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Truly naked spherically symmetric and distorted black holes

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We demonstrate the existence of spherically symmetric truly naked black holes (TNBH) for which the Kretschmann scalar is finite on the horizon but some curvature components including those responsible for tidal forces as well as the energy density $\bar{\rho}$ measured by a free-falling observer are infinite. We choose a rather generic powerlike asymptotics for the metric functions and analyze possible types of a horizon depending on the behavior of curvature components in the free-falling frame. It is also shown in a general case of distorted black holes that $\bar{\rho}$ and tidal forces are either both finite or both infinite. The general approach developed in the article includes previously found examples and, in particular, TNBHs with an infinite area of a horizon. The fact that the detection of singularity depends on a frame may be relevant for a more accurate definition of the cosmic censorship conjecture. TNBHs may be considered as a new example of so-called nonscalar singularities for which the scalar curvature invariants are finite but some components of the Riemann tensor may diverge in certain frames.

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I. INTRODUCTION

Usually, the regular or singular character of points or surfaces in spacetime reveals itself as an inner property inherent to the manifold and does not depend on the frame in which it is described. In particular, the value Kretschmann scalar is finite or infinite, whatever frame is used for its calculation. Nonetheless, as was pointed out in [1,2], in the vicinity of black holes regular and singular features may entangle in a nontrivial way. It turns out that in some cases the curvature components in the free-falling frame are enhanced significantly with respect to their static values to the extent that they are finite nonzero in spite of in the static frame they are negligible. A similar observation was made for particular types of black holes with an infinite horizon area in some Brans-Dicke theories [3,4]. The reason lies in the singular character of the static frame itself: on the horizon a timelike static observer becomes null and the frame by itself fails. The term "naked black holes" was suggested in [1,2] for such objects. Strictly speaking, the word "naked" is not quite exact here since all curvature components remain finite in the free-falling frame. One may ask whether it is possible to take the next step and find the horizons for which the distinction between both frames is even more radical in the following sense. In the static frame all curvature components are finite but in the free-falling one some of them as well as corresponding tidal forces diverge. The answer is positive [5]. The explanation of how to reconcile some infinite components of the Riemann tensor in the orthonormal free-falling frame with the finiteness of the Kretschmann scalar is connected with the Lorentz signature of the spacetime. In the static frame all components of the curvature enter the Kretschmann scalar with the same sign but in the free-falling one this is not the case. As a result, different divergent terms cancel each other and the net outcome is finite. (From a more general viewpoint, the systems discussed in our paper represent examples of so-called nonscalar polynomial curvature singularities [6] (Chap. 8), [7], as will be clear below.) To distinguish black holes with infinite tidal forces on the horizon in the free-falling frame from naked ones [1,2] the term "truly naked black holes" (TNBHs) was suggested in [5]. It is also worth noting that the crucial difference between a static frame and a free-falling one on the horizon reveals itself not only in the dynamic effects (such as tidal force) but also in the algebraic structure of the gravitational field. Thus, the Petrov type of the field on the horizon detected by a free-falling observer can differ from the limit measured by a sequence of static observers in the near-horizon limit [5].

TNBHs considered in [5] are entirely due to a distorted, nonspherical character of the metric. In doing so, certain Weyl scalar (quantities obtained by the projection of the Weyl tensor to the null tetrad attached to an observer) diverge on the horizon [5]. Meanwhile, the full set of quantities that determines gravitational field includes, apart from Weyl scalars, also components of the Ricci tensor. In cases of distorted TNBHs considered in [5] both types of quantities diverge on the horizon. In general, this is not necessarily so. In the spherically symmetrical case there is only one nonvanishing Weyls scalar (this is connected with the fact that a generic spherically symmetric gravitational field is of type D [8]). Its value is finite and coincides in both frames (static and free-falling). Then, the only potential source of divergencies is the Ricci tensor in a free-falling frame and the corresponding components of the stress-energy tensor. Therefore, there remains an open window for the existence of TNBHs even in the spherically symmetrical case in spite of the finiteness of Weyl scalars.

The aim of the present paper is to fill this gap and show that spherically symmetrical TNBHs do exist. We demonstrate that this is possible in the case of extremal and so-

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called ultraextremal horizons whereas for the nonextremal case we return to the situation already discussed in [1,2]. As the key role is played by the asymptotic behavior of the metric near the horizon, we suggest a general approach which relies on this asymptotics only and does not require the knowledge of the metric everywhere. The corresponding general approach includes cases considered in [1–4]. We also enlarge the previous analysis carried out in [5]. We analyze explicitly the behavior of all curvature components for nonextremal, extremal, and ultraextremal horizons and conclude that divergencies in tidal transverse forces is the sufficient criterion to include an object to the class of TNBHs. Apart from this, we also show for the generic distorted horizon that divergencies of the energy density in the free-falling frame may be compatible with the finiteness of the Kretschmann scalar.

II. SPHERICALLY SYMMETRICAL CASE

A. Static frame

Let us consider the spherically symmetric metric

$$ds^2 = -dt^2 U + V^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (1)$$

supported by the stress-energy tensor having the form

$$T_\mu^\nu = \mathrm{diag}(-\rho, p_r, p_\perp, p_\perp). \quad (2)$$

It follows from 00 and rr Einstein equations that

$$U = V\exp(2\psi), \quad (3)$$

$$\psi = 4\pi \int^r d\bar{r}\, \bar{r}\, Q, \qquad Q = \frac{(p_r + \rho)}{V}. \quad (4)$$

The nonvanishing components of the curvature tensor read

$$R_{0r}^{0r} = E = -\frac{V'\Phi'}{2} - V(\Phi'' + \Phi'^2), \qquad U \equiv \exp(2\Phi), \quad (5)$$

$$R_{0\theta}^{0\theta} = \bar{E} = -\frac{V\Phi'}{r}, \qquad R_{\phi\theta}^{\phi\theta} = F = \frac{1-V}{r^2}, \quad (6)$$

$$R_{r\theta}^{r\theta} = \bar{F} = -\frac{V'}{2r}. \quad (7)$$

For what follows, it is convenient to introduce the quantity

$$Z = \bar{F} - \bar{E} = \frac{\psi' V}{r} = 4\pi Q V = 4\pi(p_r + \rho). \quad (8)$$

B. Free-falling frame

Consider also the boosted frame moving in the radial direction with the boost angle α. Then the curvature components (hat stands for the orthonormal frame) are transformed according to [1]

$$\hat{R}_{0'r'0'r'} = \hat{R}_{0r0r}, \qquad \hat{R}_{0'\theta' r'\theta'} = -\cosh\alpha \sinh\alpha Z, \quad (9)$$

$$\hat{R}_{0'\theta' 0'\theta'} = \hat{R}_{0\theta 0\theta} + \sinh^2\alpha Z, \qquad \hat{R}_{r'\theta' r'\theta'} = \hat{R}_{r\theta r\theta} + \sinh^2\alpha Z, \quad (10)$$

and similarly for components with θ replaced by ϕ. Here $\cosh\alpha = \frac{\varepsilon}{\sqrt{N}}$, ε is the energy per unit mass. (We choose $\alpha > 0$, then our definition differs from that in [1] by the sign.)

As the difference between the static and boosted frame reveals itself for all components (except the $0r0r$ one) in a similar way, the analysis in [1] was mainly restricted to the component $R_{0\theta 0\theta}$ that has a clear physical meaning, being responsible for tidal forces in the transverse directions. It is somewhat more convenient to deal with the combination of two components Z that includes the effect of tidal forces. From geodesics equations, one can obtain easily (cf. [9] for the two-dimensional analogue) that the quantity Z is related to the energy density of the source $\bar{\rho}$ measured by a free-falling observer:

$$\bar{\rho} = T_{\mu\nu} u^\mu u^\nu = \frac{\varepsilon^2 Z}{4\pi U} - T_r^r\left(1 + \frac{L^2}{r^2}\right) + \frac{T_\phi^\phi L^2}{r^2}. \quad (11)$$

ε is the energy of a particle per unit mass along the geodesics. It follows from the transformation laws (9) and (10) that the quantity Z transforms as

$$\tilde{Z} = Z\left(2\frac{\varepsilon^2}{U} - 1\right) = 8\pi Q \varepsilon^2 \frac{V}{U} - Z \quad (12)$$

(see also derivation in a more general case of an arbitrary static metric in Sec. IV below). Thus,

$$\bar{\rho} = \frac{\bar{Z} + Z}{8\pi} - T_r^r\left(1 + \frac{L^2}{r^2}\right) + \frac{T_\phi^\phi L^2}{r^2}. \quad (13)$$

It follows from (13) that $\bar{\rho}$ diverges on the horizon if and only if $\bar{Z}$ does so. In other words, on the horizon $\bar{\rho}$ is infinite for TNBHs and finite for usual and naked black holes.

We suppose that there is a horizon at $r = r_0$. We restrict ourselves by the simple asymptotics

$$V \approx a(r - r_0)^p, \qquad U \approx b(r - r_0)^q \quad (14)$$

(more general discussion of the behavior of the metric functions near the regular horizon can be found in [10]). As $r = r_0$ corresponds to the horizon, we must have $q > 0$, $p > 0$, as usual.

It follows from the finiteness of $\bar{E}$ and $\bar{F}$ on the horizon that $p \geq 1$. Then, E is the only potentially diverging term in the vicinity of the horizon:

$$E = -\frac{aq(r - r_0)^{p-2}}{4}(p + q - 2) + O((r - r_0)^{p-1}). \quad (15)$$

If $p = 1$, the regularity of the spacetime selects the only value $q = 1$. If $p \geq 2$, the geometry is regular on the horizon for any q.

Then, by direct substitution, we obtain that near the horizon

$$Z \sim (q-p)(r-r_0)^{p-1}, \qquad \tilde{Z} \sim (q-p)(r-r_0)^{p-1-q}, \tag{16}$$

$$Q \approx \frac{(q-p)}{4\pi r_0(r-r_0)} + Q_0, \tag{17}$$

where Q_0 is a constant. If $p > 1$, we obtain at once from (16) that $Z \to 0$ on the horizon. If $p = 1 = q$, the leading terms of the order unity mutually cancel in (16), so that the main contribution has the order $r - r_0$ and, again, $Z \to 0$ on the horizon.

In what follows, we also need the proper time of motion (for timelike geodesics) or affine parameter (for lightlike ones). In both cases we will denote it as τ. Then, from the equations of motions and conservation of energy we have

$$\tau = \int \frac{dr}{\sqrt{Y}}, \qquad Y = V\left(\frac{E^2}{U} - \frac{L^2}{r^2} + \delta\right), \tag{18}$$

where $L = u_\phi$ is the angular momentum, $\delta = -1$ for timelike geodesics, and $\delta = 0$ for lightlike ones.

In our case the time needed to reach the horizon diverges as

$$\tau \sim (r-r_0)^c, \qquad c = \frac{2+q-p}{2}, \qquad p-q > 2, \tag{19}$$

or

$$\tau \sim -\ln(r-r_0), \qquad p-q = 2, \tag{20}$$

and is finite if $p - q < 2$.

We want to examine under what conditions (1) the horizon is regular, (2) its nature is elucidated. As far as point (2) is concerned, we distinguish three cases:

(i) $Z \to 0$, $\tilde{Z} \to 0$ (by definition, "usual"), (ii) $Z \to 0$, $\tilde{Z} \to \text{const} \neq 0$ ("naked"), (iii) $Z \to 0$, $Z \to \infty$ ("truly naked").

According to p. (1), we want to eliminate leading and subleading divergencies in (15). If $p < 2$, we must choose $p + q = 2$. For the subleading divergencies to be absent, we must also choose $p = 1$ if, as usual, only integer power degrees are allowed (in some special cases this is not necessary if the coefficient at the term of the order $p - 1$ also vanishes but this depends strongly on the details of a system and we do not discuss this case), so that $q = 1$. If, for generality, not-integer p is also allowed, one may take $1 < p < 2$ and $q = 2 - p$. Then, $\tilde{Z} \sim (r - r_0)^{2p-3}$. In all other cases we assume that $p \geq 2$ (cases (5)–(10) below). By definition, $p = 2$ represents the extremal case and $p > 2$ corresponds to the ultraextremal one.

Then, the set of possibilities can be collected in Table I.

The Schwarzschild and Reissner-Nordström black holes belong to class (1) with $Q = 0$, whereas the examples considered in [1,2] fall into the same class with $Q \neq 0$. Case (6) includes, for example, the Reissner-Nordström–de Sitter ultracold horizon [11]. In cases (2)–(5) and (7)–(10) the quantity Q is infinite on the horizon but, nonetheless, the horizon is regular. Moreover, in cases (4), (9), and (10) the horizon is usual in spite of divergencies in Q. Even if $\psi' \sim Q$ is infinite on the horizon, in the product $\psi' V$ the second factor overcomes divergencies in all cases, so that $Z = 0$ and $p_r + \rho = 0$ on any regular spherically symmetrical horizons—not only nonextremal but also extremal and ultraextremal ones. For comparison, it is instructive to mention that for the distorted extremal TNBHs the analog of the latter equality does not in general hold that violates the horizon structure of the Enistein tensor [5] typical of the usual (not naked) black holes [12]. In cases (2), (5), and (7) the energy density $\bar{\rho}$ measured by a free-falling observer is infinite but the horizon is regular (naked or truly naked).

It is worth noting that the parameter τ needed to reach the horizon is finite except case (10). This can occur for usual horizons only. Then, the horizon is at infinite proper distance and τ is also infinite. In this sense, this is null infinity rather than the horizon. If only integers p and q are considered (as takes place usually), cases (2)–(4), (7), and (9) are absent.

TABLE I. Types of horizons with finite area.

		Type of horizon	Q	τ
1	$p = q = 1$	usual ($Q = 0$) or naked ($Q \neq 0$)	finite	finite
2	$1 < p < \frac{3}{2}$	truly naked	infinite	finite
3	$p = \frac{3}{2}$	naked	infinite	finite
4	$\frac{3}{2} < p < 2$	usual	infinite	finite
5	$p < q$	truly naked	infinite	finite
6	$p = q \geq 2$	usual ($Q = 0$) or naked ($Q \neq 0$)	finite	finite
7	$q < p < q + 1$	truly naked	infinite	finite
8	$p = q + 1$	naked	infinite	finite
9	$q + 1 < p < q + 2$	usual	infinite	finite
10	$p \geq q + 2$	usual	infinite	infinite

C. Behavior of tidal forces and exotic matter

The equality $p_r + \rho = 0$ which has to be satisfied on the horizon means that the null energy condition (NEC) is satisfied on the verge. In recent years, there is much interest in systems for which NEC is violated ($p_r + \rho < 0$) since, on one hand, such a kind of matter leads to acceleration of the Universe [13,14] and, on the other hand, it is a necessary ingredient for the existence of wormholes [15,16]. In the present context, it follows from (8) that the validity or violation of NEC are determined by the same quantity Z that determines also the behavior of tidal forces and

$$p_r + \rho \sim (q - p)(r - r_0)^{p-1}. \tag{21}$$

Thus, if $q < p$, there is a region adjacent to the horizon inside which the matter is exotic. According to Table I, TNBH can occur both with the exotic $p_r + \rho < 0$, case (7) and normal ($p_r + \rho > 0$, case (5) matter in the vicinity of the horizon in the external region.

III. INFINITE AREA OF HORIZON

It was tacitly assumed in the above consideration that, as usual, the horizon radius r_0 is finite. Meanwhile, there are cases when this condition is violated and the asymptotics of metric functions has the form

$$U \sim r^q, \qquad V \sim r^p, \tag{22}$$

where $q < 0$ and p are not necessarily integers, $r \to \infty$. For example, it occurs for some kinds of wormholes supported by phantom matter, with $q = -1$ and $p = 0$ [17] or black holes in Brans-Dicke theory (see below). In doing so,

$$Z \sim \bar{F} \sim r^{p-2} \sim \bar{E} \sim E, \tag{23}$$

$$F \sim r^{p-2}, \qquad p \geq 0 \quad \text{or} \quad r^{-2}, \qquad p < 0, \tag{24}$$

$$\tilde{Z} \sim r^{-2c}, \tag{25}$$

where c is defined in (19).

The system is regular if $p \leq 2$, then the proper distance to such a "horizon" l is infinite. Repeating our analysis we obtain the following list of possible cases (see Table II).

The fact that Z does not vanish on the horizon in the case $p = 2$ is due to an infinite area because of which dependence of Z on r changes as compared to the case of finite r_0 considered in the previous section. In case (3) a naked horizon is combined with an infinite proper time. Then, an observer never reaches it and the horizon reveals itself as null infinity rather than a horizon in its strict sense.

Now, for illustration, we remind here two examples of exact solutions in Brans-Dicke theory [3,4,18–20]. As the theory differs from general relativity, Eqs. (3) and (4) are no longer valid since they rely on Einstein equations but the rest of the formulas retain their meaning,

TABLE II. Types of horizons with infinite area.

		Z	$\tilde{Z}$	Type of horizon	τ
1	$p = 2$	finite	infinite	truly naked	finite
2	$2 - \lvert q\rvert < p < 2$	0	infinite	truly naked	finite
3	$p = 2 - \lvert q\rvert$	0	finite	naked	infinite
4	$p < 2 - \lvert q\rvert$	0	0	usual	infinite

(1)
$$ds^2 = -\left(1 - \frac{\rho_+}{\rho}\right)^{Q-\chi} dt^2 + \left(1 - \frac{\rho_+}{\rho}\right)^{-Q} d\rho^2 + \rho^2\left(1 - \frac{\rho_+}{\rho}\right)^{1-Q}(d\theta^2 + \sin^2\theta d\phi^2). \tag{26}$$

Let $\rho \to \rho_+$, then $l \to \infty$, $r \sim \rho_+(1 - \frac{\rho_+}{\rho})^{(1-Q)/2}$. The horizon exists, if $Q > \chi$. If $Q > 1$, the area of the horizon at $\rho = \rho_+$ is infinite. Regularity demands that $Q \geq 2$. Now $|q| = \frac{2(Q-\chi)}{Q-1}$, $p = \frac{2}{Q-1}$. Then, according to the table, we have a truly naked black hole ($Q \geq 2$, $\chi < 2$), naked ($Q > 2$, $\chi = 2$), or usual ($Q > \chi > 2$),

(2)
$$ds^2 = e^{-su}\left[-e^{-2bu}dt^2 + \frac{e^{2bu}}{u^2}\left(\frac{du^2}{u^2} + d\omega^2\right)\right], \quad s \neq 0. \tag{27}$$

This case combines a powerlike and exponential asymptotics, so the analysis is slightly modified. Spatial infinity corresponds to $u = 0$. Regular horizon exists at $u \to +\infty$ if

$$b > 0, \qquad -2b < s < 2b. \tag{28}$$

As the quantities b, s are related to the Brans-Dicke parameter ω according to $s^2(\omega + \frac{3}{2}) = -2b^2$ (see the aforementioned references for details), it is implied that $s \neq 0$. It was noticed in [3,4] that $\tau \sim \int \frac{du}{u^2}\exp(-su)$ may be infinite depending on the parameter s. We want to add to this observation the relationship between Z and $\tilde{Z}$. Then,

$$Z \sim \exp[u(s - 2b)] \to 0, \qquad \tilde{Z} \sim \exp(2su), \tag{29}$$

and we obtain the following set of variants (see Table III).

Thus, examples considered in [3,4] blend with our general scheme. It was pointed out in [3,4] that in corresponding black hole solutions infinite tidal forces appear only with the combination with finite proper time. It is seen

TABLE III. Types of horizons in Brans-Dicke theory (case (2)).

		Z	$\tilde{Z}$	Type of horizon	τ
1	$s < 0$	0	0	usual	infinite
2	$s > 0$	0	infinite	truly naked	finite

from Tables I, II, and III that such a combination persists in a general case of black holes with a finite or infinite horizon area and the behavior of metric functions described by (14) or (22).

IV. DISTORTED HORIZONS

The generic static metric may be written in the Gauss normal coordinates as

$$ds^2 = -N^2dt^2 + dn^2 + \gamma_{ab}dx^a dx^b, \qquad (30)$$

where $x^1 = n$, $a, b = 2, 3$. The horizon corresponds to $N = 0$.

Let an observer move in the n direction only. We attach the tetrad to him that includes the vector of the four-velocity u^μ, the vector e^μ lying in the $t - n$ submanifold and orthogonal to it. For a static observer, $u^\mu = N^{-1}(1, 0, 0, 0)$, $e^\mu = (0, 1, 0, 0)$. If an observer moves along the geodesics with the energy ε per unit mass, the local Lorentz boost reads

$$\bar{u}^\mu = u^\mu \cosh\alpha - e^\mu \sinh\alpha, \qquad (31)$$

$$\bar{e}^\mu = e^\mu \cosh\alpha - u^\mu \sinh\alpha \qquad (32)$$

where it follows from the conservation law that $\cosh\alpha = \frac{\varepsilon}{N}$ and it is chosen $\alpha > 0$. Then, the transformation of the curvature components under the local boosts to the orthonormal frame of an infalling observer ($u^1 < 0$) read

$$\hat{R}_{1'a'1'b'} = \hat{R}_{1a1b} + \sinh^2\alpha Z_{ab}, \qquad (33)$$

$$\hat{R}_{a'b'c'd'} = \hat{R}_{abcd}, \qquad (34)$$

$$\hat{R}_{1'a'b'c'} = \cosh\alpha \hat{R}_{1abc}, \qquad (35)$$

$$\hat{R}_{0'a'b'c'} = -\sinh\alpha \hat{R}_{1abc}, \qquad (36)$$

$$\hat{R}_{0'a'1'b'} = -\cosh\alpha \sinh\alpha Z_{ab}, \qquad (37)$$

$$\hat{R}_{0'1'a'b'} = 0, \qquad (38)$$

$$\hat{R}_{0'1'1'a'} = \hat{R}_{010a} \sinh\alpha, \qquad (39)$$

$$\hat{R}_{0'1'0'1'} = \hat{R}_{0101}, \qquad (40)$$

$$\hat{R}_{0'a'0'b'} = \hat{R}_{0a0b} + \sinh^2\alpha Z_{ab} \qquad (41)$$

$$\hat{R}_{0'1'0'a'} = \hat{R}_{010a} \cosh\alpha, \qquad (42)$$

$$\bar{Z}_{ab} = Z_{ab}(2\cosh^2\alpha - 1), \qquad (43)$$

where the combination

$$Z_{ab} = R_{\mu a \nu b}(u^\mu u^\nu + e^\mu e^\nu) \qquad (44)$$

and similarly for $\bar{Z}_{ab}$. One can check that (33)–(44) agree with formulas (9), (10), and (12) for the spherically symmetric case.

Using explicit formulas for the curvature tensor (see, e.g., the collection of useful formulas in [12], extended slightly in [5]) we obtain that

$$\hat{R}_{1abc} = K_{ac;b} - K_{ab;c}, \qquad (45)$$

$$\hat{R}_{010a} = \frac{R_{010a}}{N^2} = \frac{\partial_n N_{;a} + K_a^b N_{;b}}{N}, \qquad (46)$$

$$Z_{ab} = \frac{N_{;a;b}}{N} - \frac{K_{ab}N'}{N} + \frac{\partial K_{ab}}{\partial n} + (K^2)_{ab}. \qquad (47)$$

Here $(\ldots)_{;a}$ denotes a covariant derivative with respect to the two-dimensional metric γ_{ab}, the tensor of the extrinsic curvature $K_{ab} = -\frac{1}{2}\frac{\partial \gamma_{ab}}{\partial n}$, $(K^2)_{ab} = K_{ac}K_b^c$. As an observer approaches the horizon, $\cosh\alpha \to \infty$ but, typically, $Z_{ab} \to 0$ and the behavior of the product in (43) is not obvious in advance.

Let us denote $Z \equiv \frac{1}{2}Z_{ab}\gamma^{ab}$. It follows from (47) that

$$2Z = \frac{\Delta_2 N}{N} + K' - K\frac{N'}{N} - SpK^2, \qquad K = K_a^a, \qquad SpK^2 = K_{ab}K^{ab}. \qquad (48)$$

Further, one can check, using explicit formulas for the curvature tensor [12] that the combination (48) reduces to $G_n^n - G_0^0 = 8\pi(T_n^n - T_0^0) \equiv 8\pi(\rho + p_n)$ where $\rho = T_{\mu\nu}u^\mu u^\nu$ is the energy density measured by a static observer, $p_n = T_{\mu\nu}e^\mu e^\nu$ is the corresponding pressure in the n direction. Then, we obtain from (31) and (32) that

$$\bar{\rho} = T_{\mu\nu}\bar{u}^\mu \bar{u}^\nu = \frac{\bar{Z} + Z}{8\pi} - p_n, \qquad (49)$$

which is the direct analog of Eq. (13) (with $L = 0$ for radial motion) which applies now to observers moving along geodesics in the n direction in the generic spacetime (30). The pressure transforms in a similar way, so that

$$\bar{p}_n = \frac{\bar{Z} + Z}{8\pi} - \rho, \qquad \bar{p}_n = T_{\mu\nu}\bar{e}^\mu \bar{e}^\nu \qquad (50)$$

that is equivalent to $\tilde{Z} = Z(2\frac{\varepsilon^2}{U} - 1)$ in accordance with (12).

It is clearly seen from (49) that in the generic case of distorted horizons $\bar{\rho}$ (energy density) and $\bar{Z}$ (tidal forces) are either both infinite or both finite in a free-falling frame. (The same statement concerns also the relationship between the pressure $\bar{p}_n$ and $\bar{Z}$.) This applies to nonextremal, extremal, or ultraextremal horizons and enlarges on the observation made in [5] due to explicitly examining different asymptotical behavior of the metric near the horizon. In the spherically symmetric case we return to (13). Apart from this, one should take into account the behavior of components (35), (36), (39), and (42) which were absent for spherically symmetrical metrics.

Below we discuss different types of horizons separately.

A. Nonextremal case

It follows that the finiteness of the Kretschmann scalar on the horizon requires that the relevant metric functions have the asymptotic expansions [12]

$$N = \kappa_H n + \frac{\kappa_2(x^a)}{3!} n^3 + \frac{\kappa_3(x^a)}{4!} n^4 + O(n^5), \tag{51}$$

$$\gamma_{ab} = [\gamma_H]_{ab}(x^a) + \frac{[\gamma_2]_{ab} n^2}{2!} + \frac{[\gamma_3]_{ab} n^3}{3!} + O(n^4), \tag{52}$$

$$\begin{aligned} K_{ab} &= K^{(1)}_{ab} n + \frac{K^{(2)}_{ab}}{2!} n^2 + \frac{K^{(3)}_{ab}}{3!} n^2 + O(n^4), \\ K^{(1)}_{ab} &= -\frac{[\gamma_2]_{ab}}{2}, \qquad K^{(2)}_{ab} = -\frac{[\gamma_3]_{ab}}{2}, \end{aligned} \tag{53}$$

n is the proper distance from the horizon, the constant κ_H has the meaning of the surface gravity. Then, one obtains that near the horizon

$$Z_{ab} = \frac{K^{(2)}_{ab}}{2} n + O(n^2), \tag{54}$$

so

$$\bar{Z}_{ab} = \frac{\varepsilon^2}{\kappa_H^2} K^{(2)}_{ab} n^{-1} + Y_{ab}, \tag{55}$$

where $Y_{ab} = \text{const}$. It follows from (35) and (46) that $\hat{R}_{1abc} \sim \hat{R}_{010a} \sim n$. As $\cosh\alpha \sim \sinh\alpha \sim N^{-1} \sim n^{-1}$, we obtain that in the boosted frame the components $\hat{R}_{1'a'b'c'}$, $\hat{R}_{0'a'b'c'}$, $\hat{R}_{0'1'1'a'}$, and $\hat{R}_{0'1'0'a'}$ are finite and in general nonvanishing. The component $\hat{R}_{0'a'1'b'}$ behaves like $\bar{Z}_{ab}$ (the latter applies to other types of the horizons as is seen from (37) and (43)).

In the spherically symmetric case $r - r_0 \sim n^2$, so that the expansion (53) contains only odd powers, $K^{(2)}_{ab} = 0$ and $\bar{Z}_{ab}$ is finite. Thus, spherically symmetric nonextremal TNBHs do not exist in accordance with Table I, line "$p = q = 1$." However, simply naked black holes are possible in accordance with [1,2].

B. Ultraextremal case

Now we consider the metric which reads [5]

$$N = \frac{A_1(x^a)}{n^m} + \frac{A_2(x^a)}{n^{m+1}} + O(n^{-m-2}), \tag{56}$$

$$\gamma_{ab} = \gamma^{(0)}_{ab} + \frac{\gamma_{(1)ab}}{n^s} + O(n^{-s-1}), \qquad s > 0, \qquad m > 0, \tag{57}$$

$$K_{ab} = \frac{s\gamma_{(1)ab}}{2n^{s+1}} + O(n^{-s-2}), \tag{58}$$

the horizon is at infinite proper distance, and the Kretschmann scalar is finite on the horizon. In the spherically symmetrical case it reduces to (14) with

$$p = 2 + \frac{2}{s} > 2, \qquad q = \frac{m}{s}, \tag{59}$$

$$s = \frac{2}{p-2}, \qquad m = \frac{2q}{p-2}, \qquad p > 2. \tag{60}$$

Then, previous consideration applies—see Table I.

In the distorted case one finds that on the horizon

$$Z_{ab} = \frac{A_{1a;b}}{A_1}. \tag{61}$$

Then, if $(A_1)_{;a} \neq 0$,

$$\bar{Z}_{ab} \sim \frac{Z_{ab}}{N^2} \sim n^{2m} \to \infty. \tag{62}$$

The crucial point is that this contribution comes from the terms which were absent in the spherically symmetrical case but dominate now. In doing so, $\hat{R}_{1abc} \sim n^{-s-1}$, $\hat{R}_{010a} \sim n^{-1}$, $\hat{R}_{0'1'0'a'} \sim n^{m-1} \sim \hat{R}_{0'1'1'a'}$, $\hat{R}_{1'a'b'c'}$, $\hat{R}_{0'a'b'c'} \sim n^{m-s-1}$. All such black holes are truly naked in agreement with [5]. But even if $A_1 = \text{const}$, this does not guarantee the absence of the TNBH. Indeed, in this case we must take into account first corrections. Let A_1 be a constant but A_2 not. The first term in right-hand side (r.h.s.) of (47) dominates and, as a result,

$$Z_{ab} \sim n^{-1}. \tag{63}$$

Then,

$$\bar{Z}_{ab} \sim \frac{Z_{ab}}{N^2} \sim n^{2m-1}. \tag{64}$$

If $m > \frac{1}{2}$ tidal forces diverge in the boosted frame.

Let us consider now the behavior of the rest of the components. We have $\cosh\alpha \sim \sinh\alpha \sim n^m$, $\hat{R}_{1abc} \sim n^{-s-1}$, $\hat{R}_{010a} \sim n^{-2}$. As a result, $\hat{R}_{0'1'0'a'} \sim \hat{R}_{0'1'1'a'} \sim n^{m-2}$, $\hat{R}_{1'a'b'c'}$, $\hat{R}_{0'a'b'c'} \sim n^{m-s-1}$ and diverge if $m > s + 1$. This criterion is more tight than for tidal transverse forces. Therefore, it is the behavior of $\bar{Z}_{ab}$ that forces us to include a black hole into the class of TNBHs. We have the usual ($m < \frac{1}{2}$), naked ($m = \frac{1}{2}$), or truly naked ($m > \frac{1}{2}$) black hole.

C. Extremal case, $p = 2$

Consider another typical case that corresponds to the finite Kretschmann scalar on the horizon [5]:

$$N = B_1(x^a)\exp\left(-\frac{n}{n_0}\right) + B_2(x^a)\exp\left(-\frac{2n}{n_0}\right) + B_3(x^a)\exp\left(-\frac{3n}{n_0}\right) + \ldots, \quad (65)$$

$n_0 > 0$ is a constant,

$$\gamma_{ab} = \gamma_{ab}^{(0)} + \gamma_{(1)ab}\exp\left(-\frac{n}{n_0}\right) + O\left(\exp\left(-\frac{2n}{n_0}\right)\right), \quad (66)$$

$n \to \infty$.

Then, near the horizon

$$Z_{ab} = \frac{1}{2}\lim_{n\to\infty}\frac{N_{;a;b}}{N}, \qquad \bar{Z}_{ab} \sim \frac{Z_{ab}}{N^2}. \quad (67)$$

If (a) $(B_1)_{;a} \neq 0$, it turns out that $Z_{ab} \neq 0$ is finite and $\bar{Z}_{ab}$ is infinite (TNBH) and diverges like N^{-2}. If (b) $(B_1)_{;a} = 0$, $(B_2)_{;a} \neq 0$, $Z_{ab} = 0$ but $\bar{Z}_{ab}$ is still infinite (TNBH) and diverges like N^{-1}. If (c) $(B_1)_{;a} = 0 = (B_2)_{;a}$, $(B_3)_{;a} \neq 0$, $Z_{ab} = 0$, $\bar{Z}_{ab}$ is finite (naked black hole). If (d) $(B_1)_{;a} = (B_2)_{;a} = (B_3)_{;a} = 0$, $(B_4)_{;a} \neq 0$, $Z_{ab} = 0 = \bar{Z}_{ab}$ (usual).

As far as the other curvature components are concerned, $\hat{R}_{1abc} \sim \exp(-\frac{n}{n_0}) \sim N$, so that $\hat{R}_{1'a'b'c'}$, $\hat{R}_{0'a'b'c'}$ are finite due to the compensating factor $\cosh\alpha$ or $\sinh\alpha \sim N^{-1}$ in (35) and (36). However, the component $\hat{R}_{010a}$ is now nonvanishing in general, according to (46), so it follows from (39) and (42) that in the boosted frame the components $\hat{R}_{0'1'0'a'}$ and $\hat{R}_{0'1'1'a'}$ diverge like N^{-1}. It is weaker than for $\bar{Z}_{ab}$ in case (a) and is the same in case (b). We see again, that the behavior of tidal forces in the transverse direction is sufficient to conclude that the object belongs to the class of TNBHs.

V. SUMMARY AND DISCUSSION

In practice, the regularity of the energy encountered by a free-falling observer is often considered as a criterion of the regularity of spacetime at the horizon. In particular, the typical test exploited for examining the existence (or nonexistence) of quantum-corrected extremal or ultraextremal black holes consists of determining whether or not this quantity is finite [21]. Meanwhile, we saw that, actually, there are three different criteria: (i) the finiteness of the Kretschmann scalar which is a standard condition of the regularity of the geometry, (ii) the finiteness of the energy density, and (iii) the finiteness of separate curvature components in a given frame (in particular, the finiteness of tidal forces). We saw that the conditions (ii) and (iii) are equivalent to each other but, in general, they are not equivalent to (i). More precisely, if we are interested in the spacetimes regular in the sense (i) only, it entails the validity of (ii) and (iii) for a static observer. However, for a free-falling observer (ii) and (iii) may be violated without violation of (i).

The corresponding object called "truly naked black holes" is the ultimate extension of "naked black holes" of Refs. [1,2] for which tidal forces experienced by a free-falling observer are enhanced with respect to the static frame but remain finite. This extension became possible due to consideration or spherically symmetrical ultraextremal and extremal black holes or distorted horizons. The class of these objects includes also examples with the infinite area of a horizon found earlier for particular cases in [3,4]. In the present article we also shown that TNBHs may be spherically symmetrical, not only distorted [5]. As far as the structure of the Riemann tensor is concerned, now divergencies in the free-falling frame are not due to Weyl scalars (as it was for distorted TNBHs considered in [5]) but entirely due to the Ricci tensor (or corresponding energy density $\bar{\rho}$ and pressure $\bar{p}_n$). In principle, in the most general case divergencies may occur in both types of quantities. We also found which components are enhanced to infinity and which remain finite. In particular, for distorted TNBHs some new diverging components appear that have no analogue in the spherically symmetrical case.

From a more general viewpoint, the objects discussed in our paper are intimately connected with so-called nonscalar polynomial curvature singularities [6] (Chap. 8), [7] (Sec. 4.1, 4.2). They appeared when there are no diverging scalar fields but components of the curvature tensor in some frames behave badly. This happens if local Lorentz boosts from one frame to another also behave badly. In the present context, a static observer which resides near the horizon, becomes badly determined in the near-horizon limit. The force which is needed to support him in the rest, grows unbound and, roughly speaking, a timelike observer tends to a lightlike one. Correspondingly, the local Lorentz boost from a static frame to a free-falling one, becomes ill-defined, its angle parameter $\cosh\alpha$ diverges in the horizon limit. Thus, black hole physics supplies us with the mechanism in which the singularities under discussion are generated in a natural way. However, by itself the presence of a horizon does not lead to such singularities. It depends also on the rate under which the boost parameter diverges and details of the asymptotic behavior of the curvature tensor near the horizon. Depending on these properties, one obtains usual black holes (like the Schwarzschild or the Reissner-Norström ones) where there is no enhancement of the curvature components at all, their enhancement from zero to finite values (like in naked black holes of [1,2]) or truly naked black holes considered in our paper. Only in the third case can one speak about singularities in the above sense.

The existence of TNBH configurations discussed in the present article as well as in the previous one [5] points to some potential rooms in scenarios of gravitational collapse which need further consideration. It also hints that the cosmic censorship should be somehow reformulated to take into account these subtleties. It was pointed out in

[1] the existence of the naked black hole may affect the issue of information loss and black hole entropy since large tidal forces disturb significantly the matter falling into a black hole. The more so, this factor becomes important in the case of TNBHs when tidal forces are not simply large but infinite on the horizon.

We examined the nonextremal, extremal, and ultraextremal types of a horizon and found that the situation when tidal forces in the transverse direction are finite but other curvature components diverge is impossible. Therefore, actually it is behavior of tidal transverse forces that enables us to classify an object as a TNBH. We also demonstrated that in the spherically symmetrical case such a black hole should be extremal or ultraextremal. In the latter case the null energy condition may be violated in some vicinity of a horizon in the outward direction.

To summarize, there are three types of horizons in the aspect under discussion: usual (in both frames curvature components are finite, tidal forces are zero), naked (in both frames curvature components are finite, tidal forces are zero for a static observer but finite nonvanishing for a free-falling observer), and "truly naked" (some curvature components are infinite for a free-falling observer). In the context of the backreaction problem, all examples analyzed in [21] fall into the first class ($p = q = 2$ for the extremal subcase and $p = q = 3$ for the ultraextremal one) in the unperturbed case. However, according to line 6 of Table I, corresponding quantum-corrected metrics represent naked black holes, so in this sense quantum backreaction is able to change the type of an extremal or ultraextremal horizon.

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