becomes[3]:

$$S_0(-w_1, w_2)S_0(w_1, w_2) = \sigma^2(-w_1, w_2)\sigma^2(w_1, w_2) = \left(\frac{(w_1^- + w_2^-)(w_1^- - w_2^+)}{(w_1^+ + w_2^-)(w_1^+ - w_2^+)}\right)^2,$$
$$\text{where } S_0(w_1, w_2) = \frac{(w_1^- - w_2^+)(w_1^+ + w_2^-)}{(w_1^+ - w_2^-)(w_1^- + w_2^+)}\sigma^2. \quad (2.9)$$

The crossing transformation itself maps $w^\pm \to -w^\pm$ along a path that we will specify later. It is amusing to note that the constraint (2.3) looks very similar to the constraint between energy and momentum in a relativistic theory in $1 + 1$ dimensions. Thus one may introduce a rapidity variable η via[4]

$$w^+ = e^{i\pi/4}\cosh\eta, \qquad w^- = e^{i\pi/4}\sinh\eta. \quad (2.10)$$

In fact, the variable η, which starts out living on a cylinder, is what remains after taking the near-flat-space limit of Janik's torus [10].

One can attempt to solve the crossing equation directly by looking for a meromorphic (but not periodic) function in η. The right-hand side of the crossing Eq. (2.9), however, has a structure that excludes any such solution. (For a detailed argument see Appendix A.) One is therefore forced to introduce branch cuts somewhere. In fact, the recently proposed solutions in Refs. [5,6,15] all have branch cuts, as we will see later. A similar result is also true for the full crossing-symmetry equation presented in Ref. [10]. Namely, *there is no meromorphic solution as a function of the coordinates of the torus* z_1, z_2, even after going to the covering spaces of the two tori (which comprise two complex planes).

III. LAGRANGIAN IN THE NEAR-FLAT-SPACE LIMIT

In this section we consider the near-flat-space limit of the sigma model. The limiting sigma model is a well defined system on its own, which looks simpler than the original system. We investigate its properties with the hope that it will prove easier than the full sigma model to solve directly.

A. The $O(N)$ case

As an exercise, let us consider the $O(N)$ sigma model. The target space of this sigma model is a sphere S^{N-1}. Let us consider a state with a constant spin density $J = J_{12}$,

[3]The conventions we employ in defining S_0 and σ are the same as those in Ref. [6].

[4]The system does not have relativistic invariance under $\eta \to \eta+$ constant. For example, the right-hand side of (2.9) is not simply a function of $\eta_1 - \eta_2$.

where J_{kl} are rotation generators in the kl plane. We denote the corresponding angle on the sphere by φ. Starting with a classical analysis, we see that the lowest energy state is given simply by $\dot{\varphi} = \text{const}$. For simplicity, we can take the case with $N = 3$ and parametrize the S^2 according to

$$S = \frac{R^2}{4\pi}\int \cos^2\theta(\partial\varphi)^2 + (\partial\theta)^2, \qquad \frac{R^2}{4\pi} = g. \tag{3.1}$$

Starting with a solution for which $\dot{\varphi} = 1$, $\theta = 0$, we can perform a boost on the world sheet coordinates $\tilde{\sigma}^{\pm} = \tilde{\sigma}^0 \pm \tilde{\sigma}^1$ and expand in small fluctuations around the constant-spin-density solution

$$\begin{aligned} &\tilde{\sigma}^+ = 2\sqrt{g}\sigma^+, \qquad \tilde{\sigma}^- = \frac{\sigma^-}{2\sqrt{g}}, \\ &\varphi = \frac{\tilde{\sigma}^+ + \tilde{\sigma}^-}{2} + \frac{\delta}{\sqrt{g}} = \sqrt{g}\sigma^+ + \frac{\chi}{\sqrt{g}}, \\ &\chi = \frac{\sigma^-}{4} + \delta, \qquad \theta = \frac{y}{\sqrt{g}}, \qquad g \to \infty, \end{aligned} \tag{3.2}$$

where $\sigma^{\pm}$ are the coordinates after performing the boost.

Note that we will be interested in solutions where $\chi = \frac{1}{4}\sigma^- + \delta$, with δ representing a small fluctuation. Under these rescalings, the action, up to constant and total derivative terms, is finite:

$$S = 4\int \partial_-\chi\partial_+\chi + \partial_+ y\partial_- y - y^2\partial_-\chi. \tag{3.3}$$

By setting $\chi = \frac{1}{4}\sigma^- + \delta$, we see that to leading order we have a massless field δ and a massive field y. Long wavelength fluctuations in δ are simply sound waves, or spin-density waves propagating along the system. The massive field y is the massive field that arises in the plane-wave limit. We thus find that the system is described by a simple Lagrangian (3.3) in the near-flat-space limit. Notice that the only nontrivial interaction is a cubic interaction that breaks Lorentz invariance. To generalize to the case of an $O(N)$ (as opposed to $O(3)$) sigma model, all we need to do is replace $y^2 \to \vec{y}^2$ in the above Lagrangian, where $\vec{y}$ has $N - 2$ components.

From the point of view of the $O(N)$ sigma model, this limit has the following interpretation. The ground state with constant spin density gives rise to a sort of Fermi sea of particles, all with the same spin. Since the particles are interacting, it is not exactly a Fermi sea, but in the thermodynamic limit there is a sharp cutoff in the momentum of the particles [16]. The theory described by Eq. (3.3) corresponds to considering excitations around the ground state that are moving with a momentum close to one of the Fermi momenta[5] $\pm p_f$.

[5]From this point of view, this limit is reminiscent of the double-scaling limit of matrix models, where one focuses near the end point of a distribution of eigenvalues (see, e.g., Ref. [17] for a review).

The equations of motion for the theory in Eq. (3.3) appear as

$$\begin{aligned} &0 = \partial_- j_+, \qquad j_+ \equiv \partial_+\chi - \frac{y^2}{2}, \\ &0 = \partial_-\partial_+ y + y\partial_-\chi, \end{aligned} \tag{3.4}$$

where the first equation implies the existence of a chiral conserved current j_+. It is useful to understand what happens to the Virasoro generators in this limit. The T_{--} generator is

$$T_{--} = (\partial_-\chi)^2 + (\partial_- y)^2, \tag{3.5}$$

and it is conserved according to Eq. (3.4): $\partial_+ T_{--} = 0$. In fact, we see that the action (3.3) is right-conformal invariant. In other words, it is invariant under $\sigma^- \to f(\sigma^-)$. However, the action is not invariant under left-moving conformal transformations. The original left-moving conformal symmetry has become a chiral $U(1)$ symmetry acting as $\chi \to \chi + \epsilon(\sigma^+)$. The right-moving stress tensor of the original theory takes the form

$$T_{++} = \frac{g}{2} + j_+ + o(1/g), \tag{3.6}$$

where j_+ was defined in Eq. (3.4).

In the full theory we will impose Virasoro constraints after adding a timelike direction, and we consider a solution with $\dot{t} = 1$, or $t = \tilde{\sigma}^0 = \sqrt{g}\sigma^+ + \frac{\sigma^-}{4}$ [see Eq. (3.2)]. The leading-order term in Eq. (3.6) should be equated with the contribution of the timelike direction to the stress tensor. The zeroth-order term in Eq. (3.6) should be set to zero, which leads to the constraint $j_+ = 0$. Similarly, Eq. (3.5) should be equated with the timelike contribution, which leads to the constraint $T_{--} = \frac{1}{16}$. We see that both constraints are attainable classically, since they are related to symmetries of the theory.

After adding a trivial time direction and imposing the two Virasoro constraints

$$j_+ = 0, \qquad T_{--} = \tfrac{1}{16}, \tag{3.7}$$

we can move to a gauge-fixed Lagrangian by defining new coordinates

$$x^+ \equiv \sigma^+, \qquad x^- \equiv \frac{\sigma^-}{2} + 2\chi = \sigma^- + 2\delta. \tag{3.8}$$

The derivatives then transform as

$$\begin{aligned} &\partial_{\sigma^+} = \partial_{x^+} + 2(\partial_{\sigma^+}\chi)\partial_{x^-} = \partial_{x^+} + y^2\partial_{x^-}, \\ &\partial_{\sigma^-} = 2(\tfrac{1}{4} + \partial_-\chi)\partial_{x^-}, \end{aligned} \tag{3.9}$$

where we used the constraint $j_+ = 0$. After using the Virasoro condition $T_{--} = \frac{1}{16}$, one can check that the gauge-fixed Lagrangian becomes[6]

[6]Essentially, we perform the change of variables in the equations of motion and write the Lagrangian from which they follow.

$$S = 4\int dx^+ dx^- [\partial_+ y \partial_- y - \tfrac{1}{4}y^2 + y^2(\partial_- y)^2]. \quad (3.10)$$

The momentum is given by

$$k = \sqrt{g}p = \sqrt{g}\int (d\varphi - d\sigma^0) = \int d\chi - \frac{1}{4}d\sigma^- = \int d\delta$$
$$= \int \bigg[-2(\partial_- y)^2 dx^- + \bigg(\frac{y^2}{2} + 2y^2(\partial_- y)^2\bigg)dx^+ \bigg], \quad (3.11)$$

where the derivatives in the second line are taken with respect to $x^\pm$. Note that we can understand the momentum both as an angle[7] and as a Noether charge under x^- translations. In the limit of interest, the momentum k is not periodic but is defined on the semi-infinite line $(0, \infty)$. The expression for the energy takes the form

$$-k_+ = \hat{\epsilon} = -4\int [(\partial_+ y)^2 + (\partial_+ \chi)^2] d\sigma^+ - y^2 d\chi$$
$$= 4\int \bigg[\frac{y^2}{4} - y^2(\partial_- y)^2\bigg] dx^-$$
$$- [(\partial_+ y)^2 + 2y^2 \partial_+ y \partial_- y] dx^+, \quad (3.12)$$

where $\partial_+ = \partial_{\sigma^+}$ in the first line, and $\partial_\pm = \partial_{x^\pm}$ in the second.

It is interesting to consider giant-magnon solutions to this Lagrangian. For this purpose we consider the $O(N)$ case with $N \geq 4$, and we focus on the first two components of $\vec{y}$. We can start with the solution carrying an additional angular momentum J_2 in the complex $y_1 + iy_2$ plane [18–20], and then take the limit. We obtain

$$\chi = \frac{1}{4}\sigma^- + \delta, \qquad \delta = \frac{k}{2}\tanh u,$$
$$y^1 + iy^2 = e^{iv}\frac{k}{2\cosh u}, \qquad u \equiv -\frac{k^3}{J_2^2 + k^4}\sigma^- + \frac{k}{4}\sigma^+,$$
$$v \equiv \frac{J_2 k}{J_2^2 + k^4}\sigma^- + \frac{J_2}{4k}\sigma^+. \quad (3.13)$$

To think about the *elementary* giant magnon we can set $J_2 = 1$; the other solutions can be thought of as bound states of the elementary solution [18]. The energy of these states is given by

$$\hat{\epsilon} = \frac{J_2^2}{2k} - \frac{k^3}{6}, \quad (3.14)$$

which, for $J_2 = 1$, reduces to Eq. (2.7). Notice that the velocity of these particles can be computed as

$$v_- = \frac{d\sigma^-}{d\sigma^+} = -\frac{d\hat{\epsilon}}{dk} = \frac{1}{2}\bigg(\frac{1}{k^2} + k^2\bigg),$$
$$v = \frac{d\sigma^1}{d\sigma^0} = \frac{1 - v_-}{1 + v_-} = -\frac{(\frac{1}{k} - k)^2}{(\frac{1}{k} + k)^2}. \quad (3.15)$$

[7]Note that in this limit δ ceases to be periodic.

Notice also that they always move with a speed that is less than the speed of light, and they all move to the left.[8]

Near $k \sim 0$, the above theories (3.3) and (3.10) are weakly coupled, while as k increases toward the region $k \sim 1$, the theories become strongly coupled. We expect that the elementary excitations become giant-magnon solutions (3.13) for large k. In this latter region the giant magnon is extended, so that the classical description becomes appropriate. In fact, we can see that the size of the solution is of order k for large k. We can find the giant-magnon solution for the Lagrangian (3.10) by defining the coordinates x^- as above (3.8). The equation is invertible since $\frac{dx^-}{d\sigma^-} = 1 + 2\partial_- \delta > 0$.

Because they are limits of integrable theories, the above Lagrangians (3.3) and (3.10) are themselves integrable. In Appendix B we display explicitly the Lax connection for $O(N)$ theories in this limit. We can ask whether the theory (3.3) remains right-conformal after we take into account quantum corrections. In Appendix B we argue that right-conformal symmetry is broken if $N \neq 2$, but for $N = 2$ the theory remains conformal. Thus, the theories (3.3) and (3.10) will not be equivalent as quantum theories. We know, however, that the theory (3.3) is quantum integrable, since it is a limit of the integrable $O(N)$ sigma model. We could therefore solve the quantum theory (3.3) by taking a limit of the $O(N)$ quantum theory [21], but we will not do so here. However, let us mention one result. If one computes the scattering amplitude for giant magnons in the region where $k_1 - k_2 \ll k_{1,2} \gg 1$, one finds that the branch cut in the semiclassical scattering amplitude, which is the same for all N, becomes a sequence of single poles for $N \neq 2$ and a sequence of double poles for $N = 2$. For $N = 2$, the model can be viewed as a limit of an $OSP(M + 2|M)$ theory, as explained in Ref. [22].

The theory in Eq. (3.10) may or may not be integrable at the quantum level. However, for $N = 4$, (3.10) is also a limit of the Fadeev-Reshetikin theory [23,24], so it must be quantum integrable in this case. These theories are reminiscent of the chiral Potts model, in that they are integrable and they break Lorentz symmetry.[9]

B. The near-flat-space limit of the $AdS_5 \times S^5$ sigma model

We can now consider the full $AdS_5 \times S^5$ sigma model, starting with the Green-Schwarz action as written in Ref. [2]. We parametrize by φ the angle on S^5 that is shifted by the action of the generator J. We also pick t to be the coordinate on AdS_5 whose shift corresponds on the field theory side to the action of the gauge theory dilatation operator Δ. We then perform the following rescalings

[8]Before performing the boost in Eq. (3.2), they traveled to the right (2.8).

[9]We thank B. McCoy for a discussion on this topic.

$$t = \sqrt{g}\sigma^+ + \frac{\tau}{\sqrt{g}}, \qquad \varphi = \sqrt{g}\sigma^+ + \frac{\chi}{\sqrt{g}},$$
$$\vec{\theta} = \vec{y}/\sqrt{g}, \qquad \vec{\rho} = \vec{z}/\sqrt{g}, \tag{3.16}$$
$$\Theta^1 \sim \frac{\psi_-}{g^{1/4}}, \qquad \Theta^2 \sim \frac{\psi_+}{g^{3/4}}, \qquad g \to \infty,$$

where y denotes the four transverse directions in the S^5 subspace, z denotes the transverse coordinates on the AdS_5 subspace, and Θ^i denote two ten-dimensional Weyl spinors of type IIB string theory after fixing kappa symmetry (so that $\psi_\pm$ are $SO(8)$ spinors).

Upon taking the limit we obtain the Lagrangian

$$\begin{aligned}\mathcal{L} = 4\{&-\partial_+\tau\partial_-\tau + \partial_+\chi\partial_-\chi + \partial_+\vec{z}\partial_-\vec{z} + \partial_+\vec{y}\partial_-\vec{y} \\ &- \vec{y}^2\partial_-\chi - \vec{z}^2\partial_-\tau + i\psi_+\partial_-\psi_+ \\ &+ 2i(\partial_-\tau + \partial_-\chi)\psi_-\partial_+\psi_- + 2i(\partial_-\tau + \partial_-\chi)\psi_-\Pi\psi_+ \\ &- i\psi_-(\partial_- z^j\Gamma^j + \partial_- y^{j'}\Gamma^{j'})(z^i\Gamma^i - y^{i'}\Gamma^{i'})\psi_- \\ &+ \tfrac{1}{12}\partial_-(\tau+\chi)[\psi_-\Gamma^{ij}\psi_-\psi_-\Gamma^{ij}\psi_- \\ &- \psi_-\Gamma^{i'j'}\psi_-\psi_-\Gamma^{i'j'}\psi_-]\},\end{aligned} \tag{3.17}$$

where $\psi_\pm$ are real, positive-chirality $SO(8)$ spinors, Γ^i are real $SO(8)$ gamma matrices, and $\Pi \equiv \Gamma^1\Gamma^2\Gamma^3\Gamma^4$ is the product of the first four gamma matrices.[10] The indices i, j run over the four transverse directions in AdS_5, and the indices i', j' run over the four transverse directions in S^5.

As in the $O(N)$ case, we have the following chiral conserved currents

$$\begin{aligned} &j_+^\chi = \partial_+\chi - \frac{y^2}{2} + i\psi_-\partial_+\psi_- + 2i\psi_-\Pi\psi_+ + 4\text{ fermi}, \\ &\partial_- j_+^\chi = 0, \\ &j_+^\tau = \partial_+\tau + \frac{z^2}{2} - i\psi_-\partial_+\psi_- - 2i\psi_-\Pi\psi_+ - 4\text{ Fermi}, \\ &\partial_- j_+^\tau = 0,\end{aligned} \tag{3.18}$$

where we have not explicitly recorded the four-fermion terms proportional to the last term in (3.17). These currents are conserved due to the equations of motion for τ and χ. In addition, we have the conserved stress tensor

$$\begin{aligned} T_{--} = &-(\partial_-\tau)^2 + (\partial_-\chi)^2 + (\partial_- z)^2 + (\partial_- y)^2 \\ &+ 2i(\partial_-\tau + \partial_-\chi)\psi_-\partial_-\psi_-, \end{aligned} \tag{3.19}$$

obeying $\partial_+ T_{--} = 0$. So, as above, we have a field theory that is conformal for the right movers. In this case, we expect that the field theory remains right-conformal, even after we include quantum corrections.

Following the example above, we can now gauge-fix by imposing the conditions

$$0 = j_+^\chi + j_+^\tau = \partial_+(\tau+\chi) + \frac{z^2 - y^2}{2}, \qquad 0 = T_{--}. \tag{3.20}$$

We choose the coordinates[11]

$$x^+ \equiv \sigma^+, \qquad x^- \equiv 2(\tau + \chi), \tag{3.21}$$

so that the derivatives become

$$\begin{aligned}\partial_{\sigma^+} &= \partial_{x^+} + (z^2 - y^2)\partial_{x^-}, \\ \partial_{\sigma^-} &= 2[\partial_{\sigma^-}(\tau+\chi)]\partial_{x^-}.\end{aligned} \tag{3.22}$$

We thereby obtain the following gauge-fixed Lagrangian:

$$\begin{aligned}\mathcal{L} = 4\{&\partial_+\vec{z}\partial_-\vec{z} + \partial_+\vec{y}\partial_-\vec{y} - \tfrac{1}{4}(\vec{y}^2 + \vec{z}^2) + (\vec{y}^2 - \vec{z}^2) \\ &\times[(\partial_-\vec{z})^2 + (\partial_-\vec{y})^2] + i\psi_+\partial_-\psi_+ + i\psi_-\partial_+\psi_- \\ &+ i\psi_-\Pi\psi_+ + i(\vec{y}^2 - \vec{z}^2)\psi_-\partial_-\psi_- - i\psi_-(\partial_- z^j\Gamma^j \\ &+ \partial_- y^{j'}\Gamma^{j'})(z^i\Gamma^i - y^{i'}\Gamma^{i'})\psi_- + \tfrac{1}{24}[\psi_-\Gamma^{ij}\psi_-\psi_-\Gamma^{ij}\psi_- \\ &- \psi_-\Gamma^{i'j'}\psi_-\psi_-\Gamma^{i'j'}\psi_-]\},\end{aligned} \tag{3.23}$$

where we have also performed a simple rescaling of ψ_+.

Even though this action looks complicated, it is much simpler than the full gauge-fixed Lagrangian that was written in Ref. [3]. These theories ought to have the full extended $SU(2|2)^2 \times R^2$ symmetry algebra described in Ref. [4], so it is useful to study the form of this algebra in the limit we are considering. We start with the supersymmetry algebra in Ref. [4], which can be written as

$$\begin{aligned} &\{Q_-^i, Q_-^j\} = \delta^{ij}\kappa_-, \\ &\kappa_\pm = \kappa^0 \mp \kappa_1, \\ &\{Q_-^i, Q_+^j\} = \delta^{ij}\kappa_2 + SU(2)\text{ currents}, \\ &\{Q_+^i, Q_+^j\} = \delta^{ij}\kappa_+.\end{aligned} \tag{3.24}$$

We can think of κ^μ as a $2+1$ dimensional momentum [9], $\kappa^0 = \epsilon$ as the ordinary energy, and $\kappa_{1,2}$ are the two central charges introduced in Ref. [4].

For the short representation corresponding to the elementary magnon, these charges obey

$$\kappa_+\kappa_- - \kappa_2^2 = (\kappa^0)^2 - \kappa_1^2 - \kappa_2^2 = 1. \tag{3.25}$$

For a single particle with momentum k, one obtains the values

$$\kappa^1 + i\kappa^2 = -i2g e^{ip_l}(e^{ip} - 1), \tag{3.26}$$

where p_l is the sum of the momenta to the left of the excitation. The appearance of this phase factor reflects the Hopf algebra structure of the problem [25].[12] Notice that, according to Eq. (3.25), the physical energy does not

[10]We can redefine $\psi_+ \to \Pi\psi_+$ to get rid of this matrix, which only appears in the fermion mass term.

[11]These coordinates are similar to the ones chosen in Ref. [3].

[12]The presence of this phase is easy to understand by drawing the pictures described in Ref. [9]. In that work the magnons are represented as line segments that join points on a circle, and the momentum is the angle subtended by these points. The segment itself can be viewed as the complex number $\kappa_1 + i\kappa_2$. The fact that the phase depends on the number of magnons to the left is then clear: the magnon in question has to be positioned on the circle at a point where the previous magnon ends, and thus its orientation depends on the total angle, or momentum, subtended by all the magnons to its left.

depend on the overall phase in Eq. (3.26), and it is therefore well-defined for each individual magnon. In the near-flat-space limit that we are considering, we can rescale the charges as $Q_{\pm} \to g^{\pm 1/4}\hat{Q}_{\pm}$ and introduce similarly rescaled quantities $\hat{\kappa}_{\pm}$ (we do not need to rescale κ_2). After these operations, the final algebra takes the same form as in Eq. (3.24), but expressed in terms of rescaled quantities. The expressions for the rescaled central charges in terms of the momentum are

$$\hat{\kappa}_- = 4k, \quad \hat{\kappa}_+ = \frac{1}{4k} + \frac{k^3}{4} + k^2 k_l + k k_l^2, \quad \kappa_2 = k(k + 2k_l). \tag{3.27}$$

We see that the central charges acting on a single magnon state still depend on the momentum of the magnons to their left (k_l), so they retain their Hopf algebra character.

At first sight it is a bit surprising that the energy, which is conjugate to x^+, can be negative [as seen for large k in Eq. (3.14)]. In a supersymmetric system we might have expected the energy to be positive. In fact, the right-hand side of the $\hat{Q}_+$ anticommutator is $\hat{\kappa}_+$, and is not the energy (or $i\partial_+$). Indeed, $\hat{\kappa}_+$ is always positive.

Notice that k is proportional to $\hat{\kappa}_-$. In fact, the right-moving supercharges $\hat{Q}_-$ act in the ordinary way on the Lagrangian (3.23). One can write the Lagrangian (3.23) in terms of (0, 2) superfields by realizing explicitly two of the 8 right-moving supercharges. The action of the $\hat{Q}_+$ supercharges will be more nontrivial. The authors of Ref. [26] have expanded the action around the plane-wave limit up to terms quartic in the fields, so a limit of their computation should give the proper symmetries of Eq. (3.23). (See also the discussion in Ref. [27] for further details.) We have not done this analysis here, and it would be interesting to check explicitly that this Lagrangian indeed admits the full symmetry algebra.

C. The S matrix in the weakly-coupled regions

The above theory becomes simple when the momenta of the excitations are small or large. For small momenta the excitations are described by ordinary weakly-coupled massive quanta. In the limit of very small momenta, they are free and the S matrix goes to the identity for $k_i \to 0$. The leading correction away from this limit is given by the computation done in Ref. [11,27] (see also Ref. [1]). In fact, it is simply the high-energy limit of the result in Ref. [11,27]. More explicitly, their result for the σ factor is

$$\sigma^2 = 1 + \frac{i}{2g}\tilde{p}_1^2\tilde{p}_2^2\left[\frac{\tilde{p}_2}{1+\epsilon_2} - \frac{\tilde{p}_1}{1+\epsilon_1}\right] \times \frac{1}{(\epsilon_1+1)(\epsilon_2+1) - \tilde{p}_1\tilde{p}_2}, \tag{3.28}$$

where $\tilde{p}_1$ are the momenta rescaled to the plane-wave region

$$\tilde{p} = 2gp, \qquad \epsilon = \sqrt{1+\tilde{p}^2}. \tag{3.29}$$

By taking the near-flat-space limit (2.2) we find

$$\sigma^2 = 1 - i2k_1k_2\frac{k_1 - k_2}{k_1 + k_2}. \tag{3.30}$$

This result is valid for small k, where the correction is small. As we increase k we should also consider higher-order corrections.

Similarly, we can consider the large-k region. In this region the magnons can be approximated by classical solitons, and their scattering amplitude is a simple limit of the one computed in Ref. [9] for the full theory.[13] The limit produces the following scattering phase:

$$\log\sigma^2 \sim -i(k_1^2 - k_2^2)\log\left(\frac{k_1 - k_2}{k_1 + k_2}\right). \tag{3.31}$$

This result is valid as long as the right-hand side is large, which occurs for large k.

Both of the results in Eqs. (3.30) and (3.31) are summarized by the Arutyunov-Frolov-Staudacher (AFS) phase factor [12], which in this limit can be written as

$$\sigma^2 = \frac{(w_1^- + w_2^+)^2}{(w_1^+ + w_2^-)^2}\left(\frac{(w_1^+ + w_2^-)(w_1^- + w_2^+)}{(w_1^+ + w_2^+)(w_1^- + w_2^-)}\right)^{i2(u_1 - u_2)}. \tag{3.32}$$

However, we should emphasize that this is *not* the leading-order answer in the region where $k \sim 1$ or $w^{\pm} \sim 1$, despite the fact that g is large. In fact, in this region all higher-order corrections should be taken into account, as we show more explicitly in the next section.

We digress briefly to discuss the branch cut present in the semiclassical amplitude (3.31) at $k_1 - k_2 = 0$. In principle, the exact phase can have branch cuts, but we do not expect that the exact phase would have branch cuts traversing regions of physical (real) momenta, or at least momenta that are very close to the real axis. So, for the purpose of this discussion, let us assume that the function $S(z)$, where $z = k_1 - k_2$, is meromorphic. We expect that, as in other integrable theories, the branch cuts would be replaced by poles.

We know that the function S has the approximate expression

$$\log S \sim h(z)\log z. \tag{3.33}$$

This expression should hold on the real axis and should fail when we approach the line of zeros or poles. Since the function should obey a unitarity condition $S(1,2) = 1/S(2,1)$, the existence of a pole in the upper half plane would imply a zero in the lower half plane. Let us assume, therefore, that there is single line of zeros in the upper half plane and a single line of poles in the lower half plane, both starting at $z = 0$. Here, by a line of poles we mean a very

[13] Note that $S_0^{\text{here}} = (S_0^{[9]})^{-1}$.

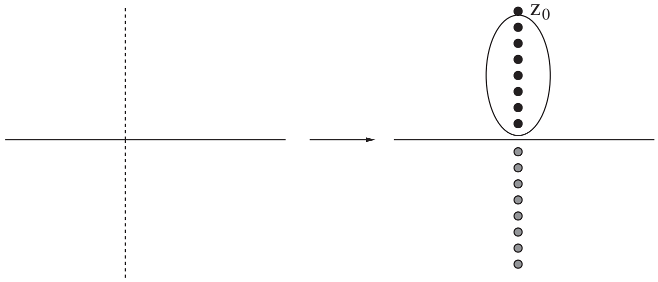

FIG. 2. A closely spaced sequence of zeros or poles should replace the branch cuts in the semiclassical scattering amplitude (3.31). We expect to have zeros in the upper half plane and poles in the lower half plane.

closely spaced sequence. Thus, we will put half of the branch cut in Eq. (3.33) in the upper half plane and the other half in the lower half plane. As we cross the branch cut in the upper half plane the function jumps by $2\pi i h(z)$, and when we cross the lower one it jumps by $-i\pi h(-z)$. After we use that $h(z) = h(-z)$, we see that the total jump of the function as we go around a circle centered at $z = 0$ is indeed $2\pi i h(z)$. (As an example, the reader could think about the function $S = \frac{\Gamma(-iz)}{\Gamma(iz)}$.) We now consider a contour that goes through $z = 0$ and then circles around the line of zeros in the upper half plane, as depicted in Fig. 2. We then perform the following integral:

$$N_p - N_z = \frac{1}{2\pi i}\oint d\log S \sim \frac{1}{2}h(z_0), \tag{3.34}$$

where $N_{p,z}$ represents the number of poles or zeros enclosed by the contour. For this to make sense we need that $h(z_0)$ is real, which constrains the possible locations of zeros. Applied to the above problem, this logic would indicate the appearance of zeros in the upper half plane, where

$$N_z = -\frac{i}{2}(k_1^2 - k_2^2), \tag{3.35}$$

and N_z is taken to be real and positive. The formula (3.35) should be interpreted as giving the mean density of zeros, not their precise location. As discussed in Ref. [9], the sigma model has localized time-dependent solutions, and one might be tempted to interpret these poles as physical bound states. Note, however, that this is not the only possible interpretation. In fact, if the poles turn out to be double poles, as we will see below (based on the BES/BHL guess [5,6]), then another possible interpretation is that of "anomalous thresholds" [28].[14] This issue will be addressed in a future publication.

[14]This interpretation was suggested to us by N. Dorey.

IV. CONJECTURED S MATRIX IN THE NEAR-FLAT-SPACE LIMIT

As usual, we will parametrize the contribution to the scattering amplitude as

$$\begin{aligned} \sigma^2 &= e^{2i\delta_{12}}, \\ \delta_{12} &= \chi(w_1^+, w_2^+) - \chi(w_1^-, w_2^+) - \chi(w_1^+, w_2^-) \\ &\quad + \chi(w_1^-, w_2^-) - (1 \leftrightarrow 2), \\ \chi &= \sum_{n=0}^{\infty} \chi^n, \end{aligned} \tag{4.1}$$

where the χ^n represent contributions to the phase at n loops.

A. Tree level and one-loop order

As mentioned above, the tree-level contribution in the scaling limit becomes [12]

$$\chi^0(w_1, w_2) - \chi^0(w_2, w_1) = -(w_1^2 - w_2^2)\log(w_1 + w_2). \tag{4.2}$$

The one-loop contribution turns out to be the simplest solution of the iterated crossing Eq. (2.9). In fact, when written in the appropriate variables, the one-loop result in this limit is the same as the one-loop result in the full theory. We will begin by discussing a couple of aspects of the one-loop answer before taking the limit.

If one defines the variables [6,29]

$$x^{\pm} = \tanh\frac{\theta^{\pm}}{2}, \tag{4.3}$$

then the double crossing[15] equation becomes

$$\begin{aligned} &\sigma^2_{1\text{-loop}}(\theta_1^+ + 2i\pi, \theta_1^- + 2i\pi, \theta_2^+, \theta_2^-) \\ &\quad = \frac{\tanh^2\frac{\theta_1^+ - \theta_2^+}{2}\tanh^2\frac{\theta_1^- - \theta_2^-}{2}}{\tanh^2\frac{\theta_1^+ - \theta_2^-}{2}\tanh^2\frac{\theta_1^- - \theta_2^+}{2}}\sigma^2_{1\text{-loop}}(\theta_1^+, \theta_1^-, \theta_2^+, \theta_2^-), \end{aligned} \tag{4.4}$$

where σ^2 is the one-loop contribution to the phase factor. The simplest solution to this equation is [29]

$$\sigma^2_{1\text{-loop}} = \frac{h(\theta_1^+ - \theta_2^+)h(\theta_1^- - \theta_2^-)}{h(\theta_1^+ - \theta_2^-)h(\theta_1^- - \theta_2^+)}, \tag{4.5}$$

where h is a function of the form

$$\begin{aligned} h(\theta) &= \prod_{n=-\infty}^{\infty}\left(\frac{(\theta - 2\pi i n)^2}{(\theta - 2\pi i(n+\frac{1}{2}))(\theta - 2\pi i(n-\frac{1}{2}))}\right)^n, \\ h(-\theta) &= 1/h(\theta), \qquad h(\theta + 2\pi i) = \tanh^2\frac{\theta}{2}h(\theta). \end{aligned} \tag{4.6}$$

[15]As in Ref. [6], "double crossing" means that we iterate the crossing transformation $x \to 1/x \to x$ along some particular path.

As shown in Ref. [6], this solution is actually the same as the one-loop contribution to the S matrix [30].

Now, as we take the near-flat-space limit, we can expand

$$x^{\pm} = \tanh\frac{\theta^{\pm}}{2} \sim 1 - 2e^{-\theta^{\pm}}, \qquad \hat{\theta}^{\pm} = \theta^{\pm} - \log(2\sqrt{g}) + i\pi, \qquad w^{\pm} = e^{-\hat{\theta}^{\pm}}. \tag{4.7}$$

Note that certain quantities such as $\theta_1^{\pm} - \theta_2^{\pm} = \hat{\theta}_1^{\pm} - \hat{\theta}_2^{\pm}$ remain the same in the limit. Thus the one-loop answer is the same as Eq. (4.5), except that we replace $\theta^{\pm} \to \hat{\theta}^{\pm}$, which is related to $w^{\pm}$ through Eq. (4.7).

B. Higher orders

In BES/BHL [5,6], a concrete proposal for the S matrix was made. The answer was expressed as a series expansion in $1/g$ and $1/x_1$, $1/x_2$. Since we are interested in the region near $x_1 \sim 1$, we will need to sum the series expansion in $1/x$. After this is done (for details, see Appendix D) we obtain

$$\tilde{\chi}^n = -\frac{\zeta(n)}{(-2\pi)^n}\frac{1}{x_1 x_2^2}\Gamma(n-1)\Gamma\left(\frac{n}{2}\right)\left[g\left(1-\frac{1}{x_2^2}\right)\left(1-\frac{1}{x_1x_2}\right)\right]^{1-n}\sum_{l=0}^{n-2}\sum_{q=0}^{l}\sum_{m=0}^{n-2}(-1)^{q+m}\frac{\Gamma(\frac{n}{2}+l-q)}{q!\Gamma(1+l-q)\Gamma(\frac{n}{2}-q)}$$
$$\times\frac{\Gamma(n-1+l-q-m)}{\Gamma(\frac{n}{2}+l-q-m)m!\Gamma(n-1-m)}\left(\frac{1-\frac{1}{x_1x_2}}{1-\frac{1}{x_2^2}}\right)^l\left(1-\frac{1}{x_2^2}\right)^{m+q}. \tag{4.8}$$

These expressions manifestly display the singularities at $x_i = 1$, and are valid for $n \geq 2$.

We can now take our near-flat-space limit in Eq. (2.2), and we find that only the leading singular terms with $q = w = 0$ in Eq. (4.8) contribute:

$$\chi^n(w_1, w_2) \equiv \lim_{g\to\infty}\tilde{\chi}^n(x_1, x_2),$$
$$\chi^n(w_1, w_2) = -\frac{\zeta(n)}{(-2\pi)^n}\frac{1}{(w_1+w_2)^{n-1}(2w_2)^{n-1}} \times \sum_{l=0}^{n-2}\frac{\Gamma(n+l-1)}{\Gamma(l+1)}\frac{(w_1+w_2)^l}{(2w_2)^l}. \tag{4.9}$$

As anticipated, we need to keep all orders in α' in this regime. The weak coupling expansion corresponds to the expansion in powers of $1/w$. The expressions in Eqs. (4.8) or (4.9) are such that if $x_1 \sim 1$, but $x_2 \sim -1$, then there is no contribution as $g \to \infty$. Similarly, the one-loop contribution also vanishes in this limit as $\theta_1^{\pm} - \theta_2^{\pm} \to \infty$ in Eq. (4.5), and we use that $h \to e^{-i\pi/4}$ in that limit, so that $\sigma^2_{1\text{-loop}} \to 1$. Then in the region $x_1 \sim 1$ and $x_2 \sim -1$, the leading contribution is just the tree-level contribution (4.2), which was shown in [12] to reproduce precisely the flat-space spectrum. In other words, the BES/BHL guess [5] has a structure that renders the computation of the flat-space spectrum in Ref. [12] valid.[16]

One nice aspect of this strong-coupling series is that it is Borel summable. As shown in detail in Appendix D, we can write the full sum over n as

$$\hat{\chi}(w_1, w_2) = \sum_{n=2}^{\infty}\chi^n(w_1, w_2),$$
$$\hat{\chi}(w_1, w_2) = \frac{1}{2\pi}\int_0^{\infty}d\tau\frac{1}{w_1+w_2+\tau} \times \log[1 - e^{-2\pi\tau^2 - 2\pi(2w_2)\tau}]. \tag{4.10}$$

This integral defines the sum exactly for all values of $w^{\pm}$. Note that the odd-n contributions were essential to be able to Borel sum the expression.[17] The full phase factor is then $\chi = \chi^0 + \chi^1 + \hat{\chi}$.

It is also possible to write a combined integral expression for χ of the form

$$\chi(w_1, w_2) - \chi(w_2, w_1) = \frac{1}{2\pi}\int_0^{\infty}\frac{d\tau}{w_1+w_2+\tau} \times \log\left\{\frac{\sinh[\pi\tau(\tau+2w_2)]}{\sinh[\pi\tau(\tau+2w_1)]} \times \frac{(\tau+2w_1)}{(\tau+2w_2)}\right\}. \tag{4.11}$$

One can perform a few checks on this expression. First, one would like to check that the expression obeys the crossing-symmetry equation. All the contributions from loop orders two and beyond, constructed from Eq. (4.10), should not change under double crossing, but should change in a very specific way under single crossing [6]. We can see that this is indeed the case, provided we choose the crossing contour shown in Fig. 3. We can characterize the contour in terms of the variable η introduced in Eq. (2.10) by stipulating that $\eta = \epsilon + it$, with ϵ small and positive. The crossing transformation corresponds to $t \to t + \pi$, while double crossing takes $t \to t + 2\pi i$. Of course, the starting point for the crossing transformation

[16] This of course holds with the caveat we mentioned above regarding the need to take $J \sim \sqrt{g}$, which might make the asymptotic analysis unreliable.

[17] Another double-scaling limit was considered in Ref. [31], where only even n were summed, which led to singularities.

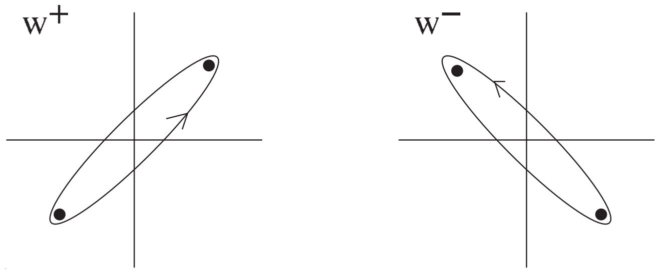

FIG. 3. Contours that we should choose for performing the crossing transformation. We need to enforce the the constraint $(w^+)^2 - (w^-)^2 = i$, and move w^+ and w^- together as we do the crossing transformation. The dots sit at $\pm e^{\pm i\pi/4}$.

need not be on this contour, in which case we deform the contour in such a way that we go around any possible branch cuts. This choice of contours is important because the integrand has singularities at

$$\tau = -w_2 \pm \sqrt{w_2^2 + in}, \tag{4.12}$$

for all integers $n \in \mathbf{Z}$, and these might lead to additional contributions. These points lead to branch cuts in $\hat{\chi}$ at $w_2 = \pm\sqrt{in}$. These branch cuts were not present in each of the individual terms in Eq. (4.8) and, for this reason, it is nontrivial that there exists a contour that allows the crossing transformation to work. The explicit check of the crossing equation is discussed in more detail in Appendix D.

Note that the final result is a smooth function without singularities as long as $Re(w_1)$, $\mathrm{Re}(w_2) > 0$. This condition is obeyed at the physical (real) values of the momenta (2.3). Thus, we smoothly interpolate between the small k and large k regimes. At this point it should be checked that the large-k answer is indeed the one we expect from Eq. (3.31), and we want to see how the branch cut in Eq. (3.31) is replaced by poles. At first sight we seem to have a problem, since the leading-order result (3.32) has a branch cut at $w_1 + w_2 \sim 0$. In fact, Eq. (4.10) also contains a branch cut that precisely cancels the leading-order result (3.32). This branch cut arises from the pole of Eq. (4.10) at $\tau = -w_1 - w_2$. This cancellation is most easily seen by looking at Eq. (4.11), which is manifestly nonsingular at $\tau = -w_1 - w_2$.

In addition, we can see that we will get additional singularities whenever two singularities of the integrand, which are at $\tau = -w_2 + \sqrt{w_2^2 + in}$, pinch the integration contour. All these singularities arise away from the real physical values of momenta. However, for large momenta they can lie very close the physical values. Focusing on the singularities that are very close to the physical subspace, we can show that the full factor σ^2 contains double poles at

$$\sqrt{(w_2^+)^2 + in} + w_1^- = 0, \tag{4.13}$$

where $n > 0$. Appendix D gives the derivation of this result. Here we define the branch of the square root in such a way that its real part is positive when the real part of w_2^+ is positive. Of course, we also have double zeroes at

$$\sqrt{(w_1^+)^2 + in} + w_2^- = 0. \tag{4.14}$$

We also find that the one-loop expression has a single zero at $w_2^+ + w_1 = 0$, which is cancelled by a pole in the sum of all the higher-order terms. The net result is that the σ^2 factor does not have a zero or pole at $w_2^+ + w_1^- = 0$ (see Appendix D). In fact, it is analytic at this point. This is again most easily seen by looking at Eq. (4.11).[18]

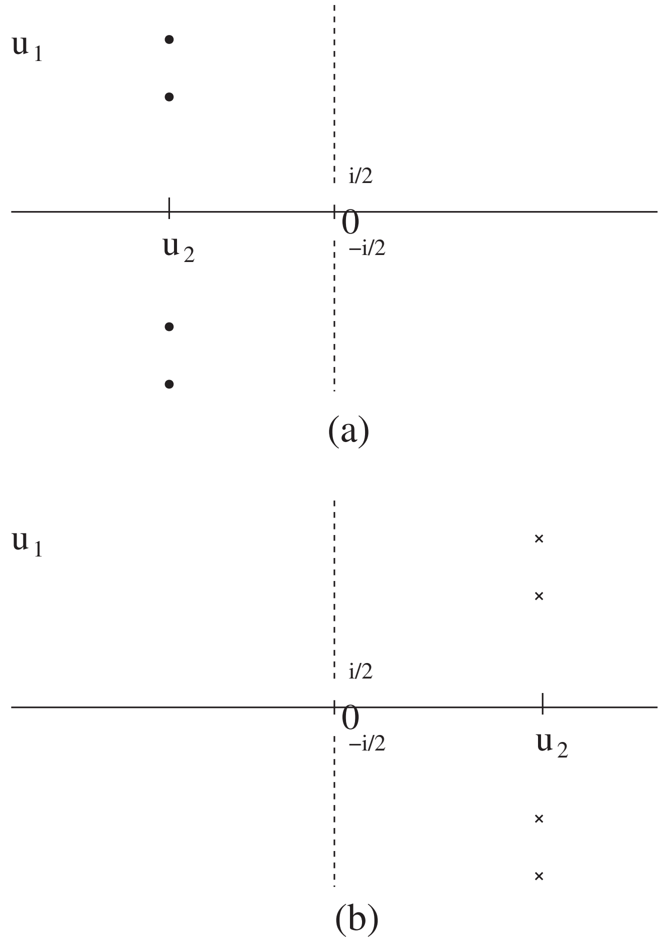

FIG. 4. We fix a real value for u_2 and display the poles and zeros in the complex u_1 plane. There is a branch cut in the amplitude starting at $u_1 = \pm i/2$, and possible additional branch cuts appear at larger imaginary values. If $u_2 < 0$, as in (a), the poles are immediately accessible by moving u_1 in the complex direction. They are denoted by dots. If we start with $u_2 > 0$, as in (b), the poles are in a second branch that is accessible only after taking u_1 through the branch cut. We denoted these by crosses. Thus, for $u_2 > 0$ we do not encounter poles when we move u_1 in the imaginary direction.

[18]The first version of this paper had an incorrect statement on this point.

We can also express Eq. (4.13) as

$$u_1 - u_2 = in, \qquad n > 1, \tag{4.15}$$

but here we are losing information since we do not distinguish between the two possible signs in the square root on Eq. (4.13). In fact, we do not find these poles if we start with large positive $u_1 \sim u_2$ and analytically continue in the imaginary direction. We only find them if we start with *negative* u_i and analytically continue. This structure is shown in Fig. 4. We get a mean density of poles consistent with the one sketched at the end of Sec. III C.[19]

Note that it is a nontrivial check of BES/BHL [5,6] that we get the expected results in the giant-magnon region. In fact, some of the other guesses in Ref. [6] do not have this property. Note, in particular, that the elementary excitations and the giant magnon only differ by their momentum, and they are continuously connected and do not constitute two different kinds of excitations.

V. CONCLUSIONS

In this article we have studied some aspects of an especially interesting limit of the world sheet S matrix of type IIB string theory on $AdS_5 \times S^5$. Since the limit is one in which the spacetime curvature radius is taken to infinity, one might have thought that we would arrive at a free theory. Nevertheless, the effects of the leading deviation away from flat space are still important in this limit. The reason is that the excitations have a long time to interact: the effects of small curvature are thus compounded, producing large contributions.

In this limit, the theory has many of the features that are encoded in the full S matrix, such as the fact that the magnon scattering is off-diagonal and admits a nontrivial crossing-symmetry equation. Moreover, we have found that the Lagrangian in this limit looks fairly simple, and might be exactly solvable in some fashion. Since the string theory ultimately becomes weakly coupled, one would expect that there is a clever way to solve this model directly.

We also studied the BES/BHL [5,6] proposal for the S matrix and found that it correctly reproduces the salient properties of the giant-magnon region, at least in this limit. We also found that we have double poles in this region. The physical interpretation of these double poles will be addressed in a future publication.

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APPENDIX A: ANALYTIC PROPERTIES IMPLIED BY THE CROSSING RELATION

We introduce the variable η defined via

$$w^+ = i^{1/4}\cosh\eta, \qquad w^- = i^{1/4}\sinh\eta. \tag{A1}$$

Crossing symmetry shifts $\eta \to \eta + i\pi$. Let us write the crossing symmetry Eq. (2.9) [10] in terms of these variables. We find

$$\begin{aligned}\sigma^2(\eta_1 + i\pi, \eta_2)\sigma^2(\eta_1, \eta_2) &= \frac{(\sinh\eta_1 - \cosh\eta_2)^2}{(\cosh\eta_1 + \sinh\eta_2)^2} \times \frac{(\sinh\eta_1 + \sinh\eta_2)^2}{(\cosh\eta_1 - \cosh\eta_2)^2} \\ &= \frac{(\sinh\eta_1 - \cosh\eta_2)^2}{(\cosh\eta_1 + \sinh\eta_2)^2} \times \frac{1}{\tanh^2\frac{\eta_1 - \eta_2}{2}},\end{aligned} \tag{A2}$$

where we have simplified the second factor. We can iterate the equation once to obtain the equation

$$\begin{aligned}\sigma^2(\eta_1, \eta_2 - 2\pi i) &= \sigma^2(\eta_1 + 2i\pi, \eta_2) \\ &= \sigma^2(\eta_1, \eta_2)\frac{(\cosh\eta_1 + \sinh\eta_2)^2}{(\sinh\eta_1 - \cosh\eta_2)^2} \times \frac{(\sinh\eta_1 + \cosh\eta_2)^2}{(-\cosh\eta_1 + \sinh\eta_2)^2} \times \tanh^4\frac{\eta_1 - \eta_2}{2},\end{aligned} \tag{A3}$$

where we have used the equations for crossing of η_1 and η_2. So we see that σ^2 should pick up the above factor when we shift its argument by $2\pi i$. If we had just the last factor involving the hyperbolic tangent it would be easy to find a meromorphic solution to Eq. (A3).[20] The problem arises from the first factor. We will now show that there exists no meromorphic solution, as a function of η_1 and η_2, which solves (A3).

First, we focus on a particular pole developed by one of the factors in Eq. (A3):

$$\sinh\eta_1 - \cosh\eta_2 = 0. \tag{A.4}$$

Solving for η_2 as a function of η_1, we see that we have $\mathbf{Z}_2$ branch points where $\sinh\eta_1 = \pm 1$. If we take a contour in the complex η_1 plane going around these two branch points, we see that $\eta_2(\eta_1)$ will change by $2\pi i$. This contour is depicted in Fig. 5.

[19]Note that $u_1 - u_2 \sim -\frac{1}{4}(k_1^2 - k_2^2)$ for large $k_i \gg 1$.

[20]The solution is just h^2, with h in Eq. (4.6).

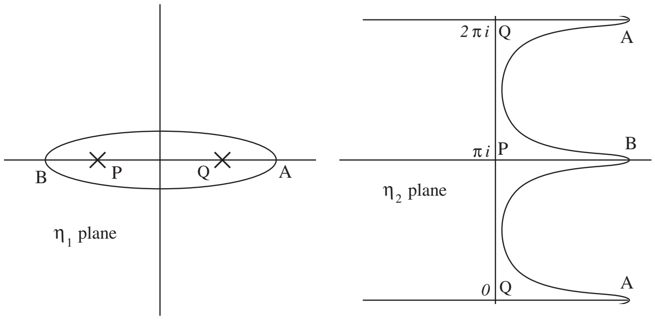

FIG. 5. Contour in the η_1 plane that leads to a shift in η_2 when we impose the equation $\sinh\eta_1 - \cosh\eta_2 = 0$. Letters indicate points that are mapped to each other, and the contour on the left maps into the contour on the right.

Now, let us imagine that we obtain a putative meromorphic solution $\sigma^2(\eta_1, \eta_2)$. Consider the holomorphic function $f(\eta_1) = \sigma^2(\eta_1, \eta_2(\eta_1))$, and assume that for some value of η_1 the function f is neither zero nor infinite. Then, by assumption, we will get a meromorphic function $f(\eta_1)$, which could have branch points where $\eta_2(\eta_1)$ exhibits branch points. However, by continuing η_1 along the contour in Fig. 5, so that $\eta_2(\eta_1) \to \eta_2(\eta_1) + 2\pi i$, we find that $f(\eta_1)$ becomes identically zero by virtue of the crossing relations. This is a contradiction, because we would find that f becomes zero after going through a branch cut. If $\sigma^2(\eta_1, \eta_2)$ has a finite-order pole or zero at $\eta_2 = \eta_2(\eta_1)$, then we can repeat this argument after we go around the contour a sufficient number of times so as to cancel the given pole or zero.

To evade this contradiction we need to either allow branch cuts in σ^2 or allow essential singularities along the regions where Eq. (A.4) is obeyed.[21] The option with essential singularities appears to be incorrect, since the line of essential singularities approaches the real physical line, for example, as $w_1^- = w_2^+ \to \infty$.

Even though we have made this argument in this special, near-flat-space limit, the same argument can be made for the full crossing-symmetry equation in Ref. [10]. In this case one may start with the assumption of a meromorphic solution on the two planes that result from going to the cover of the two tori appearing in the description used in Ref. [10].

APPENDIX B: SOME PROPERTIES OF THE $O(N)$ SIGMA MODEL IN THE NEAR-FLAT-SPACE LIMIT

1. Classical integrability

Consider the theory

$$S = \int \partial_+ \vec{y} \partial_- \vec{y} + \partial_+ \chi \partial_- \chi - \vec{y}^2 \partial_- \chi. \tag{B1}$$

[21] Solutions with essential singularities do exist [29].

The Lax connection can be obtained by taking a simple limit of the connection for the $O(N)$ theory [32]. To write it down explicitly, let us select one of the $O(N)$ generators, J_{12}, and consider the off-diagonal generators that mix the (1, 2) plane with the rest: we call them $J^{\pm i}$, where $i = 1, \cdots, N-2$. We will need the following commutation relations:

$$[J^{12}, J^{\pm i}] = \pm J^{\pm i}, \qquad [J^{+i}, J^{-j}] = -\delta_{ij} J^{12} - J^{ij}, \qquad [J^{-i}, J^{+j}] = \delta_{ij} J^{12} - J^{ij}. \tag{B2}$$

We then construct the flat connection $\mathcal{A}(w)$, where w is the spectral parameter

$$\begin{aligned} d + \mathcal{A} &= d + \mathcal{A}_+ d\sigma^+ + \mathcal{A}_- d\sigma^-, \\ d\mathcal{A} + \mathcal{A}^2 &= 0, \\ \mathcal{A}_+ &= \frac{i}{\sqrt{2}}[e^{-i\sigma^+ w} y^i J^{+i} + e^{+i\sigma^+ w} y^i J^{-i}], \\ \mathcal{A}_- &= \frac{1}{w}\bigg[-i\partial_- \chi J^{12} - \frac{1}{\sqrt{2}} e^{-i\sigma^+ w} \partial_- y^i J^{+i} + \frac{1}{\sqrt{2}} e^{i\sigma^+ w} \partial_- y^i J^{-i}\bigg]. \end{aligned} \tag{B3}$$

We can now use the new variables (3.8) and the gauge-fixing conditions $j_+ = T_{--} = 0$ to derive the Lax connection for the gauge-fixed Lagrangian (3.10). It will be convenient to employ the relations

$$d\sigma^+ = dx^+, \qquad d\sigma^- 2(\tfrac{1}{4} + \partial_{\sigma^-}\chi) = dx^- - y^2 dx^+. \tag{B4}$$

Using these equations and the constraints we find the new connection

$$\begin{aligned} & d + \mathcal{A}_+ dx^+ + \tilde{\mathcal{A}}(dx^- - y^2 dx^+), \\ & \tilde{\mathcal{A}} \equiv \frac{1}{w}\bigg[-i\bigg(\frac{1}{4} - (\partial_{x^-} y)^2\bigg) J^{12} - \frac{1}{\sqrt{2}} e^{-ix^+ w} \partial_{x^-} y^i J^{+i} + \frac{1}{\sqrt{2}} e^{ix^+ w} \partial_{x^-} y^i J^{-i}\bigg], \end{aligned} \tag{B5}$$

and $\mathcal{A}_+$ is the same as what we had above in Eq. (B3).[22]

One can perform a gauge transformation to remove the constant part of the connection

$$\mathcal{A} \to \mathcal{A}' = g^{-1}\mathcal{A} g + g^{-1} dg, \qquad g = e^{i(1/4w)x^- J^{12}}. \tag{B6}$$

We then find that the quantities in the above equation become

[22] Note that the "+" component of the connection in these variables is $\mathcal{A}_+ - y^2 \tilde{\mathcal{A}}$.

$$d + \mathcal{A}'_+ dx^+ + \tilde{\mathcal{A}}'(dx^- - y^2 dx^+),$$
$$\mathcal{A}'_+ = \frac{i}{\sqrt{2}}[e^{-ix^+ w - i(1/4w)x^-} y^i J^{+i} + e^{ix^+ w + i(1/4w)x^-} y^i J^{-i}],$$
$$\tilde{\mathcal{A}}' = \frac{1}{w}\left[i(\partial_{x^-} y)^2 J^{12} - \frac{1}{\sqrt{2}} e^{-ix^+ w + i(1/4w)x^-} \partial_{x^-} y^i J^{+i} + \frac{1}{\sqrt{2}} e^{ix^+ w + i(1/4w)x^-} \partial_{x^-} y^i J^{-i}\right]. \quad \text{(B7)}$$

2. The quantum theory

We now consider the quantum theory based on Eq. (B1). To avoid IR problems, it is convenient to expand the theory around a vacuum where $\partial_- \chi = m^2$. In this way the field y becomes massive. We obtain a divergent diagram from the one-loop self energy of y of the form

$$I(m^2) = (N-2)\frac{1}{2}\int \frac{d^2p}{(2\pi)^2}\log(p^2 + m^2). \quad \text{(B8)}$$

We can see that

$$\partial_{m^2} I = \frac{(N-2)}{2}\int \frac{d^2p}{(2\pi)^2}\frac{1}{p^2+m^2} \sim \frac{(N-2)}{8\pi}\log(\Lambda^2/m^2). \quad (73)$$

Thus it is clear that $I \sim m^2 \log m^2$. This violates the right-conformal symmetry $\sigma^- \longrightarrow \lambda\sigma^-$ that rescales m^2. This is perhaps not surprising, given that the original $O(N)$ sigma model is not conformal once quantum corrections are taken into account. Thus the theory is not right-conformal in general.

Even though the quantum theory is not conformal it is still integrable, and one can solve it using the known solution for the S matrix for the $O(N)$ sigma model [21]. Note that, for the particular case of $N = 2$, the theory is conformal. At first sight this theory is trivial, since in that case we do not have any y. However, as stressed in Ref. [22], we can view this theory as one based on the $OSp(M + 2|M)$ supergroup. In this case we would have M bosonic y fields and M fermionic y fields. The $OSp(M + 2|M)$ theory was studied in Ref. [22,33] as a toy model for the full $AdS_5 \times S^5$ sigma model. One can similarly study it in the limit we consider here, where some of the equations in Ref. [22] simplify a bit. Using those results, one can show that the scattering of impurities in this model gives rise to double poles.

APPENDIX C: A PROOF OF FORMULA (4.8)

In this section we consider the sums that appear in the definition of the phase. The starting expressions will be the conjectured forms in BES/BHL [5,6], for $n \geq 2$:

$$\chi^n = (-1)\frac{\zeta(n)}{(-2\pi)^n}\frac{1}{g^{n-1}\Gamma(n-1)}\frac{x_1}{(x_1x_2)^2}\hat{\chi};$$
$$\hat{\chi} \equiv \sum_{t=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(t+1+m+\frac{n}{2})\Gamma(m+\frac{n}{2})}{\Gamma(t+3+m-\frac{n}{2})\Gamma(2+m-\frac{n}{2})}\frac{1}{x^t y^m},$$
$$r_{[6]} = t+2, \qquad s_{[6]} = r+1+2m,$$
$$x \equiv x_1x_2, \qquad y \equiv x_2^2. \quad \text{(C1)}$$

In the last line we related the summation indices in Ref. [6] to our own. Using Mathematica, we can easily show that Eq. (C1) is equal to Eq. (4.8), for low values of n. These were also summed in Ref. [6]. One can give a general proof as follows. When the sums are performed one typically generates hypergeometric functions. We will use the following identity:

$$\frac{1}{\Gamma(\tilde{m}+b-\tilde{n})}F(\tilde{m}, b, \tilde{m}+b-\tilde{n}, z) = \frac{\Gamma(1-\tilde{m}+\tilde{n})}{\Gamma(\tilde{m})\Gamma(b-\tilde{n})\Gamma(b)}\sum_{s=0}^{\tilde{n}-\tilde{m}}\frac{\Gamma(b-\tilde{n}+s)\Gamma(\tilde{n}-s)}{s!\Gamma(1-\tilde{m}+\tilde{n}-s)} \times (1-z)^{s-\tilde{n}}, \quad \text{(C2)}$$

which is true if $\tilde{m}$, $\tilde{n}$ are integers, and $0 < \tilde{m} \leq \tilde{n}$.

The first step will be to do the sum over t in Eq. (C1). This gives

$$\sum_{t=0}^{\infty}\frac{\Gamma(1+t+m+n/2)}{\Gamma(t+3+m-n/2)}x^{-t} = \frac{\Gamma(n-1)}{\Gamma(2+m-\frac{n}{2})} \times \sum_{s=0}^{n-2}\frac{\Gamma(2+m-\frac{n}{2}+s)}{s!} \times \left(1-\frac{1}{x}\right)^{s+1-n}, \quad \text{(C3)}$$

after using Eq. (C2) with $\tilde{m} = 1$ and $\tilde{n} = n-1$. The result for χ is therefore

$$\hat{\chi} = \Gamma(n-1)\sum_{s=0}^{n-2}\frac{1}{s!}\left(1-\frac{1}{x}\right)^{s+1-n} Y(s,n,y),$$
$$Y(s,n,y) \equiv \sum_{m=0}^{\infty}\frac{\Gamma(2+m-\frac{n}{2}+s)\Gamma(m+\frac{n}{2})}{\Gamma(2+m-\frac{n}{2})^2}y^{-m} = \frac{\Gamma(2-\frac{n}{2}+s)\Gamma(\frac{n}{2})}{\Gamma(2-\frac{n}{2})^2}\,{}_3F_2\left(1, 2-\frac{n}{2}+s, \frac{n}{2}; 2-\frac{n}{2}, 2-\frac{n}{2}; 1/y\right). \quad \text{(C4)}$$

We now use the identity

$${}_3F_2(a_1,a_2,a_3;b_1,b_2,z) = \frac{\Gamma(b_2)}{\Gamma(a_3)\Gamma(b_2-a_3)}\int_0^1 dt t^{a_3-1} \times (1-t)^{b_2-a_3-1}\,{}_2F_1(a_1,a_2,b_1;tz). \quad \text{(C5)}$$

Applying this formula and choosing $a_1 = 1$, we find that the parameters of the hypergeometric function ${}_2F_1$ are related in such a way that we can apply Eq. (C2) with

$$\tilde{m} = a_1 = 1, \qquad \tilde{n} = a_1 + a_2 - b_1 = 1 + s, \qquad a_3 = \frac{n}{2}. \tag{C6}$$

(we take n to be a real number and s to be integer). We then find

$$\begin{aligned} {}_3F_2\Big(1, 2) - \frac{n}{2} + s, \frac{n}{2}; 2 - \frac{n}{2}, 2 - \frac{n}{2}, 1/y\Big) &= \frac{\Gamma(2-\frac{n}{2})^2}{\Gamma(\frac{n}{2})\Gamma(2-n)} \int_0^1 dt t^{(n/2)-1}(1-t)^{1-n} \frac{1}{\Gamma(2-\frac{n}{2})} \\ &\quad \times {}_2F_1\Big(1, 2 - \frac{n}{2} + s, 2 - \frac{n}{2}, t/y\Big) \\ &= \frac{\Gamma(2-\frac{n}{2})^2}{\Gamma(\frac{n}{2})\Gamma(2-n)} \int_0^1 dt t^{(n/2)-1}(1-t)^{1-n} \frac{\Gamma(s+1)}{\Gamma(1-\frac{n}{2})\Gamma(2-\frac{n}{2}+s)} \\ &\quad \times \sum_{q=0}^{s} \frac{\Gamma(1-\frac{n}{2}+q)}{q!}(1-t/y)^{q-s-1} \\ &= \frac{\Gamma(2-\frac{n}{2})^2}{\Gamma(\frac{n}{2})\Gamma(2-n)} \frac{\Gamma(s+1)}{\Gamma(1-\frac{n}{2})\Gamma(2-\frac{n}{2}+s)} \sum_{q=0}^{s} \frac{\Gamma(1-\frac{n}{2}+q)}{q!} \\ &\quad \times \int_0^1 dt t^{(n/2)-1}(1-t)^{1-n}(1-t/y)^{q-s-1}, \end{aligned} \tag{C7}$$

and

$$\int_0^1 dt t^{(n/2)-1}(1-t)^{1-n}(1-t/y)^{q-s-1} = \frac{\Gamma(\frac{n}{2})\Gamma(2-n)}{\Gamma(2-\frac{n}{2})} {}_2F_1\Big(\frac{n}{2}, 1-q+s, 2-\frac{n}{2}; 1/y\Big). \tag{C8}$$

Let us now go back to the sum we wanted to evaluate Eq. (C4):

$$Y(s,n,y) = \frac{\Gamma(\frac{n}{2})}{\Gamma(2-\frac{n}{2})} \frac{\Gamma(s+1)}{\Gamma(1-\frac{n}{2})} \sum_{q=0}^{s} \frac{\Gamma(1-\frac{n}{2}+q)}{q!} \times {}_2F_1\Big(\frac{n}{2}, 1-q+s, 2-\frac{n}{2}; 1/y\Big). \tag{C9}$$

Using Eq. (C2), with $\tilde{m} = 1 - q + s$, $b = n/2$, and $\tilde{n} = n - 1 + s - q$, we obtain

$$\begin{aligned} Y(s,n,y) &= \frac{\Gamma(s+1)\Gamma(n-1)}{\Gamma(1-\frac{n}{2})} \sum_{q=0}^{s} \frac{\Gamma(1-\frac{n}{2}+q)}{q!\Gamma(1+s-q)\Gamma(-n/2+1-s+q)} \\ &\quad \times \sum_{k=0}^{n-2} \frac{\Gamma(-n/2+1-s+q+k)\Gamma(n-1+s-q-k)}{k!\Gamma(n-1-k)}\Big(1-\frac{1}{y}\Big)^{k-n+1+q-s}. \end{aligned} \tag{C10}$$

So the final answer takes the form

$$\begin{aligned} \hat{\chi}(x,y) &= \frac{\Gamma(n-1)^2}{\Gamma(1-\frac{n}{2})} \sum_{s=0}^{n-2} \sum_{q=0}^{s} \sum_{k=0}^{n-2} \frac{\Gamma(1-\frac{n}{2}+q)}{q!\Gamma(1+s-q)\Gamma(-n/2+1-s+q)} \frac{\Gamma(-n/2+1-s+q+k)\Gamma(n-1+s-q-k)}{k!\Gamma(n-1-k)} \\ &\quad \times \Big(1-\frac{1}{y}\Big)^{k-n+1+q-s}\Big(1-\frac{1}{x}\Big)^{s+1-n}, \end{aligned} \tag{C11}$$

with

$$\begin{aligned} \chi &= -\frac{\zeta(n)}{(-2\pi)^n} \frac{1}{x_1 x_2^2} \frac{\Gamma(n-1)}{\Gamma(1-\frac{n}{2})} \Big[g\Big(1-\frac{1}{x_2^2}\Big)\Big(1-\frac{1}{x_1 x_2}\Big)\Big]^{1-n} \sum_{s=0}^{n-2} \sum_{q=0}^{s} \sum_{k=0}^{n-2} \frac{\Gamma(1-\frac{n}{2}+q)}{q!\Gamma(1+s-q)\Gamma(-\frac{n}{2}+1-s+q)} \\ &\quad \times \frac{\Gamma(-\frac{n}{2}+1-s+q+k)\Gamma(n-1+s-q-k)}{k!\Gamma(n-1-k)} \Big(\frac{1-\frac{1}{x_1 x_2}}{1-\frac{1}{x_2^2}}\Big)^s \Big(1-\frac{1}{x_2^2}\Big)^{k+q}. \end{aligned} \tag{C12}$$

Note that we have expressed the full phase in the second expression. The leading-order answer in the scaling limit comes from $q = k = 0$. In that case one can see that this expression reduces to Eq. (4.9), after a relabeling of the indices. When n is even, we can rewrite the last result in a way that is slightly more concise:

$$\chi = -\frac{\zeta(n)}{(-2\pi)^n}\frac{1}{x_1x_2^2}\Gamma(n-1)\Gamma\left(\frac{n}{2}\right)\times\left[g\left(1-\frac{1}{x_2^2}\right)\left(1-\frac{1}{x_1x_2}\right)\right]^{1-n}\times\sum_{s=0}^{n-2}\sum_{q=0}^{s}\sum_{k=0}^{n-2}(-1)^{q+k}\frac{\Gamma(\frac{n}{2}+s-q)}{q!\Gamma(1+s-q)\Gamma(\frac{n}{2}-q)}\times\frac{\Gamma(n-1+s-q-k)}{\Gamma(\frac{n}{2}+s-q-k)k!\Gamma(n-1-k)}\left(\frac{1-\frac{1}{x_1x_2}}{1-\frac{1}{x_2^2}}\right)^s\times\left(1-\frac{1}{x_2^2}\right)^{k+q}. \tag{C13}$$

APPENDIX D: COMMENTS ON THE ANALYTIC STRUCTURE OF THE PHASE

We consider the antisymmetric combination

$$\chi_a(w_1, w_2) = \chi(w_1, w_2) - \chi(w_2, w_1), \tag{D1}$$

since χ always appears in this combination. We then define $w_c \equiv w_1 + w_2$ and $w_r \equiv w_1 - w_2$, and add the tree-level result in Eq. (4.2) to the above expression. We define by χ'_a the antisymmetric combination of the form in Eq. (D1), including all contributions except that at one-loop order. Up to terms that will cancel in the full phase, we find

$$\partial_{w_r}\chi'_a = -\int_0^\infty d\tau\left\{\frac{\tau}{\tau+w_c}\left[\frac{1}{e^{2\pi\tau(\tau+w_c-w_r)}-1}+\frac{1}{e^{2\pi\tau(\tau+w_c+w_r)}-1}+1\right]-1+\frac{w_c}{\alpha+\tau}\right\}, \tag{D2}$$

where the last two terms are introduced to make the integral finite. The quantity α is a constant that will drop out of the combinations appearing in the total phase (4.1). In the end we find that these last two terms can essentially be ignored.

We are interested in understanding the analytic structure of this function. We see that the integrand is an analytic function of τ with poles at values that depend on w_c, w_r. Let us first understand the branch points. These will arise when any of the singularities of the integrand approaches $\tau = 0$. In general, if we have an integral of the form

$$\int_0^\infty d\tau\frac{1}{\tau-a}h(\tau), \tag{D3}$$

then we find a branch cut at $a = 0$ of the form

$$h(a)\log a. \tag{D4}$$

Following this logic, one might expect that there is a branch cut at $w_c = 0$. This branch cut is actually not present because the residue on this pole vanishes. This happens because we combined the $n \geq 2$ contribution with the tree-level answer (4.2). This shows that the branch cut at $w_1 + w_2$, which is present in the tree-level answer, is removed in the exact answer. Note also that the one-loop answer does not have a branch cut at this position. This, in turn, implies that the branch cut in Eq. (3.31) is removed.

We can then look at points where the exponentials in Eq. (D2) lead to poles in the integrand. These occur at $\tau = \tau_n^\pm(w_1)$, $\tau_n^\pm(w_2)$, where

$$\tau_n^\pm(w_2) = -w_2 \pm \sqrt{w_2^2 + \text{in}}. \tag{D5}$$

In these cases the residues do not vanish, and we find branch cuts that arise when we move along a contour in w_2, in such a way that $\tau^\pm$ encloses the origin. If τ^+ circles around the origin in an anticlockwise manner, then we get the following shift in the phase:

$$\Delta\chi'_a|_{\tau_n^\pm\to e^{2\pi i}\tau_n^\pm} \sim i\log(w_1 + w_2 + \tau_n^\pm(w_2)). \tag{D6}$$

We find similar terms with $w_2 \to w_1$, and a flip of the overall sign. Note that we have integrated the result obtained from analyzing the singularities in Eq. (D2).

The case corresponding to $\tau_0^-(w_2) = -2w_2$ deserves special attention, since its partner $\tau_0^+ = 0$ apparently does not encircle the origin. In this case we get

$$\Delta\chi(w_1, w_2^+)|_{w_2^+\to e^{2\pi i}w_2^+} = -i\log\left(\frac{w_1+w_2^+}{w_1-w_2^+}\right). \tag{D7}$$

We get a similar expression for w_2^- enclosing the origin:

$$\Delta\chi(w_1, w_2^-)|_{w_2^-\to e^{2\pi i}w_2^-} = -i\log\left(\frac{w_1+w_2^-}{w_1-w_2^-}\right). \tag{D8}$$

If we thought that this was the only contribution obtained from double crossing, then we would find a contribution coming from the one-loop answer that precisely cancels this.

However, if we follow the path in Fig. 3, then there is an additional contribution arising in $\chi(w_1, w_2^+)$ from the term involving $\tau_{-1}^\pm(w_2^+) = -w_2^+ \pm w_2^-$. We find that $\tau_{-1}^+(w_2^+) = -w_2^+ + w_2^-$ encircles the origin in a clockwise fashion, while $\tau_{-1}^-(w_2^+) = -w_2^+ - w_2^-$ cycles around the origin in an anticlockwise fashion. This can be seen most easily by using the variable η in Eq. (2.10) so that $-w_2^+ \pm w_2^- \sim e^{\mp\eta}$, and remembering that $\eta \to \eta + 2\pi i$ under the double-crossing transformation. The net contribution from these two terms is

$$\Delta\chi(w_1, w_2^+)|_{\tau_{-1}^\pm\to e^{\mp 2\pi i}\tau_{-1}^\pm} = -i\log\left(\frac{w_1+w_2^-}{w_1-w_2^-}\right). \tag{D9}$$

When we consider the contour for w_1^-, we find that $\tau_1^\pm(w_1^-) = -w_2^- \pm w_2^+$ enclose the origin. In fact, these are the same combinations that we considered above, producing a contribution of the form

$$\Delta\chi(w_1, w_2^-)|_{\tau_1^\pm\to e^{\mp 2\pi i}\tau_1^\pm} = -i\log\left(\frac{w_1 + w_2^+}{w_1 - w_2^+}\right). \quad (D10)$$

With the contour choice we have made, all the other $\tau_n^\pm$, which were not explicitly considered, do not encircle the origin and therefore do not give rise to a shift in χ. We see that when we add all of these contributions together, the shift of $\Delta[\chi(w_1, w_2^+) - \chi(w_1, w_2^-)]$ vanishes. Here we will have to sum over $w_1 \to w_1^\pm$, but each one vanishes on its own. Thus we see that, with the contour choice that we have made, the double-crossing equation is obeyed. Note, however, that the branch cuts at $w^\pm = 0$ in the one-loop expression seem to be canceled by the branch cut coming form higher-order contributions. Of course, it would be nice to understand the analytic properties of the full phase more completely.

1. A check of the single crossing equation

The contribution to the phase from the terms with $n \geq 2$ can be written as

$$\theta = \frac{1}{2\pi}\int_{w_2^+}^{w_2^-} d\tau\left(\frac{1}{\tau + w_1^+} - \frac{1}{\tau + w_1^-}\right) \times \log[1 - e^{-2\pi(\tau^2-(w_2^+)^2)}] - \frac{1}{2\pi}\int_{w_1^+}^{w_1^-} d\tau\left(\frac{1}{\tau + w_2^+} - \frac{1}{\tau + w_2^-}\right) \times \log[1 - e^{-2\pi(\tau^2-(w_1^+)^2)}], \quad (D11)$$

where we have used that $e^{2\pi(w^+)^2} = e^{2\pi(w^-)^2}$. To check the crossing relation, we want to evaluate $\bar\theta + \theta \equiv \theta(-w_1, w_2) + \theta(w_1, w_2)$. Let us first ignore possible terms that are acquired under continuation. We therefore write this expression as follows:

$$\bar\theta + \theta = \frac{1}{2\pi}\int_{w_2^+}^{w_2^-} 2\tau d\tau\left(\frac{1}{\tau^2 - (w_1^+)^2} - \frac{1}{\tau^2 - (w_1^-)^2}\right) \times \log[1 - e^{-2\pi(\tau^2-w_2^2)}] - \frac{1}{2\pi}\int_{w_1^+}^{w_1^-} 2\tau d\tau\left(\frac{1}{\tau^2 - (w_2^+)^2} - \frac{1}{\tau^2 - (w_2^-)^2}\right) \times \log[1 - e^{-2\pi(\tau^2-w_1^2)}], \quad (D12)$$

where we have combined the term scaling like $1/(\tau + w_1^+)$ from θ with the term scaling like $1/(\tau - w_1^+)$ from $\bar\theta$. In the second line we see that $\bar\theta$ involves an integral in the interval $[-w_1^+, -w_1^-]$. Under a change of integration variables $\tau \to -\tau$, we recover an integral with the same limits as the one we had in θ, but with an extra minus sign in the denominator. It is assumed that in all of these manipulations we do not cross the contour or acquire any extra contributions to the expressions.

We can now invoke a change of variables $u = \tau^2 - (w_2^+)^2$ in the first integral, along with a similar change in the second integral. Imposing the constraint between $w_i^\pm$ in Eq. (2.3), we get

$$\bar\theta + \theta = \frac{1}{2\pi}\int_0^{-i} du\left(\frac{1}{u-\delta} - \frac{1}{u-\delta+i}\right)\log[1 - e^{-2\pi u}] - \frac{1}{2\pi}\int_0^{-i} du\left(\frac{1}{u+\delta} - \frac{1}{u+\delta+i}\right)\log[1 - e^{-2\pi u}],$$
$$\delta \equiv (w_1^+)^2 - (w_2^-)^2. \quad (D13)$$

Note that $\mathrm{Re}(\delta) \ll 0$ in the region $|w_2^+| \gg |w_1^\pm|$, with w_2^+ almost real. By changing variables $u' = -(u + i)$ in the second line, we find that the limits of integration of the second term are the same as in the first term. In addition, the integrand looks very similar. After combining terms we obtain

$$\bar\theta + \theta = \frac{1}{2\pi}\int_0^{-i} du\left(\frac{1}{u-\delta} - \frac{1}{u-\delta+i}\right) \times (\log[1 - e^{-2\pi u}] - \log[1 - e^{2\pi u}]), \quad (D14)$$

where

$$\log[1 - e^{-2\pi u}] - \log[1 - e^{2\pi u}] = -2\pi u - i\pi. \quad (D15)$$

Note that the $-i\pi$ is correct in the difference of logs, since the first log is defined such that it is real for $u > 0$, and the second so that it is real for $u < 0$. We then get

$$\bar\theta + \theta = -(\delta + i/2)\log\frac{-i-\delta}{-\delta} + (\delta - i/2)\log\frac{-\delta}{-\delta+i}. \quad (D16)$$

We now compute the change in the tree-level expression (4.2):

$$\bar\theta^0 + \theta^0 = -2\delta\log(-\delta) - (-\delta + i)\log(-\delta + i) - (-\delta - i)\log(-\delta - i). \quad (D17)$$

When we add this to the results of Eq. (D14) we find

$$(\bar\theta + \bar\theta^0) + (\theta + \theta^0) = -\frac{i}{2}\log\frac{-\delta + i}{-\delta - i}. \quad (D18)$$

After taking into account the change in the one-loop answer, the total change should indeed be (D18) (see Ref. [6] for further details).

2. Poles

Let us now focus on the poles in the amplitude. It is simplest to consider the expression involving a derivative (D2), and the poles in the S matrix will give rise to poles in Eq. (D2). These poles arise when two singularities of the integrand come close together. In principle, they have to pinch the integration contour, but it seems that by analytically continuing in a suitable manner we will get a singularity on some of the branches discussed above.

An interesting set of singularities arises when

$$-w_c = \tau_n^\pm(w_2) \quad \text{or} \quad 0 = w_1 \pm \sqrt{(w_2)^2 + in}. \quad (D19)$$

At these singularities we find that $w_r w_c = in$. We can therefore take $w_r = in/w_c + \epsilon$ and expand the integral for small ϵ. When we do this we see that the contour can be pinched, and we can find a singularity in the phase of the form

$$\chi_a' = -i\log\epsilon \sim -i\log(w_1 \pm \sqrt{(w_2)^2 + in}). \quad \text{(D20)}$$

The analysis thus far does not immediately tell us whether we get a singularity in the region close to the physical space. We expect that some general principle will tell us what the physical region is. For the moment, we will just define it as having values of $w\pm$, which are close to the physical values and far from the branch points.

In particular, we can consider the scattering amplitude in the giant-magnon region, where $k_i \gg 1$. In this regime, w_i are large and almost purely imaginary, and we find that all branch points are far from the physical line. We would then like to know if there are any poles that arise as we analytically continue around this region by a small amount. For this to happen, we need to have values of $\tau_n^{\pm}(w_2)$ and $w_1 + w_2$ that can pinch the contour. In particular, we would like to have $w_1 + w_2$ be small. This happens only if we consider the combinations (w_1^+, w_2^-) and (w_1^-, w_2^+). We can then find that when we analytically continue in the region where $k_1 \sim k_2 \gg 1$, we pinch the contour and get poles that, by our definitions, are close to the physical line.

A simple way to see this is to start with the expression in Eq. (D2), for the case with $w_1 = w_1^-$ and $w_2 = w_2^+$, with $k_1, k_2 \gg 1$. We can then rotate the contour to the line $\tau = e^{i(\pi/4)}y$, with real and positive y. The integral along the new contour gives an exponentially small contribution (up to analytic terms in the phase). So the only remaining contribution arises from the poles that we pick up when we rotate the contour. These poles are at

$$\tau = \tau_n^+(w_2^+) = -w_2^+ + \sqrt{(w_2^+)^2 + in}, \qquad n > 0, \quad \text{(D21)}$$

where the branch of the square root is chosen so that, for physical values of w^+, the real part of the square root is positive. The contribution at each pole is given in terms of the residue of the integrand, and is equal to

$$-i\frac{\tau_n^+(w_2^+)}{\tau_n^+(w_2^+) + w_2^+ + w_1^-}\frac{1}{2(\tau_n^+(w_2^+) + w_2^+)} = -\partial_{w_r} i\log(\tau_n^+(w_2^+) + w_2^+ + w_1^-). \quad \text{(D22)}$$

There is a similar contribution from the terms with (w_1^+, w_2^-). These two combine to give

$$\sigma^2 \sim e^{-2i\chi_a'(w_1^-, w_2^+) + 2i\chi_a'(w_2^-, w_1^+)} \sim \prod_{n=1}^{\infty}\frac{(\sqrt{(w_1^+)^2 + in} + w_2^-)^2}{(\sqrt{(w_2^+)^2 + in} + w_1^-)^2}, \sim \prod_{n=1}^{\infty}\frac{\sqrt{(w_2^+)^2 + in} + w_2^-)(\sqrt{(w_2^-)^2 - in} + w_1^+)}{(\sqrt{(w_2^+)^2 + in} + w_1^-)(\sqrt{(w_1^-)^2 - in} + w_2^+)}. \quad \text{(D23)}$$

These products in the right-hand side are defined only up to analytic terms which are divergent.[23] We have written a couple of different expressions whose poles are the same in the region of interest. This expression is similar to the one written for the so called "giant magnon" phase in Ref. [6]. In fact one can add other contributions to obtain an approximate expression that looks like the "giant" guess in Ref. [6]. However, the full phase is not the same as the "giant magnon" phase in Ref. [6], since the expression (4.10) has a branch cut at $w \pm 2 = 0$, while the expression in the "giant magnon" phase in Ref. [6] does not. In terms of the variable u introduced in Eq. (2.5), the poles lie at

$$u_1 - u_2 = in, \qquad n > 1. \quad \text{(D24)}$$

Notice that displaying the poles as in Eq. (D24) is misleading, since in the region $u_i \gg 0$ we do not have poles near the physical region. This is the plane wave region. On the other hand, for $u_i \ll 0$, which is the giant-magnon region, these poles are close to the physical region. The reason that this happens is that there are many branch cuts starting from the one at $u_i = \pm i/2$ (see Fig. 4).

It turns out that at $u_1 - u_2 = i$ the factor σ^2 is finite, with no zeros or poles. We should simply note that Eq. (4.11) is analytic at $w_1^+ + w_2^- = 0$.

The following is a longer argument showing the same fact. To analyze possible poles at $u_1 - u_2 = i$, or, more precisely, at $w_1^+ + w_2^- = 0$, we need to analyze the one-loop term in Eq. (4.5). So far we have not discussed the one-loop term because, due to its form in Eq. (4.6), we see that it can produce only poles or zeros for $u_1 - u_2 = 0$ or $u_1 - u_2 = \pm i$. Most of these poles are in other branches that one reaches after going through the cuts at $u_i = \pm i/2$. We are only going to discuss poles in the main branch, which is directly connected to the physical values of u. For this purpose, it is useful to notice that as we change the physical momentum k from zero to infinity, the imaginary part of $\hat{\theta}^+$ in Eq. (4.7) goes between zero and $-i\pi/2$. Similarly, the imaginary part of $\hat{\theta}^-$ goes between zero and $+i\pi/2$. When we do an analytic continuation in the neighborhood of the physical values of $w^{\pm}$, the imaginary part of $\hat{\theta}^{\pm}$ will not change by a large amount. Since differences between $\hat{\theta}_i^{\pm}$ can only be at most $\pm i\pi$, we only need to worry about the following terms in h [see Eq. (4.6)]:

$$h(\theta) \sim \frac{(\theta + i\pi)}{(\theta - i\pi)} \times \ldots. \quad \text{(D25)}$$

In addition, the only differences that can get close to $\pm i\pi$ are $\theta_1^+ - \theta_2^- \sim -i\pi$ or $\theta_1^- - \theta_2^+ \sim i\pi$. In the first case we

[23] To produce a formula that makes mathematical sense, we can take the log on each side and take derivatives $\partial_{w_1^+}\partial_{w_2^+}$. The divergence comes from a similar divergent term that we dropped after we rotated the contour. The expression we started from, on the left-hand side, is finite.

see from Eqs. (D25) and (4.5) that we will get a pole in σ^2 as $w_1^- + w_2^+ \sim 0$. In the second case we get a zero in σ^2 as $w_1^- + w_2^+ \sim 0$.

When we analyzed the contributions from the higher-loop terms coming from Eq. (D2), we considered derivatives with respect to $w_r = w_1 - w_2$ for fixed $w_1 + w_2$, so we could have missed contributions involving purely $w_1 + w_2$. In order to find such contributions from the higher-loop terms, we would be tempted to simply set $w_1 = w_2$ first (which amounts to setting $w_r = 0$), and then we would find that there is no singularity at $w_c = 0$. But we should be more careful since the function has branch cuts at $w_i = 0$. It is safer just to keep w_r to be finite, as it would be in the physical region (giant magnon region) we are exploring and then compute the behavior of the function for small w_c. Approximating the integral for small w_c, we find that singularities come from small τ so that we also expand the integral for small τ to find

$$\chi_a^l \sim \frac{1}{2\pi} \int \frac{d\tau}{\tau + w_c} \log\left[\frac{\tau - w_r}{\tau + w_r}\right] = -\frac{i}{2} \log w_c, \quad \text{(D26)}$$

where we had to use

$$\log\frac{-w_r}{w_r} = \log\frac{w_2^+ - w_1^-}{w_1^- - w_2^+} \sim i\pi, \quad \text{(D27)}$$

when we start from physical particles and then we go to the limit where $w_2^+ + w_1^- \sim 0$.

Thus we see that the higher-loop contributions to σ^2 produce a single pole of the form $\sigma^2 \sim (w_2^+ + w_1^-)^{-1}$. This pole cancels the one-loop term and gives no net pole or zero in σ^2 for $w_2^+ + w_1^- \sim 0$.

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[1] M. Staudacher, J. High Energy Phys. 05 (2005) 054; J. A. Minahan and K. Zarembo, J. High Energy Phys. 03 (2003) 013.
[2] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. **B533**, 109 (1998).
[3] S. Frolov, J. Plefka, and M. Zamaklar, J. Phys. A **39**, 13037 (2006).
[4] N. Beisert, arXiv:hep-th/0511082.
[5] N. Beisert, B. Eden, and M. Staudacher, arXiv:hep-th/0610251.
[6] N. Beisert, R. Hernandez, and E. Lopez, arXiv:hep-th/0609044.
[7] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower, and V. A. Smirnov, arXiv:hep-th/0610248.
[8] D. Berenstein, J. M. Maldacena, and H. Nastase, J. High Energy Phys. 04 (2002) 013.
[9] D. M. Hofman and J. M. Maldacena, J. Phys. A **39**, 13095 (2006).
[10] R. A. Janik, Phys. Rev. D **73**, 086006 (2006).
[11] C. G. Callan, Jr., T. McLoughlin, and I. J. Swanson, Nucl. Phys. **B700**, 271 (2004); **B694**, 115 (2004); C. G. Callan, Jr., H. K. Lee, T. McLoughlin, J. H. Schwarz, I. J. Swanson, and X. Wu, Nucl. Phys. **B673**, 3 (2003); T. McLoughlin and I. J. Swanson, Nucl. Phys. **B702**, 86 (2004); I. J. Swanson, arXiv:hep-th/0505028.
[12] G. Arutyunov, S. Frolov, and M. Staudacher, J. High Energy Phys. 10 (2004) 016.
[13] N. Beisert, V. Dippel, and M. Staudacher, J. High Energy Phys. 07 (2004) 075.
[14] N. Beisert, Fortschr. Phys. **53**, 852 (2005).
[15] N. Beisert, arXiv:hep-th/0606214.
[16] See, for example, V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge Univ. Press, Cambridge, England, 1993).
[17] P. H. Ginsparg and G. W. Moore, arXiv:hep-th/9304011.
[18] N. Dorey, J. Phys. A **39**, 13119 (2006); H. Y. Chen, N. Dorey, and K. Okamura, J. High Energy Phys. 09 (2006) 024.
[19] M. Spradlin and A. Volovich, J. High Energy Phys. 10 (2006) 012; C. Kalousios, M. Spradlin, and A. Volovich, arXiv:hep-th/0611033.
[20] M. Kruczenski, J. Russo, and A. A. Tseytlin, J. High Energy Phys. 10 (2006) 002.
[21] A. B. Zamolodchikov and A. B. Zamolodchikov, Ann. Phys. (N.Y.) **120**, 253 (1979).
[22] N. Mann and J. Polchinski, Phys. Rev. D **72**, 086002 (2005).
[23] L. D. Faddeev and N. Y. Reshetikhin, Ann. Phys. (N.Y.) **167**, 227 (1986).
[24] T. Klose and K. Zarembo, J. Stat. Mech. 05 (2006) P006.
[25] C. Gomez and R. Hernandez, arXiv:hep-th/0608029; J. Plefka, F. Spill, and A. Torrielli, Phys. Rev. D **74**, 066008 (2006).
[26] G. Arutyunov, S. Frolov, J. Plefka, and M. Zamaklar, arXiv:hep-th/0609157.
[27] T. Klose, T. McLoughlin, R. Roiban, and K. Zarembo, arXiv:hep-th/0611169.
[28] S. R. Coleman and H. J. Thun, Commun. Math. Phys. **61**, 31 (1978).
[29] R. Janik (unpublished).
[30] R. Hernandez and E. Lopez, J. High Energy Phys. 07 (2006) 004.
[31] C. Gomez and R. Hernandez, arXiv:hep-th/0611014.
[32] N. Beisert, V. A. Kazakov, and K. Sakai, Commun. Math. Phys. **263**, 611 (2006).
[33] N. Mann and J. Polchinski, arXiv:hep-th/0408162.