Viscous hydrodynamics relaxation time from AdS/CFT correspondence

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(Received 17 April 2007; published 27 July 2007)

We consider an expanding boost-invariant plasma at strong coupling using the AdS/CFT correspondence for $\mathcal{N} = 4$ super Yang-Mills theory. We determine the relaxation time in second order viscous hydrodynamics and find that it is around 30 times shorter than weak coupling expectations. We find that the nonsingularity of the dual geometry in the string frame necessitates turning on the dilaton which leads to a nonvanishing expectation value for $\mathrm{tr}F^2$ behaving like $\tau^{-10/3}$.

DOI: 10.1103/PhysRevD.76.025027 PACS numbers: 12.38.Mh, 11.25.Tq

I. INTRODUCTION

It is believed that the quark-gluon plasma (QGP) produced at RHIC in heavy-ion collisions is strongly coupled (see e.g. [1]) and is well described by almost perfect fluid hydrodynamics [2]. Therefore it is very interesting to develop methods for studying its properties nonperturbatively from first principles. A very powerful tool in the study of the dynamics of gauge theory at strong coupling is the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [3]. Although so far there does not exist a version of the correspondence with a gauge theory which would have all the features of QCD, even the simplest version for $\mathcal{N} = 4$ super Yang-Mills theory was argued to share a lot of properties *at finite temperature* with QCD plasma. But of course one has to keep in mind the differences which may not be important for some features of the dynamics but which may be crucial for other phenomena. In this paper we thus work exclusively with hot expanding plasma in the $\mathcal{N} = 4$ SYM theory.

Extensive work has been done in the understanding of transport properties of the plasma at a fixed temperature like calculating the shear viscosity [4,5]. Much less is known about the properties of more dynamical time-dependent processes. On a qualitative level, thermalization has been suggested to correspond to black hole formation in the bulk of the five-dimensional dual geometry [6], while cooling was advocated to correspond to black hole motion in the 5th direction [7].

In [8] a quantitative framework has been proposed for studying the time-dependent expansion of a boost-invariant plasma system. The criterion of nonsingularity of the dual geometry was shown to predict almost perfect fluid hydrodynamic expansion [8] with leading deviations coming from shear viscosity [9] with the shear viscosity coefficient being exactly equal to the one derived in the static case in [4]. Further work in this framework includes [10–14]. The aim of this paper is to investigate in more detail the hydrodynamic expansion and to determine the remaining parameter in second order viscous hydrodynamics [15,16]—the relaxation time τ_Π. This requires going one order higher in the subasymptotic expansion of the geometry.

The plan of this paper is as follows. In Sec. II we will describe the kinematic regime of longitudinal boost invariance. Then in Sec. III we will briefly review second order viscous hydrodynamics. In Sec. IV we describe the AdS/CFT methods used here and review, in the following section, the results obtained so far and a method of determining the relaxation time τ_Π. In Sec. VI we give final results for τ_Π and in Sec. VII we analyze the incorporation of the dilaton and calculate the expectation value of $\mathrm{tr}F^2$. We close the paper with a discussion.

II. BOOST-INVARIANT KINEMATICS

An interesting kinematical regime of the expanding plasma is the so-called central rapidity region. There, as was suggested by Bjorken [17], one assumes that the system is invariant under longitudinal boosts. This assumption is in fact commonly used in realistic hydrodynamic simulations of QGP [2]. If in addition we assume no dependence on transverse coordinates (a limit of infinitely large nuclei) the dynamics simplifies enormously.

In order to study boost-invariant plasma configurations it is convenient to pass from Minkowski coordinates $(x^0, x^1, x_\perp)$ to proper-time/spacetime rapidity ones $(\tau, y, x_\perp)$ through

$$x^0 = \tau \cosh y, \qquad x^1 = \tau \sinh y. \tag{1}$$

The object of this work is to describe the spacetime dependence of the energy-momentum tensor of a boost-invariant plasma in $\mathcal{N} = 4$ SYM theory at strong coupling. The symmetries of the problem reduce the number of independent components of $T_{\mu\nu}$ to three. Energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ and tracelessness $T^\mu_\mu = 0$ allow to express all components in terms of just

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a single function—the energy density $\varepsilon(\tau)$ in the local rest frame. Explicitly we have [8]

$$T_{\tau\tau} = \varepsilon(\tau), \tag{2}$$

$$T_{yy} = -\tau^2\Big(\varepsilon(\tau) + \frac{d}{d\tau}\varepsilon(\tau)\Big), \tag{3}$$

$$T_{xx} = \varepsilon(\tau) + \frac{1}{2}\tau\frac{d}{d\tau}\varepsilon(\tau). \tag{4}$$

Gauge theory dynamics should now pick out a definite function $\varepsilon(\tau)$. The aim of this paper is to find its behavior at large proper times, up to subsubasymptotic terms, and to interpret this behavior in terms of parameters of second order hydrodynamics which we will describe in the next section.

III. SECOND ORDER VISCOUS HYDRODYNAMICS

The object of a hydrodynamic model is to determine the spacetime dependence of the energy-momentum tensor for an expanding (plasma) system. The simplest dynamical assumption is that of a perfect fluid. This amounts to assuming that the energy momentum has the form

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu + p\eta_{\mu\nu}, \tag{5}$$

where u^μ is the local 4-velocity of the fluid ($u^2 = -1$), ε is the energy density, and p is the pressure. In the case of $\mathcal{N} = 4$ SYM theory that we consider here $T^\mu_\mu = 0$ and hence $\varepsilon = 3p$. The equation of motion that one obtains from energy conservation in the boost-invariant setup is

$$\partial_\tau\varepsilon = -\frac{\varepsilon + p}{\tau} \equiv -\frac{4}{3}\frac{\varepsilon}{\tau} \tag{6}$$

whose solution is the celebrated Bjorken result

$$\varepsilon = \frac{1}{\tau^{4/3}}. \tag{7}$$

Once one wants to include dissipative effects coming from shear viscosity, the description becomes more complex. In a first approximation one adds to the perfect fluid tensor a dissipative contribution $\eta(\nabla_\mu u_\nu + \nabla_\nu u_\mu)$, where η is the shear viscosity of the fluid. The resulting equations of motion get modified to

$$\partial_\tau\varepsilon = -\frac{4}{3}\frac{\varepsilon}{\tau} + \frac{4\eta}{3\tau^2}. \tag{8}$$

Note that in the above equation the shear viscosity is generically temperature dependent ($\eta \propto T^3$ in the $\mathcal{N} = 4$ case) and hence τ dependent. In order to have a closed system of equations we have to incorporate this dependence through

$$\eta = A \cdot \varepsilon^{3/4} \tag{9}$$

with A being some numerical coefficient.

However this so-called first order formalism suffers from a number of problems. First it is inconsistent with relativistic invariance (causality)—excitations may propagate at speeds faster than light. Second these equations suffer from some unphysical behavior (see e.g. [16]).

In order to cure these problems Israel and Stewart introduced a second order theory [15] with an additional parameter—the so-called relaxation time τ_Π which can overcome the problems with causality. This theory has found applications in modelling heavy-ion collisions [18]. The corresponding equations of motion, again in the Bjorken regime which we are considering here, are now

$$\partial_\tau\varepsilon = -\frac{4}{3}\frac{\varepsilon}{\tau} + \frac{\Phi}{\tau}, \tag{10}$$

$$\tau_\Pi\partial_\tau\Phi = -\Phi + \frac{4\eta}{3\tau}, \tag{11}$$

where Φ is related to the dissipative part of the energy-momentum tensor $\Pi_{\mu\nu}$ through $\Phi = -\tau^2\Pi^{yy}$. Let us note that when $\tau_\Pi \to 0$ the equations reduce to the first order formalism case. In the above equations one has to keep in mind that the shear viscosity η is again given by (9). Hence the only remaining independent parameter is therefore τ_Π.

In the derivation of the second order viscous hydrodynamics from Boltzmann equations one finds that the ratio of τ_Π to η is given in terms of the pressure p [16]

$$\tau_\Pi^{\mathrm{Boltzmann}} = 2\beta_2\eta = \frac{3\eta}{2p}. \tag{12}$$

This is the value usually used in viscous hydrodynamic simulations. The aim of this work is to determine the relaxation time in the strong coupling regime using the AdS/CFT correspondence. To this end let us parametrize the relaxation time at strong coupling as

$$\tau_\Pi = r \cdot \tau_\Pi^{\mathrm{Boltzmann}}, \tag{13}$$

where $\tau_\Pi^{\mathrm{Boltzmann}}$ is *defined* by the expression (12) and r is a numerical coefficient. The aim of this paper is to determine r.

IV. ADS/CFT DESCRIPTION OF AN EXPANDING BOOST-INVARIANT PLASMA

The procedure adopted in [8] to describe, using AdS/CFT, an expanding system of plasma in $\mathcal{N} = 4$ SYM is (i) consider a family of possible behaviors of the spacetime expectation value of the energy-momentum tensor $\langle T_{\mu\nu}\rangle$, (ii) for each of those $\langle T_{\mu\nu}\rangle$'s find the dual geometry which generically will be singular, and (iii) use the criterion of nonsingularity of the constructed geometry to pick out the *physical* spacetime profile of $\langle T_{\mu\nu}\rangle$.

For the plasma configuration considered in this paper, as described in Sec. II, all possible boost-invariant profiles of

$\langle T_{\mu\nu}\rangle$ can be expressed in terms of a single function $\varepsilon(\tau)$ which is just the energy density in the local rest frame.

The construction of a dual geometry then proceeds as follows [19]. First we adopt the Fefferman-Graham coordinates [20] for the 5-dimensional metric

$$ds^2 = \frac{\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dz^2}{z^2}, \tag{14}$$

where the z coordinate is the "fifth" coordinate while μ is a 4D index. $\tilde{g}_{\mu\nu}$ is here a function both of the 4D spacetime coordinates *and* of the "fifth" coordinate z. Then we have to solve Einstein equations with negative cosmological constant

$$E_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}R - 6g_{\alpha\beta} = 0 \tag{15}$$

with a boundary condition for $\tilde{g}_{\mu\nu}$ around $z = 0$:

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + z^4\tilde{g}^{(4)}_{\mu\nu} + \ldots, \tag{16}$$

where the fourth order term is related to the expectation value of the energy-momentum tensor through

$$\langle T_{\mu\nu}\rangle = \frac{N_c^2}{2\pi^2}g^{(4)}_{\mu\nu}. \tag{17}$$

It is convenient to ignore the factor $N_c^2/(2\pi^2)$ throughout the calculation and only reinstate it in the final result when e.g. writing the energy density or shear viscosity directly in terms of the temperature.

For the boost-invariant geometries relevant here, the most general metric in the Fefferman-Graham coordinates takes the form

$$ds^2 = \frac{1}{z^2}(-e^{a(z,\tau)}d\tau^2 + e^{b(z,\tau)}\tau^2dy^2 + e^{c(z,\tau)}dx_\perp^2) + \frac{dz^2}{z^2} \tag{18}$$

with three coefficient functions $a(z, \tau)$, $b(z, \tau)$, and $c(z, \tau)$. The energy density defining the physics is simply related to the boundary asymptotics of the $a(z, \tau)$ coefficient

$$\varepsilon(\tau) = -\lim_{z\to 0}\frac{a(z, \tau)}{z^4}. \tag{19}$$

Of course it is too difficult to explicitly perform this construction for arbitrary functions $\varepsilon(\tau)$. What one does in practice is to perform an expansion of $\varepsilon(\tau)$ for (large) proper times τ and determine one by one the subsequent terms in the expansion. In the following section we will review the results obtained so far in carrying out this program [8–10].

V. REVIEW OF THE DUAL GEOMETRY UP TO $\mathcal{O}(\tau^{-(4/3)})$

In [8] it was shown that in order to study the large proper-time limit of the metric one is led to introduce a scaling variable

$$v = \frac{z}{\tau^{1/3}} \tag{20}$$

and take the limit $\tau \to \infty$ with v fixed. In order to study subleading terms in the metric we have to perform an expansion around this limit.

Let us now expand the metric coefficient functions $a(z, \tau)$, $b(z, \tau)$, and $c(z, \tau)$ in a series of the following form:

$$a(z, \tau) = a_0(v) + a_1(v)\frac{1}{\tau^{2/3}} + a_2(v)\frac{1}{\tau^{4/3}} + a_3(v)\frac{1}{\tau^2} + \ldots \tag{21}$$

and similar expressions for the other coefficients. The motivation for choosing such specific powers of τ is twofold. First, on the gauge theory side once we have viscous hydrodynamics with $\eta \propto T^3$, the asymptotic expansion of the energy density $\varepsilon(\tau)$ exactly corresponds to the above decomposition of the metric coefficients [21] (21). Second, one can show directly from the gravity side without assuming anything on viscous dynamics in gauge theory, that a correction at order $1/\tau^{2/3}$ has to occur. Namely suppose that we start from the leading order solution with a *generic* first correction

$$a(z, \tau) = a(v) + a_r(v)\frac{1}{\tau^r} + \ldots \tag{22}$$

Then we find that the square of the Riemann tensor is nonsingular at the leading and $1/\tau^r$ orders but will *always* have a singularity at order $1/\tau^{4/3}$. There will also be a singularity at order $1/\tau^{2r}$. The only possibility of obtaining a nonsingular geometry is that $2r = \frac{4}{3}$ and that these two singularities cancel, which is indeed what occurs. This fixes the power of the first subleading correction to be $1/\tau^{2/3}$ which is exactly what is expected for the form of corrections due to shear viscosity with the coefficient behaving like $\eta \propto T^3$.

The procedure is now to insert the expansion (21) into Einstein's equations and solve them order by order. At each order a new integration constant (free parameter) will occur and we will determine it by requiring that a similar expansion of the square of the Riemann tensor will be nonsingular, i.e.

$$\begin{aligned}\mathfrak{R}^2 &\equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \\ &= R_0(v) + R_1(v)\frac{1}{\tau^{2/3}} + R_2(v)\frac{1}{\tau^{4/3}} + R_3(v)\frac{1}{\tau^2} + \ldots\end{aligned} \tag{23}$$

with all $R_i(v)$ being nonsingular.

In order to present the solutions it turns out to be convenient to define

$$\tilde{b}_i(v) \equiv b_i(v) + 2c_i(v). \tag{24}$$

The leading order solution found in [8] is

$$a_0(v) = \log\frac{(1-v^4/3)^2}{1+v^4/3},\qquad \tilde{b}_0(v) = \log(1+v^4/3)^3,\qquad c_0(v) = \log(1+v^4/3), \tag{25}$$

and the resulting $\mathfrak{R}^2$ coefficient is nonsingular:

$$R_0 = \frac{8(5v^{16}+60v^{12}+1566v^8+540v^4+405)}{(3+v^4)^4}. \tag{26}$$

From this expression, using the similarity with the static black hole metric we may read off the temperature from the position of the horizon to obtain at this order:

$$T(\tau) = \frac{\sqrt{2}}{3^{1/4}\pi\tau^{1/3}}. \tag{27}$$

We will use this expression later in the paper. The first subleading correction was found in [10] and reads in our conventions

$$a_1(v) = 2\eta_0\frac{(9-v^4)v^4}{9-v^8},\qquad \tilde{b}_1(v) = -6\eta_0\frac{v^4}{3+v^4},\qquad c_1(v) = -2\eta_0\frac{v^4}{3+v^4} - \eta_0\log\frac{3-v^4}{3+v^4}, \tag{28}$$

where η_0 is an undetermined integration constant (which has the physical interpretation as the coefficient of shear viscosity). However it turns out that η_0 is undetermined at this order from nonsingularity of $\mathfrak{R}^2$ since

$$R_1 = \frac{41\,472(v^4-3)v^8}{(3+v^4)^5}\cdot\eta_0 \tag{29}$$

and is nonsingular for *any* value of η_0. Therefore in order to fix η_0 one has to go one order higher. This was done in [9] with the result

$$\begin{aligned}
a_2(v) &= \frac{(9+5v^4)v^2}{6(9-v^8)} - C\frac{(9+v^4)v^4}{36(9-v^8)} + \eta_0^2\frac{(-1053-171v^4+9v^8+7v^{12})v^4}{3(9-v^8)^2} + \frac{1}{4\sqrt{3}}\log\frac{\sqrt{3}-v^2}{\sqrt{3}+v^2} - \frac{3}{2}\eta_0^2\log\frac{3-v^4}{3+v^4},\\
\tilde{b}_2(v) &= \frac{v^2}{2(3+v^4)} + C\frac{v^4}{12(3+v^4)} + \eta_0^2\frac{(39+7v^4)v^4}{(3+v^4)^2} + \frac{1}{4\sqrt{3}}\log\frac{\sqrt{3}-v^2}{\sqrt{3}+v^2} + \frac{3}{2}\eta_0^2\log\frac{3-v^4}{3+v^4},\\
c_2(v) &= -\frac{\pi^2}{144\sqrt{3}} + \frac{v^2(9+v^4)}{6(9-v^8)} + C\frac{v^4}{36(3+v^4)} - \eta_0^2\frac{(-9+54v^4+7v^8)v^4}{3(3+v^4)(9-v^8)} + \frac{1}{4\sqrt{3}}\log\frac{\sqrt{3}-v^2}{\sqrt{3}+v^2}\\
&\quad + \frac{1}{36}(C+66\eta_0^2)\log\frac{3-v^4}{3+v^4} + \frac{1}{12\sqrt{3}}\left(\log\frac{\sqrt{3}-v^2}{\sqrt{3}+v^2}\log\frac{(\sqrt{3}-v^2)(\sqrt{3}+v^2)^3}{4(3+v^4)^2} - \mathrm{li}_2\left(-\frac{(\sqrt{3}-v^2)^2}{(\sqrt{3}+v^2)^2}\right)\right).
\end{aligned} \tag{30}$$

Here C is a *new* free integration constant appearing at this order and li_2 is the dilogarithm function. A calculation of the $\mathfrak{R}^2$ coefficient at this order gives

$$\begin{aligned}
R_2 = &-\frac{576(v^4-3)v^8}{(3+v^4)^5}C + \frac{6912(5v^{24}-60v^{20}+2313v^{16}-6912v^{12}+26\,487v^8-18\,468v^4+13\,851)v^8\eta_0^2}{(3-v^4)^4(3+v^4)^6}\\
&-\frac{4608(5v^{16}+6v^{12}+162v^8+54v^4+405)v^{10}}{(3-v^4)^4(3+v^4)^5}.
\end{aligned} \tag{31}$$

We see that there is a fourth order pole singularity. It may be cancelled when the viscosity coefficient η_0 takes the value [9]

$$\eta_0 = \frac{1}{2^{1/2}3^{3/4}} \tag{32}$$

which reproduces, in the boost-invariant expanding setup, the exact viscosity coefficient of $\mathcal{N}=4$ SYM calculated in the static case in [4].

We see however, that at this order the new integration constant C is still undetermined. By analogy with the case of η_0 we may expect that it will be fixed at the next order of the series expansion. Before we proceed to do this let us first discuss the physical interpretation of this coefficient.

The physical interpretation of C

The energy density extracted from the metric expanded in the scaling limit up to $\mathcal{O}(\tau^{-(4/3)})$ through (19) is given by

$$\varepsilon(\tau) = \left(\frac{N_c^2}{2\pi^2}\right)\cdot\frac{1}{\tau^{4/3}}\left\{1 - \frac{2\eta_0}{\tau^{2/3}} + \left(\frac{10}{3}\eta_0^2 + \frac{C}{36}\right)\frac{1}{\tau^{4/3}} + \ldots\right\}. \tag{33}$$

Let us now suppose that this behavior can be described by the equations of second order viscous hydrodynamics (10) and (11) with some parameters η and τ_Π. We will use this hypothesis to extract these parameters [22] which may then be used in hydrodynamic simulations for generic $3+1$ evolving plasma systems.

For ease of computation we note that Eqs. (10) and (11) can be divided out by the factor $N_c^2/(2\pi^2)$, thus we can drop this factor from ε, η, and Φ. Let us now insert Φ from (10) into (11) and obtain a differential equation expressed sorely in terms of $\varepsilon(\tau)$ and the *numerical* coefficients A and r.

We may now extract these numerical coefficients by plugging in the energy density from the AdS/CFT calculation (33) into the resulting equation.

The result is

$$A = \eta_0, \tag{34}$$

$$r = \frac{-C - 66\eta_0^2}{324\eta_0^2}. \tag{35}$$

A is precisely equal to η_0 which is not unexpected since in the leading order $\varepsilon(\tau) = 1/\tau^{4/3}$ and hence $\eta = \eta_0/\tau$ may be written as $\eta = \eta_0\varepsilon^{3/4}$. The value of η_0 is determined from nonsingularity to this order [9] to be given by (32). The new information is r defining the relaxation time τ_Π. Plugging in the value (32) for η_0 we may express r in terms of C

$$r = -\frac{11}{54} - \frac{C}{18\sqrt{3}}. \tag{36}$$

Hence the relaxation time τ_Π will be found once we know C (we note that thus we must have $C < 0$). Again, as was the case with η_0 one has to go one order higher in order to determine the value of the coefficient. We will proceed to do it in the following section.

VI. DETERMINATION OF C AND THE RELAXATION TIME τ_Π

It is straightforward to obtain equations for the third order metric coefficients $a_3(v)$, $b_3(v)$, and $c_3(v)$ using a computer algebra system, albeit these equations appear to be prohibitively complex at first sight.

However one can notice that the zz component of the Einstein equations $E_{zz} = 0$ (15) gives an equation just for the combination

$$d_3'(z) \equiv a_3'(v) + \tilde{b}_3'(v) \equiv a_3'(v) + b_3'(v) + 2c_3'(v), \tag{37}$$

where the prime stands for the ordinary derivative. Then expressing $b_3'(v)$ in terms of the other functions and plugging the result into the $\tau\tau$ component $E_{\tau\tau} = 0$ of the Einstein equation yields an equation purely for the derivative $a_3'(v)$. So at this stage we have analytic expressions [23] for $a_3'(v)$ and (say) $\tilde{b}_3'(v)$.

Fortunately, the expression for the $\mathcal{O}(\tau^{-2})$ coefficient R_3 of $\mathfrak{R}^2$ can be expressed sorely in terms of $a_3'(v)$, $\tilde{b}_3'(v)$ and their derivatives. The expression is quite lengthy so we do not give it here [23]. Substituting $a_3'(v)$ and $\tilde{b}_3'(v)$ into R_3, we find again that there is a fourth order pole in R_3 at $v = 3^{1/4}$ which is canceled exactly when

$$C = \frac{6\log 2 - 17}{\sqrt{3}}. \tag{38}$$

For this value the poles of lower orders are also canceled. However a new feature arises here, namely, there is a leftover *logarithmic* singularity:

$$R_{3\|C=(6\log 2-17)/\sqrt{3}} = \text{finite} + 8\cdot 2^{1/2}3^{3/4}\log(3 - v^4). \tag{39}$$

We will show in the next section that if we turn on the dilaton this singularity may be canceled without modifying the value of C determined above. Let us now determine the coefficient of the relaxation time r from (36). We find that

$$r = \tfrac{1}{9}(1 - \log 2) \sim 0.034 \sim \tfrac{1}{30} \tag{40}$$

which shows that the relaxation time of $\mathcal{N} = 4$ SYM at strong coupling is much smaller than one would expect. In particular the second order viscous hydrodynamics seem to be much closer to the first order behavior. If we express the relaxation time in terms of the proper time and then relate it to the temperature through the leading order expression (27), we obtain

$$\tau_\Pi = \frac{1 - \log 2}{6\pi T}. \tag{41}$$

VII. THE DILATON AND trF^2

In this section we will show how to cancel the remaining logarithmic divergence in R_3 by turning on the dilaton field. On the gauge theory side, this means that we generate a nonzero expectation value for trF^2 i.e. electric and magnetic modes are no longer equilibrated. Physically this may well happen since we are considering a regime where dissipative effects are important.

If we include the dilaton ϕ, the equations of motion become those of a coupled Einstein-dilaton system which read (in the Einstein frame):

$$R_{\mu\nu} + 4g_{\mu\nu} = \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi, \tag{42}$$

$$\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) = 0. \tag{43}$$

In analogy to the metric coefficients (21) we would like to make a large τ expansion of the dilaton. This requires a judicious choice of the powers of τ appearing in that expansion. We fix those powers by the following argument.

Suppose that $\phi(z, \tau) \sim \tilde{\phi}(v)\tau^{-r}$. Then one can check using (42) that the dilaton source term will contribute to the metric coefficients at order τ^{-2r}. The above considerations suggest the following expansion of the dilaton field:

$$\phi(z, \tau) = \sum_{i=1} \phi_i(v) \frac{1}{\tau^{i/3}}. \tag{44}$$

The $i = 0$ component is absent as has already been checked in [12].

If a spacetime has a nontrivial dilaton profile, in string theory one has two distinguished metrics—the Einstein frame metric considered above, and the string frame metric defined as

$$g_{\mu\nu}^{\text{string}} = e^{(1/2)\phi} g_{\mu\nu} \tag{45}$$

which is the natural metric from the point of view of *strings* propagating in the spacetime. Therefore a natural question arises to which of the two metrics we should apply the nonsingularity criterion. A natural guess would be that $\Re^2(g_{\mu\nu}^{\text{string}})$ should be nonsingular; we will however for the moment keep both possibilities open.

Let us now go back to the expansion (44). The only terms that would contribute to the metric coefficients [24] $a_1(v)$, $a_2(v)$, and $a_3(v)$ are $\phi_1(v)$, $\phi_2(v)$, and $\phi_3(v)$. So only these terms would influence $\Re^2$ in the Einstein frame to the order considered in the previous sections. If on the other hand we would require the nonsingularity of $\Re^2$ in the *string* frame, then we must consider in addition $\phi_4(v)$, $\phi_5(v)$, and $\phi_6(v)$.

Finally let us give one gauge theoretical interpretation of a nontrivial dilaton profile. It might give rise to an expectation value for trF^2 through [25,26]

$$\frac{1}{4g_{YM}^2} \langle \text{tr} F^2 \rangle = \frac{N^2}{2\pi^2} \cdot \lim_{z \to 0} \frac{\phi(z, \tau)}{z^4}. \tag{46}$$

In the following we will consider various possible leading behaviors of the dilaton according to the expansion (44) and consider $\Re^2$ in string frame (and in the Einstein frame where relevant) *ϕ at $\mathcal{O}(\tau^{-1/3})$*. At this order the leading behavior of ϕ can be easily obtained as

$$\phi_1(v) = k \log \frac{3 - v^4}{3 + v^4}. \tag{47}$$

In the string frame $\Re^2_{\text{string}}$ will have a new contribution at order $\tau^{-1/3}$. This contribution has a piece which is proportional to $k \log(3 - v^4)$. Therefore we have to set $k = 0$ and hence $\phi_1(v) = 0$. Let us note that the natural appearance of a logarithmic divergence gives a hope that indeed the dilaton might cancel the leftover divergence in (39). We will see below that this indeed happens.

In the Einstein frame the calculations are more complicated. ϕ_1 will modify the metric coefficients $a_1(v)$, $b_1(v)$, and $c_1(v)$. Then the Einstein frame $\Re^2$ will also get modified and (29) will be replaced by

$$R_1 = \frac{41\,472(v^4 - 3)v^8}{(3 + v^4)^5} \cdot \eta_0 - \frac{1152(13v^{12} - 99v^8 + 27v^4 + 27)v^8}{(3 + v^4)^5(3 - v^4)^2} \cdot k^2. \tag{48}$$

Requiring nonsingularity again gives $k = 0$ and hence $\phi_1(v) = 0$.

ϕ at $\mathcal{O}(\tau^{-2/3})$. The leading part $\phi_2(v)$ has a functional form identical to (47). Again the string frame $\Re^2_{\text{string}}$ will have a logarithmic singularity but now at order $\tau^{-2/3}$, hence $\phi_2(v)$ has to vanish at this order if we assume string frame nonsingularity.

In the Einstein frame, the coefficients $a_2(v)$, $b_2(v)$, and $c_2(v)$ will get modified. $\Re^2$ will still have a fourth order pole at $\mathcal{O}(\tau^{-4/3})$ which will be canceled by taking (32) but an additional second order pole proportional to k will persist. Hence also requiring nonsingularity in the Einstein frame will lead to $\phi_2(v) = 0$.

ϕ at $\mathcal{O}(\tau^{-1})$. Here again in the string frame one has a logarithmic singularity hence $\phi_3(v) = 0$. The calculations in Einstein frame are very tedious but also rule out a nonvanishing $\phi_3(v)$.

ϕ at $\mathcal{O}(\tau^{-4/3})$. At this order, the metric coefficients are not modified, hence it makes sense to consider only the string frame $\Re^2_{\text{string}}$. A logarithmic singularity which appears at order $\tau^{-4/3}$ requires $\phi_4(v) = 0$.

ϕ at $\mathcal{O}(\tau^{-5/3})$. Again a logarithmic singularity requires $\phi_5(v) = 0$.

ϕ at $\mathcal{O}(\tau^{-2})$. This is the relevant order at which a cancellation of the logarithmic singularity may occur. At this order ϕ_6 has again the form

$$\phi_6(v) = k \log \frac{3 - v^4}{3 + v^4}. \tag{49}$$

The string frame $\Re^2_{\text{string}}$ at $\mathcal{O}(\tau^{-2})$ will get modified by

$$R_3^{\text{string}} = R_3 - k \cdot \frac{8(5v^{16} + 60v^{12} + 1566v^8 + 540v^4 + 405)}{(3 + v^4)^4} \times \log \frac{3 - v^4}{3 + v^4}, \tag{50}$$

where R_3 is the Einstein frame coefficient calculated earlier. Therefore we see that by a suitable choice of k we may exactly cancel the leftover logarithmic singularity without changing the value of C obtained above which came from canceling a fourth order pole. Indeed performing the expansion of R_3^{string} at $v = 3^{1/4}$ we get

$$R_3^{\text{string}} = \text{finite} + (8 \cdot 2^{1/2} 3^{3/4} - 112k) \log(3 - v^4) \tag{51}$$

which gives

$$\phi_6(v) = \frac{3^{3/4}}{7\sqrt{2}} \log\frac{3 - v^4}{3 + v^4}. \tag{52}$$

Now we may evaluate the resulting expectation value of $\langle \mathrm{tr} F^2 \rangle$ from (46):

$$\frac{1}{4g_{YM}^2}\langle \mathrm{tr} F^2 \rangle = \frac{N^2}{2\pi^2} \cdot -\frac{\sqrt{2}}{7 \cdot 3^{1/4}} \cdot \frac{1}{\tau^{10/3}}. \tag{53}$$

We see that this expectation value is *negative* which signifies that magnetic modes become dominant.

VIII. DISCUSSION

In this paper we have considered the subasymptotic proper-time evolution of an infinite expanding boost-invariant plasma in $\mathcal{N} = 4$ SYM at strong coupling. We have used the AdS/CFT correspondence to construct dual geometries and used the criterion that the unique nonsingular geometry corresponds to the physical evolution of the plasma. Then the proper-time dependence of the energy density $\varepsilon(\tau)$ can be read off from the form of the 5-dimensional metric.

We have found subleading terms of $\varepsilon(\tau)$ which are sensitive to the relaxation time τ_Π appearing, in addition to the shear viscosity η, as a new element in second order viscous hydrodynamics. This served to determine the relaxation time (40) and (41) which turned out to be about 30 times shorter than the one expected from Boltzmann kinetic theory estimates (12). Thus second order hydrodynamics appears to be much closer to the ordinary first order formalism.

In addition we have found that canceling a remaining logarithmic divergence requires turning on the dilaton field [27]. A nonvanishing dilaton field in turn implies a nonzero expectation value (53) for $\mathrm{tr} F^2$ with the proper-time scaling $\tau^{-10/3}$. This means that electric and magnetic modes are not exactly equilibrated. A similar difference of electric and magnetic modes appears to be generic for plasma instabilities (albeit at weak coupling) [28].

It would be very interesting to understand from a microscopic perspective the very short relaxation time. In particular it would be interesting to understand the weak coupling corrections to the Boltzmann value $\tau_\Pi^{\mathrm{Boltzmann}}$. Another intriguing question would be to estimate by other means the τ dependence of $\langle \mathrm{tr} F^2 \rangle$. On the strong coupling side it would be interesting to understand the dual metric from the point of view of dynamical horizons in general relativity [29] and analyze the relevant thermodynamics.

ACKNOWLEDGMENTS

R. J. would like to thank Dongsu Bak, Robi Peschanski, and participants of the GGI workshop "High density QCD" for discussions and comments. R. J. would like to thank the Galileo Galilei Institute for Theoretical Physics for the hospitality and the INFN for partial support during the completion of this work. This work has been supported in part by the Polish Ministry of Science and Information Society Technologies grants No. 1P03B02427 (2004-2007), No. 1P03B04029 (2005-2008), RTN network ENRAGE MRTN-CT-2004-005616, and the Marie Curie Actions Transfer of Knowledge project COCOS (Contract No. MTKD-CT-2004-517186).

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[22] Since viscosity is known, the new information is the relaxation time τ_Π.
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