Unconventional stringlike singularities in flat spacetime

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The conical singularity in flat spacetime is mostly known as a model of the cosmic string or the wedge disclination in solids. Another, equally important, function is to be a representative of quasiregular singularities. From all these points of view it seems interesting to find out whether there exist other similar singularities. To specify what "similar" means I introduce the notion of the stringlike singularity, which is, roughly speaking, an absolutely mild singularity concentrated on a curve or on a 2-surface $\mathcal{S}$ (depending on whether the space is three- or four-dimensional). A few such singularities are already known: the aforementioned conical singularity, its two Lorentzian versions, the "spinning string," the "screw dislocation," and Tod's spacetime. In all these spacetimes $\mathcal{S}$ is a straight line (or a plane) and one may wonder if this is an inherent property of the stringlike singularities. The aim of this paper is to construct stringlike singularities with less trivial $\mathcal{S}$. These include flat spacetimes in which $\mathcal{S}$ is a spiral, or even a loop. If such singularities exist in nature (in particular, as an approximation to gravitational field of strings), their cosmological and astrophysical manifestations must differ drastically from those of the conventional cosmic strings. Likewise, being realized as topological defects in crystals, such loops and spirals will probably also have rather unusual properties.

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I. INTRODUCTION

Consider the spacetime $\mathcal{M}_1$:

$$\begin{aligned} ds^2 &= -dt^2 + dz^2 + d\rho^2 + \rho^2 d\phi^2, \\ t, z &\in \mathbb{R}, \qquad \rho > 0, \qquad \phi = \phi + a \end{aligned} \tag{1}$$

(the last formula means that ϕ parametrizes the circle and is defined modulo a only). When $a = 2\pi$ the spacetime is merely the Minkowski space $\mathbb{L}^4$ in which the timelike plane $\rho = 0$ is deleted. However, if a takes any other nonzero value, $\mathcal{M}_1$ becomes a quite nontrivial spacetime often referred to as a "straight cosmic string." $\mathcal{M}_1$ is evidently singular and it is the singularities of this type that are discussed in this paper under the general name of "stringlike singularities" (the words "this type" need some elucidation of course and it will be given in due course). These singularities are important in many ways:

Cosmology. It is widely believed that the phase transitions in the early universe could result in formation of cosmic strings—infinitely long and at the same time very thin solutions of the combined system of Einstein, Higgs and gauge field equations (see, e.g., [1] for reviews and references). No exact solutions of that system are known, but Vilenkin [2] argued that in some approximation a universe with a static cylindrically-symmetric cosmic string, when in addition the metric is invariant under boosts along the string, is described by a spacetime U that coincides with $\mathcal{M}_1$ at sufficiently large ρ: $\rho > \rho_0$. In this sense $\mathcal{M}_1$ is an approximation—useful when ρ_0 can be neglected—of the singularity-free string spacetime U; the latter can be called a "thickening" of $\mathcal{M}_1$.

The singularities considered in this paper are defined by two properties: they (in the sense yet to be elaborated) are surfaces of codimension 2 and the spacetimes harboring them are regular (flat in most cases). So, they can be regarded as generalization of $\mathcal{M}_1$ to less symmetric case (each of them must of course lack some of the symmetries mentioned above, e.g., the "spinning string" $\mathcal{M}_4^-$ is not boost-invariant, while all curved singularities are not cylindrically-symmetric), and that is why they are called stringlike.

All timelike singularities discussed below admit thickening, but it may happen that for some particular type the thickenings cannot describe cosmic strings, because, say, the properties of the matter required (by the Einstein equations) for their existence are unrealistic. With this reservation, however, every new class of stringlike singularities must be of great importance to the cosmic string theory, being an approximation to the gravitational field of (closed, curved, accelerated, etc.) strings.

Solid state physics. The spacetime (1) bears much resemblance to what is called wedge disclination in condensed matter physics (see, e.g., [3]). Another stringlike singularity is similar to screw dislocation (see Sec. II C). Though the analogy between the spacetime singularities and the defects from the theory of elasticity is delusive sometimes (see Sec. III C and Appendix B), it can be stretched further—Puntigam and Soleng used the Volterra construction to classify the stringlike singularities [4]. This suggests that, vice versa, the properties of stringlike singularities might be important in condensed matter physics.

Relativity. What makes the stringlike singularities especially interesting is, in my view, their relation to the most fundamental problems of general relativity. Whether so mild singularities exist in nature is, in a sense, a more important question than, say, whether there is a singularity inside the Schwarzschild horizon. Indeed the singularities in discussion satisfy (see definition 2) the condition—let us call it *absolute mildness*—that there be a finite open covering of the spacetime M,

$$M = \bigcup_{i=1,\dots m} U_i \tag{2}$$

such that every U_i can be extended to a nonsingular spacetime M_i (that is, there are isometries ω_i mapping U_i to $\omega_i(U_i) \subset M_i$). In fact, the absolutely mild singularities are—at least in flat spacetimes—a subclass of quasiregular ones [5,6], the difference being essential only in exotic situations, when infinitely many quasiregular singularities accumulate in a spacetime (see Appendix B of [5]). The peculiarity of these types of singularities is that however close one of them is approached the geometry remains perfectly nice. This makes their presence in relativity ruinous for its predictive force: even if a spacetime is initially globally hyperbolic its evolution cannot be predicted from the Cauchy surface, because at any moment a singularity (say, a "branching" singularity discussed in Sec. III B) can form, nullifying all our predictions. At the same time it is absolutely unclear how to exclude such singularities from the theory. Unless forbidden by some *ad hoc* global postulate the same branching singularity would apparently present in any geometrical theory, regardless of the dynamical equations for the geometry, its relation to the matter source, or the properties of that source.

1. Remark. The seriousness of the problem is often underestimated. For example, in their pioneering paper on quasiregular singularities [5] Ellis and Schmidt speaking through Salviati say: "We know lots of examples of quasiregular singularities, all constructed by cutting and gluing together decent space-times; and because of this construction, we know that these examples are not physically relevant." The argument is emphatically untenable: *any* spacetime can be constructed by cutting and gluing together some other decent spacetimes and *any* of them can be constructed otherwise. The spacetimes with the singularities in discussion are absolutely no different in this respect from the others. Correspondingly, no reasons are seen to regard $\mathcal{M}_1$ and suchlike less physically relevant than any other spacetime.

To sum up, there are many reasons for studying stringlike singularities and, in particular, those occurring in flat spacetimes (supposedly they are the simplest). The first question that one may ask is: What form do they have? So far only a few such singularities have been considered in the literature and all of them (with a possible exception of the "branching disclination," discussed in Sec. III B) have very dull form: they are flat surfaces of codimension 2. In other words, they (or rather their thickenings) correspond to straight strings moving at constant velocities. The main aim of the present paper is to provide examples of flat spacetimes with *different* stringlike singularities including those corresponding to curved—and even closed—strings and strings moving with acceleration. This will be done in Sec. III after some general consideration in Sec. II, where I define stringlike singularities and (roughly) classify them.

II. GENERAL CONSIDERATION

A. Stringlike singularities

In trying to build a curved or otherwise unusual stringlike singularity, one immediately comes up against the problem of definition. It is customary, for example, to refer to the singularity in the spacetime [from now on the word *spacetime* stands for smooth connected (pseudo-) Riemannian manifold]

$$\begin{aligned} \mathcal{M}^+_{1'}: \quad & ds^2 = dz^2 + d\rho^2 + \rho^2 d\phi^2, \\ \rho > 0, \quad & \phi = \phi + a, \quad a \neq 0, 2\pi \end{aligned} \tag{3}$$

as to the "straight line" while the singularity in $\mathcal{M}_1$ is, in these terms, a "plane" or "a straight line at rest." But what *exactly* is meant by that? The metric, and hence the spacetime, cannot be extended to the z-axis, straight or not. But if the z-axis is missing,[1] then just what is straight or bent? In fact, this naive question has no good answer at present, being a particular case of a notoriously hard problem of assigning in a natural way a topology (never mind geometry) to singularities [8]. Fortunately, when the condition (2) is satisfied, it is possible to give at least a *tolerable* working definition to the relevant entity.

Consider to this end the set Γ of geodesics $\gamma(\tau)$: $[-1, 0) \to M$ which cannot be extended to the zero value of the affine parameter τ. Denote, further, by Γ_i the subset of Γ which consists of the geodesics lying, at least when $|\tau|$ is sufficiently small, in U_i [here U_i is a member of the covering (2)]. Below we are only interested in spacetimes and coverings such that

$$\Gamma = \bigcup_{i=1,\dots m} \Gamma_i \tag{4}$$

(this does not follow automatically from (2) as can be seen by example of Misner's space). Though the geodesics $\gamma \in \Gamma_i$ do not have the endpoints $\gamma(0)$, their images $\omega_i \circ \gamma$ in M_i do. We shall denote such endpoints by s with corresponding indices:

$$s_{\gamma,i} = \omega_i \circ \gamma(0).$$

Since γ may lie in more than one U_i, it may happen that two different points s, $s' \in \bigcup_i M_i$ are generated by the same geodesic. We shall write $s_1 \sim s_2$ in such cases:

$$s \sim s' \Leftrightarrow \exists i_1, i_2, \gamma: \quad s = s_{\gamma,i_1}, \qquad s' = s_{\gamma,i_2}.$$

It would be natural to identify s and s' and to associate the singularity with the quotient of $\mathcal{S}_\Gamma = \bigcup s_{\gamma,i}$ over $\sim$, but unfortunately in the general case $\sim$ is not an equivalence

[1] One can try to retain it in the spacetime by developing "distributional geometry," see [7] and references therein.

relation. Therefore, we introduce the equivalence relation $\eqsim$ as the transitive closure of $\sim$:

$$s_1 \eqsim s_2 \quad \Leftrightarrow \quad s_1 \sim s_{k_1} \sim s_{k_2} \ldots \sim s_{k_m} \sim s_2$$

and use $\eqsim$ in checking whether a candidate set represents the entire singularity.

2. Definition. Let conditions (2) and (4) hold in a spacetime M. Then a set $\mathcal{S} \subset \mathcal{S}_\Gamma$ is said to *represent* the singularity of M if for any $s \in \mathcal{S}_\Gamma$ there is $s' \in \mathcal{S}$ such that $s' \eqsim s$.

$\mathcal{S}$ is considered as subspace of $\bigcup M_i$ (not just a set of points) and correspondingly it can be straight, or curved, timelike or not, etc. So, the definition seems to capture the idea of a singularity being of a particular form.

3. Remark. The price to be paid is some arbitrariness. First, depending on the choice of U_i, the same singularity can be represented by different sets. Furthermore, one can argue that geometrically it would be more consistent to consider the singularity itself, defined, say, as $\mathcal{S}_\Gamma/\eqsim$, rather than a set representing it. On the other hand, in considering strings, i.e., thickenings of the singularities, $\mathcal{S}$ seems to be more adequate. The difference between the two objects is exemplified by the "spinning string," see Sec. II C. $\mathcal{S}$ in that case is a plane, while $\mathcal{S}_\Gamma/\eqsim$ is a *cylinder*.

Now we can at last delineate our subject more specifically.

4. Definition. A singularity is *stringlike* if it can be represented by a surface of codimension 2.

5. Notation. In what follows *three*-dimensional spacetimes with stringlike—represented by *curves* in this occasion—singularities are denoted $\mathcal{M}_{1'}$, $\mathcal{M}_{2'}$, etc. Some of them differ only in the signature in the sense that they are obtained by the same, explicitly prescribed, manipulations applied either to the Euclidean space $\mathbb{E}^3$ or to the Minkowski space $\mathbb{L}^3$ (by the "same" manipulations I mean that their verbal descriptions become the same after the words z-axis and t-axis are interchanged; hence the notation ϑ in the figures—it stands for "z or t"). To such spacetimes the same numbers will be given and they will be denoted by $\mathcal{M}^+_{k'}$ and $\mathcal{M}^-_{k'}$, correspondingly. The *four*-dimensional spacetimes will be denoted similarly but without primes: $\mathcal{M}_1$, $\mathcal{M}_2$, etc. And the correspondence rule is: given an $\mathcal{M}^{(\pm)}_{k'}$ one obtains its four-dimensional version (i.e., $\mathcal{M}^{(\pm)}_{k}$) by simply multiplying the former by the relevant axis. For example,

$$\mathcal{M}_1 = \mathcal{M}^-_{1'} \times \mathbb{E}^1 \quad \text{or} \quad \mathcal{M}^+_4 = \mathcal{M}^+_{4'} \times \mathbb{L}^1, \quad \text{etc.}$$

The first family of stringlike singularities was constructed by Ellis and Schmidt [5] who produced them from the sets of fixed points $\mathcal{F}$ of discrete isometries ζ acting on the Minkowski space (see Appendix A,) much as one obtains the usual two-dimensional cone by identifying the points on a plane related by a rotation by some fixed angle. The point is that the spacetime $(\mathbb{L}^4 - \mathcal{F})/\zeta$ is singular (the geodesics which in the Minkowski space terminated at $\mathcal{F}$ now have no endpoints) and its singularity (which is irremovable due to the nature of $\mathcal{F}$, cf. Sec. 5.8 in [8]) is represented by $\mathcal{F}$. Three ζ were considered in [5]—rotation, boost, and boost + rotation—to obtain in each case a stringlike singularity represented by a plane. So, it might have appeared that

(i) The problem of determining all the elementary stringlike singularities in flat space is essentially equivalent to finding all the discrete subgroups of the Lorentz group which have two-dimensional surfaces $\mathcal{F}$ as their sets of the fixed points [5], and

(ii) All such singularities are straight, i.e., $\mathcal{F}$ are planes [9].

In fact, however, neither is true. Counterexamples to (ii) are built in the next section (an obvious one is the double covering of $\mathbb{E}^3 - \mathcal{F}$, where $\mathcal{F}$ is an arbitrary curve), and that (i) is not the case is seen from the fact that even the spacetime (1) with $a > 1$ cannot be obtained in that manner (instead of $\mathbb{L}^4$ one could have started from a covering of $\mathbb{L}^4 - \mathcal{F}$ in this case, but the relevant isometries have no fixed points).

The reasons why the requirements to the isometries can be weakened will become evident from examples in Sec. II C (roughly speaking one can produce the desired singularities from the discontinuing set of an isometry rather from its set of the fixed points), to which we shall turn after introducing (or, rather, formalizing—it is well known and widely used) a more visual method of constructing spacetimes.

B. Cut-and-paste surgery

Given V is an open subset of a spacetime one can construct a new spacetime W in the following way. Pick a pair $p_{1,2}$ of different points in the boundary $\mathcal{B} \equiv \mathrm{Bd}V$ and let $O_{1,2}$ be disjoint neighborhoods of $p_{1,2}$. Either of the neighborhoods can be split into three disjoint sets

$$O_{j\vee} = O_j \cap V, \qquad \mathcal{B}_j = O_j \cap \mathcal{B},$$
$$O_{j\wedge} = O_j - B_j - O_{j\vee}, \qquad j = 1, 2$$

(see Fig. 1). Suppose now there is an isometry κ: $O_1 \to O_2$ such that

$$\kappa(O_{1\vee}) = O_{2\wedge}, \qquad \kappa(\mathcal{B}_1) = \kappa(\mathcal{B}_2), \qquad \kappa(O_{1\wedge}) = O_{2\vee}.$$

Then an equivalence relation $\approx$ can be defined

$$p \approx q \quad \Leftrightarrow \quad p = q, \text{ or } \kappa(q), \text{ or } \kappa^{-1}(q)$$

and the spacetime W constructed by identifying equivalent points:

$$W = (V \cup O_1 \cup O_2)/\approx = (V \cup \mathcal{B}_1 \cup \mathcal{B}_2)/\approx.$$

Pictorially speaking W is obtained by first attaching to V two parts of its boundary, $\mathcal{B}_1$ and $\mathcal{B}_2$, and by then gluing these parts together (remarkably, as long as $\mathcal{B}_1$ and $\mathcal{B}_2$ are

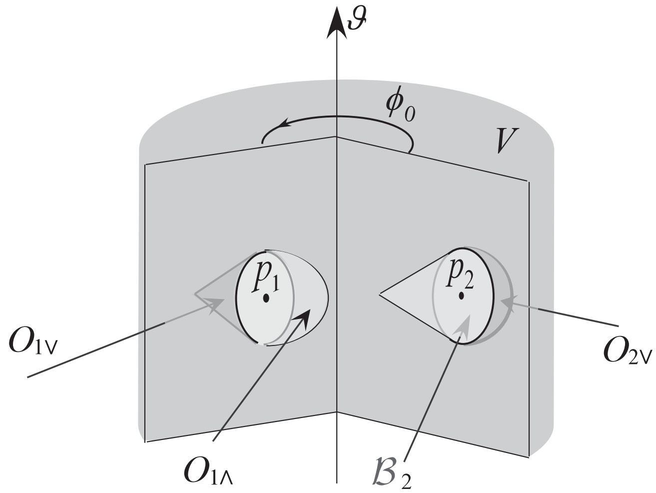

FIG. 1. The tear-shaped regions are O_1 and O_2. If O_1, O_2 are chosen otherwise (namely, to be the sectors $-\delta < \phi < \delta$ and $\phi_0 - \delta < \phi < \phi_0 + \delta$), then $\mathcal{B}_1$ and $\mathcal{B}_2$ are half-planes bounded by the ϑ-axis. Gluing them with an appropriate isometry one obtains all (but $\mathcal{M}_3$ and $\mathcal{M}_5$) singular spacetimes listed in this section and Appendix A.

kept fixed the choice of the points $p_{1,2}$ and their neighborhoods $O_{1,2}$ does not affect the result; they only are needed to guarantee the smoothness of W). The isometry κ has to be explicitly pointed out sometimes because there may be more than one way of gluing $\mathcal{B}_1$ to $\mathcal{B}_2$.

6. Example. Take V to be the region $0 < \hat{\phi} < a$ in

$$\hat{M}: \quad ds^2 = d\hat{z}^2 + d\hat{\rho}^2 + \hat{\rho}^2 d\hat{\phi}^2, \qquad \hat{z}, \hat{\phi} \in \mathbb{R}^1, \qquad \hat{\rho} > 0 \tag{5}$$

and glue its boundaries $\mathcal{B}_1$ and $\mathcal{B}_2$ (i.e., the surfaces $\hat{\phi} = 0$ and $\hat{\phi} = a$, respectively) with the translation κ: $\hat{\phi} \mapsto \hat{\phi} + a$ (in Sec. II C we shall glue the same surfaces with other isometries as well). The result is the conical space $\mathcal{M}^+_{1'}$, see (3).

7. Remark. Of course the spacetime W can be described as well in terms of quotient spaces—it is, for instance, a quotient of its universal covering $\tilde{W}$. Vice versa, given $\tilde{W}$ we can build W by cutting a fundamental region from $\tilde{W}$ and gluing appropriately its boundaries. So, it is just a matter of convenience, which language to use. In particular, the Ellis-Schmidt singularities, see Appendix A, are also easily constructed by cut-and-paste surgery.

Below we shall construct spacetimes much like in example 6. Denote by $\mathbb{M}^n$ the Minkowski or Euclidean n-dimensional space

$$\mathbb{M}^n = \mathbb{L}^n, \mathbb{E}^n \qquad n = 3, 4$$

(which exactly will be indicated explicitly, when important). Pick an $(n-1)$-dimensional simply connected surface $H \subset \mathbb{M}^n$ such that

(i) H is invariant under an isometry κ' (the isometry is understood to act in some neighborhood of H), and

(ii) $\mathcal{S} \equiv \overline{H} - H$ is an $(n-2)$-dimensional surface and $M \equiv \mathbb{M}^n - \mathcal{S}$ is non-simply connected with the fundamental group $\pi_1 = \mathbb{Z}$ (so, in the three-dimensional case H can be, for example, a half-plane or a disk but not an infinite cylinder; in example 6 H is the half-plane $\phi = \text{const}$). The universal covering of M will be denoted by $\hat{M}$ and the natural projection $\hat{M} \to M$ by π.

Take V to be $\mathbb{M}^n - \overline{H}$. The spacetime V is extendible and we shall consider it (not as a spacetime in itself, or a part of M, but) as a part of $\hat{M}$ and to indicate this the coordinates in V will be labeled with hats. The boundary of V in $\hat{M}$ is two disjoint copies of H, which we denote by $\mathcal{B}_1$ and $\mathcal{B}_2$.

Further, $\pi_1(M)$ is generated by a single element—the homotopy class of a curve ℓ which circles $\mathcal{F}$ once. Thus an isometry χ acting on $\hat{M}$ is defined by the conditions that for any $x \in \hat{M}$, first, $\pi(x) = \pi(\chi(x))$ and, second, there is a path α from x to $\chi(x)$ such that $\pi(\alpha)$ is a loop homotopic to ℓ (in example 6, χ is the translation $\hat{\phi} \mapsto \hat{\phi} + 2\pi$). Evidently, $\mathcal{B}_2 = \chi(\mathcal{B}_1)$ and we construct the desired spacetime $\mathcal{M}$ from $\overline{V}$ by gluing $\mathcal{B}_1$ and $\mathcal{B}_2$ with the isometry

$$\kappa = \kappa' \circ \chi$$

(gluing them with $\kappa = \chi$ we would get merely M). In other words, we scissor H from $\mathbb{M}^n$, move in a special way one bank of the cut with respect to the other (the motion is devised so as to keep the boundary of the cut, i.e., $\mathcal{S}$, at place) and glue the banks together again. The spacetime $\hat{M}$ was introduced in this procedure only for giving a rigorous sense to the notion of banks.

C. Examples

Let H be the half-plane $(\phi = 0, \rho > 0)$ in the three-dimensional Minkowski space. Then $\mathcal{S}$ is the z-axis, $\hat{M}$ is the spacetime

$$ds^2 = -d\hat{t}^2 + d\hat{\rho}^2 + \hat{\rho}^2 d\hat{\phi}^2, \qquad \hat{t}, \hat{\phi} \in \mathbb{R}^1, \qquad \hat{\rho} > 0,$$

and V is the region $0 < \hat{\phi} < 2\pi$ bounded by the half-planes $\mathcal{B}_1$ and $\mathcal{B}_2$, which are defined by the equations $\hat{\phi} = 0$ and $\hat{\phi} = 2\pi$, respectively. If one glued $\mathcal{B}_1$ to $\mathcal{B}_2$ with χ: $\hat{\phi} \mapsto \hat{\phi} + 2\pi$, one would just restore the initial Minkowski space. But if the same surfaces are glued with the isometry

$$\kappa: \quad \hat{\phi} \mapsto \hat{\phi} + 2\pi, \qquad \hat{t} \mapsto \hat{t} + \hat{t}_0$$

(i.e., $\kappa = \kappa' \circ \chi$, where κ' is the translation by $\hat{t}_0$ in the $\hat{t}$-direction), the result [10] is a singular spacetime $\mathcal{M}^-_{4'}$. Its four-dimensional counterpart $\mathcal{M}^-_4 = \mathbb{E}^1 \times \mathcal{M}^-_{4'}$ discovered in [11,12] is often called "the spinning string." Topologically $\mathcal{M}^-_4$ is equivalent to the straight string $\mathcal{M}_1$. They both are everywhere flat and their singularities are both represented by flat planes. Nevertheless the two spacetimes differ significantly in some respects. For one, in $\mathcal{M}_4^-$ chronology is violated. Another difference is that a tetrad parallel transported along ℓ returns rotated in $\mathcal{M}_1$ but not so in $\mathcal{M}_4^-$. As we discuss below, there is also a more local difference.

One also can repeat the procedure just described, starting this time from $\mathbb{E}^3$ instead of $\mathbb{L}^3$ and, correspondingly, shifting $\mathcal{B}_1$—before it is glued to $\mathcal{B}_2$—in the z-direction instead of the t-direction

$$\kappa\colon \quad \hat\phi \mapsto \hat\phi + 2\pi, \qquad \hat z \mapsto \hat z + \hat z_0. \tag{6}$$

In this case the result is the spacetimes $\mathcal{M}_{4'}^+$ and $\mathcal{M}_4^+$, called *screw dislocations* for their similarity to the corresponding distortion [13]. $\mathcal{M}_4^+$ differs both from $\mathcal{M}_1$ and $\mathcal{M}_4^-$ though its singularity is also represented by a plane.

Finally, one can start from $\mathbb{L}^4$ (in this case $\mathcal{B}_j$ are given by the same equations, but now they are three-dimensional half-spaces) and choose κ to be a combination of χ with a boost in the z-direction [14]. Thus obtained spacetime $\mathcal{M}_5$ has a number of curious properties. For example, it is not even stationary, even though its every simply connected region is static.

Sometimes stringlike singularities can be built by "superposing" elementary ones. Take, for example, V to be the sector $0 < \hat\phi < a$ in $\hat M$ and κ to be the superposition of translations: by a in the $\hat\phi$-direction, by $\hat t_0$ in the $\hat t$-direction, and by $\hat z_0$ in the $\hat z$-direction. The result is the spacetime [13,14]

$$\mathrm{d}s^2 = -(\mathrm{d}\hat t - a^{-1}\hat t_0 \mathrm{d}\hat\phi)^2 + (\mathrm{d}\hat z^2 - a^{-1}\hat z_0 \mathrm{d}\hat\phi)^2 + \mathrm{d}\hat\rho^2 + \hat\rho^2 \mathrm{d}\hat\phi^2, \qquad \hat\phi = \hat\phi + a, \tag{7}$$

which combines the properties of the three spacetimes.

D. The strength of the singularities

In discussing singular spacetimes it is often hard to decide whether a particular property should be considered as a characteristic of the singularity or of the "regular part" of the spacetime. For example, it seems natural to classify the stringlike singularities according to their holonomies [4,14]. On the other hand, example 8 below suggests that such a classification may be misleading. Fortunately, the simplicity of the spacetimes at hand, allows a quantity to be found which seemingly relates just to the singularity. The cost is that the corresponding classification is quite rough—the singularities are divided only into three categories. One of them contains the singularities from Appendix A and the other contains those from Sec. II C.

Let s_γ be a singular point, $\gamma(\tau)$ be a geodesic defining this point as was discussed above, and $\{\boldsymbol{e}_{(i)}\}$ be an orthonormal frame in $\gamma(-1)$. Now if a curve $\alpha(\xi)\colon [\xi_1, \xi_2] \to M$ starts from some point of γ,

$$\alpha(\xi_1) = \gamma(\tau_1),$$

it is possible to assign to it the "b-length" $B(\alpha)$. To this end one defines $\{\boldsymbol{e}_{(i)}\}(\xi)$ to be the frame in $\alpha(\xi)$ obtained by parallel transportation of $\{\boldsymbol{e}_{(i)}\}$, first, along γ to the point $\gamma(\tau_1)$ and then along α. Then $B(\alpha)$ (it is the length of α in the generalized affine parameter, see [8]) is defined as follows:

$$B(\alpha) = \int_{\xi_1}^{\xi_2} \Big[\sum_i \langle \partial_\xi, \boldsymbol{e}_{(i)} \rangle^2 \Big]^{1/2} \mathrm{d}\xi.$$

Now we can attach a number

$$\Delta(s_\gamma) \equiv \lim_{\tau\to 0} \inf_{\text{noncontractible } \alpha \text{ through } \gamma(\tau)} B(\alpha)$$

to every point s_γ. Of course the value of $\Delta(s_\gamma)$ for a given s_γ may depend on γ and $\{\boldsymbol{e}_{(i)}\}$, but not when

$$\Delta(s_\gamma) = 0. \tag{$*$}$$

The property ($*$) holds for *all* (equivalent) γ and all $\{\boldsymbol{e}_{(i)}\}$ if it holds for some.

The spacetimes in which ($*$) is true for all $s \in \mathcal{S}$ will be called *disclinations* after the spacetime $\mathcal{M}_1$, which is often called so (by analogy with the theory of elasticity). In fact, all spacetimes constructed in example 6 and Appendix A are disclinations. On the contrary, the spacetimes $\mathcal{M}_4^\pm$, $\mathcal{M}_5$ all are characterized by the opposite property: ($*$) holds in none of $s \in \mathcal{S}$. I shall use the word *dislocation*[2] as a common name for all such spacetimes.

Absolutely mild singularities are often referred to as "topological." However, when it concerns the disclinations such a name may be a bit misleading. What makes these singularities "true" (i.e., irremovable) is the purely *geometrical* requirement that the metric should be nondegenerate. In a hypothetical theory in which this requirement is relaxed[3] there would be no singularity at all except maybe a "coordinate singularity" like that on the horizon of the Schwarzschild black hole or in the origin of the polar coordinates. Indeed, the spacetime (3), for instance, can be extended to $\mathbb{R}^2 \times \mathbb{S}^1$ by simply letting ρ vary over the entire real axis. The only pathology is that $g = 0$ at $\rho = 0$.

Dislocations in this sense are stronger singularities. As is seen from the definition, $\mathcal{S}$ cannot be retained in the spacetime even at the cost of the metric degeneration (as long, that is, as only continuous metrics are allowed).

III. UNCONVENTIONAL SINGULARITIES

In this section a few stringlike singularities are constructed with rather unusual properties. To my knowledge none of them, except $\mathcal{M}_{10}^\pm$, have been considered in the literature.

[2]At variance with Puntigam and Soleng, who divided the spacetimes into disclinations and dislocations according to their *global* properties [4].

[3]Such a theory would differ significantly from general relativity.

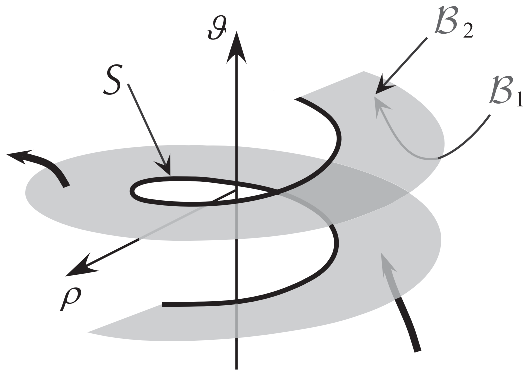

FIG. 2. The spacetime $\mathcal{M}^+_{6'}$. The thick directed line is continuous.

A. Curved, closed, and accelerating dislocations

In the Euclidean space $\mathbb{M}^3 = \mathbb{E}^3$ consider the surface H,

$$\phi = bz \quad \mod 2\pi, \qquad \rho > \rho_0 > 0, \qquad b \neq 0.$$

H is a half of a helicoid without the core, see Fig. 2, and is bounded by the spiral

$$\mathcal{S}: \quad \phi = bz \quad \mod 2\pi, \qquad \rho = \rho_0.$$

With these $\mathbb{M}^3$ and H let us carry out the procedure described in the end of Sec. II B. The boundary of $\mathbb{M}^3 - H$ in $\hat{M}$ consists of $\mathcal{B}_1$ and $\mathcal{B}_2 = \chi(\mathcal{B}_1)$ (two disjoint copies of H) and there is an obvious isometry

$$\kappa': \quad \hat{\phi} \mapsto \hat{\phi} + b\hat{z}_0, \qquad \hat{z} \mapsto \hat{z} + \hat{z}_0 \tag{8}$$

that maps $\mathcal{B}_2$ to itself. So, gluing it to $\mathcal{B}_1$ with $\kappa = \kappa' \circ \chi$ we obtain a spacetime $\mathcal{M}^+_{6'}$ with a singularity represented by the spiral $\mathcal{S}$. (In other words, we have made a cut in $\mathbb{E}^3$ along the helical surface, rotated the lower bank of the slit—let it be $\mathcal{B}_1$ for definiteness—counterclockwise shifting it at the same time upward so that $\mathcal{B}_1$ slides over $\mathcal{B}_2$, and pasted the banks together again into a single surface.) The singularity $\mathcal{S}$ satisfies the relation

$$\Delta(s) = \sqrt{(2\rho_0 \sin\tfrac{1}{2}b\hat{z}_0)^2 + \hat{z}_0^2} \quad \forall\ s \in \mathcal{S},$$

being thus a dislocation. I shall call it *spiral* (not to be confused with helical).

To realize the structure of the spiral dislocation it is instructive to depict $\mathcal{M}^+_{6'}$ in the coordinates z, ρ, ϕ as in Fig. 2. These coordinates are invalid, of course, on $\mathcal{B}_{1,2}$; that is why a smooth curve in $\mathcal{M}^+_{6'}$ looks discontinuous in the picture. It is easy to see that at $b\hat{z}_0 \neq 2\pi$ the geometry of the space outside the cylinder $\rho \leq \rho_0$ is exactly the same as in (the three-dimensional version of) the Gal'tsov-Letelier (GL) space[4] (7) with $a = b\hat{z}_0 - 2\pi$. The relation of the two spacetimes becomes even more evident when the spacetime is considered which is obtained exactly as $\mathcal{M}^+_{6'}$ but with the surface H',

$$H': \quad \phi = bz \quad \mod 2\pi, \qquad 0 < \rho < \rho_0,$$

taken instead of H. In contrast to H, H' is bounded by $\mathcal{S}$ *and* the z-axis. Thus the spacetime [see Fig. 3(a) ignoring for the moment the "ripples" on $\mathcal{S}$] has two singularities, of which $\mathcal{S}$ is spiral and $\mathcal{S}'$ (the former z-axis) is of the GL type. At $\rho > \rho_0$ the spacetime is just $\mathbb{E}^3$, so the spiral singularity *shields* the GL one. And in exactly the same sense two equal spiral singularities can shield each other. The spacetime of this type is built by taking H to be the central part of a helicoid, see Fig. 3(b).

The most striking feature of $\mathcal{M}^+_{6'}$ is of course the form of the singularity. From all the preceding examples it might seem that stringlike singularities in flat spacetime by some reason have to be straight. And now we see that this is not the case—they may well be curved. Note that $\mathcal{S}$ does not need to be a perfect spiral—instead of H we could take another surface as long as it is invariant under the isometry (8) and its boundary, exemplified by the undulate line in Fig. 3(a), will represent a singularity of exactly the same type. Moreover, $\mathcal{S}$ can even make a *loop*. Indeed, pick a closed curve $\mathcal{S} \subset \mathbb{E}^3$ bounding a surface H invariant under the rotation

$$\kappa': \quad \phi \mapsto \phi + \phi_0.$$

(H and $\mathcal{S}$ needn't be a surface of revolution and a circle, respectively, if $\phi_0 = 2\pi/m$, $m \in \mathbb{N}$). Proceeding as before (i.e., making an incision along $\overline{H}$ and gluing the banks together after rotating one of them by ϕ_0), we obtain a space $\mathcal{M}^+_{7'}$ with a *closed* stringlike singularity, see Fig. 4(a). $\mathcal{M}^+_{7'}$ can be viewed as a limit case of the spiral singularity corresponding to $b = \infty$. Another limit, $b = \rho_0 = 0$ is the space $\mathcal{M}^+_{8'}$ depicted in Fig. 4(b). It is a pure screw dislocation, but *curved*. This space is built exactly as

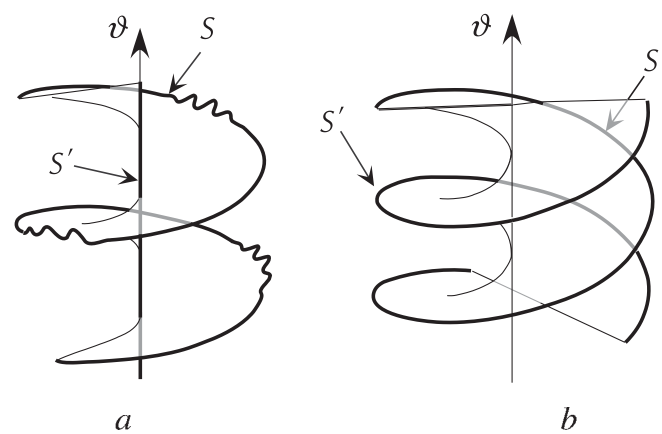

FIG. 3. (a) A spiral singularity shielding a GL one. (b) Two equal spiral singularities shielding each other.

[4] The difference in presentation yields an interesting byproduct: the Gal'tsov-Letelier space is not defined at $a = 0$, while $\mathcal{M}^+_{6'}$ is a nice spacetime for any $\hat{z}_0$

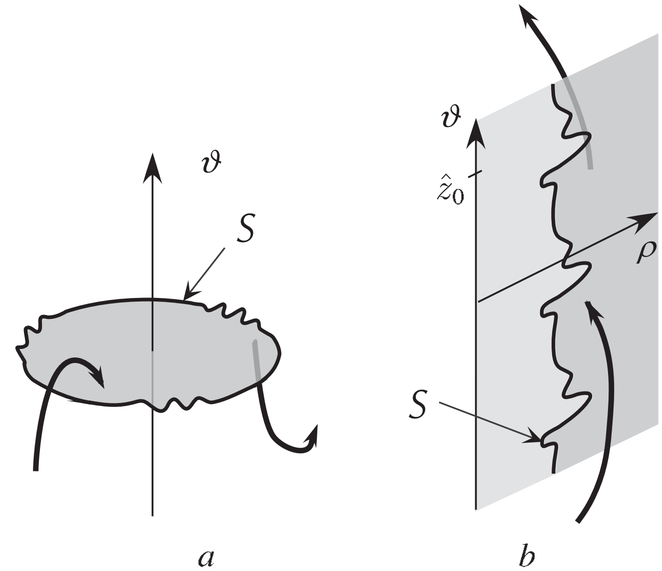

FIG. 4. (a) A loop singularity with $\phi_0 = \frac{2}{3}\pi$. (b) Curved screw dislocation.

$\mathcal{M}_5^+$ with the only difference that instead of the entire half-plane $\phi = 0$ one takes H to be the part of that half-plane lying above the graph of a periodic function $\rho(z)$,

$$\rho(z) \geq 0, \qquad \rho(z) = \rho(z + \hat{z}_0).$$

A different family of dislocations, $\mathcal{M}^-_{6',7',8'}$, is obtained when the surgery employed in constructing $\mathcal{M}^+_{6',7',8'}$ is applied to $\mathbb{L}^3$ instead of $\mathbb{E}^3$. Correspondingly, their four-dimensional versions $\mathcal{M}^-_{6,7,8}$ are obtained by interchanging the z- and t-axes in $\mathcal{M}^+_{6,7,8}$. Of these, especially interesting is $\mathcal{M}_6^-$, see Fig. 2 with $\vartheta = t$. At $\rho > \rho_0$ it is just a spinning string, but taken as a whole it has two important distinctions. First, $\mathcal{M}_6^-$ in a large range of its parameters b, $\hat{z}_0$, and ρ_0 is *causal*. And, second, the singularity there is represented by a straight line moving in quite a bizarre manner: it *circles around nothing*.

Yet another accelerated singularity—let us call it $\mathcal{M}_9$—results from choosing H to be the 3-space $x > \sqrt{t^2 + c^2}$, $y = 0$ in $M = \mathbb{L}^4$ and κ' to be boost in the x-direction. The singularity in $\mathcal{M}_9$ is represented by a straight line parallel to the z-axis and moving with a constant acceleration in the x-direction. If such a string passes between two observers, which initially are at rest w.r.t. each other, either of them would suddenly discover that the other has acquired some speed in the x-direction, even though no apparent forces were involved.

B. Curved disclination

In an arbitrary spacetime M, pick a surface $\mathcal{S}$ of codimension 2 such that curves wrapping around it are noncontractible in $M - \mathcal{S}$. Consider the i-fold covering of $M - \mathcal{S}$. Irrespective of what M, $\mathcal{S}$ and i are chosen (i must be finite, though) the covering has a stringlike singularity represented by $\mathcal{S}$. It is easy to see that this "branching singularity" is a disclination corresponding to $\zeta = \mathrm{id}$ in terms of Appendix A. As such, the singularities of this type have received surprisingly little attention in the literature, however, implicitly they are present in a number of known spacetimes.

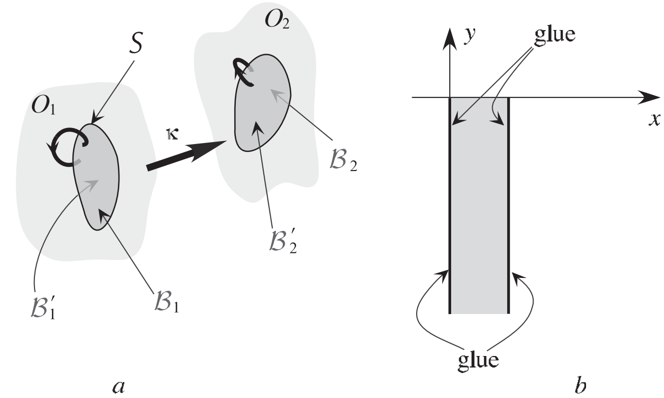

FIG. 5. (a) $\mathcal{B}_1$ is glued to $\mathcal{B}_2$ and $\mathcal{B}_1'$ to $\mathcal{B}_2'$. The thick curve is, in fact, continuous (and closed). (b) The thick vertical lines are slits.

Let κ be an isometry sending an open subset O_1 of a spacetime M to a subset O_2 disjoint with O_1. To construct a new spacetime M', pick a two-sided hypersurface H lying in O_1 together with its closure $\overline{H}$, remove H and $\kappa(H)$ from M and glue the corresponding banks of the two thus obtained slits as is shown in Fig. 5(a). Then M' has a singularity represented by $\mathcal{S} = \overline{H} - H$ which is just a branching dislocation[5]—the gray region in the figure being nothing but the double covering of $O_1 - \mathcal{S}$.

8. Example. When H is a spacelike disc in the Minkowski space and κ is a timelike translation combined with the time reflection, $\mathcal{M}^-_{10} = M'$ is the Deutsch-Politzer space. It contains closed causal curves and due to its simplicity is used extensively in time machine theory [15]. Its Euclidean analog $\mathcal{M}^+_{10}$ is a "loop-based wormhole" constructed (in terms of "delta function Riemann tensor") in [16]. For its thickened version see [17] (curiously, in $\mathcal{M}^-_{10}$ the singularity *cannot* be thickened [18]).

If nothing else, the branching singularity is a wonderful source of counterexamples. It shows, in particular, that in the *general* case:

(1) The presence of a stringlike singularity puts *no* restrictions on the stress-energy tensor of the hosting spacetime.

(2) A stringlike singularity can take any form and change it arbitrarily (though in a smooth manner, of course) with time. It also can appear and disap-

[5]To give a precise meaning to the word "banks" and to make the procedure rigorous in every way, one should consider $M - \overline{H} - \kappa(\overline{H})$ as a region in a covering of $M - \mathcal{S} - \kappa(\mathcal{S})$ and proceed as in Sec. II B.

pear at will (for example, $\mathcal{S}$ can be a circle in the (t, x)-plane multiplied by the z-axis). So, there are no "laws of motion" for a general singularity (and, in particular, it does not have to be a plane, contrary to what is asserted in [9]).

(3) Holonomies do not characterize stringlike singularities, even disclinations. Indeed, in constructing $\mathcal{M}_{10}$ we could vary κ (combining, say, the translation with a rotation) and obtain spacetimes with different holonomies even though the singularities remain the same.

Incidentally, the first two facts mean that the Cosmic Censorship Conjecture can be proved *only* if general relativity is complemented by an additional global postulate, like hole-freeness (cf. [5]).

C. Mixed singularities

Remove from the Euclidean plane $\mathbb{E}^2$ the region

$$0 < x < 1, \qquad y \leq 0$$

[the gray strip in Fig. 5(b)] and glue its vertical boundaries:

$$(0, y) \mapsto (1, y)$$

to obtain a new 2D spacetime $\mathcal{M}_e$. One might think that $\mathcal{M}_e^4 = \mathbb{L}^2 \times \mathcal{M}_e$ is a spacetime with a stringlike singularity of yet another type, which could have been called "edge dislocation" for its similarity to the corresponding distortion in solid state science (and that is, indeed, how $\mathcal{M}_e^4$—or, rather, its part—was called in [4], see Appendix B). In fact, however, this is not the case, because $\mathcal{M}_e$ is *extendible* and hence its singularities, formally speaking, are not even absolutely mild. It is easy to find an inextendible extension of $\mathcal{M}_e$: such is, for example, the spacetime $\mathcal{M}_e^{\text{ext}}$ obtained by making a pair of slits

$$x = 0, 1, \qquad y \leq 0$$

in $\mathbb{E}^2$ and gluing their banks as is shown in Fig. 5(b). It is seen that there *is* a singularity in $\mathcal{M}_e^{\text{ext}}$, but this is just a branching singularity discussed above (the only difference between $\mathcal{M}_e^{\text{ext}}$ and a 2D loop-based wormhole is that the slits are semi-infinite in the former and finite in the latter).

The interrelation between the edge dislocation and the branching singularity suggests (again by analogy with the solid state physics) that the latter can "transform" into a screw dislocation forming thus a *mixed* singularity. And this is the case. To build an example, remove the plane $y = 0$ from $\mathbb{E}^3$ and attach two banks—each is a copy of the plane—to the cut as explained in Sec. II B. On one of the banks draw a curve f which is the graph of a smooth monotone function $z(x)$:

$$z = \begin{cases} -1 & x < -3, \\ 0 & |x| < 1, \\ 1 & x > 3, \end{cases}$$

$$z(x) = 1 + z(x - 3) \quad \text{at } x \in (0, 3)$$

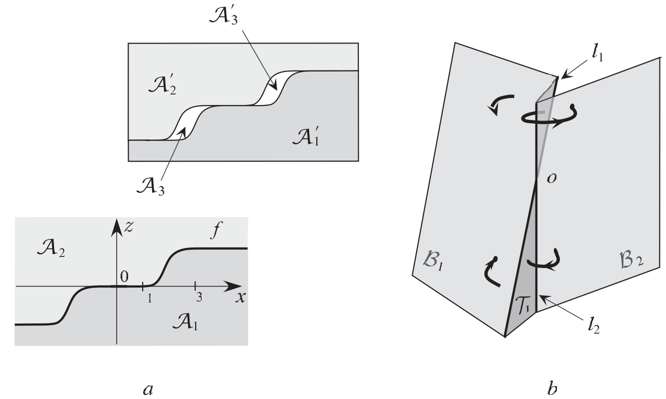

FIG. 6. (a) Each $\mathcal{A}_m$ is glued to $\mathcal{A}'_m$. (b) V' is the part of the space separated from us by $\mathcal{B}$. The curves with arrows make in fact a single smooth path.

see Fig. 6(a). This curve splits the bank into two regions which I denote $\mathcal{A}_1$ and $\mathcal{A}_2$. On the other bank draw two lines—one is f and the other is obtained from it by a horizontal shift $x \to x + \frac{1}{2}$. The bank thus is split into four regions denoted $\mathcal{A}'_1$, $\mathcal{A}'_2$, $\mathcal{A}_3$ and $\mathcal{A}'_3$. To assemble the spacetime $\mathcal{M}^+_{11'}$, remove all three copies of f and paste each $\mathcal{A}_m$ to the corresponding $\mathcal{A}'_m$, $m = 1, \ldots 3$ (κ for $m = 3$ is a combination of the shift $(x, z) \mapsto (x + 3, z + 1)$ and reflection $y \mapsto -y$). As can be easily seen, $\mathcal{M}^+_{11}$ has a stringlike singularity represented by f. The singularity is mixed: in particular, it is a (branching) disclination at $x = 2$ and a screw dislocation at $x = 4$.

The last example is built as follows. In $\mathbb{E}^3$ pick two intersecting straight lines l_1 and l_2 and attach a half-plane to either of them as shown in Fig. 6(b). Together these half-planes $\mathcal{B}_{1,2}$ and the angles $\mathcal{T}_{1,2}$ bounded by the lines $l_{1,2}$ form a surface $\mathcal{B}$ that divides the space into two disjoint regions V and V'. There are two obvious rotations, one of which (denoted κ_1) maps $\mathcal{B}_1$ to $\mathcal{B}_2$ and the other—κ_2—maps $\mathcal{T}_1$ to $\mathcal{T}_2$. The spacetime $\mathcal{M}_{12'}$ is obtained by throwing away V' and pasting $\mathcal{B}_1$ to $\mathcal{B}_2$ with κ_1 and $\mathcal{T}_1$ to $\mathcal{T}_2$ with κ_2. The singularity in $\mathcal{M}_{12}$ is represented by a plane $l_1 \times t$-axis and yet it is none of the singularities considered in Sec. II. In particular, it is neither a disclination nor a dislocation, since $\Delta(s) = 0$ at $s = l_1 \cap l_2$ and $\neq 0$ otherwise.

ACKNOWLEDGMENTS

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APPENDIX A

Denote by $\mathcal{F}$ a straight line or a plane (depending on whether $n = 3$ or 4) which is the set of fixed points of an isometry ζ: $\mathbb{M}^n \to \mathbb{M}^n$:

$$\mathcal{F} = \{p \in \mathbb{M}^n \colon \zeta(p) = p\}.$$

Let $\hat{M}$ be the universal covering of $M \equiv \mathbb{M}^n - \mathcal{F}$. Define

$$\hat{\zeta}_k\colon \hat{M} \to \hat{M}$$

to be a lift of $\zeta \circ \pi$ (here $k \in \mathbb{Z}$, since every point of the fiber defines a new lift). If, for example, ζ is the rotation by a in $\mathbb{E}^3$, then $\hat{M}$ is given by (5) and $\hat{\zeta}_k$: $\hat{\phi} \mapsto \hat{\phi} + a + 2\pi k$.

It is the quotients $\mathcal{M} = \hat{M}/\hat{\zeta}_k$ that Ellis and Schmidt tested for absolutely mild singularities. Indeed, the construction of $\mathcal{M}$ is a generalization of that producing a cone or the Misner space, so it is reasonable to expect (though not *guaranteed*) that $\mathcal{M}$ has a stringlike singularity represented by $\mathcal{F}$.

One spacetime of that kind is $\mathcal{M}_{1'}^{+}$, which already has been constructed in example 6. The four-dimensional spacetime $\mathcal{M}_1$ with a stringlike singularity (in this case $\mathcal{M}_1 = \mathcal{M}_1^+ = \mathcal{M}_1^-$) is obtained as the product $\mathcal{M}_{1'}^+ \times \mathbb{L}^1$. Two more spaces of this type were found in [5]; let us denote them $\mathcal{M}_2$ and $\mathcal{M}_3$. The spacetime $\mathcal{M}_2$ is obtained by taking $\mathcal{F}$ to be a spacelike surface in the Minkowski space and ζ to be a boost in the direction perpendicular to $\mathcal{F}$. [Interestingly enough not *all* k are equally appropriate in this case: one particular k_0 (that for which $\hat{M}/\hat{\zeta}_{k_0} = M/\zeta$) must be excluded, because the quotient is non-Hausdorff.] Finally, $\mathcal{M}_3$ is built exactly as $\mathcal{M}_2$ but with $\mathcal{F}$ being null and ζ being, correspondingly, the combination boost + rotation which leaves all points of $\mathcal{F}$ fixed. Clearly, all three spacetimes $\mathcal{M}_{1,2,3}$ have stringlike singularities represented by the planes $\mathcal{F}$.

APPENDIX B

In their paper [4] Puntigam and Soleng employed the Volterra process to obtain flat spacetimes with unusual holonomies and thus with singularities. Two of them (see entries 1 and 2 of Table 2) are called "edge dislocation." The goal of this appendix is to demonstrate that these two spacetimes (they are isometric) are, in fact, regions in the spacetime $\mathbb{L}^2 \times \mathcal{M}_e$ considered in the beginning of Sec. III C.

The spacetimes in discussion are $\mathbb{L}^2 \times \mathcal{M}_{PS}$, where the metric of $\mathcal{M}_{PS}$ is

$$ds^2 = dx'^2 + dy^2 + 2\frac{\Theta^1}{2\pi r^2} dx'(x'dy - ydx') + \left(\frac{\Theta^1}{2\pi r^2}\right)^2 (x'dy - ydx')^2, \qquad r^2 \equiv x'^2 + y^2.$$

To analyze the structure of $\mathcal{M}_{PS}$ let us first rewrite the metric in a more transparent way:

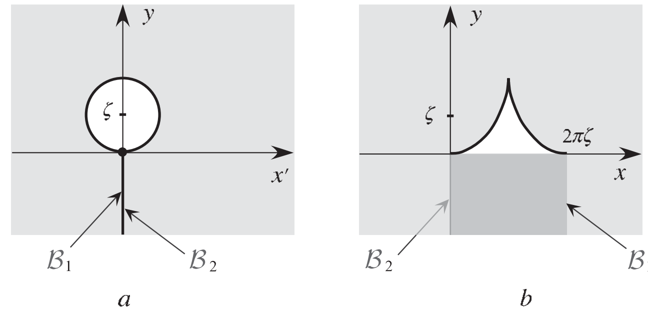

FIG. 7. $\mathcal{M}_{PS}$ is the manifold on the left with the metric (B1) or, equivalently, the manifold on the right [the ∧-like curve is the cycloid to which the transformation (B3) sends the circle (B2); the dark gray indicates that we see one sheet through another] with the metric (B4). $\mathcal{B}_1$ and $\mathcal{B}_2$ in both cases are glued.

$$ds^2 = (1 + \theta^2 x'^2)dy^2 + (1 - \theta y)^2 dx'^2 + 2\theta x'(1 - \theta y)dx'dy$$
$$\theta \equiv 2\zeta/r^2, \qquad \zeta \equiv \frac{1}{4\pi}\Theta^1. \tag{B1}$$

It is easy to see now that the metric diverges at $r = 0$ and degenerates at $\theta y = 1$, i.e., on the circle

$$(y - \zeta)^2 + x'^2 = \zeta^2. \tag{B2}$$

Its domain consists thus of two *disjoint* regions (since the metric *must* be nondegenerate). Restricting ourselves to the larger region (i.e., to the exterior of the circle) we conclude that the spacetime $\mathcal{M}_{PS}$ is the manifold

$$N = \mathbb{R}^2 - \{(y - \zeta)^2 + x'^2 \le \zeta^2\}$$

endowed with the metric (B1). In the coordinates x', y the manifold N has the appearance shown by gray in Fig. 7(a). It is instructive, however to introduce a new coordinate x

$$x(x', y) \equiv x' + 2\zeta \operatorname{arctg} y/x', \qquad (x', y) \in N', \tag{B3}$$

where N' is N without the semi-axis $\{x' = 0, y < 0\}$.

9. Remark The cut is necessary to make arctg well defined, but it is made in the domain of the function x, not in the spacetime being discussed.

In the coordinates x, y the metric takes the form

$$ds^2 = dy^2 + dx^2, \tag{B4}$$

while N' becomes the surface shown in Fig. 7(b) (and N ensues when $\mathcal{B}_1$ is glued to $\mathcal{B}_2$).

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