String triality, black hole entropy, and Cayley's hyperdeterminant

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(Received 3 May 2007; published 19 July 2007)

The four-dimensional $N = 2$ STU model of string compactification is invariant under an $SL(2, Z)_S \times SL(2, Z)_T \times SL(2, Z)_U$ duality acting on the dilaton/axion S, complex Kahler form T, and the complex structure fields U, and also under a string/string/string triality $S \leftrightarrow T \leftrightarrow U$. The model admits an extremal black hole solution with four electric and four magnetic charges whose entropy must respect these symmetries. It is given by the square root of the hyperdeterminant introduced by Cayley in 1845. This also features three-qubit quantum entanglement.

DOI: 10.1103/PhysRevD.76.025017 PACS numbers: 04.65.+e, 04.70.Dy, 11.25.Mj

I. INTRODUCTION

An interesting subsector of string compactification to four dimensions is provided by the STU model whose low energy limit is described by $N = 2$ supergravity coupled to three vector multiplets. One may regard it as a truncation of an $N = 4$ theory obtained by compactifying the heterotic string on T^6 where S, T, U correspond to the dilaton/axion, complex Kahler form, and complex structure fields, respectively. It exhibits an $SL(2, Z)_S$ strong/weak coupling duality and an $SL(2, Z)_T \times SL(2, Z)_U$ target space duality. By string/string duality, this is equivalent to a type IIA string on $K3 \times T^2$ with S and T exchanging roles [1–3]. Moreover, by mirror symmetry this is in turn equivalent to a type IIB string on the mirror manifold with T and U exchanging roles. Hence the truncated theory has a combined $[SL(2, Z)]^3$ duality and complete S-T-U triality symmetry [4]. Alternatively, one may simply start with this $N = 2$ theory directly as an interesting four-dimensional supergravity in its own right, as described in Sec. II.

The model admits extremal black holes solutions carrying four electric and magnetic charges. In Sec. III we organize these 8 charges into the $2 \times 2 \times 2$ *hypermatrix* and display the S-T-U symmetric Bogomol'nyi mass formula [4].

Associated with this hypermatrix is a *hyperdeterminant*, discussed in Sec. IV, first introduced by Cayley in 1845 [5].

The black hole entropy, first calculated in [6], is quartic in the charges and must be invariant under $[SL(2, Z)]^3$ and under triality. The main result of the present paper, given in Sec. V, is to show that this entropy is given by the square root of Cayley's hyperdeterminant.

The hyperdeterminant also makes its appearance in quantum information theory [7] as the measure of three-qubit entanglement known as the *3-tangle* [8], which we briefly review in Sec. VI.

II. THE STU MODEL

Consider the three complex scalars axion/dilaton field S, the complex Kahler form field T, and the complex structure field U

$$S = S_1 + iS_2, \qquad T = T_1 + iT_2, \qquad U = U_1 + iU_2. \tag{2.1}$$

This complex parametrization allows for a natural transformation under the various $SL(2, Z)$ symmetries. The action of $SL(2, Z)_S$ is given by

$$S \to \frac{aS + b}{cS + d}, \tag{2.2}$$

where a, b, c, d are integers satisfying $ad - bc = 1$, with similar expressions for $SL(2, Z)_T$ and $SL(2, Z)_U$. Defining the matrices $\mathcal{M}_S$, $\mathcal{M}_T$, and $\mathcal{M}_U$ via

$$\mathcal{M}_S = \frac{1}{S_2}\begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix}, \tag{2.3}$$

the action of $SL(2, Z)_S$ now takes the form

$$\mathcal{M}_S \to \omega_S{}^T \mathcal{M}_S \omega_S, \tag{2.4}$$

where

$$\omega_S = \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \tag{2.5}$$

with similar expressions for $\mathcal{M}_T$ and $\mathcal{M}_U$. We also define the $SL(2, Z)$ invariant tensors

$$\epsilon_S = \epsilon_T = \epsilon_U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.6}$$

Starting from the heterotic string, the bosonic action for the graviton $g_{\mu\nu}$, dilaton η, two-form $B_{\mu\nu}$ four $U(1)$ gauge fields A_S^a, and two complex scalars T and U is [4]

$$\begin{aligned} I_{STU} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}e^{-\eta}\Big[& R_g + g^{\mu\nu}\partial_\mu\eta\partial_\nu\eta \\ & - \frac{1}{12}g^{\mu\lambda}g^{\nu\tau}g^{\rho\sigma}H_{\mu\nu\rho}H_{\lambda\tau\sigma} \\ & + \frac{1}{4}\mathrm{Tr}(\partial\mathcal{M}_T{}^{-1}\partial\mathcal{M}_T) + \frac{1}{4}\mathrm{Tr}(\partial\mathcal{M}_U{}^{-1}\partial\mathcal{M}_U) \\ & - \frac{1}{4}F_{S\mu\nu}{}^T(\mathcal{M}_T \times \mathcal{M}_U)F_S{}^{\mu\nu}\Big], \end{aligned} \tag{2.7}$$

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where the metric $g_{\mu\nu}$ is related to the four-dimensional canonical Einstein metric $g^c_{\mu\nu}$ by $g_{\mu\nu} = e^\eta g^c{}_{\mu\nu}$ and where

$$H_{\mu\nu\rho} = 3(\partial_{[\mu} B_{\nu\rho]} - \tfrac{1}{2} A_{S[\mu}{}^T (\epsilon_T \times \epsilon_U) F_{S\nu\rho]}). \tag{2.8}$$

This action is manifestly invariant under T duality and U duality, with

$$\begin{aligned} F_{S\mu\nu} &\to (\omega_T{}^{-1} \times \omega_U{}^{-1}) F_{S\mu\nu}, \\ \mathcal{M}_{T/U} &\to \omega_{T/U}^T \mathcal{M}_{T/U} \omega_{T/U}, \end{aligned} \tag{2.9}$$

and with η, $g_{\mu\nu}$, and $B_{\mu\nu}$ inert. Its equations of motion and Bianchi identities (but not the action itself) are also invariant under S duality (2.2), with T and $g^c{}_{\mu\nu}$ inert and with

$$\begin{pmatrix} F_{S\mu\nu}{}^a \\ \tilde{F}_{S\mu\nu}{}^a \end{pmatrix} \to \omega_S^{-1} \begin{pmatrix} F_{S\mu\nu}{}^a \\ \tilde{F}_{S\mu\nu}{}^a \end{pmatrix}, \tag{2.10}$$

where

$$\begin{aligned} \tilde{F}_{S\mu\nu}{}^a = &-S_2[(\mathcal{M}_T{}^{-1} \times \mathcal{M}_U{}^{-1})(\epsilon_T \times \epsilon_U)]^a{}_b * \\ &\times F_{S\mu\nu}{}^b - S_1 F_{S\mu\nu}{}^a, \end{aligned} \tag{2.11}$$

where the axion field a is defined by

$$\epsilon^{\mu\nu\rho\sigma} \partial_\sigma a = \sqrt{-g} e^{-\eta} g^{\mu\sigma} g^{\nu\lambda} g^{\rho\tau} H_{\sigma\lambda\tau}, \tag{2.12}$$

and where $S = S_1 + iS_2 = a + ie^{-\eta}$.

Thus T duality transforms Kaluza-Klein electric charges ($F_S{}^3, F_S{}^4$) into winding electric charges ($F_S{}^1, F_S{}^2$) (and Kaluza-Klein magnetic charges into winding magnetic charges), U duality transforms the Kaluza-Klein and winding electric charge of one circle ($F_S{}^3, F_S{}^2$) into those of the other ($F_S{}^4, F_S{}^1$) (and similarly for the magnetic charges) but S duality transforms Kaluza-Klein electric charge ($F_S{}^3, F_S{}^4$) into winding magnetic charge ($\tilde{F}_S{}^3, \tilde{F}_S{}^4$) (and winding electric charge into Kaluza-Klein magnetic charge). In summary we have $SL(2, Z)_T \times SL(2, Z)_U$ and $T \leftrightarrow U$ off shell but $SL(2, Z)_S \times SL(2, Z)_T \times SL(2, Z)_U$ and an S-T-U interchange on shell.

One may also consider the type IIA action I_{TUS} and the type IIB action I_{UST} obtained by cyclic permutation of the fields S, T, U. Finally, one may consider an action [6] where the S, T, and U fields enter democratically with a prepotential

$$F = STU \tag{2.13}$$

which off shell has the full STU interchange but none of the $SL(2, Z)$. All four versions are on-shell equivalent.

III. THE BOGOMOL'NYI SPECTRUM

Following [4], it is now straightforward to write down an S-T-U symmetric Bogomol'nyi mass formula. Let us define electric and magnetic charge vectors α_S^a and β_S^a associated with the field strengths $F_S{}^a$ and $\tilde{F}_S{}^a$ in the standard way. The electric and magnetic charges Q_S^a and P_S^a are given by

$$F_{S0r}{}^a \sim \frac{Q_S^a}{r^2}, \qquad *F_{S0r}{}^a \sim \frac{P_S^a}{r^2}, \tag{3.1}$$

giving rise to the charge vectors

$$\begin{pmatrix} \alpha_S^a \\ \beta_S^a \end{pmatrix} = \begin{pmatrix} S_2^{(0)} \mathcal{M}_T^{-1} \times \mathcal{M}_U^{-1} & S_1^{(0)} \epsilon_T \times \epsilon_U \\ 0 & -\epsilon_T \times \epsilon_U \end{pmatrix}^{ab} \begin{pmatrix} Q_S^b \\ P_S^b \end{pmatrix}. \tag{3.2}$$

For our purpose it is useful to define a $2 \times 2 \times 2$ array a_{ijk} via

$$\begin{pmatrix} a_{000} \\ a_{001} \\ a_{010} \\ a_{011} \\ a_{100} \\ a_{101} \\ a_{110} \\ a_{111} \end{pmatrix} = \begin{pmatrix} -\beta_S^1 \\ -\beta_S^2 \\ -\beta_S^3 \\ -\beta_S^4 \\ \alpha_S^1 \\ \alpha_S^2 \\ \alpha_S^3 \\ \alpha_S^4 \end{pmatrix}, \tag{3.3}$$

transforming as

$$a^{ijk} \to \omega_S{}^i{}_l \omega_T{}^j{}_m \omega_U{}^k{}_n a^{lmn}. \tag{3.4}$$

Then the mass formula is

$$\begin{aligned} m^2 = \tfrac{1}{16} a^T (&\mathcal{M}_S{}^{-1} \mathcal{M}_T{}^{-1} \mathcal{M}_U{}^{-1} - \mathcal{M}_S{}^{-1} \epsilon_T \epsilon_U \\ &- \epsilon_S \mathcal{M}_T{}^{-1} \epsilon_U - \epsilon_S \epsilon_T \mathcal{M}_U{}^{-1}) a. \end{aligned} \tag{3.5}$$

This is consistent with the general $N = 2$ Bogomol'nyi formula [9]. Although all theories have the same mass spectrum, there is clearly a difference of interpretation with electrically charged elementary states in one picture being solitonic monopole or dyon states in the other.

This $2 \times 2 \times 2$ array a_{ijk} is an example of a "hypermatrix," a term coined by Cayley in 1845 [5] where he also introduced a "hyperdeterminant."

IV. THE CAYLEY HYPERDETERMINANT

In analogy with the determinant of a 2×2 matrix a^{ij}

$$\det a_2 = \tfrac{1}{2} \epsilon^{ij} \epsilon^{lm} a_{il} a_{jm} = a_{00} a_{11} - a_{01} a_{10}, \tag{4.1}$$

the hyperdeterminant of a_{ijk} is defined to be

$$\begin{aligned} \mathrm{Det} a_3 = &-\tfrac{1}{2} \epsilon^{ii'} \epsilon^{jj'} \epsilon^{kk'} \epsilon^{mm'} \epsilon^{nn'} \epsilon^{pp'} a_{ijk} a_{i'j'm} a_{npk'} a_{n'p'm'} \\ = &\, a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 \\ &- 2(a_{000} a_{001} a_{110} a_{111} + a_{000} a_{010} a_{101} a_{111} \\ &+ a_{000} a_{100} a_{011} a_{111} + a_{001} a_{010} a_{101} a_{110} \\ &+ a_{001} a_{100} a_{011} a_{110} + a_{010} a_{100} a_{011} a_{101}) \\ &+ 4(a_{000} a_{011} a_{101} a_{110} + a_{001} a_{010} a_{100} a_{111}). \end{aligned} \tag{4.2}$$

The hyperdeterminant vanishes if the following system of equations in six unknowns u^i, v^j, w^k has a nontrivial solution, not allowing any of the pairs to be both zero:

$$a_{ijk}u^i v^j = 0, \qquad a_{ijk}u^i w^k = 0, \qquad a_{ijk}v^j w^k = 0. \tag{4.3}$$

Other useful identities are provided by the polynomial symmetric under permutation of the four indices [10]:

$$\begin{aligned} P(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = {}& x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 + x_4^2y_4^2 \\ &- 4x_1x_2x_3x_4 - 4y_1y_2y_3y_4 \\ &- 2x_1y_1x_2y_2 - 2x_1y_1x_3y_3 \\ &- 2x_1y_1x_4y_4 - 2x_2y_2x_3y_3 \\ &- 2x_2y_2x_4y_4 - 2x_3y_3x_4y_4 \end{aligned} \tag{4.4}$$

which obeys

$$\begin{aligned} P(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = {}& (x_1y_1 + x_2y_2 + x_3y_3 \\ &+ x_4y_4)^2 - 4(x_1x_2 + y_3y_4) \\ &\times (x_3x_4 + y_1y_2) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &y_1^2 P(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \\ &\quad = (x_1y_1^2 - x_2y_2y_1 - x_3y_3y_1 - x_4y_4y_1 - 2x_2x_3x_4)^2 \\ &\qquad - 4(x_2x_3 + y_1y_4)(x_2x_3 + y_1y_4) \\ &\qquad \times (x_2x_4 + y_1y_3)(x_3x_4 + y_1y_2). \end{aligned} \tag{4.6}$$

Comparison with (4.2) yields

$$\mathrm{Det}\, a_3 = P(-a_{000}, a_{110}, a_{101}, a_{011}, -a_{111}, a_{001}, a_{010}, a_{100}). \tag{4.7}$$

For our purposes, the important properties of the hyperdeterminant are that it is a quartic invariant under $[SL(2, Z)]^3$ and under triality.

V. BLACK HOLE ENTROPY

The STU model admits extremal black hole solutions satisfying the Bogomol'nyi mass formula. As usual, their entropy is given by one quarter the area of the event horizon. However, to calculate this area requires evaluating the mass not with the asymptotic values of the moduli, but with their frozen values on the horizon which are fixed in terms of the charges [11]. This ensures that the entropy is moduli-independent, as it should be. The relevant calculation was carried out in [6] for the model with the STU prepotential. The electric and magnetic charges of that paper are denoted (p^0, q_0), (p^1, q_1), (p^2, q_2), (p^3, q_3) with O(2, 2) scalar products

$$p^2 = (p^0)^2 + (p^1)^2 - (p^2)^2 - (p^3)^2, \tag{5.1}$$

$$q^2 = (q_0)^2 + (q_1)^2 - (q_2)^2 - (q_3)^2, \tag{5.2}$$

$$p \cdot q = (p^0q_0) + (p^1q_1) + (p^2q_2) + (p^3q_3). \tag{5.3}$$

In these variables, the entropy is given by

$$S = \pi(W(p^\Lambda, q_\Lambda))^{1/2}, \tag{5.4}$$

where

$$\begin{aligned} W(p^\Lambda, q_\Lambda) = {}& -(p \cdot q)^2 + 4((p^1q_1)(p^2q_2) \\ &+ (p^1q_1)(p^3q_3) + (p^3q_3)(p^2q_2)) \\ &- 4p^0q_1q_2q_3 + 4q_0p^1p^2p^3. \end{aligned} \tag{5.5}$$

The function $W(p^\Lambda, q_\Lambda)$ is symmetric under transformations: $p^1 \leftrightarrow p^2 \leftrightarrow p^3$ and $q_1 \leftrightarrow q_2 \leftrightarrow q_3$. For the solution to be consistent we have to require $W > 0$, otherwise the model is not defined.

If we now make the identifications

$$\begin{pmatrix} a_{000} \\ a_{001} \\ a_{010} \\ a_{011} \\ a_{100} \\ a_{101} \\ a_{110} \\ a_{111} \end{pmatrix} = \begin{pmatrix} -p^0 \\ -p^1 \\ -p^2 \\ -q_3 \\ p^3 \\ q_2 \\ q_1 \\ -q_0 \end{pmatrix}, \tag{5.6}$$

we recognize from (4.2) that

$$W = -\mathrm{Det} a_3 \tag{5.7}$$

and hence the black hole entropy is given by

$$S = \pi\sqrt{-\mathrm{Det} a_3}. \tag{5.8}$$

Some examples of supersymmetric black hole solutions [12] are provided by the electric Kaluza-Klein black hole with $\alpha = (1, 0, 0, 0)$ and $\beta = (0, 0, 0, 0)$; the electric winding black hole with $\alpha = (0, 0, 0, -1)$ and $\beta = (0, 0, 0, 0)$; the magnetic Kaluza-Klein black hole with $\alpha = (0, 0, 0, 0)$ and $\beta = (0, -1, 0, 0)$; the magnetic winding black hole with $\alpha = (0, 0, 0, 0)$ and $\beta = (0, 0, -1, 0)$. These are characterized by a scalar-Maxwell coupling parameter $a = \sqrt{3}$. By combining these 1-particle states, we may build up 2-, 3-, and 4-particle bound states at threshold [4,12]. For example $\alpha = (1, 0, 0, -1)$ and $\beta = (0, 0, 0, 0)$ with $a = 1$; $\alpha = (1, 0, 0, -1)$ and $\beta = (0, -1, 0, 0)$ with $a = 1/\sqrt{3}$; $\alpha = (1, 0, 0, -1)$ and $\beta = (0, -1, -1, 0)$ with $a = 0$. The 1-, 2-, and 3-particle states all yield vanishing contributions to $\det a_3$. A nonzero value is obtained for the 4-particle example, however, which is just the Reissner-Nordstrom black hole.

VI. 3-QUBIT QUANTUM ENTANGLEMENT

Interestingly enough, Cayley's hyperdeterminant also makes its appearance in quantum information theory.

Let the system ABC be in a pure state $|\Psi\rangle$, and let the components of $|\Psi\rangle$ in the standard basis be a_{ijk}:

$$|\Psi\rangle = \sum_{ijk} a_{ijk}|ijk\rangle \tag{6.1}$$

or

$$|\Psi\rangle = a_{000}|000\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{011}|011\rangle + a_{100}|100\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle. \tag{6.2}$$

In this context the a_{ijk} are complex numbers rather than integers and the symmetry is $[SL(2, C)]^3$ rather than $[SL(2, Z)]^3$. The three-way entanglement of the three qubits A, B, and C is given by the *3-tangle* of Coffman, Kondu, and Wooters [8]

$$\tau_{ABC} = 2|\epsilon^{ii'}\epsilon^{jj'}\epsilon^{kk'}\epsilon^{mm'}\epsilon^{nn'}\epsilon^{pp'}a_{ijk}a_{i'j'm}a_{npk'}a_{n'p'm'}| = 4|\mathrm{Det}a_3|. \tag{6.3}$$

The 3-tangle is maximal for the Greenberger-Horne-Zeilinger (GHZ) state $|000\rangle + |111\rangle$ [13] and vanishes for the states $p|100\rangle + q|010\rangle + r|001\rangle$. The relation between three-qubit quantum entanglement and the Cayley hyperdeterminant was pointed out by Miyake and Wadati [7].

Thus Cayley's hyperdeterminant provides an interesting connection, at least at the level of mathematics, between string theory and quantum entanglement. Other mathematical similarities are provided by the division algebras [14] and by twistors [15]. What about physics? The near horizon geometry of the black holes is $\mathrm{AdS}^2 \times S^2$ and one might expect a relation between the black hole entropy and the entanglement entropy of the conformal quantum mechanics that lives on the boundary [16], although the nature of this particular anti-de Sitter/conformal field theory duality is not well understood [17]. In any event, the 3-tangle is not the same as the entropy of entanglement [18]. So the appearance of the Cayley hyperdeterminant in these two different contexts of stringy black hole entropy and 3-qubit quantum entanglement remains, for the moment, a purely mathematical coincidence.

ACKNOWLEDGMENTS

It is a pleasure to thank Chris Hull, Jerome Gauntlett, Kelly Stelle, Arkady Tseytlin, and Dan Waldram for useful conversations and Jim Liu, Anthony Sudbery, Martin Plenio, and Joachim Rahmfeld for correspondence. I am especially grateful to Peter Levay who first drew my attention to Cayley's hyperdeterminant.

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