# Gauge invariance of the action principle for gauge systems with noncanonical symplectic structures

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The effect of the gauge transformation in the action principle for Hamiltonian gauge systems formulated in terms of noncanonical symplectic structures is studied and, particularly, the compatibility between gauge conditions and boundary conditions is analyzed. It is shown that the complete set of commuting observables at the time boundary is now fixed by the boundary term and the symplectic structure. The theory is applied to two nontrivial models having $SL(2,\mathbb{R})$ and $SU(2)$ gauge symmetries, respectively, whose extended phase spaces are endowed with new interactions produced by noncanonical symplectic structures.

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## I. INTRODUCTION

Hamiltonian systems with constraints, in the sense introduced by Dirac, are described by an action principle of the form [1,2] (summation convention over repeated indices is used throughout)

$$S[q^i, p_i, \lambda^a, \lambda^\alpha] = \int_{\tau_1}^{\tau_2} (p_i \dot{q}^i - H - \lambda^a \gamma_a - \lambda^\alpha \chi_\alpha) d\tau, \tag{1}$$

$i = 1, \ldots, N$, where $H$ is taken to be a first-class Hamiltonian and the $\gamma$'s are first-class constraints, while the $\chi$'s are second class, i.e., [3]

$$\{\gamma_a, \gamma_b\} = C_{ab}{}^c \gamma_c + T_{ab}{}^{\alpha\beta} \chi_\alpha \chi_\beta, \tag{2a}$$

$$\{\gamma_a, \chi_\alpha\} = C_{a\alpha}{}^b \gamma_b + C_{a\alpha}{}^\beta \chi_\beta, \tag{2b}$$

$$\{H, \gamma_a\} = V_a{}^b \gamma_b + V_a{}^{\alpha\beta} \chi_\alpha \chi_\beta, \tag{2c}$$

$$\{H, \chi_\alpha\} = V_\alpha{}^b \gamma_b + V_\alpha{}^\beta \chi_\beta. \tag{2d}$$

Also,

$$\{\chi_\alpha, \chi_\beta\} = C_{\alpha\beta}, \qquad \det(C_{\alpha\beta}) \neq 0. \tag{3}$$

The Poisson brackets in Eqs. (2) and (3),

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}, \tag{4}$$

are computed with respect to the canonical symplectic structure

$$\Omega = dp_i \wedge dq^i. \tag{5}$$

The Hamiltonian action (1) is obtained from a Lagrangian action through the systematic implementation of Dirac's method, i.e., the starting point is a Lagrangian action from which the momenta $p_i$ canonically conjugate to the configuration variables $q^i$ are defined. From the definition of the momenta, primary constraints usually arise which are evolved until all the constraints are obtained, which are then classified into first class and second class. By construction, the canonical symplectic structure (5) plays a central role in Dirac's method.

Nevertheless, the action (1) is *not* the most general action allowed to describe a constrained Hamiltonian system. In fact, constrained dynamical systems with a finite number of degrees of freedom can be written in a Hamiltonian form by means of the dynamical equations

$$\dot{x}^\mu = \omega^{\mu\nu}(x)\left(\frac{\partial H}{\partial x^\nu} + \lambda^a \frac{\partial \gamma_a}{\partial x^\nu} + \lambda^\alpha \frac{\partial \chi_\alpha}{\partial x^\nu}\right) = \omega^{\mu\nu}(x)\frac{\partial H_E}{\partial x^\nu}, \qquad \mu, \nu = 1, 2, \ldots, 2N, \tag{6}$$

where $H_E = H + \lambda^a \gamma_a + \lambda^\alpha \chi_\alpha$ is the extended Hamiltonian and

$$\gamma_a(x) \approx 0, \qquad \chi_\alpha(x) \approx 0, \tag{7}$$

are the constraints, which define the constraint surface $\Sigma$ embedded in the extended phase space $\Gamma$. $\Gamma$ is a symplectic manifold endowed with the symplectic structure

$$\omega = \frac{1}{2}\omega_{\mu\nu}(x) dx^\mu \wedge dx^\nu, \tag{8}$$

where $(x^\mu)$ are the coordinates that locally label the points of $\Gamma$, which is considered as a single entity, i.e., $\Gamma$ need not be necessarily interpreted as the cotangent bundle of a configuration space $\mathcal{C}$. Now, instead of Eqs. (2) and (3), the following equations hold:

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$$\{\gamma_a, \gamma_b\}_\omega = C_{ab}{}^c\gamma_c + T_{ab}{}^{\alpha\beta}\chi_\alpha\chi_\beta, \tag{9a}$$

$$\{\gamma_a, \chi_\alpha\}_\omega = C_{a\alpha}{}^b\gamma_b + C_{a\alpha}{}^\beta\chi_\beta, \tag{9b}$$

$$\{H, \gamma_a\}_\omega = V_a{}^b\gamma_b + V_a{}^{\alpha\beta}\chi_\alpha\chi_\beta, \tag{9c}$$

$$\{H, \chi_\alpha\}_\omega = V_\alpha{}^b\gamma_b + V_\alpha{}^\beta\chi_\beta. \tag{9d}$$

Also,

$$\{\chi_\alpha, \chi_\beta\}_\omega = C_{\alpha\beta}, \qquad \det(C_{\alpha\beta}) \neq 0, \tag{10}$$

where the Poisson brackets involved in Eqs. (9) and (10) are computed using the symplectic structure $\omega$ on $\Gamma$:

$$\{f, g\}_\omega = \frac{\partial f}{\partial x^\mu}\omega^{\mu\nu}(x)\frac{\partial g}{\partial x^\nu}. \tag{11}$$

The dynamical equations of motion (6) and the constraints (7) can be obtained from the action principle

$$S[x^\mu, \lambda^a, \lambda^\alpha] = \int_{\tau_1}^{\tau_2}(\theta_\mu(x)\dot{x}^\mu - H_E)d\tau, \tag{12}$$

with $\omega = d\theta$ where $\theta = \theta_\mu(x)dx^\mu$ is the symplectic potential 1-form.

Once the differences between these two approaches to Hamiltonian systems have been recalled, the problem studied in this paper is set down. In Refs. [4–6] the issue of the change of the action of Eq. (1) due to the change of the canonical variables induced by the gauge transformation generated by the first-class constraints was analyzed. There, the analysis was restricted to infinitesimal gauge transformations. Later on, the analysis was extended to include the full change of the action induced by *finite* gauge transformations [7], which are relevant both classically and quantum mechanically because the latter include also the "large" gauge transformations that are not included in the infinitesimal procedure developed in Refs. [4–6]. Moreover, the infinitesimal case was also developed in Ref. [7], where some new aspects of this case were reported, among others the differential equation that the boundary term must satisfy in order to have gauge-invariant actions. In all these papers, the analysis was carried out by using the action (1), i.e., the extended phase space $\Gamma$ is endowed with canonical symplectic structures from the very beginning. Moreover, the analysis has been applied to general relativity formulated in terms of Ashtekar variables and Polyakov's action defined on manifolds $\mathcal{M}$ having the topology $\mathcal{M} = \Sigma \times \mathbb{R}$ with $\Sigma$ compact and without a space boundary [8,9].

In this paper, the issue of the gauge invariance of the action principle is analyzed for an action of the form (12). This topic is relevant for the path integral quantization of gauge systems [3–6,10], when new interactions are introduced through the noncanonical symplectic structure.

## II. THEORETICAL FRAMEWORK

The gauge invariance of the action can be analyzed from two viewpoints depending on how the action (12) is interpreted: as a Hamiltonian action or as a Lagrangian action. These two possible interpretations of the action lead to two alternative approaches. In this paper the action will be interpreted as a Hamiltonian one, and in the Concluding Remarks section some comments regarding the Lagrangian viewpoint will be made.

Therefore, it is assumed that the action (12) has the Hamiltonian form from the very beginning. For the sake of completeness, a function $B(x, \tau)$ will be added at the time boundary

$$S_B := S[x^\mu, \lambda^a] - B(x, \tau)|_{\tau_1}^{\tau_2}, \tag{13}$$

just to choose the variables that are going to be fixed at $\tau_1$ and $\tau_2$. The change of the action (13) under the infinitesimal gauge transformation of the $x$'s,

$$\delta_\varepsilon x^\mu = \{x^\mu, \varepsilon^a\gamma_a\}_\omega = \omega^{\mu\nu}(x)\frac{\partial G}{\partial x^\nu}, \tag{14}$$

where $G := \varepsilon^a\gamma_a$, and $\varepsilon^a$ are the infinitesimal gauge parameters, and the infinitesimal gauge transformations of the Lagrange multipliers $\delta_\varepsilon\lambda^a$ and $\delta_\varepsilon\lambda^\alpha$ is

$$\delta_\varepsilon S_B = \int_{\tau_1}^{\tau_2}[(\delta_\varepsilon\theta_\mu)\dot{x}^\mu + \theta_\mu\delta_\varepsilon\dot{x}^\mu - \delta_\varepsilon H - \gamma_a\delta_\varepsilon\lambda^a - \lambda^a\delta_\varepsilon\gamma_a - \chi_\alpha\delta_\varepsilon\lambda^\alpha - \lambda^\alpha\delta_\varepsilon\chi_\alpha]d\tau - \delta_\varepsilon B|_{\tau_1}^{\tau_2}. \tag{15}$$

Integrating by parts the term $\theta_\mu\delta_\varepsilon\dot{x}^\mu$, by plugging (6) and

$$\delta_\varepsilon\theta_\mu = \{\theta_\mu, G\}_\omega = \frac{\partial\theta_\mu}{\partial x^\alpha}\omega^{\alpha\beta}\frac{\partial G}{\partial x^\beta}, \qquad \dot{\theta}_\mu = \{\theta_\mu, H_E\}_\omega = \frac{\partial\theta_\mu}{\partial x^\alpha}\omega^{\alpha\beta}\frac{\partial H_E}{\partial x^\beta}, \tag{16}$$

into the right-hand side of $\delta_\varepsilon S_B$ and using $\omega_{\mu\nu} = \partial_\mu\theta_\nu - \partial_\nu\theta_\mu$, leads to

$$\delta_\varepsilon S_B = \int_{\tau_1}^{\tau_2}[\{H_E, G\}_\omega - \delta_\varepsilon H - \gamma_a\delta_\varepsilon\lambda^a - \lambda^a\delta_\varepsilon\gamma_a - \chi_\alpha\delta_\varepsilon\lambda^\alpha - \lambda^\alpha\delta_\varepsilon\chi_\alpha]d\tau + (\theta_\mu\delta_\varepsilon x^\mu - \delta_\varepsilon B)|_{\tau_1}^{\tau_2}. \tag{17}$$

Using (9) to write explicitly the right-hand side of $\delta_\varepsilon H = \{H, G\}_\omega = \varepsilon^a\{H, \gamma_a\}_\omega + \gamma_a\{H, \varepsilon^a\}_\omega$ and inserting the result (and doing the same for $\delta_\varepsilon\gamma_a$ and $\delta_\varepsilon\chi_\alpha$), leads to

$$\begin{aligned}\delta_\varepsilon S_B = \int_{\tau_1}^{\tau_2}&[\varepsilon^a\{H_E, \gamma_a\}_\omega - \varepsilon^a(V_a{}^b\gamma_b + V_a{}^{\alpha\beta}\chi_{\alpha\beta}) \\ &- \lambda^a\varepsilon^b(C_{ab}{}^c\gamma_c + T_{ab}{}^{\alpha\beta}\chi_\alpha\chi_\beta) + \lambda^\alpha\varepsilon^a(C_{a\alpha}{}^b\gamma_b \\ &+ C_{a\alpha}{}^\beta\chi_\beta) - \gamma_a\delta_\varepsilon\lambda^a - \chi_\alpha\delta_\varepsilon\lambda^\alpha]d\tau \\ &+ (\theta_\mu\delta_\varepsilon x^\mu - \delta_\varepsilon B)|_{\tau_1}^{\tau_2}.\end{aligned} \tag{18}$$

On the other hand, if the Lagrange multipliers $\lambda^a$ are simultaneously transformed as [3]

$$\delta_\varepsilon \lambda^a = \frac{d\varepsilon^a}{d\tau} + \lambda^c \varepsilon^b C_{bc}{}^a + \lambda^\alpha \varepsilon^b C_{b\alpha}{}^a - \varepsilon^b V_b{}^a, \qquad \delta_\varepsilon \lambda^\alpha = \lambda^c \varepsilon^b T_{bc}{}^{\alpha\beta}\chi_\beta - \varepsilon^b V_b{}^{\alpha\beta}\chi_\beta + \lambda^\beta \varepsilon^b C_{b\beta}{}^\alpha, \tag{19}$$

then

$$\delta_\varepsilon S_B = \int_{\tau_1}^{\tau_2}\left(\varepsilon^a\{H_E, \gamma_a\}_\omega - \gamma_a \frac{d\varepsilon^a}{d\tau}\right)d\tau + (\theta_\mu \delta_\varepsilon x^\mu - \delta_\varepsilon B)|_{\tau_1}^{\tau_2}. \tag{20}$$

Integrating by parts the second integrand leads to

$$\delta_\varepsilon S_B = \int_{\tau_1}^{\tau_2}(\varepsilon^a\{H_E, \gamma_a\}_\omega + \varepsilon^a \dot{\gamma}_a)d\tau + (\theta_\mu \delta_\varepsilon x^\mu - G - \delta_\varepsilon B)|_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2} \varepsilon^a \frac{\partial \gamma_a}{\partial \tau} d\tau + (\theta_\mu \delta_\varepsilon x^\mu - G - \{B, G\}_\omega)|_{\tau_1}^{\tau_2}. \tag{21}$$

Assuming that the $\gamma$'s do not depend explicitly on $\tau$, $\frac{\partial \gamma_a}{\partial \tau} = 0$.[1] So, by using (14), finally

$$\delta_\varepsilon S_B = \left[\theta_\mu(x)\omega^{\mu\nu}(x)\frac{\partial G}{\partial x^\nu} - G - \{B, G\}_\omega\right]\Bigg|_{\tau_1}^{\tau_2} = \left[\left(\theta_\mu(x) - \frac{\partial B(x,\tau)}{\partial x^\mu}\right)\omega^{\mu\nu}(x)\frac{\partial G}{\partial x^\nu} - G\right]\Bigg|_{\tau_1}^{\tau_2} \tag{22}$$

is the change of the action (13) at first order in the gauge parameters $\varepsilon^a$ induced by the gauge transformation of the dynamical variables generated by the first-class constraints $\gamma_a$. Equation (22) clearly expresses the fact that there are five objects which contribute to the boundary term (22): the symplectic potential $\theta = \theta_\mu(x)dx^\mu$, the inverse of the symplectic structure $\omega^{\mu\nu}$, the gauge parameters $\varepsilon^a$, the first-class constraints $\gamma_a$, and the boundary term $-B|_{\tau_1}^{\tau_2}$.

Some remarks follow:

(1) if $B$ in (13) does not depend explicitly on $\tau$, $B(x)$, then its contribution to the action (13) and therefore to the boundary term (22) can be absorbed by choosing the new symplectic potential $\vartheta_\mu(x) = \theta_\mu - \partial B/\partial x^\mu$ [the symplectic potential, by hypothesis, does not depend explicitly on $\tau$; see Eq. (12)].

(2) in the previous approaches to the subject [5–7], the discussion about the objects that contribute to the boundary term was focused on the dependency of the first-class constraints $\gamma_a$, the gauge parameters, and the boundary term $-B|_{\tau_1}^{\tau_2}$, simply because the symplectic structure and the potential were fixed and tied to canonical coordinates from the very beginning. Thus the roles of the symplectic potential and the symplectic structure were not fully appreciated.

(3) Equation (22) was obtained allowing the possibility that the gauge parameters $\varepsilon$ could depend on the phase space variables $x$: $\delta_\varepsilon F(x) = \{F, G\}_\omega = \varepsilon^a\{F, \gamma_a\}_\omega + \gamma_a\{F, \varepsilon^a\}_\omega$. Usually, however, the gauge parameters are allowed to depend on $\tau$ only and so $\delta_\varepsilon F = \{F, G\}_\omega = \varepsilon^a\{F, \gamma_a\}$. In this last case, Eq. (22) is unaltered and reduces to

$$\delta_\varepsilon S_B = \left(\varepsilon^a(\tau)\left[\left(\theta_\mu(x) - \frac{\partial B}{\partial x^\mu}\right) \times \omega^{\mu\nu}(x)\frac{\partial \gamma_a}{\partial x^\nu} - \gamma_a\right]\right)\Bigg|_{\tau_1}^{\tau_2}. \tag{23}$$

Under this assumption, the boundary term vanishes because either (a) the gauge parameters vanish at the time boundaries, $\varepsilon^a(\tau_1) = 0 = \varepsilon^a(\tau_2)$, (b) the terms inside the curly brackets vanish at the time boundaries, or (c) a combination of both (a) and (b).

(4) regarding the particular case given in Eq. (23) where the gauge parameters depend on $\tau$ only, if the gauge transformation is allowed at the time boundaries, $\varepsilon^a(\tau_2) \neq 0 \neq \varepsilon^a(\tau_1)$, then the boundary term can vanish even for first-class constraints $\gamma_a$ quadratic in the variables $x^\mu$ provided that an appropriate choice for the geometrical objects ($\theta_\mu$, $B$, $\omega^{\mu\nu}$, and $\gamma_a$) is made, i.e., gauge-invariant actions $S_{\rm inv}$ can be constructed by properly handling these geometrical objects. (See the examples in the next section.)

(5) if the original choice of the gauge potential $\theta_\mu$ and the function $B$ is such that the action is not gauge invariant, it is still possible to add another function at the time boundary in such a way that the new action is invariant (assuming that $\omega^{\mu\nu}$ and $\gamma_a$ have been fixed).

(6) note that the complete set of commuting observables fixed at the time boundaries $\tau_1$ and $\tau_2$ depend not just on the boundary term $B$ but also on the symplectic structure, a property not fully appreciated when a canonical symplectic structure is used.

Coming back to the general discussion, once the gauge invariance of the action has been analyzed, it just remains to say some words about the compatibility between the gauge conditions and the boundary conditions. Note that good gauge conditions must take into account the symplectic structure in the sense that the matrix of their Poisson brackets with the first-class constraints must have a nonvanishing determinant. Like in the case when symplectic structures and symplectic potentials having the canonical form are employed [4–7], the boundary conditions of the action (13) might not be compatible with the choice of a particular gauge condition without it mattering if the action is invariant or if it is not. If this were the case, the procedure for how to achieve such compatibility is

[1] Alternatively, the $\tau$-dependence of the $\gamma$'s can be handled by parametrizing the theory and considering $(\tau, p_\tau)$ as new variables thus enlarging the phase space as it was done in the canonical case [11].

essentially the same as that discussed in Refs. [4–7]. For the benefit of the readers, such a procedure is applied to the examples contained in the next section.

Finally, for the sake of completeness, Noether's theorem is discussed. If the action (13) transforms (without using the equations of motion) at first order as

$$\delta S_B = \int_{\tau_1}^{\tau_2} \frac{d\mathcal{F}}{d\tau} d\tau \tag{24}$$

under the infinitesimal transformations

$$x'^{\mu}(\tau') = x^{\mu}(\tau) + \delta x^{\mu}, \qquad \lambda'^{a}(\tau') = \lambda^{a}(\tau) + \delta\lambda^{a}, \\ \lambda'^{\alpha}(\tau') = \lambda^{\alpha}(\tau) + \delta\lambda^{\alpha}, \qquad \tau' = \tau + \delta\tau, \tag{25}$$

then the corresponding Noether's condition $\frac{\delta \mathcal{L}'}{\delta x^{\mu}}\tilde{\delta}x^{\mu} + \frac{\delta \mathcal{L}'}{\delta \lambda^{a}}\tilde{\delta}\lambda^{a} + \frac{\delta \mathcal{L}'}{\delta \lambda^{\alpha}}\tilde{\delta}\lambda^{\alpha} + \frac{d}{d\tau}(\frac{\partial \mathcal{L}'}{\partial \dot{x}^{\mu}}\tilde{\delta}x^{\mu} + \mathcal{L}'\delta\tau - \mathcal{F}) = 0$ with $\delta x^{\mu} = \tilde{\delta}x^{\mu} + \frac{dx^{\mu}}{d\tau}\delta\tau$ (and so on for the other variables) and $\mathcal{L}' = (\theta_{\mu} - \frac{\partial B}{\partial x^{\mu}})\dot{x}^{\mu} - \frac{\partial B}{\partial \tau} - H - \lambda^{a}\gamma_{a} - \lambda^{\alpha}\chi_{\alpha}$ becomes

$$\left(\omega_{\mu\nu}\dot{x}^{\nu} - \frac{\partial H_E}{\partial x^{\mu}}\right)\tilde{\delta}x^{\mu} - \gamma_a\tilde{\delta}\lambda^{a} - \chi_{\alpha}\tilde{\delta}\lambda^{\alpha} \\ + \frac{d}{d\tau}\left(\left(\theta_{\mu} - \frac{\partial B}{\partial x^{\mu}}\right)\tilde{\delta}x^{\mu} + \mathcal{L}'\delta\tau - \mathcal{F}\right) = 0, \tag{26}$$

and so, if the equations of motion (6) and the first- and second-class constraints are satisfied,

$$\mathcal{O} = \left(\theta_{\mu} - \frac{\partial B}{\partial x^{\mu}}\right)\tilde{\delta}x^{\mu} - \mathcal{F} \\ + \left[\left(\theta_{\mu} - \frac{\partial B}{\partial x^{\mu}}\right)\omega^{\mu\nu}\frac{\partial H_E}{\partial x^{\nu}} - H - \frac{\partial B}{\partial \tau}\right]\delta\tau \tag{27}$$

is constant along evolution in $\tau$. It is understood that all terms in the right-hand side of the last equation are evaluated on the surface of first- and second-class constraints.

## III. EXAMPLES THAT INVOLVE NONCANONICAL SYMPLECTIC STRUCTURES

In this section, the ideas developed in Sec. II are applied to two nontrivial models. In both cases, the extended phase space is $\Gamma = \mathbb{R}^8$, and its points are labeled by $(x^{\mu}) = (x^1, x^2, \ldots, x^8) = (u^1, u^2, v^1, v^2, p_1, p_2, \pi_1, \pi_2)$. The symplectic geometry on $\Gamma = \mathbb{R}^8$ is given by the nondegenerate closed 2-form $\omega = \frac{1}{2}\omega_{\mu\nu}dx^{\mu} \wedge dx^{\nu} = dp_1 \wedge du^1 + dp_2 \wedge du^2 + d\pi_1 \wedge dv^1 + d\pi_2 \wedge dv^2 + \theta dp_1 \wedge dp_2 + \phi d\pi_1 \wedge d\pi_2$. Equivalently, the inverse of the noncanonical symplectic structure is

$$(\omega^{\mu\nu}) = \{x^{\mu}, x^{\nu}\} \\ = \begin{pmatrix} 0 & \theta & 0 & 0 & 1 & 0 & 0 & 0 \\ -\theta & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \phi & 0 & 0 & 1 & 0 \\ 0 & 0 & -\phi & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{28}$$

where $\theta$ and $\phi$ are fixed constant real parameters. The difference between the two models comes from the concrete form for the constraints that the phase space variables must satisfy in each case.

One of the lessons learned from these models is that gauge-invariant actions can be built in spite of the fact that the constraints are quadratic in the phase space variables provided that an appropriate choice for the symplectic potential is made.

### A. $SL(2, \mathbb{R})$ model

This model was introduced in Ref. [12] in the context of noncommutative quantum theory. Here, however, the model is interpreted as a usual gauge system of the type mentioned in Secs. I and II.

The phase space variables must satisfy the constraints

$$\mathcal{C}_1 := \frac{1}{2}[(p_1)^2 + (p_2)^2 - (v^1)^2 - (v^2)^2] - \phi v^1 \pi_2 \\ - \frac{1}{2}\phi^2(\pi_2)^2 \approx 0, \\ \mathcal{C}_2 := \frac{1}{2}[(\pi_1)^2 + (\pi_2)^2 - (u^1)^2 - (u^2)^2] - \theta u^1 p_2 \\ - \frac{1}{2}\theta^2(p_2)^2 \approx 0, \\ \mathcal{V} := u^i p_i - v^i \pi_i + \theta p_1 p_2 - \phi \pi_1 \pi_2 \approx 0, \tag{29}$$

with $i = 1, 2$. It turns out that the constraints of Eq. (29) are first class with respect to the Poisson brackets computed with the symplectic structure of Eq. (28). The resulting algebra of constraints is

$$\{\mathcal{C}_1, \mathcal{C}_2\}_{\omega} = \mathcal{V}, \qquad \{\mathcal{C}_1, \mathcal{V}\}_{\omega} = -2\mathcal{C}_1, \\ \{\mathcal{C}_2, \mathcal{V}\}_{\omega} = 2\mathcal{C}_2, \tag{30}$$

which is isomorphic to the $\mathfrak{sl}(2, r)$ Lie algebra.

By plugging (28) and (29), $(\lambda^1, \lambda^2, \lambda^3) = (N, M, \lambda)$, and $(\gamma_1, \gamma_2, \gamma_3) = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{V})$ into Eqs. (6), the dynamical Eqs. (6) acquire the form

$$\begin{aligned}
\dot{u}^1 &= Np_1 - M\theta u^2 + \lambda(u^1 + 2\theta p_2),\\
\dot{u}^2 &= Np_2 + \lambda u^2,\\
\dot{v}^1 &= M\pi_1 - \lambda(v^1 + 2\phi\pi_2) - \phi N v^2,\\
\dot{v}^2 &= M\pi_2 - \lambda v^2,\\
\dot{p}_1 &= M(u^1 + \theta p_2) - \lambda p_1,\\
\dot{p}_2 &= Mu^2 - \lambda p_2,\\
\dot{\pi}_1 &= N(v^1 + \phi\pi_2) + \lambda\pi_1,\\
\dot{\pi}_2 &= Nv^2 + \lambda\pi_2.
\end{aligned} \tag{31}$$

Using (31), the evolution of the constraints (29) yields

$$\dot{\mathcal{C}}_1 = M\mathcal{V} - 2\lambda\mathcal{C}_1, \qquad \dot{\mathcal{C}}_2 = -N\mathcal{V} + 2\lambda\mathcal{C}_2, \qquad \dot{\mathcal{V}} = -2M\mathcal{C}_2 + 2N\mathcal{C}_1, \tag{32}$$

in agreement with the $\mathfrak{sl}(2, r)$ Lie algebra of Eq. (30).

*Gauge transformation*. The finite gauge transformation of the dynamical variables is

$$\begin{pmatrix} u'^1 \\ u'^2 \\ p'_1 \\ p'_2 \end{pmatrix} = \begin{pmatrix} \alpha & -\theta\gamma & \beta & \theta(\alpha-\delta) \\ 0 & \alpha & 0 & \beta \\ \gamma & 0 & \delta & \theta\gamma \\ 0 & \gamma & 0 & \delta \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \\ p_1 \\ p_2 \end{pmatrix}, \tag{33}$$

and

$$\begin{pmatrix} \pi'_1 \\ \pi'_2 \\ v'^1 \\ v'^2 \end{pmatrix} = \begin{pmatrix} \alpha & \phi\beta & \beta & 0 \\ 0 & \alpha & 0 & \beta \\ \gamma & \phi(\delta-\alpha) & \delta & -\phi\beta \\ 0 & \gamma & 0 & \delta \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ v^1 \\ v^2 \end{pmatrix}, \tag{34}$$

where $\alpha, \beta, \delta, \gamma \in \mathbb{R}$ with $\alpha\delta - \beta\gamma = 1$, while the Lagrange multipliers transform as

$$\begin{aligned}
N' &= \alpha^2 N - \beta^2 M - 2\alpha\beta\lambda + \alpha\dot{\beta} - \dot{\alpha}\beta,\\
M' &= -\gamma^2 N + \delta^2 M + 2\gamma\delta\lambda + \dot{\gamma}\delta - \gamma\dot{\delta},\\
\lambda' &= -\alpha\gamma N + \beta\delta M + (\alpha\delta + \beta\gamma)\lambda + \dot{\alpha}\delta - \dot{\beta}\gamma.
\end{aligned} \tag{35}$$

Notice that when the parameters $\theta$ and $\phi$ are turned off, $\theta = 0 = \phi$, the current model reduces to the $SL(2, \mathbb{R})$ model introduced in Ref. [13].

It is time to implement on this model the theory developed in Sec. II. This will be done in the following two subsections by choosing different symplectic potentials which amounts to choosing different boundary conditions.

### *1. A noninvariant action*

The equations of motion of the model (29) and (31) can, for instance, be obtained from the action principle

$$S[x^\mu, N, M, \lambda] = \int_{\tau_1}^{\tau_2} d\tau[p_1(\dot{u}^1 + \theta\dot{p}_2) + p_2\dot{u}^2 + \pi_1(\dot{v}^1 + \phi\dot{\pi}_2) + \pi_2\dot{v}^2 - N\mathcal{C}_1 - M\mathcal{C}_2 - \lambda\mathcal{V}] \tag{36}$$

under the boundary conditions

$$(u^1 + \theta p_2)(\tau_\alpha) = U^1_\alpha, \qquad u^2(\tau_\alpha) = U^2_\alpha,$$
$$(v^1 + \phi\pi_2)(\tau_\alpha) = V^1_\alpha, \qquad v^2(\tau_\alpha) = V^2_\alpha, \qquad \alpha = 1, 2, \tag{37}$$

with $U^1_\alpha$, $U^2_\alpha$, $V^1_\alpha$, and $V^2_\alpha$ specified real numbers.

The change of the action (36) under the finite gauge transformation (33)–(35) is

$$\begin{aligned}
S[x'^\mu, N', M', \lambda'] = S[x^\mu, N, M, \lambda] &+ \beta\gamma(\vec{u}\cdot\vec{p} + \vec{v}\cdot\vec{\pi} + \theta p_1 p_2 + \phi\pi_1\pi_2)\\
&+ \frac{1}{2}(\alpha\gamma)[\vec{u}\cdot\vec{u} + \vec{\pi}\cdot\vec{\pi} + 2\theta u^1 p_2 + \theta^2(p_2)^2] + \frac{1}{2}(\beta\delta)[\vec{v}\cdot\vec{v} + \vec{p}\cdot\vec{p} + 2\phi v^1\pi_2 + \phi^2(\pi_2)^2],
\end{aligned} \tag{38}$$

and so the action is not invariant. Independently of this property of the action (36), the choice of specific gauge conditions could imply a gauge orbit whose end points at $\tau_1$ and $\tau_2$ might additionally not satisfy the boundary conditions (37) already specified [4–7]. For instance, the gauge conditions

$$C_1 := u^1 + \theta p_2 = 0, \qquad C_2 := v^1 + \phi\pi_2 = 0, \qquad C_3 := u^2 = c_3 \neq 0, \tag{39}$$

with $c_3$ a constant, are in general incompatible with the boundary conditions (37). In fact, Eqs. (39) are good gauge conditions in the sense that they fix the Lagrange multipliers to be $N = M = \lambda = 0$ (and thus the dynamics is "frozen"). Moreover, the Poisson brackets between the gauge conditions and the first-class constraints are

$$(\{C_a, \gamma_b\}_\omega) = \begin{pmatrix} p_1 & 0 & u^1 + \theta p_2 \\ 0 & \pi_1 & -(v^1 + \phi\pi_2) \\ p_2 & 0 & u^2 \end{pmatrix} = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & \pi_1 & 0 \\ p_2 & 0 & c_3 \end{pmatrix}, \tag{40}$$

where the gauge conditions (39) were used to get the second equality. The determinant of this matrix is $c_3 p_1 \pi_1$, and does not vanish for generic values of the variables.

The incompatibility between the gauge conditions (39) and the boundary conditions (37) can be removed by modifying both the action as well as the boundary conditions. The idea is to impose the gauge conditions in the gauge-transformed variables

$$u'^1 + \theta p'_2 = 0, \qquad v'^1 + \phi\pi'_2 = 0, \qquad u'^2 = c_3 \neq 0, \tag{41}$$

i.e., the gauge conditions retain their functional form in the gauge-transformed variables. The substitution of (33) and (34) into the left-hand side of (41) implies precise forms for the gauge parameters:

$$\begin{aligned}\alpha &= -\frac{c_3 p_1}{(u^1+\theta p_2)p_2 - p_1 u^2},\\ \beta &= \frac{c_3(u^1+\theta p_2)}{(u^1+\theta p_2)p_2 - p_1 u^2},\\ \gamma &= -\frac{(v^1+\phi\pi_2)[(u^1+\theta p_2)p_2 - p_1u^2]}{c_3[(u^1+\theta p_2)(v^1+\phi\pi_2) - p_1\pi_1]},\\ \delta &= \frac{\pi_1[(u^1+\theta p_2)p_2 - p_1u^2]}{c_3[(u^1+\theta p_2)(v^1+\phi\pi_2) - p_1\pi_1]}.\end{aligned} \tag{42}$$

On the other hand, by using (33) and (34), the unprimed variables in the left-hand side of the boundary conditions (37) are replaced in terms of the primed variables and the gauge parameters. In the expressions thus obtained,

$$\begin{aligned}(\delta u'^1 + \theta\delta p'_2 - \beta p'^1)(\tau_\alpha) &= U^1_\alpha,\\ (\alpha v'^1 + \phi\alpha\pi'_2 - \gamma\pi'_1)(\tau_\alpha) &= V^1_\alpha,\\ (\delta u'_2 - \beta p'_2)(\tau_\alpha) &= U^2_\alpha,\\ (-\gamma\pi'_2 + \alpha v'^2)(\tau_\alpha) &= V^2_\alpha,\end{aligned} \tag{43}$$

the parameters given in (42) must be inserted to obtain the right boundary conditions compatible with the gauge conditions and with the corresponding action

$$\begin{aligned}S[x'^\mu, N', M', \lambda'] - \Big[&-(\beta\gamma)(\vec{u}'\cdot\vec{p}' + \vec{v}'\cdot\vec{\pi}' + \theta p'_1 p'_2\\ &+ \phi\pi'_1\pi'_2) + \frac{1}{2}(\gamma\delta)(\vec{u}'\cdot\vec{u}' + \vec{\pi}'\cdot\vec{\pi}' + 2\theta u'^1 p'_2\\ &+ \theta^2(p'_2)^2) + \frac{1}{2}(\alpha\beta)(\vec{v}'\cdot\vec{v}' + \vec{p}'\cdot\vec{p}'\\ &+ 2\phi v'^1\pi'_2 + \phi^2(\pi'_2)^2)\Big],\end{aligned} \tag{44}$$

which is (the original action $S[x^\mu, N, M, \lambda]$ but expressed in terms of the primed variables and it is) obtained from (38) [7].

### *2. An invariant action*

As mentioned in Sec. II, there exists the possibility of adding a boundary term to the action (36) in such a way that the resulting action is gauge invariant [7]. The simplest action is

$$\begin{aligned}S_{\rm inv}[x^\mu, N, M, \lambda] = \int_{\tau_1}^{\tau_2} d\tau[&p_1(\dot{u}^1 + \theta\dot{p}_2) + p_2\dot{u}^2\\ &+ \pi_1(\dot{v}^1 + \phi\dot{\pi}_2) + \pi_2\dot{v}^2 - NC_1\\ &- MC_2 - \lambda\mathcal{V}] - \frac{1}{2}[(u^1+\theta p_2)p_1\\ &+ u^2p_2 + (v^1+\phi\pi_2)\pi_1 + v^2\pi_2]|^{\tau_2}_{\tau_1}.\end{aligned} \tag{45}$$

In fact, a straightforward computation shows that

$$S_{\rm inv}[x'^\mu, N', M', \lambda'] = S_{\rm inv}[x^\mu, N, M, \lambda] \tag{46}$$

under the gauge transformation (33)–(35). Therefore, $S_{\rm inv}[x^\mu, N, M, \lambda]$ is fully invariant because the noninvariant action (36) and the added noninvariant boundary term combine exactly to make $S_{\rm inv}[x^\mu, N, M, \lambda]$ strictly invariant.

Equivalently, due to the fact the boundary term in (45) is $\tau$-independent, its contribution can be understood as a different choice for the symplectic potential. By introducing this boundary term into the integrand,

$$S_{\rm inv}[x^\mu, N, M, \lambda] = \int_{\tau_1}^{\tau_2} d\tau[\Theta_\mu\dot{x}^\mu - NC_1 - MC_2 - \lambda\mathcal{V}], \tag{47}$$

with

$$\begin{aligned}\Theta = \frac{1}{2}[&p_1du^1 + p_2du^2 + \pi_1dv^1 + \pi_2dv^2\\ &- (u^1+\theta p_2)dp_1 + (\theta p_1 - u^2)dp_2\\ &- (v^1+\phi\pi_2)d\pi_1 + (\phi\pi_1 - v^2)d\pi_2].\end{aligned} \tag{48}$$

That is to say, if the dynamical system were defined by the action (47) from the very beginning, there would be no need to add a boundary term because such an action is already invariant under the gauge transformation, in complete agreement with Eq. (23):

$$\Theta_\mu\omega^{\mu\nu}\frac{\partial\gamma_a}{\partial x^\nu} - \gamma_a = 0. \tag{49}$$

The action (45) or (47) yields the equations of motion provided that the boundary conditions

$$\begin{aligned}&\frac{1}{2}\ln\Big(\frac{u^1+\theta p_2}{p_1}\Big)(\tau_\alpha) = Q^1_\alpha, \qquad \frac{1}{2}\ln\Big(\frac{u^2}{p_2}\Big)(\tau_\alpha) = Q^2_\alpha,\\ &\frac{1}{2}\ln\Big(\frac{v^1+\phi\pi_2}{\pi_1}\Big)(\tau_\alpha) = Q^3_\alpha, \qquad \frac{1}{2}\ln\Big(\frac{v^2}{\pi_2}\Big)(\tau_\alpha) = Q^4_\alpha,\\ &\alpha = 1, 2\end{aligned} \tag{50}$$

are satisfied.

Like in the the case of the noninvariant action, the choice of a particular gauge condition might lead to a solution of the equations of motion (a gauge orbit) whose values at the end points $\tau_1$ and $\tau_2$ might not match the specified values in the right-hand side of the boundary conditions (50) (see Fig. 1). For instance, the gauge conditions

$$\begin{aligned}C_1 &:= u^1 + \theta p_2 = c_1 \neq 0,\\ C_2 &:= v^1 + \phi\pi_2 = c_2 \neq 0,\\ C_3 &:= u^2 - p_2 = 0,\end{aligned} \tag{51}$$

with $c_1$ and $c_2$ constants, are not compatible with the boundary conditions (50). These conditions are good gauge conditions in the sense that they fix the Lagrange multi-

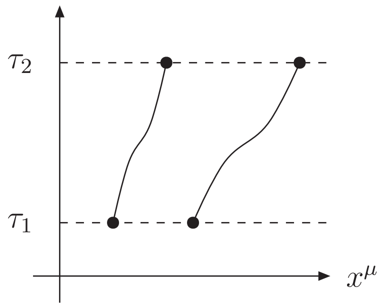


FIG. 1. The path in the right-hand side (RHS) matches the gauge conditions, but its end points are incompatible with the original boundary conditions for the action principle. Keeping the gauge conditions (and so the path in the RHS) forces us to modify the boundary conditions. The right modification can be achieved by using the path in the left-hand side (LHS) which is obtained from the path in the RHS by a gauge transformation that does not vanish at the end points. The path in the LHS is not compatible with the gauge conditions, but its end points are compatible with the original boundary conditions.

pliers to be $\lambda = N = M = 0$. Additionally,

$$(\{C_a, \gamma_b\}_\omega) = \begin{pmatrix} p_1 & 0 & u^1 + \theta p_2 \\ 0 & \pi_1 & -(v^1 + \phi\pi_2) \\ p_2 & 0 & u^2 \end{pmatrix} = \begin{pmatrix} p_1 & 0 & c_1 \\ 0 & \pi_1 & -c_2 \\ p_2 & -p_2 & 2p_2 \end{pmatrix}, \tag{52}$$

where the gauge conditions were used to get the second equality. The determinant of the matrix is $p_2[p_1(2\pi_1 - c_2) - c_1\pi_1]$.

Such a compatibility can be achieved by imposing the gauge conditions in the gauge-related variables

$$u'^1 + \theta p'_2 = c_1 \neq 0, \qquad v'^1 + \phi\pi'_2 = c_2 \neq 0, \qquad u'^2 - p'_2 = 0, \tag{53}$$

from which, using (33) and (34), the gauge parameters are fixed:

$$\alpha = \frac{p_2[c_1(v^1 + \phi\pi_2) - c_2 p_1]}{[(u^1 + \theta p_2)p_2 - p_1 u^2](v^1 + \phi\pi_2)} - \frac{\gamma p_1[u^2(v^1 + \phi\pi_2) - p_2\pi_1]}{[(u^1 + \theta p_2)p_2 - p_1 u^2](v^1 + \phi\pi_2)}, \tag{54}$$

$$\beta = \frac{c_2(u^1 + \theta p_2)p_2 - c_1 u_2(v^1 + \phi\pi_2)}{[(u^1 + \theta p_2)p_2 - p_1 u^2](v^1 + \phi\pi_2)} + \frac{\gamma(u^1 + \theta\pi_2)[u^2(v^1 + \phi\pi_2) - p_2\pi_1]}{[(u^1 + \theta p_2)p_2 - p_1 u^2](v^1 + \phi\pi_2)}, \qquad \delta = \frac{c_2 - \gamma\pi_1}{(v^1 + \phi\pi_2)} \tag{55}$$

(recall that $\alpha\delta - \beta\gamma = 1$). On the other hand, using (33) and (34), the original boundary conditions (50) are rewritten in terms of the primed variables and the gauge parameters:

$$\begin{aligned} \frac{1}{2}\ln\left(\frac{u'^1 + \theta p'_2}{p'_1}\right)(\tau_\alpha) &= \frac{1}{2}\ln\left(\frac{\beta(\tau_\alpha) + \alpha(\tau_\alpha)e^{2Q^1_\alpha}}{\delta(\tau_\alpha) + \gamma(\tau_\alpha)e^{2Q^1_\alpha}}\right), \\ \frac{1}{2}\ln\left(\frac{u'^2}{p'_2}\right)(\tau_\alpha) &= 0, \\ \frac{1}{2}\ln\left(\frac{v'^1 + \phi\pi'_2}{\pi'_1}\right)(\tau_\alpha) &= \frac{1}{2}\ln\left(\frac{\beta(\tau_\alpha) + \alpha(\tau_\alpha)e^{2Q^3_\alpha}}{\delta(\tau_\alpha) + \gamma(\tau_\alpha)e^{2Q^3_\alpha}}\right), \\ \frac{1}{2}\ln\left(\frac{v'^2}{\pi'_2}\right)(\tau_\alpha) &= \frac{1}{2}\ln\left(\frac{\beta(\tau_\alpha) + \alpha(\tau_\alpha)e^{2Q^4_\alpha}}{\delta(\tau_\alpha) + \gamma(\tau_\alpha)e^{2Q^4_\alpha}}\right), \end{aligned} \tag{56}$$

where $\alpha = 1, 2$, which are by construction compatible with the gauge conditions. The corresponding action is the same one given in (45) or (47) but expressed in terms of the primed variables: $S_{\rm inv}[x'^\mu, M', M', \lambda']$. Once the goal has been achieved, one can simply drop the apostrophe "$'$" both in the action as well as in the boundary conditions to have an action principle written in the usual form.

The difference between the case of the noninvariant action and the case of the fully gauge-invariant action lies in the fact that in the latter there is no need to modify the action, but just to handle the boundary conditions.

## B. $SU(2)$ model

Now, a second model having an $SU(2)$ gauge symmetry is given. The dynamical variables must satisfy the constraints

$$\begin{aligned} H_1 &:= u^1u^2 + v^1v^2 + p_1p_2 + \pi_1\pi_2 + \theta u^2p_2 + \phi v^2\pi_2 \approx 0, \\ H_2 &:= (u^1)^2 + (v^1)^2 + (p_1)^2 + (\pi_1)^2 - (u^2)^2 - (v^2)^2 \\ &\quad - (p_2)^2 - (\pi_2)^2 + 2\theta u^1 p_2 + 2\phi v^1\pi_2 + \theta^2(p_2)^2 \\ &\quad + \phi^2(\pi_2)^2 \approx 0, \\ D &:= u^1p_2 + v^1\pi_2 - u^2p_1 - v^2\pi_1 + \theta(p_2)^2 + \phi(\pi_2)^2 \approx 0. \end{aligned} \tag{57}$$

The Poisson brackets computed with respect to the symplectic structure of Eq. (28) among these constraints yield

$$\{H_1, H_2\}_\omega = -4D, \qquad \{H_1, D\}_\omega = H_2, \qquad \{H_2, D\}_\omega = -4H_1, \tag{58}$$

and so they are first class. The algebra turns out to be isomorphic to the $\mathfrak{su}(2)$ Lie algebra. This can be easily accomplished by rescaling the constraints $J_1 := H_1/2$, $J_2 := -H_2/4$, and $J_3 := D/2$, which satisfy $\{J_i, J_j\} = \varepsilon_{ij}{}^k J_k$ with $\varepsilon_{ijk}$ the three-dimensional Levi-Civita symbol $\varepsilon_{123} = +1$.

Like in the noncommutative $SL(2, \mathbb{R})$ model, when the noncommutative parameters are turned off, $\theta = 0 = \phi$, the resulting model still involves an $SU(2)$ gauge symmetry.

By plugging (28) and (57), $(\lambda^1, \lambda^2, \lambda^3) = (N, M, \lambda)$, and $(\gamma_1, \gamma_2, \gamma_3) = (H_1, H_2, D)$ into Eqs. (6), the dynamical equations become

$$\begin{aligned}
&\dot{u}^1 = N(p_2 + \theta u^1 + \theta^2 p_2) + 2M(p_1 - \theta u^2) - \lambda(u^2 + \theta p_1), \qquad \dot{u}^2 = Np_1 - 2Mp_2 + \lambda(u^1 + \theta p_2),\\
&\dot{v}^1 = N(\pi_2 + \phi v^1 + \phi^2 \pi_2) + 2M(\pi_1 - \phi v^2) - \lambda(v^2 + \phi \pi_1), \qquad \dot{v}^2 = N\pi_1 - 2M\pi_2 + \lambda(v^1 + \phi\pi_2),\\
&\dot{p}_1 = -Nu^2 - 2M(u^1 + \theta p_2) - \lambda p_2, \qquad \dot{p}_2 = -N(u^1 + \theta p_2) + 2Mu^2 + \lambda p_1,\\
&\dot{\pi}_1 = -Nv^2 - 2M(v^1 + \phi\pi_2) - \lambda\pi_2, \qquad \dot{\pi}_2 = -N(v^1 + \phi\pi_2) + 2Mv^2 + \lambda\pi_1.
\end{aligned} \tag{59}$$

*Gauge transformation*. The finite gauge transformation of the phase space variables is

$$X' = AX, \qquad Y' = BY,$$
$$X = \begin{pmatrix} u^1 \\ u^2 \\ p_1 \\ p_2 \end{pmatrix}, \qquad Y = \begin{pmatrix} v^1 \\ v^2 \\ \pi_1 \\ \pi_2 \end{pmatrix}, \tag{60}$$

where the matrix $A$ and $B$ are given by

$$A = \begin{pmatrix} a - \theta d & -b + \theta c & -c - \theta b & -d(1 + \theta^2) \\ b & a & -d & c + \theta b \\ c & d & a & -b + \theta c \\ d & -c & b & a + \theta d \end{pmatrix},$$
$$B = \begin{pmatrix} a - \phi d & -b + \phi c & -c - \phi b & -d(1 + \phi^2) \\ b & a & -d & c + \phi b \\ c & d & a & -b + \phi c \\ d & -c & b & a + \phi d \end{pmatrix}, \tag{61}$$

where $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 + c^2 + d^2 = 1$, while the Lagrange multipliers transform as

$$\begin{aligned}
N' &= [1 - 2(b^2 + c^2)]N + 4(ab + cd)M + 2(ac - bd)\lambda \\
&\quad + a\dot{b} + d\dot{a} - a\dot{d} - b\dot{c},\\
M' &= [1 - 2(b^2 + d^2)]M - (bc + ad)\lambda + (cd - ab)N \\
&\quad + \frac{1}{2}(c\dot{a} + b\dot{d} - d\dot{b} - a\dot{c}),\\
\lambda' &= [1 - 2(c^2 + d^2)]\lambda + 4(ad - bc)M - 2(bd + ac)N \\
&\quad + a\dot{b} + c\dot{d} - b\dot{a} - d\dot{c}.
\end{aligned} \tag{62}$$

*Dirac Observables*. The following functions are invariant under the gauge transformation (60)–(62):

$$\begin{aligned}
\mathcal{O}_1 &= \frac{1}{2}[(u^1)^2 + (u^2)^2 + (p_1)^2 + (p_2)^2] + \theta u^1 p_2 + \frac{1}{2}\theta^2 (p_2)^2,\\
\mathcal{O}_2 &= u^1 v^1 + u^2 v^2 + p_1\pi_1 + p_2\pi_2 + \phi u^1 \pi_2 + \theta v^1 p_2 + \theta\phi(p_2)^2,\\
\mathcal{O}_3 &= u^1 v^2 - u^2 v^1 + p_2\pi_1 - p_1\pi_2 + \theta v^2 p_2 - \phi u^2 \pi_2,\\
\mathcal{O}_4 &= u^1\pi_1 + u^2\pi_2 - v^1 p_1 - v^2 p_2 + \theta p_2\pi_1 - \phi v^1 \pi_2,\\
\mathcal{O}_5 &= u^1\pi_2 - u^2\pi_1 - v^1 p_2 + v^2 p_1 + \theta p_2\pi_2 - \phi p_2 \pi_2,\\
\mathcal{O}_6 &= \frac{1}{2}[(v^1)^2 + (v^2)^2 + (\pi_1)^2 + (\pi_2)^2] + \phi v^1 \pi_2 + \frac{1}{2}\phi^2(\pi_2)^2.
\end{aligned} \tag{63}$$

The Poisson brackets among them are

$$\begin{aligned}
&\{\mathcal{O}_1, \mathcal{O}_2\}_\omega = \mathcal{O}_4, \qquad \{\mathcal{O}_1, \mathcal{O}_3\}_\omega = -\mathcal{O}_5,\\
&\{\mathcal{O}_1, \mathcal{O}_4\}_\omega = -\mathcal{O}_2, \qquad \{\mathcal{O}_1, \mathcal{O}_5\}_\omega = \mathcal{O}_3,\\
&\{\mathcal{O}_2, \mathcal{O}_4\}_\omega = 2\mathcal{O}_1 - 2\mathcal{O}_6, \qquad \{\mathcal{O}_2, \mathcal{O}_6\}_\omega = \mathcal{O}_4,\\
&\{\mathcal{O}_3, \mathcal{O}_5\}_\omega = 2\mathcal{O}_1 + 2\mathcal{O}_6, \qquad \{\mathcal{O}_3, \mathcal{O}_6\}_\omega = \mathcal{O}_5,\\
&\{\mathcal{O}_4, \mathcal{O}_6\}_\omega = -\mathcal{O}_2, \qquad \{\mathcal{O}_5, \mathcal{O}_6\}_\omega = -\mathcal{O}_3.
\end{aligned} \tag{64}$$

A straightforward computation shows that this algebra of observables is isomorphic to the $\mathfrak{su}(2) \oplus \mathfrak{so}(2, 1)$ Lie algebra [14].

### 1. A noninvariant action

Now, the effect of the gauge transformation in the action principle will be analyzed. The equations of motion of the model (57) and (59) can, for instance, be obtained from the action principle

$$\begin{aligned}
S[x^\mu, N, M, \lambda] = \int_{\tau_1}^{\tau_2} d\tau[&p_1(\dot{u}^1 + \theta\dot{p}_2) + p_2\dot{u}^2 \\
&+ \pi_1(\dot{v}^1 + \phi\dot{\pi}_2) + \pi_2\dot{v}^2 - NH_1 \\
&- MH_2 - \lambda D],
\end{aligned} \tag{65}$$

under the boundary conditions

$$(u^1+\theta p_2)(\tau_\alpha)=U^1_\alpha, \qquad u^2(\tau_\alpha)=U^2_\alpha,$$
$$(v^1+\phi\pi_2)(\tau_\alpha)=V^1_\alpha, \qquad v^2(\tau_\alpha)=V^2_\alpha,$$
$$\alpha=1,2, \tag{66}$$

with $U^1_\alpha$, $U^2_\alpha$, $V^1_\alpha$, and $V^2_\alpha$ specified real numbers.

The change of the action (65) under the finite gauge transformation (60)–(62) yields

$$S[x'^\mu,N',M',\lambda']=S[x^\mu,N,M,\lambda]+\frac{(ac+bd)}{2}\times[H_2+2(p_1)^2-2(p_2)^2+2(\pi_1)^2-2(\pi_2)^2]+(ad-bc)[H_1-2p_1p_2-2\pi_1\pi_2]-(c^2+d^2)[\vec{u}\cdot\vec{p}+\vec{v}\cdot\vec{\pi}+\theta p_1p_2+\phi\pi_1\pi_2], \tag{67}$$

and so the action (65) is not gauge invariant.

#### 2. An invariant action

Once again, it is possible to build gauge-invariant actions, the simplest of which is

$$S_{\rm inv}[x^\mu,N,M,\lambda]=\int_{\tau_1}^{\tau_2}d\tau[p_1(\dot{u}^1+\theta\dot{p}_2)+p_2\dot{u}^2+\pi_1(\dot{v}^1+\phi\dot{\pi}_2)+\pi_2\dot{v}^2-NH_1-MH_2-\lambda D]-\frac{1}{2}[(u^1+\theta p_2)p_1+u^2p_2+(v^1+\phi\pi_2)\pi_1+v^2\pi_2]|^{\tau_2}_{\tau_1}. \tag{68}$$

In fact, a straightforward computation using (60)–(62) shows that

$$S_{\rm inv}[x'^\mu,N',M',\lambda']=S_{\rm inv}[x^\mu,N,M,\lambda]. \tag{69}$$

Therefore, $S_{\rm inv}[x^\mu,N,M,\lambda]$ is strictly invariant. Like in the case of the $SL(2,\mathbb{R})$ model, introducing the boundary term into the integrand

$$S_{\rm inv}[x^\mu,N,M,\lambda]=\int_{\tau_1}^{\tau_2}d\tau[\Theta_\mu\dot{x}^\mu-NH_1-MH_2-\lambda D], \tag{70}$$

with

$$\Theta=\frac{1}{2}[p_1du^1+p_2du^2+\pi_1dv^1+\pi_2dv^2-(u^1+\theta p_2)dp_1+(\theta p_1-u^2)dp_2-(v^1+\phi\pi_2)d\pi_1+(\phi\pi_1-v^2)d\pi_2], \tag{71}$$

leads to the introduction of a new symplectic potential and

$$\Theta_\mu\omega^{\mu\nu}\frac{\partial\gamma_a}{\partial x^\nu}-\gamma_a=0, \tag{72}$$

as expected.

The analysis of a possible incompatibility between the boundary conditions

$$\frac{1}{2}\ln\left(\frac{u^1+\theta p_2}{p_1}\right)(\tau_\alpha)=Q^1_\alpha, \qquad \frac{1}{2}\ln\left(\frac{u^2}{p_2}\right)(\tau_\alpha)=Q^2_\alpha,$$
$$\frac{1}{2}\ln\left(\frac{v^1+\phi\pi_2}{\pi_1}\right)(\tau_\alpha)=Q^3_\alpha, \qquad \frac{1}{2}\ln\left(\frac{v^2}{\pi_2}\right)(\tau_\alpha)=Q^4_\alpha,$$
$$\alpha=1,2, \tag{73}$$

of the action (68) or (70) with chosen gauge conditions can be carried out along the same steps made for the case of the $SL(2,\mathbb{R})$ model.

## IV. CONCLUDING REMARKS

The issue of the gauge invariance of the action principle for Hamiltonian gauge systems whose extended phase space is described in terms of arbitrary symplectic structures has been studied assuming that the action is already in a Hamiltonian form. One of the main results reported in this paper is the fact that an action featuring first-class constraints quadratic in the phase space variables can be strictly gauge invariant. The gauge invariance of the action (13) can also be analyzed from a Lagrangian viewpoint. In this last approach, the action (13) is assumed to have a Lagrangian form and Dirac's method is applied systematically, i.e, one first defines momenta canonically conjugate to the variables $x^\mu$, $\lambda^a$, and $\lambda^\alpha$, increasing the number of variables that label the points of the extended phase space which is also equipped with a symplectic structure having, by construction, the usual canonical form. If this approach were followed, the gauge invariance of the resulting action could be handled with the tools developed in Refs. [4–7]. Sometimes, however, it is not convenient to enlarge the phase space, but to work the theory assuming that the action already has a Hamiltonian form, and then the analysis of the gauge invariance of the action must be carried out along the ideas studied in this paper. (An example of the Hamiltonian viewpoint can be found in the three-dimensional Chern-Simons theory.)

Finally, the results of this paper can also be used to extend the analyses developed in Refs. [15–17] by introducing new interactions through the symplectic structure.

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