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Scalar and vector perturbations in quantum cosmological backgrounds

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Generalizing a previous work concerning cosmological linear tensor perturbations, we show that the Lagrangians and Hamiltonians of cosmological linear scalar and vector perturbations can be put in simple form through the implementation of canonical transformations and redefinitions of the lapse function, without ever using the background classical equations of motion. A similar result was obtained by Langlois in the case of a scalar field, but we generalize it for any perfect fluid. In such case, i.e., when the matter content of the Universe is a perfect fluid, we can go further and show that the Hamiltonian of scalar perturbations can be reduced, as usual, to a Hamiltonian of a scalar field with variable mass depending on background functions, independently of the fact that these functions satisfy the background Einstein classical equations. These simple Lagrangians and Hamiltonians can then be used in situations where the background metric is also quantized, hence providing a substantial simplification over the direct approach originally developed by Halliwell and Hawking.

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I. INTRODUCTION

In the theory of linear cosmological perturbations, simple evolution equations for the perturbations have been obtained [1]. Lagrangians and Hamiltonians describing the dynamics of scalar, vector, and tensor perturbations coming from the Einstein-Hilbert Lagrangian have been greatly simplified in different cosmological scenarios under the assumption that the background metric satisfies Einstein classical field equations, and after taking out total derivatives in space and time [1]. Once these simple Lagrangians and Hamiltonians are obtained, the quantization of linear cosmological perturbations becomes easy, with a quite simple interpretation: they can be seen as quantum fields which behave essentially as scalar fields with a time dependent effective mass. The time varying background scale factor which is responsible for this ''mass'' acts as a pump field [2], creating or destroying modes of the perturbations. In this framework, one can assume an initial vacuum state for the perturbations, yielding primordial perturbation spectra which can be compared with observations. In the cosmological inflationary scenario [3], the resulting spectrum for scalar perturbations is in good agreement with the data [4].

However, this state of affairs is rather incomplete: the overwhelming majority of classical backgrounds possess an initial singularity at which the classical theory is expected to break down, and one needs to justify the initial conditions for inflation and quantum perturbations. Hence, a full quantum treatment including the background must be constructed. The first approach in this direction was made in Ref. [5], where the canonical quantization of the perturbations and background was implemented through the derivation of the super-Hamiltonian constraint of the whole system and its consequent Wheeler-DeWitt equation $H(\hat{A}, \hat{P}_A, \hat{X}, \hat{P}_X)\Psi = 0$, where A and P_A represent the phase space background variables, and X and P_X the perturbation phase space variables. They claim that the no boundary proposal can set the initial conditions for inflation and the vacuum initial state for the perturbations. Then, through the imposition of the ansatz on the wave functional $\Psi(A, X, t) = \varphi(A, t)\psi(A, X, t)$, they could manage to separate the quantum effects in the background from the quantum perturbations, where the wave function for the background $\varphi(A, t)$ obeys an independent quantum minisuperspace description where backreaction terms from the quantum perturbations are negligible. The singularity is bypassed through a Euclideanization of space-time near it, and a consequent beginning of time when (or where) the geometry passes from the Euclidean signature to the Lorentzian one. The quantum perturbations are described in the oscillatory part of the background wave function, where a WKB approximation can be used (see also Ref. [6] for further details). Then, the evolution of the scale factor in time may be obtained through the equation $\dot{a} \propto \partial S/\partial a$, where S is a solution of the classical Hamilton-Jacobi equation. Hence, this evolution is the classical one, and we are back to a semiclassical description of the perturbations.

In parallel to that, the possibility that the singularity could be avoided through a bounce connecting the present expanding phase with a preceding contracting phase has been explored. In this case, the Universe is eternal: there is no beginning of time, nor horizons. Many frameworks where bounces may occur have been proposed [7–9]. These new features of the background introduce a new picture for the evolution of cosmological perturbations: vacuum initial conditions may now be imposed when the Universe was very big and almost flat, and effects due to the contracting and bouncing phases, which are not present in models with a beginning of time, may change the sub-

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sequent evolution of perturbations in the expanding phase. Because of that, the evolution of cosmological perturbations in bouncing models has been the cause of intense debate [10].

In the framework of quantum cosmology in minisuperspace models, bouncing models had also been proposed where the bounce occurs due to quantum effects in the background [11–17]. Some approaches have used an ontological interpretation of quantum mechanics, the Bohm-de Broglie [18] one, to interpret the results [14–16]. In this interpretation, quantum Bohmian trajectories, the quantum evolution of the scale factor $a_q(t)$ at zeroth order, can be defined through the relation $\dot{a} \propto \partial S/\partial a$, where S is now the phase of the background wave function $\varphi(A, t)$ without any approximation: it is not a solution of the classical Hamilton-Jacobi equation. In fact it satisfies a modified Hamilton-Jacobi equation derived from the Wheeler-DeWitt equation for $\varphi(A, t)$, and hence $a_q(t)$ is not the classical trajectory: in the regions where the quantum effects cannot be neglected, the quantum trajectory $a_q(t)$ performs a bounce which connects two asymptotic classical regions where the quantum effects are negligible. One then has in hand a definite function of time for the background, even at the quantum level, which realizes a soft transition from the contracting phase to the expanding one. Because of the results of Ref. [5], where the background minisuperspace Wheeler-DeWitt equation for $\varphi(A, t)$ continues to hold when quantum perturbations are present because backreaction terms are negligible (which can also be justified through other ansatz for the wave function or verified "*a posteriori*"), this background quantum function $a_q(t)$ is sufficient to describe all quantum features of the background. The natural question to ask is what happens with the perturbations when it passes through this well-defined and regular quantum bounce. One could then use the Hamiltonian H of Ref. [5] to investigate the evolution of quantum perturbations in this quantum background. However, the structure of H is rather complicated, making it difficult to obtain any detailed result about the spectra of perturbations, especially the scalar ones. Also, a simplification of H using the zeroth order classical equations, as done in Ref. [1] and described in the beginning of this section, is not possible because the background is also quantized and it does not satisfy the classical Einstein equations. This state of affairs motivated us to find a way to simplify the Hamiltonian of Ref. [5], without ever recurring to the background classical equations, and apply it to these quantum systems.

Recently, we have managed to put the Hamiltonian of tensor perturbations into a very simple form through the implementation of canonical transformations and redefinitions of the lapse function only, without recurring to any classical equations of motion [19]. Its consequences were explored in Ref. [20]. However, tensor perturbations are very special (they are automatically gauge invariant; their equations do not depend on the matter background) and it remains to investigate if it would be possible to do the same procedures to simplify the Hamiltonian of scalar and vector perturbations. Note that such perturbations are not gauge invariant from the beginning, and they have contributions from the matter perturbations, which renders the calculations much more intricate.

In Ref. [21], D. Langlois has shown, using a Hamilton-Jacobi technique, that one can indeed considerably simplify the Hamiltonian of scalar perturbations when the matter content is constituted by a scalar field, without ever using the background classical equations. However, a full quantization of the system is still involved, as long as the mass term of the perturbations contains negative and fourth powers of the background momenta, which renders the Wheeler-DeWitt equation for the perturbations and background still complicated. In order to describe the evolution of these perturbations in terms of linear quantum scalar fields with a simple time variable mass, as in Mukhanov's approach [1], one still needs to use the background equations at the final step.

The aim of this paper is to extend the results of Ref. [21] to the case of hydrodynamical fluids and study their consequences. We show that it is also possible to put the complicated Hamiltonians of scalar and vector perturbations for these fluids into a very simple form, and, in a second step, to put them in the form of Ref. [1] (quantum scalar fields with time dependent mass), also without ever using any classical background equations.

For the first step, working in the Hamiltonian framework, we will exhibit the canonical transformations and lapse function redefinitions which achieve our goal. The simplified constraints obtained do not contain any nontrivial couplings between background momenta and perturbation degrees of freedom, as it happens in the scalar field case, and they have direct physical interpretations. Because of that, the full quantization of the theory yields a very simple Wheeler-DeWitt equation for the perturbations and background, which can now be used with whatever interpretation of quantum mechanics and gauge time choice one makes.

In the second step, as the Wheeler-DeWitt equation is simple and assumes a Schrödinger form in the case of perfect fluids, a further simplification can be achieved through the use of an ansatz for the total wave functional, yielding a separate Wheeler-DeWitt equation for the background, and provided one uses the ontological interpretation of Bohm and de Broglie [18]. As in this case a quantum Bohmian trajectory $a_q(t)$ at zeroth order can be defined, a time dependent unitary transformation can be implemented in the scalar perturbation sector using this $a_q(t)$, and, like in Ref. [1], the Hamiltonian for the perturbations can be further simplified, rendering equations governing the scalar perturbations which are formally equivalent to simple equations for a scalar field with an

effective mass depending on the quantum solution for the scale factor $a_q(t)$, the quantum "pump field."

The paper is organized as follows. In the following section, we specify the action and Hamiltonian by restricting attention to the particular case of a Friedmann-Lemaître-Robertson-Walker (FLRW) background and perturbations around it, without yet making any separation in scalar, vector, and tensor perturbations. In Secs. III and IV we analyze the cases of vector and scalar perturbations, respectively, for the hydrodynamical fluid case. In Sec. IV we present all the steps to simplify the scalar part of the Hamiltonian. In Sec. V we quantize this system. After separating the background Schrödinger equation from the perturbed one, we show how to use the Bohm-de Broglie interpretation in order to perform the last canonical transformations which yield quantum equations for the perturbations with the same form as those presented in Ref. [1]. Finally, Sec. VI ends this paper with some general conclusions. In Appendix A, we exhibit an alternative derivation of the simpler Hamiltonian for scalar fields calculated in Ref. [21], using the same Hamiltonian methods employed in Sec. IV for hydrodynamical fluids. Appendix B presents the explicit canonical transformations used in Sec. IV.

II. LINEAR COSMOLOGICAL PERTURBATIONS

Let the geometry of space-time be given by

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

where $g^{(0)}_{\mu\nu}$ represents a homogeneous and isotropic cosmological background,

$$ds^2 = g^{(0)}_{\mu\nu}dx^\mu dx^\nu = N^2(t)dt^2 - a^2(t)\gamma_{ij}dx^i dx^j, \tag{2}$$

where γ_{ij} is the spatial metric of the spacelike hypersurfaces with constant curvature $K = 0, \pm 1$, and the $h_{\mu\nu}$ are the linear perturbations, which we decompose into

$$h_{00} = 2N^2\phi, \qquad h_{0i} = -NaA_i, \qquad h_{ij} = a^2\epsilon_{ij}. \tag{3}$$

Substituting Eq. (3) into the Einstein-Hilbert action

$$S_{gr} = -\frac{1}{6l^2}\int d^4x\sqrt{-g}R, \tag{4}$$

where $l^2 = 8\pi G/3$, yields the zeroth and second order actions

$$S^{(0)}_{\rm gr} = \frac{1}{6l^2}\int d^4x N\gamma^{1/2}a^3\left(-\frac{6\dot{a}^2}{a^2N^2} + \frac{6K}{a^2}\right), \tag{5}$$

$$\begin{aligned}\delta_2 S_{\rm gr} = \frac{1}{6l^2}\int d^4x N\gamma^{1/2}a^3\Bigg[&\frac{1}{4N^2}\dot{\epsilon}^{ij}\dot{\epsilon}_{ij} - \frac{1}{4N^2}\dot{\epsilon}^2 + \frac{1}{aN}\dot{A}_i\epsilon^{ij}{}_{|j} - \frac{1}{aN}\dot{\epsilon}A^i{}_{|i} + \frac{\dot{a}}{a^2N}(-4\phi A^i{}_{|i} + 2A_i\epsilon^{ij}{}_{j}) \\ &+ \frac{\dot{a}}{aN^2}(-\epsilon\dot{\epsilon} - 2\dot{\epsilon}\phi + 2\dot{\epsilon}_{ij}\epsilon^{ij}) + \frac{\dot{a}^2}{a^2N^2}\left(-3\epsilon\phi - 9\phi^2 + 3A^iA_i - \frac{3}{4}\epsilon^2 + \frac{3}{2}\epsilon^{ij}\epsilon_{ij}\right) + \frac{1}{a^2}A^{i|j}A_{[i|j]} \\ &- \frac{1}{4a^2}\epsilon_{ij|k}\epsilon^{ij|k} + \frac{1}{2a^2}\epsilon^{ij}{}_{|j}\epsilon_i{}^k{}_{|k} + \frac{1}{a^2}\phi_{|i}\epsilon^{ij}{}_{|j} - \frac{1}{2a^2}\epsilon_{|i}\epsilon^{ij}{}_{|j} - \frac{1}{a^2}\phi_{|i}\epsilon^{|i} + \frac{1}{4a^2}\epsilon_{|i}\epsilon^{|i} \\ &+ \frac{K}{a^2}\left(A^iA_i - \epsilon^{ij}\epsilon_{ij} - 3\phi^2 - \epsilon\phi + \frac{1}{4}\epsilon^2\right)\Bigg].\end{aligned} \tag{6}$$

The first order action was discarded because we are assuming that the mean value of the perturbations over the spatial sections is null:

$$\int d^3x\gamma^{1/2}h_{\mu\nu} = 0. \tag{7}$$

These are the actions for the gravitational sector. Let us now focus on the action for the matter sector. We will concentrate on the perfect fluid case, leaving the scalar field for Appendix A as a verification of the results of Ref. [21] using a different method.

We will restrict the construction of the Hamiltonian to the $K = 0$ case and postpone the $K \neq 0$ case, which is more intricate, to a future publication.

Following the approach of Ref. [1], the Lagrangian density of the perfect fluid is

$$\mathcal{L}_m = -\varepsilon, \tag{8}$$

where

$$\varepsilon = \rho[m_0 + \Pi(p, \rho)], \tag{9}$$

and

$$\Pi(p, \rho) = \int_0^\rho \frac{dp}{d\rho'}\frac{d\rho'}{\rho'} - \frac{p}{\rho}, \tag{10}$$

where ρ is the number density of particles, m_0 is their rest mass, and p is an arbitrary function of ρ, which will be identified with the pressure. The particle number density ρ is given by

$$\rho = \frac{F(\mathbf{a}^i)\sqrt{g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\sigma}}}{\sqrt{-g}J}, \tag{11}$$

where F is an arbitrary function of Lagrangian variables, σ is a time parameter along the particle world lines, and J is the Jacobian of the transformation from the Lagrangian variable to Eulerian ones.

The energy-momentum tensor of the fluid reads

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} = \varepsilon V^\mu V^\nu - p(g^{\mu\nu} - V^\mu V^\nu), \tag{12}$$

where it is clear that ε and p correspond to the energy density and pressure, respectively. The sound velocity c_s is defined by

$$c_s^2 = \frac{\partial \varepsilon}{\partial p}. \tag{13}$$

Perturbations displace the particles from their background positions x_0^μ to the x^μ positions given by

$$x_0^\alpha \to x^\alpha = x_0^\alpha + \xi^\alpha(x_0), \tag{14}$$

meaning a change into their Eulerian position, which implies modifications in the Jacobian,

$$J = J_0(1 + \dot{\xi}^0 + \xi^i{}_{,i} + \tfrac{1}{2}\xi^i{}_{,i}\xi^j{}_{,j} + \dot{\xi}^0\xi^i{}_{,i} - \tfrac{1}{2}\xi^i{}_{,j}\xi^j{}_{,i} - \xi^0{}_{,i}\dot{\xi}^i), \tag{15}$$

in the determinant,

$$\sqrt{-g}(x_0+\xi) = \sqrt{-g^{(0)}}(x_0)\Big(1 + \phi - \frac{1}{2}\epsilon + \frac{\dot{N}}{N}\xi^0 + \frac{3\dot{a}}{a}\xi^0 - \frac{1}{2}\phi^2 - \frac{1}{2}\epsilon\phi + \frac{1}{2}A^iA_i - \frac{1}{4}\epsilon^{ij}\epsilon_{ij} + \frac{1}{8}\epsilon^2 + \frac{\dot{N}}{N}\phi\xi_0 - \frac{1}{2}\frac{\dot{N}}{N}\epsilon\xi^0 + \frac{3\dot{a}}{a}\xi^0\phi - \frac{3}{2}\frac{\dot{a}}{a}\epsilon\xi^0 + \dot{\phi}\xi^0 + \phi_i\xi^i - \frac{1}{2}\dot{\epsilon}\xi^0 + \frac{3\dot{N}\dot{a}}{Na}\xi^0 + \frac{3\dot{a}^2}{a^2}\xi^0 + \frac{\ddot{N}}{2N}\xi^0 + \frac{3\ddot{a}}{2a}\xi^0 - \frac{1}{2}\epsilon_{,i}\xi^i\Big), \tag{16}$$

and in

$$\sqrt{g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\sigma}} = \sqrt{g^{(0)}_{\mu\nu}\frac{\partial x_0^\mu}{\partial\sigma}\frac{\partial x_0^\nu}{\partial\sigma}}\Big(1 + \phi + \frac{\dot{N}}{N}\xi^0 + \dot{\xi}^0 + \frac{\dot{N}}{N}\phi\xi^0 + \dot{\phi}\xi^0 + \phi_{|i}\xi^i + \frac{\ddot{N}}{2N}\xi_0^2 + \phi\dot{\xi}_0 + \frac{\dot{N}}{N}\dot{\xi}^0\xi^0 - \frac{a}{N}A_i\dot{\xi}^i - \frac{1}{2}\frac{a^2}{N^2}\gamma_{ij}\dot{\xi}^i\dot{\xi}^j - \frac{1}{2}\phi^2\Big). \tag{17}$$

The particle number density at point x_0 is then given by

$$\rho(x_0) = \rho_0\Big(1 + \frac{1}{2}\epsilon - \xi^i{}_{|i} + \dot{\xi}^i{}_{|i}\xi^0 + \xi^i{}_{|j|i}\xi^i - \frac{a}{N}A_i\dot{\xi}^i - \frac{1}{2}\frac{a^2}{N^2}\gamma_{ij}\dot{\xi}^i\dot{\xi}^j - \frac{1}{2}A_iA^i + \frac{1}{4}\epsilon_{ij}\epsilon^{ij} + \frac{1}{8}\epsilon^2 + \xi^0_{|i}\dot{\xi}^i + \frac{1}{2}\xi^i{}_{|i}\xi^j{}_{|j} + \frac{1}{2}\xi^i{}_{|j}\xi^j{}_{|i} - \frac{1}{2}\epsilon\xi^i{}_{|i}\Big). \tag{18}$$

Substituting all that in Eq. (10) and finally in Eq. (9) yields

$$\delta_2 S_m = -\int d^4x N a^3\gamma^{1/2}\Big[\varepsilon_0\Big(-\frac{1}{2}\phi^2 + \frac{1}{2}A^iA_i - \phi\xi^i{}_{|i}\Big) + p_0\Big(\frac{1}{2}\epsilon\phi + \frac{1}{4}\epsilon^{ij}\epsilon_{ij} - \frac{1}{8}\epsilon^2 - \phi\xi^i{}_{|i}\Big) - \frac{1}{2}(\varepsilon_0 + p_0)\Big(\frac{a^2}{N^2}\dot{\xi}^i\dot{\xi}_i + 2\frac{a}{N}A_i\dot{\xi}^i + A_iA^i\Big) + \frac{1}{2}c_s^2(\varepsilon_0 + p_0)\Big(\frac{1}{4}\varepsilon^2 + \xi^i{}_{|i}\xi^j{}_{|j} - \epsilon\xi^i{}_{|i}\Big)\Big]. \tag{19}$$

The total Lagrangian including the gravitational sector then reads

$$
\begin{aligned}
L = & -\frac{\dot{a}^2 a V}{l^2 N} - N a^3 \varepsilon_0 V + \frac{N a}{6 l^2} \int d^3x \gamma^{1/2} \Big(A^{i|j} A_{[i|j]} - \frac{1}{4} \epsilon^{ij|k} \epsilon_{ij|k} + \frac{a}{N} \dot{A}_i \epsilon^{ij}{}_{|j} + \frac{1}{2} \epsilon^{ij}{}_{|j} \epsilon_i{}^k{}_{|k} + \phi_{|i} \epsilon^{ij}{}_{|j} - \frac{1}{2} \epsilon_{|i} \epsilon^{ij}{}_{|j} \\
& - \phi_{|i} \epsilon^{|i} + \frac{1}{4} \epsilon_{|i} \epsilon^{|i} \Big) + \frac{a^3}{24 l^2 N} \int d^3x \gamma^{1/2} \dot{\epsilon}^{ij} \dot{\epsilon}_{ij} + \frac{a \dot{a}^2}{6 l^2 N} \int d^3x \gamma^{1/2} \Big(-9\phi^2 - 3\epsilon\phi - \frac{3}{4}\epsilon^2 + 3 A^i A_i + \frac{3}{2} \epsilon^{ij} \epsilon_{ij} \Big) \\
& - \frac{2 a \dot{a}}{3 l^2} \int d^3x \gamma^{1/2} \Big(\phi A^i{}_{|i} - \frac{1}{2} A_i \epsilon^{ij}{}_{|j} \Big) - \frac{a^3}{24 l^2 N} \int d^3x \gamma^{1/2} \dot{\epsilon}^2 + \frac{a^2 \dot{a}}{3 l^2 N} \int d^3x \gamma^{1/2} \Big(\epsilon^{ij} \dot{\epsilon}_{ij} - \frac{1}{2} \epsilon \dot{\epsilon} - \phi \dot{\epsilon} \Big) \\
& - \frac{a^2}{6 l^2} \int d^3x \gamma^{1/2} \dot{\epsilon} A^i{}_{|i} - N a^3 \varepsilon_0 \int d^3x \gamma^{1/2} \Big(-\frac{1}{2} \phi^2 + \frac{1}{2} A^i A_i - \phi \xi^i{}_{|i} \Big) \\
& - N a^3 p_0 \int d^3x \gamma^{1/2} \Big(\frac{1}{2} \epsilon\phi + \frac{1}{4} \epsilon^{ij} \epsilon_{ij} - \frac{1}{8} \epsilon^2 - \phi \xi^i{}_{|i} \Big) + \frac{1}{2} N a^3 (\varepsilon_0 + p_0) \int d^3x \gamma^{1/2} \Big(\frac{a^2}{N^2} \dot{\xi}^i \dot{\xi}_i + 2 \frac{a}{N} A_i \dot{\xi}^i + A_i A^i \Big) \\
& - \frac{1}{2} c_s^2 N a^3 (\varepsilon_0 + p_0) \int d^3x \gamma^{1/2} \Big(\frac{1}{4} \varepsilon^2 + \xi^i{}_{|i} \xi^j{}_{|j} - \epsilon \xi^i{}_{|i} \Big).
\end{aligned} \tag{20}
$$

The procedure of Ref. [1] to simplify Eq. (20) begins as follows: using the background equation of motion

$$
\ddot{a} = -\frac{\dot{a}^2}{2a} + \frac{\dot{a}\dot{N}}{N} - \frac{3 l^2 N^2 a}{2} p_0, \tag{21}
$$

and discarding a total time derivative

$$
\Big[\frac{a^2 \dot{a}}{6 l^2 N} \int d^3x \gamma^{1/2} \Big(\epsilon^{ij} \epsilon_{ij} - \frac{1}{2} \epsilon^2 \Big) \Big]^{\cdot}, \tag{22}
$$

we obtain

$$
\begin{aligned}
L = & -\frac{\dot{a}^2 a V}{l^2 N} - N a^3 \varepsilon_0 V + \frac{N a}{6 l^2} \int d^3x \gamma^{1/2} \Big(A^{i|j} A_{[i|j]} - \frac{1}{4} \epsilon^{ij|k} \epsilon_{ij|k} + \frac{a}{N} \dot{A}_i \epsilon^{ij}{}_{|j} + \frac{1}{2} \epsilon^{ij}{}_{|j} \epsilon_i{}^k{}_{|k} + \phi_{|i} \epsilon^{ij}{}_{|j} - \frac{1}{2} \epsilon_{|i} \epsilon^{ij}{}_{|j} - \phi_i \epsilon^{|i} \\
& + \frac{1}{4} \epsilon_i \epsilon^i \Big) + \frac{a^3}{24 l^2 N} \int d^3x \gamma^{1/2} \dot{\epsilon}^{ij} \dot{\epsilon}_{ij} + \frac{a \dot{a}^2}{6 l^2 N} \int d^3x \gamma^{1/2} (-9\phi^2 - 3\epsilon\phi + 3 A^i A_i) - \frac{a^3}{24 l^2 N} \int d^3x \gamma^{1/2} \dot{\epsilon}^2 \\
& - \frac{2 a \dot{a}}{3 l^2} \int d^3x \gamma^{1/2} \Big(\phi A^i{}_{|i} - \frac{1}{2} A_i \epsilon^{ij}{}_{|j} \Big) - \frac{a^2 \dot{a}}{3 l^2 N} \int d^3x \gamma^{1/2} \phi \dot{\epsilon} - \frac{a^2}{6 l^2} \int d^3x \gamma^{1/2} \dot{\epsilon} A^i{}_{|i} \\
& - N a^3 \varepsilon_0 \int d^3x \gamma^{1/2} \Big(-\frac{1}{2} \phi^2 + \frac{1}{2} A^i A_i - \phi \xi^i{}_{|i} \Big) - N a^3 p_0 \int d^3x \gamma^{1/2} \Big(\frac{1}{2} \epsilon\phi - \phi \xi^i{}_{|i} \Big) \\
& + \frac{1}{2} N a^3 (\varepsilon_0 + p_0) \int d^3x \gamma^{1/2} \Big(\frac{a^2}{N^2} \dot{\xi}^i \dot{\xi}_i + 2 \frac{a}{N} A_i \dot{\xi}^i + A_i A^i \Big) - \frac{1}{2} c_s^2 N a^3 (\varepsilon_0 + p_0) \int d^3x \gamma^{1/2} \Big(\frac{1}{4} \varepsilon^2 + \xi^i{}_{|i} \xi^j{}_{|j} - \epsilon \xi^i{}_{|i} \Big).
\end{aligned} \tag{23}
$$

Now using the other background equation

$$
\frac{\dot{a}^2 a}{6 l^2 N} = \frac{N a^3 \epsilon_0}{6}, \tag{24}
$$

we arrive at the simplified Lagrangian

$$
\begin{aligned}
L = & -\frac{\dot{a}^2 aV}{l^2 N} - Na^3\varepsilon_0 V + \frac{Na}{6l^2}\int d^3x\gamma^{1/2}\Big(A^{i|j}A_{[i|j]} - \frac{1}{4}\epsilon^{ij|k}\epsilon_{ij|k} + \frac{a}{N}\dot{A}_i\epsilon^{ij}{}_{|j} + \frac{1}{2}\epsilon^{ij}{}_{|j}\epsilon_i{}^k{}_{|k} + \phi_{|i}\epsilon^{ij}{}_{|j} - \frac{1}{2}\epsilon_{|i}\epsilon^{ij}{}_{|j} - \phi_{|i}\epsilon^{|i} \\
& + \frac{1}{4}\epsilon_{|i}\epsilon^{|i}\Big) + \frac{a^3}{24l^2N}\int d^3x\gamma^{1/2}\dot{\epsilon}^{ij}\dot{\epsilon}_{ij} - \frac{a^3}{24l^2N}\int d^3x\gamma^{1/2}\dot{\epsilon}^2 - \frac{a\dot{a}^2}{l^2N}\int d^3x\gamma^{1/2}\phi^2 \\
& - \frac{2a\dot{a}}{3l^2}\int d^3x\gamma^{1/2}\Big(\phi A^i{}_{|i} - \frac{1}{2}A_i\epsilon^{ij}{}_{|j}\Big) - \frac{a^2\dot{a}}{3l^2N}\int d^3x\gamma^{1/2}\phi\dot{\epsilon} - \frac{a^2}{6l^2}\int d^3x\gamma^{1/2}\dot{\epsilon}A^i{}_{|i} \\
& - Na^3(\varepsilon_0 + p_0)\int d^3x\gamma^{1/2}\Big(\frac{1}{2}\epsilon\phi - \phi\xi^i{}_{|i}\Big) + \frac{1}{2}Na^3(\varepsilon_0 + p_0)\int d^3x\gamma^{1/2}\Big(\frac{a^2}{N^2}\dot{\xi}^i\dot{\xi}_i + 2\frac{a}{N}A_i\dot{\xi}^i + A_iA^i\Big) \\
& - \frac{1}{2}c_s^2Na^3(\varepsilon_0 + p_0)\int d^3x\gamma^{1/2}\Big(\frac{1}{4}\varepsilon^2 + \xi^i{}_{|i}\xi^j{}_{|j} - \epsilon\xi^i{}_{|i}\Big).
\end{aligned}
\tag{25}
$$

Equation (25) corresponds to Eq. (10.37) of Ref. [1] if one is restricted to scalar perturbations, and if one reads β in the latter as $\beta = 3a^2l^2(\epsilon_0 + p_0)/2$. Note that it is not necessary to use Eq. (24) in order to pass from Eq. (23) to Eq. (25): the redefinition of the lapse function

$$
N =: \tilde{N}\Big[1 + \frac{1}{2V}\int d^3x\gamma^{1/2}(\epsilon\phi + \phi^2 - A^iA_i)\Big]
\tag{26}
$$

takes Eq. (23) into Eq. (25). Note that these two lapse functions related by Eq. (26) are equivalent at first order. Hence, this procedure does not modify the equations of motion at first order when we make a time gauge choice.

Let us now calculate the Hamiltonians of these Lagrangians for perfect fluids with equation of state

$$
p_0 = \lambda\varepsilon_0.
\tag{27}
$$

The Hamiltonian from Eq. (20) reads

$$
\begin{aligned}
H_T = & -\frac{Nl^2P_a^2}{4aV} + N\frac{P_T}{a^{3\lambda}} + \frac{Nl^2P_a^2}{aV^2}\int d^3x\gamma^{1/2}\Big(\frac{1}{8}\phi^2 + \frac{1}{24}\epsilon\phi - \frac{1}{8}A^iA_i - \frac{1}{32}\epsilon^2 + \frac{5}{48}\epsilon^{ij}\epsilon_{ij}\Big) \\
& + \frac{NP_a}{6V}\int d^x\gamma^{1/2}(\phi A^i{}_{|i} + \epsilon^{ij}{}_{|j}A_i) + \frac{2Nl^2P_a}{aV}\int d^3x\pi^{ij}\epsilon_{ij} - \frac{Nl^2P_a}{2a^2V^2}\int d^3x\epsilon\pi + \frac{NP_a}{12V}\int d^3x\gamma^{1/2}\epsilon A^i{}_{|i} \\
& + \frac{Nl^2P_a}{a^2V}\int d^3x\pi\phi + \frac{6Nl^2}{a^3}\int d^3x\frac{\pi^{ij}\pi_{ij}}{\gamma^{1/2}} - \frac{3l^2N}{a^3}\int d^3x\frac{\pi^2}{\gamma^{1/2}} - \frac{Na}{4l^2}\int d^3x\gamma^{1/2}A^i{}_{|i}A^j{}_{|j} - \frac{N}{a}\int d^3x\pi A^i{}_{|i} \\
& + \frac{N}{2a^5(\lambda+1)\varepsilon_0}\int d^3x\frac{\pi^i_\xi\pi_\xi}{\gamma^{1/2}} - \frac{N}{a}\int d^3x\pi^i_\xi A_i - \frac{Na}{6l^2}\int d^3x\gamma^{1/2}\Big(A^{i|j}A_{[i|j]} - \frac{1}{4}\epsilon^{ij|k}\epsilon_{ij|k} + \frac{1}{2}\epsilon^{ij}{}_{|j}\epsilon_i{}^k{}_{|k} + \phi_{|i}\epsilon^{ij}{}_{|j} \\
& - \frac{1}{2}\epsilon_{|i}\epsilon^{ij}{}_{|j} - \phi_{|i}\epsilon^{|i} + \frac{1}{4}\epsilon_{|i}\epsilon^{|i}\Big) + Na^3\varepsilon_0\int d^3x\gamma^{1/2}\Big(-\frac{1}{2}\phi^2 + \frac{1}{2}A^iA_i - \phi\xi^i{}_{|i}\Big) \\
& + Na^3\lambda\varepsilon_0\int d^3x\gamma^{1/2}\Big(\frac{1}{2}\epsilon\phi + \frac{1}{4}\epsilon^{ij}\epsilon_{ij} - \frac{1}{8}\epsilon^2 - \phi\xi^i{}_{|i}\Big) + \frac{1}{2}Na^3\lambda(\lambda+1)\varepsilon_0\int d^3x\gamma^{1/2}\Big(\frac{1}{4}\epsilon^2 + \xi^i{}_{|i}\xi^j{}_{|j} - \epsilon\xi^i{}_{|i}\Big),
\end{aligned}
\tag{28}
$$

while that from Eq. (23) is given by

$$H_T = -\frac{Nl^2P_a^2}{4aV} + N\frac{P_T}{a^{3\lambda}} + \frac{Nl^2P_a^2}{aV^2}\int d^3x\gamma^{1/2}\left(\frac{1}{8}\phi^2 + \frac{1}{8}\epsilon\phi - \frac{1}{8}A^iA_i\right) + \frac{NP_a}{6V}\int d^3x\gamma^{1/2}(\phi A^i{}_{|i} + \epsilon^{ij}{}_{|j}A_i)$$
$$+ \frac{Nl^2P_a}{a^2V}\int d^3x\pi\phi + \frac{6Nl^2}{a^3}\int d^3x\frac{\pi^{ij}\pi_{ij}}{\gamma^{1/2}} - \frac{3l^2N}{a^3}\int d^3x\frac{\pi^2}{\gamma^{1/2}} - \frac{Na}{4l^2}\int d^3x\gamma^{1/2}A^i{}_{|i}A^j{}_{|j} - \frac{N}{a}\int d^3x\pi A^i{}_{|i}$$
$$+ \frac{N}{2a^5(\lambda+1)\varepsilon_0}\int d^3x\frac{\pi^i_\xi\pi_\xi}{\gamma^{1/2}} - \frac{N}{a}\int d^3x\pi^i_\xi A_i - \frac{Na}{6l^2}\int d^3x\gamma^{1/2}\left(A^{i|j}A_{[i|j]} - \frac{1}{4}\epsilon^{ij|k}\epsilon_{ij|k} + \frac{1}{2}\epsilon^{ij}{}_{|j}\epsilon_i{}^k{}_{|k}\right.$$
$$\left. + \phi_{|i}\epsilon^{ij}{}_{|j} - \frac{1}{2}\epsilon_{|i}\epsilon^{ij}{}_{|j} - \phi_{|i}\epsilon^{|i} + \frac{1}{4}\epsilon_{|i}\epsilon^{|i}\right) + Na^3\varepsilon_0\int d^3x\gamma^{1/2}\left(-\frac{1}{2}\phi^2 + \frac{1}{2}A^iA_i - \phi\xi^i{}_{|i}\right)$$
$$+ Na^3\lambda\varepsilon_0\int d^3x\gamma^{1/2}\left(\frac{1}{2}\epsilon\phi - \phi\xi^i{}_{|i}\right) + \frac{1}{2}Na^3\lambda(\lambda+1)\varepsilon_0\int d^3x\gamma^{1/2}\left(\frac{1}{4}\epsilon^2 + \xi^i{}_{|i}\xi^j{}_{|j} - \epsilon\xi^i{}_{|i}\right). \quad (29)$$

The quantity P_T appearing in the second term of the zeroth order term of both Hamiltonians is just the kinematical constant $P_T \equiv \varepsilon_0 a^{3\lambda+3}V$. We have introduced it as a canonical momentum to a variable T which is cyclic, implying indeed that P_T is a constant. We have made an inverse Routh procedure. The variable T plays the role of time when the system is quantized. This form of the zeroth order Hamiltonian appears in other approaches to a Lagrangian formulation of fluids; see e.g. Ref. [22] for details.

One can now use the total time derivative (22) to construct the generator of canonical transformations

$$\mathcal{F} = a\tilde{P}_a - \int d^3x\pi^{ij}\tilde{\epsilon}_{ij} - \frac{\tilde{P}_a a}{12V}\int d^3x\gamma^{1/2}\left(\tilde{\epsilon}^{ij}\tilde{\epsilon}_{ij} - \frac{1}{2}\tilde{\epsilon}^2\right), \quad (30)$$

yielding

$$a = \tilde{a}\left[1 + \frac{1}{12V}\int d^3x\gamma^{1/2}\left(\epsilon^{ij}\epsilon_{ij} - \frac{1}{2}\epsilon^2\right)\right],$$
$$P_a = \tilde{P}_a\left[1 - \frac{1}{12V}\int d^3x\gamma^{1/2}\left(\epsilon^{ij}\epsilon_{ij} - \frac{1}{2}\epsilon^2\right)\right], \quad (31)$$
$$\tilde{\pi}^{ij} = \pi^{ij} + \frac{aP_a}{6V}\gamma^{1/2}\left(\epsilon^{ij}\epsilon_{ij} - \frac{1}{2}\epsilon^2\right), \qquad \tilde{\epsilon}^{ij} = \epsilon^{ij}.$$

Using the fact that $\rho \propto a^{-3}$, the particle number density transforms to

$$\rho = \tilde{\rho}\left[1 - \frac{1}{4V}\int d^3x\gamma^{1/2}\left(\tilde{\epsilon}^{ij}\tilde{\epsilon}_{ij} - \frac{1}{2}\tilde{\epsilon}^2\right)\right] =: \tilde{\rho} - \delta\rho. \quad (32)$$

Substituting this last equation into Eqs. (9) and (10) we obtain

$$\varepsilon_0 = \tilde{\varepsilon}_0 - \frac{(\tilde{\varepsilon}_0 + \tilde{p}_0)}{\tilde{\rho}}\delta\rho. \quad (33)$$

Inserting Eqs. (31) and (33) into Eq. (28), we obtain (29). Hence, in the Lagrangian point of view, one can pass from Eq. (20) to Eq. (23) without using any background equations of motion. As we have shown that we can pass from Eq. (23) to Eq. (25) just through a redefinition of N, then it is proven that Lagrangian (20) is equivalent to Lagrangian (25) at this order of approximation irrespective of the classical background equations of motion. Figure 1 shows a schematic view of what we have achieved up to now.

In order to proceed from this point, we will now separate the perturbations into scalar, vector, and tensor perturbations. We make the decomposition

$$A_i = B_{|i} + S_i,$$
$$\epsilon_{ij} = 2\psi\gamma_{ij} - 2E_{|i|j} - F_{i|j} - F_{j|i} + w_{ij} \quad (34)$$

in the gravitational sector, while the quantities w_{ij}, F_i, and S_i satisfy

$$S^i{}_{|i} = F^i{}_{|i} = 0, \qquad w^{ij}{}_{|j} = 0, \qquad w^i{}_{|i} = 0, \quad (35)$$

and

$$\xi^i = \eta^i + \zeta^{|i}, \quad (36)$$

with

$$\eta^i{}_{|i} = 0, \quad (37)$$

in the matter sector. Substituting the above decompositions into Eq. (25) leads to a separation of this Lagrangian, with some total derivatives discarded, into three independent

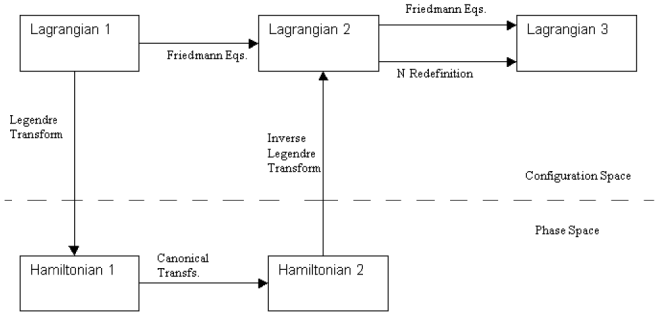

FIG. 1. Pictorial representation of the simplification schemes presented. Mukhanov follows the configuration space path, based on the Friedmann equations. We propose the configuration/phase space path based on the canonical transformation and redefinition of the lapse function. By Lagrangians 1, 2, and 3 we mean Eqs. (20), (23), and (25). By Hamiltonians 1 and 2 we mean Eqs. (28) and (29).

sectors: scalar, vector, and tensor sectors. We will focus our attention on the vector and scalar sectors because the case of tensor perturbations has already been treated in Ref. [19].

III. VECTOR PERTURBATIONS

Combining the contributions of gravitational and matter sectors and defining the gauge invariant quantities

$$V^i = S^i - \frac{a}{N}\dot{F}^i \tag{38}$$

and

$$\eta^{i(gi)} = \eta^i + F^i, \tag{39}$$

we obtain

$$L^V = \frac{Na}{12l^2}\int d^3x\gamma^{1/2}V^{i|j}V_{i|j} + \frac{1}{2}Na^3(\lambda+1)\varepsilon_0 \times \int d^3x\gamma^{1/2}\left(\frac{a}{N}\dot{\eta}^{i(gi)} + V^i\right)\left(\frac{a}{N}\dot{\eta}_i^{(gi)} + V_i\right). \tag{40}$$

When constructing the Hamiltonian, the primary constraint $\Pi_i \approx 0$ appears, where $\approx$ means a weak equality in the sense of Dirac [23], and Π_i is the momentum canonically conjugate to V^i. The Hamiltonian then reads

$$H^V = N\left[-\frac{l^2P_a^2}{4aV} + \frac{P_T}{a^{3\lambda}} + \int d^3x\left(\frac{P^iP_i}{2(1+\lambda)P_Ta^{2-3\lambda}\gamma^{1/2}} - \frac{V^iP_i}{a} - \frac{a}{12l^2}\gamma^{1/2}V^{i|j}V_{i|j}\right)\right] + \int d^3x\Lambda^i\pi_i, \tag{41}$$

where P_i is the momentum canonically conjugate to $\eta^{i(gi)}$, and the Λ^i are Lagrange multipliers.

The conservation of Π_i in time imposes a secondary constraint

$$\dot{\Pi}^i = \{\Pi^i, H^V\} = \phi_2^i = \frac{1}{a}\pi_\eta^i - \frac{a}{6l^2}\gamma^{1/2}V^{i|j}{}_{|j}. \tag{42}$$

The conservation in time of ϕ_2^i fixes the Lagrange multipliers Λ^i to the value

$$\Lambda^i = \frac{l^2P_a}{a^2V}V^i. \tag{43}$$

Then the equations of motion for V^i and $\eta^{i(gi)}$ imply that

$$V^i = \frac{V_0^i}{a^2}, \tag{44}$$

and

$$\varphi^i \equiv \left(\frac{a}{N}\dot{\eta}^{i(gi)} + V^i\right) = \frac{\nabla^2V_0^i}{(\lambda+1)P_Ta^{1-\lambda}}. \tag{45}$$

These solutions correspond to the classical result, which was obtained without recurring to the classical background equations.

The two constraints obtained are second class. After defining the corresponding Dirac brackets [23], they become strong equalities which can be used to obtain some variables from others.

IV. SCALAR PERTURBATIONS

Defining the quantities

$$F = B - \frac{a}{N}\dot{E}, \tag{46}$$

$$\zeta^{(gi)} = \zeta + E, \tag{47}$$

which is a gauge invariant quantity, and

$$\varphi = \frac{\sqrt{6}la^2\sqrt{(\lambda+1)\varepsilon_0}}{\sqrt{\lambda}}\left(\frac{a}{N}\dot{\zeta}^{(gi)} + F\right), \tag{48}$$

which can be identified with the perturbed velocity potential of the fluid particles, the scalar Lagrangian reads

$$L^E = \frac{Na}{3l^2}\int d^3x\gamma^{1/2}(\psi^{,i}\psi_{,i} - 2\phi^{,i}\psi_{,i}) - \frac{2a^2}{3l^2}\int d^3x\gamma^{1/2}\left(\dot{\psi} + \frac{\dot{a}}{a}\phi\right)F^{,i}{}_{,i} - \frac{a^3}{Nl^2}\int d^3x\gamma^{1/2}\left(\dot{\psi} + \frac{\dot{a}}{a}\phi\right)^2 - \frac{Na^3(\lambda+1)\varepsilon_0}{2}\int d^3x\gamma^{1/2}[\lambda(3\psi - \zeta^{,i(gi)}{}_{,i})^2 + 2\phi(3\psi - \zeta^{,i(gi)}{}_{,i})] + \frac{N\lambda}{12l^2a}\int d^3x\gamma^{1/2}\varphi^{,i}\varphi_{,i}. \tag{49}$$

As in the vector sector, some constraints appear, and, because of definition (48) which involves a time derivative, we have to use the Ostrogradski method [24] through the definition $\pi_\zeta = \partial L/\partial\dot{\zeta} - \dot{\pi}_\varphi$. The constraints are

$$\phi_1 = P_N; \quad \phi_2 = \pi_F; \quad \phi_3 = \pi_\phi; \quad \phi_7 = \pi_\varphi; \quad \phi_9 = P_\mu, \tag{50}$$

and the Hamiltonian is

$$H = N\left[H_0 + \int d^3x\Lambda_\phi\pi_\phi + \int d^3x\Lambda_F\pi_F + \int d^3x\Lambda_\varphi\pi_\varphi\right] + \Lambda_NP_N \tag{51}$$

where H_0 reads

$$H_0 = -\frac{l^2 P_a^2}{4aV} + \frac{P_T}{a^{3\lambda}} + \frac{(\lambda+1)P_T}{2a^{3\lambda}V}\int d^3x\gamma^{1/2}[\lambda(3\psi - \zeta^{,i(gi)}{}_{,i})^2 + 2\phi(3\psi - \zeta^{,i(gi)}{}_{,i})] + \frac{l^2 P_a}{2a^2 V}\int d^3x\phi\pi_\psi$$
$$+ \frac{1}{a}\int d^3x\pi_\zeta\left(\frac{\sqrt{V}\sqrt{\lambda}}{\sqrt{6}l\sqrt{(\lambda+1)P_T}}a^{-(1/2)(1-3\lambda)}\varphi - F\right) - \frac{a}{l^2}\int d^3x\gamma^{1/2}\left(\frac{l^2}{2a^2\gamma^{1/2}}\pi_\psi + \frac{1}{3}F^{,i}{}_{,i}\right)^2$$
$$+ \frac{\lambda}{12l^2 a}\int d^3x\gamma^{1/2}\varphi\varphi^{,i}{}_{,i}\frac{a}{3l^2}\int d^3x\gamma^{1/2}(\psi - 2\phi)\psi^{,i}{}_{,i}. \tag{52}$$

Conservation in time of the primary constraints (50) leads to the secondary constraints

$$\phi_4 = \frac{H}{N},$$
$$\phi_5 = \frac{1}{a}\pi_\zeta + \frac{2a}{3l^2}\gamma^{1/2}\left(\frac{l^2}{2a^2\gamma^{1/2}}\pi_\psi + \frac{1}{3}F^{,i}{}_{,i}\right)^{,j}{}_{,j},$$
$$\phi_6 = -\frac{(\lambda+1)P_T}{a^{3\lambda}V}\gamma^{1/2}(3\psi - \zeta^{,i}{}_{,i}) - \frac{l^2 P_a}{2a^2 V}\pi_\psi + \frac{2a}{3l^2}\gamma^{1/2}\psi^{,i}{}_{,i}, \tag{53}$$
$$\phi_8 = \frac{1}{a}\frac{\sqrt{V}}{\sqrt{6}l\sqrt{(\lambda+1)P_T}}a^{-(1/2)(1-3\lambda)}\pi_\zeta + \frac{\sqrt{\lambda}}{6l^2 a}\gamma^{1/2}\varphi^{,i}{}_{,i}.$$

Neglecting third order terms, conservations in time of ϕ_4 and ϕ_6 are identically satisfied, whereas conservation of ϕ_5 and ϕ_8 determines the Lagrange multipliers Λ_F and Λ_φ. The Lagrange multiplier Λ_F reads[1]

$$a\Lambda_F = \psi - \phi + \frac{l^2 P_a}{aV}F. \tag{54}$$

As $\Lambda_F = \dot{F}/N$, then

$$\frac{a}{N}\dot{F} = \psi - \phi - \frac{2\dot{a}}{N}F, \tag{55}$$

which, when expressed in terms of the gauge invariant Bardeen potentials, yields

$$\Phi = \Psi, \tag{56}$$

a well-known result.

The Poisson brackets among the constraints read

$$\{\phi_2(x), \phi_5(x')\} = -\frac{2a}{9l^2}\gamma^{1/2}\delta(x-x')^{,ij}{}_{,ij},$$
$$\{\phi_7(x), \phi_8(x')\} = -\frac{\sqrt{\lambda}}{6l^2 a}\gamma^{1/2}\delta(x-x')^{,i}{}_{,i},$$
$$\{\phi_6(x), \phi_5(x')\} = \frac{2}{9l^2}\gamma^{1/2}\delta(x-x')^{,ij}{}_{,ij} - \frac{l^2}{a^2 V}\pi_\psi(x)\pi_\zeta(x'), \tag{57}$$
$$\{\phi_6(x), \phi_8(x')\} = \frac{\sqrt{(\lambda+1)P_T}}{\sqrt{6}l\sqrt{V}}a^{-(3/2)(1+\lambda)}\gamma^{1/2}\delta(x-x')^{,i}{}_{,i} - \frac{l}{4\sqrt{6}\sqrt{V(\lambda+1)P_T}}(1-3\lambda)a^{-(3/2)(1+3\lambda)}\pi_\psi(x)\pi_\zeta(x').$$

Defining

$$\bar{\phi}_6 = \phi_6 + \frac{1}{a}\phi_2 + \frac{\sqrt{(\lambda+1)P_T}\sqrt{6}l}{\sqrt{\lambda}\sqrt{V}}a^{-(1/2)(1+3\lambda)}\phi_7, \tag{58}$$

one can prove that $\bar{\phi}_6$ is a first class constraint: it has zero Poisson brackets with all other constraints up to third order. We are then left with the four second class constraints, ϕ_2, ϕ_5, ϕ_7, and ϕ_8. Hence, from the 10 degrees of freedom of phase space corresponding to the variables ϕ, ψ, F, φ, and ξ, we have to extract four from the second class constraints and $2\times 2 = 4$ from the two first class constraints $\bar{\phi}_6$ and ϕ_3, leaving 2 degrees of freedom in phase space, as expected for this problem.

In order to eliminate the second class constraints, we have to define the Dirac brackets associated with them. The Dirac brackets among the variables of phase space which are not canonical are (excepting the ones involving F and π_F, which are not relevant)

$$\{\zeta^{,i(gi)}{}_{,i}(x), \varphi(x')\}^D = -\frac{\sqrt{6}l\sqrt{V}a^{-(1/2)(1-3\lambda)}}{\sqrt{\lambda}\sqrt{(\lambda+1)P_T}\gamma^{1/2}}\delta(x-x'),$$
$$\{P_a, \varphi(x)\}^D = \frac{1}{2a}(1-3\lambda)\varphi(x). \tag{59}$$

[1]The explicit value of Λ_φ is not important for what follows.

Defining the quantities

$$\begin{aligned}\varphi_{(c)} &=: a^{(1/2)(1-3\lambda)}\varphi,\\ \pi_{\varphi(c)} &=: -\frac{\sqrt{\lambda}\sqrt{(\lambda+1)P_T}}{\sqrt{6}l\sqrt{V}}\gamma^{1/2}(3\psi-\zeta^{,i(gi)}{}_{,i}),\\ \pi_{\psi(c)} &= \pi_\psi - \frac{3\sqrt{\lambda}\sqrt{(\lambda+1)P_T}}{\sqrt{6}l\sqrt{V}}\gamma^{1/2}\varphi_{(c)},\end{aligned} \tag{60}$$

we obtain that the Dirac brackets for these quantities are canonical. The Hamiltonian in terms of these new variables then reads

$$H = NH_0 - N\int d^3x\phi\phi_6 + \int d^3x\Lambda_\phi\pi_\phi \tag{61}$$

where H_0 is given by

$$\begin{aligned}H_0 = &-\frac{l^2P_a^2}{4aV}+\frac{P_T}{a^{3\lambda}}+\frac{3l^2}{a^{3\lambda}}\int d^3x\frac{\pi^2_{\varphi(c)}}{\gamma^{1/2}}-\frac{\lambda}{12l^2a^{2-3\lambda)}}\\ &\times\int d^3x\gamma^{1/2}\varphi_{(c)}\varphi^{,i}_{(c),i}-\frac{3\lambda(\lambda+1)P_T}{8a^3V}\\ &\times\int d^3x\gamma^{1/2}\varphi^2_{(c)}-\frac{\sqrt{\lambda}}{2}\sqrt{\frac{3}{2}}\frac{l\sqrt{(\lambda+1)P_T}}{a^3\sqrt{V}}\\ &\times\int d^3x\varphi_{(c)}\pi_{\psi(c)}+\frac{a}{3l^2}\int d^3x\gamma^{1/2}\psi\psi^{,i}{}_{,i}\end{aligned} \tag{62}$$

and ϕ_6 by (we omitted the bars)

$$\begin{aligned}\phi_6 = &\frac{\sqrt{6}l\sqrt{(\lambda+1)P_T}}{\sqrt{\lambda}\sqrt{V}}a^{-3\lambda}\pi_{\varphi(c)}-\frac{l^2P_a}{2a^2V}\pi_{\psi(c)}\\ &-\frac{3\sqrt{\lambda}lP_a\sqrt{(\lambda+1)P_T}}{2\sqrt{6}a^2V^{3/2}}\gamma^{1/2}\varphi_{(c)}+\frac{2a}{3l^2}\gamma^{1/2}\psi^{,i}{}_{,i}.\end{aligned} \tag{63}$$

From the second class constraints we obtain the identity

$$\frac{l^2}{2a^2\gamma^{1/2}}\pi_\psi+\frac{1}{3}F^{,i}{}_{,i}=\frac{3l^2}{2}\frac{\sqrt{\lambda}\sqrt{(\lambda+1)P_T}}{\sqrt{6}l\sqrt{V}}a^{-(3/2)(1+\lambda)}\varphi, \tag{64}$$

which in terms of the new canonical variables reads

$$\frac{l^2}{2a^2\gamma^{1/2}}\pi_{\psi(c)}+\frac{1}{3}F^{,i}{}_{,i}=0. \tag{65}$$

We will need this equation later.

We have shown that there are 2 degrees of freedom in the perturbative sector. Hence it should be possible to find a variable which renders the perturbed Hamiltonian to the simple form

$$H=\int d^3x\left[\alpha_1(t)\frac{\pi^2}{\gamma^{1/2}}+\gamma^{1/2}(\alpha_2(t)v^iv_{,i}+\alpha_3(t)v^2)\right], \tag{66}$$

where $\alpha_1(t)$, $\alpha_2(t)$, and $\alpha_3(t)$ depend on the background functions. This variable should have weakly vanishing Poisson brackets with the first order first class constraints $\phi_6\approx 0$ and $\pi_\phi\approx 0$ appearing in Eq. (61), generators of gauge transformations, in order to be gauge invariant. The unique combination, as it can be easily verified (apart an overall background function), of the configuration space variables $\varphi_{(c)}$, ϕ, and ψ which satisfies this requirement is

$$v=\frac{a^{(1/2)(3\lambda-1)}}{\sqrt{6}l}\left(\varphi_{(c)}+\frac{2\sqrt{6}\sqrt{(\lambda+1)P_T}\sqrt{V}}{lP_a\sqrt{\lambda}}a^{2-3\lambda}\psi\right), \tag{67}$$

which is just the Mukhanov-Sasaki variable. In order to obtain this new variable, one has to perform a new canonical transformation generated by

$$\begin{aligned}F_1 = &T\tilde{P}_T+a\tilde{P}_a+\int d^3x\left[\frac{1}{\sqrt{6}l}a^{-(1/2)(1-3\lambda)}\tilde{\varphi}_{(c)}\pi\right.\\ &+\psi\tilde{\pi}_\psi+\frac{2\sqrt{V}\sqrt{(\lambda+1)\tilde{P}_T}}{l^2\tilde{P}_a\sqrt{\lambda}}a^{(3/2)(1-\lambda)}\psi\pi\\ &\left.-\frac{\gamma^{1/2}}{2}\alpha\tilde{\varphi}^2_{(c)}\right]\end{aligned} \tag{68}$$

where α is given by

$$\alpha=-\frac{(\lambda+1)\tilde{P}_T}{2l^2\tilde{P}_aa}+\frac{(1-3\lambda)\tilde{P}_a}{24V}a^{-(2-3\lambda)}, \tag{69}$$

and π is the canonical momentum conjugate to v. The term proportional to α in Eq. (68) as well as its specific form given by Eq. (69) are made in order to eliminate a term proportional to $v\pi$ in the Hamiltonian. The new H_0 reads (the explicit canonical transformations are given in Appendix B)

$$
\begin{aligned}
H_0 = & -\frac{l^2 P_a^2}{4aV} + \frac{P_T}{a^{3\lambda}} + \frac{1}{2a}\int d^3x \frac{\pi^2}{\gamma^{1/2}} + \frac{\lambda}{2a}\int d^3x \gamma^{1/2} v^{,i} v_{,i} + \left[\frac{9[(\lambda+1)P_T]^2}{2P_a^2} a^{-(1+6\lambda)} - \frac{9\lambda[(\lambda+1)P_T]P_T}{2P_a^2} a^{-(1+6\lambda)}\right. \\
& \left. + \frac{(-4+18\lambda-18\lambda^2)l^4 P_a^2}{64a^3V^2} + \frac{6l^2(1+3\lambda^2)P_T}{16V} a^{-(2+3\lambda)}\right]\int d^3x \gamma^{1/2} v^2 \\
& + \frac{2\sqrt{\lambda}\sqrt{V}\sqrt{(\lambda+1)P_T}}{l^2 P_a} a^{(1/2)(1-3\lambda)}\int d^3x \gamma^{1/2} v \psi^{,i}{}_{,i} - \frac{\sqrt{\lambda}}{2}\frac{3l^2\sqrt{(\lambda+1)P_T}}{\sqrt{V}} a^{-(1/2)(5+3\lambda)}\int d^3x v \pi_\psi \\
& + \int d^3x \psi\left\{\left[\frac{18[(\lambda+1)P_T]^3 V}{l^4 P_a^4 \lambda} a^{(2-9\lambda)} - \frac{18[(\lambda+1)P_T]^2 P_T V}{l^4 P_a^4} a^{(2-9\lambda)} + \frac{(-2+9\lambda-9\lambda^2)[(\lambda+1)P_T]}{8\lambda V} a^{-3\lambda}\right.\right. \\
& \left. - \frac{3[(\lambda+1)P_T]^2}{l^2 P_a^2 \lambda} a^{(1-6\lambda)} + \frac{3(3+2\lambda+3\lambda^2)[(\lambda+1)P_T]P_T}{2l^2 P_a^2 \lambda} a^{(1-6\lambda)}\right]\gamma^{1/2}\psi \\
& + \left[\frac{(-2+3\lambda)\sqrt{(\lambda+1)P_T}}{2\sqrt{\lambda}\sqrt{V}} a^{-(1/2)(1+3\lambda)} - \frac{6\lambda\sqrt{V}\sqrt{(\lambda+1)P_T}P_T}{l^2 P_a^2\sqrt{\lambda}} a^{(1/2)(1-9\lambda)}\right]\pi \\
& + \left[-\frac{18[(\lambda+1)P_T]^{5/2}\sqrt{V}}{l^2 P_a^3\sqrt{\lambda}} a^{(1/2)(1-15\lambda)} + \frac{18\lambda[(\lambda+1)P_T]^{3/2}P_T\sqrt{V}}{l^2 P_a^3\sqrt{\lambda}} a^{(1/2)(1-15\lambda)}\right. \\
& + \frac{(2-9\lambda+9\lambda^2)l^2 P_a\sqrt{(\lambda+1)P_T}}{8V^{3/2}\sqrt{\lambda}} a^{-(3/2)(1+\lambda)} + \frac{3[(\lambda+1)P_T]^{3/2}}{\sqrt{\lambda}\sqrt{V}P_a} a^{-(1/2)(1+9\lambda)} \\
& \left. - \frac{3(3+2\lambda+3\lambda^2)\sqrt{(\lambda+1)P_T}P_T}{2\sqrt{\lambda}\sqrt{V}P_a} a^{-(1/2)(1+9\lambda)}\right]\gamma^{1/2} v + \left[\frac{a}{3l^2} - \frac{2(\lambda+1)P_T V}{l^4 P_a^2} a^{2-3\lambda}\right]\gamma^{1/2}\psi^{,i}{}_{,i} \\
& \left. + \frac{3(\lambda+1)P_T}{P_a} a^{-(1+3\lambda)}\pi_\psi\right\}. \qquad (70)
\end{aligned}
$$

This canonical transformation applied to ϕ_6 shows that v is a gauge invariant quantity. The same is not true for its momentum π, which has a nonzero Poisson bracket with the first class constraint ϕ_6. In order to obtain a gauge invariant momentum π, we must make a redefinition of π to a gauge invariant $\tilde{\pi}$, which reads

$$
\tilde{\pi} = \pi - \left[\frac{6\sqrt{V}[(\lambda+1)P_T]^{3/2}}{l^2 P_a^2\sqrt{\lambda}} a^{(3/2)(1-3\lambda)} - \frac{(1+3\lambda)\sqrt{(\lambda+1)P_T}}{2\sqrt{\lambda}\sqrt{V}} a^{(1/2)(1-3\lambda)}\right]\gamma^{1/2}\psi, \qquad (71)
$$

which induces a new canonical transformation generated by

$$
\begin{aligned}
\mathcal{F}_2 = & a\tilde{P}_a + \int d^3x\left\{\psi\tilde{\pi}_\psi + v\tilde{\pi} + \left[\frac{6\sqrt{V}[(\lambda+1)P_T]^{3/2}}{l^2 \tilde{P}_a{}^2\sqrt{\lambda}} a^{(3/2)(1-3\lambda)} - \frac{(1+3\lambda)\sqrt{(\lambda+1)P_T}}{2\sqrt{\lambda}\sqrt{V}} a^{(1/2)(1-3\lambda)}\right]\gamma^{1/2} v\psi\right. \\
& \left. + \left[-\frac{6V[(\lambda+1)P_T]^2}{l^4 \tilde{P}_a{}^3\lambda} a^{3(1-2\lambda)} + \frac{(1+3\lambda)(\lambda+1)P_T}{2l^2\tilde{P}_a\lambda} a^{2-3\lambda}\right]\gamma^{1/2}\psi^2 + \frac{2a^3 V}{3l^4\tilde{P}_a}\psi\psi^{,i}{}_{,i}\right\}. \qquad (72)
\end{aligned}
$$

Aiming at eliminating a term in v^2 proportional to P_T in the final form of the Hamiltonian, we perform the last canonical transformation

$$
\mathcal{F}_3 = a\tilde{P}_a + \frac{1}{a}\int d^3x v\tilde{\pi} - \frac{l^2\tilde{P}_a}{4aV}\int d^3x \gamma^{1/2} v^2. \qquad (73)
$$

After these two canonical transformations, the constraint ϕ_6 then reads

$$
\phi_6 = -\frac{l^2 P_a}{2a^2 V}\pi_\psi, \qquad (74)
$$

and as $-\frac{l^2 P_a}{2a^2 V}$ is not weakly zero, we can redefine the constraint ϕ_6 to be

$$
\phi_6 = \pi_\psi. \qquad (75)
$$

This constraint, in the Dirac quantization scheme, will imply that the wave functional will not depend on ψ, and the second class constraint (65) turns out to be the usual relation between ϕ and v from Ref. [1], Eq. (12.8), as we will see later on. The new H_0 is given by

$$
\begin{aligned}
H_0 &= H_0^{(0)} + H_0^{(2)} + \int d^3x[F^{(1)}\phi_6 + F^{(2)}H_0^{(0)}] \\
&\equiv -\frac{l^2P_a^2}{4aV} + \frac{P_T}{a^{3\lambda}} + \frac{1}{2a^3}\int d^3x\frac{\pi^2}{\gamma^{1/2}} + \frac{a\lambda}{2}\int d^3x\gamma^{1/2}v^{,i}v_{,i} \\
&\quad + \left\{-\frac{\sqrt{\lambda}}{2}\frac{3l^2\sqrt{(\lambda+1)P_T}}{\sqrt{V}}a^{-(3/2)(1+\lambda)}v + \frac{3(\lambda+1)P_T}{P_a}a^{-(1+3\lambda)}\psi\right\}\int d^3x\pi_\psi \\
&\quad + H_0^{(0)}\int d^3x\left\{-\frac{9(\lambda+1)P_T}{l^2P_a^2}a^{1-3\lambda}\gamma^{1/2}\psi^2 + \frac{6\sqrt{V}\sqrt{(\lambda+1)P_T}}{l^2P_a^2\sqrt{\lambda}}a^{-(1/2)(1+3\lambda)}\psi\pi\right. \\
&\quad + \left[\frac{18[(\lambda+1)P_T]^{3/2}\sqrt{V}}{l^2P_a^3\sqrt{\lambda}}a^{(3/2)(1-3\lambda)} + \frac{9(\lambda-1)\sqrt{(\lambda+1)P_T}}{2\sqrt{V}\sqrt{\lambda}P_a}a^{(1/2)(1-3\lambda)}\right]\gamma^{1/2}v\psi + \frac{2a^2V}{l^4P_a^2}\psi\psi^{,i}{}_{,i} \\
&\quad \left. + \left[-\frac{9(1+2\lambda)[(\lambda+1)P_T}{2P_a^2}a^{1-3\lambda} + \frac{9\lambda(\lambda-1)l^2}{8V}\right]v^2\right\},
\end{aligned}
\tag{76}
$$

where $H_0^{(0)}$ and $H_0^{(2)}$ are the zeroth and second order Hamiltonian constraints, and $F^{(1)}$ and $F^{(2)}$ are first and second order functions which can be read from Eq. (76).

We now make the redefinitions of N and ϕ as $\tilde{N} = N(1 + \int d^3xF^{(2)})$, which would again just imply a different irrelevant time gauge choice with terms beyond first order, and $\tilde{\phi} = N(-\frac{l^2P_a}{2a^2V}\phi + F^{(1)})$. From the inverse of the transformations (60) and (73), and definition (48), $\tilde{\phi}$ is given by

$$
\frac{l^2P_a}{2a^2V}\phi - \frac{3l^2}{2}Na(\lambda+1)\epsilon_0\left(\frac{a}{N}\dot{\xi} + B\right). \tag{77}
$$

As $\tilde{\phi}$ is, through the equations of motion, equal to $\dot{\psi}$, we obtain, imposing $N = a$, the constraint equation (10.39) of Ref. [1].

Inserting expression (76) into Eq. (61), and the above redefinitions, we obtain, omitting the tilde,

$$
\begin{aligned}
H &= N(H_0^{(0)} + H_0^{(2)}) + \Lambda_N P_N + \int d^3x\phi\phi_6 \\
&\quad + \int d^3x\Lambda_\phi\pi_\phi,
\end{aligned}
\tag{78}
$$

with

$$
H_0^{(0)} \equiv -\frac{l^2P_a^2}{4aV} + \frac{P_T}{a^{3\lambda}} \tag{79}
$$

and

$$
H_0^{(2)} \equiv \frac{1}{2a^3}\int d^3x\frac{\pi^2}{\gamma^{1/2}} + \frac{a\lambda}{2}\int d^3x\gamma^{1/2}v^{,i}v_{,i}. \tag{80}
$$

Now we are left with two first class constraints (in fact one plus ∞^3 constraints): one with the homogeneous lapse function N as its associated Lagrange multiplier, which in the quantization procedure will lead to the Wheeler-DeWitt equation, and the other ∞^3 constraints with $\phi(x^i)$ as their Lagrange multiplier, which is nothing but the inhomogeneous lapse function [see definition (3)], which, as anticipated, has been tremendously simplified to imply a simple consequence when quantized—the wave functional does not depend on ψ. The supermomentum constraint is automatically satisfied because the v variable is gauge invariant.

The connection between $\bar{v} = av$ (the Mukhanov-Sasaki variable) and Φ can be obtained from Eq. (65) which, after implementing the canonical transformations (68) and (72), reads (the bars are omitted)

$$
\begin{aligned}
&\frac{l^2}{2a^2\gamma^{1/2}}\pi_\psi + \frac{2aV}{3l^2P_a}\left(\psi + \frac{l^2P_a}{2aV}F\right)^{,i}{}_{,i} \\
&\quad + \left[\frac{3\sqrt{V}[(\lambda+1)P_T]^{3/2}}{P_a^2\sqrt{\lambda}}a^{-(1/2)(1+9\lambda)}\right. \\
&\quad \left. - \frac{(1+3\lambda)\sqrt{(\lambda+1)P_T}l^2}{4\sqrt{\lambda}\sqrt{V}}a^{-3/2(1+\lambda)}\right]v \\
&\qquad + \frac{\sqrt{(\lambda+1)P_T}\sqrt{V}}{P_a\sqrt{\lambda}\gamma^{1/2}}a^{-(1/2)(1+3\lambda)}\pi = 0.
\end{aligned}
\tag{81}
$$

Using that $P_a = -\frac{2Va\dot{a}}{Nl^2}$, we can identify the quantity $\psi + \frac{l^2P_a}{2aV}F$ with the Bardeen potential Ψ which, from Eq. (56), is equal to Φ. After some algebraic manipulations, we obtain

$$
\begin{aligned}
&\frac{3l^4P_a}{4a^3V\gamma^{1/2}}\pi_\psi + \Phi^i{}_i + \frac{3l^2\sqrt{(\lambda+1)P_T}}{2\sqrt{\lambda}\sqrt{V}} \\
&\quad \times a^{-(3/2)(1+\lambda)}\left\{\frac{\pi}{\gamma^{1/2}} + \left[\left(\frac{l^2P_a}{2aV} + \frac{3(\lambda+1)}{P_a}H_0\right)v\right]\right\} \\
&\qquad + O(3) = 0.
\end{aligned}
\tag{82}
$$

Using again that $P_a = -\frac{2Va\dot{a}}{Nl^2}$, and that $\pi = \gamma^{1/2}\dot{v}$, $H_0 \approx 0$, $\pi_\psi \approx 0$, and choosing the gauge $N = a$ (conformal time), we obtain

$$\Phi^{,i}{}_{,i} = -\frac{3l^2\sqrt{(\lambda+1)P_T}}{2\sqrt{\lambda}\sqrt{V}}a^{-(1/2)(1+3\lambda)}\left(\frac{v}{a}\right)'. \tag{83}$$

Equation (83) coincides with Eq. (12.8) of Ref. [1] relating v and Φ when the classical equations of motion are used.

V. DIRAC QUANTIZATION

In this section we will focus only on the quantization of scalar perturbations. Vector perturbations are trivial and the quantization of tensor perturbations was done in Ref. [19].

A. The functional Schrödinger equation

In the Dirac quantization procedure, the first class constraints must annihilate the wave functional $\chi[N, a, \phi(x^i), \psi(x), v(x^i), T]$, yielding

$$\frac{\partial}{\partial N}\chi = 0, \qquad \frac{\delta}{\delta\phi}\chi = 0, \qquad \frac{\delta}{\delta\psi}\chi = 0, \qquad H\chi = 0. \tag{84}$$

The first three equations impose that the wave functional does not depend on N, ϕ, and ψ: as mentioned above, N and ϕ are, respectively, the homogeneous and inhomogeneous parts of the total lapse function, which are just Lagrange multipliers of constraints, and ψ has been substituted by $v(x^i)$, the unique degree of freedom of scalar perturbations, as expected.

As P_T appears linearly in H, and making the gauge choice $N = a^{3\lambda}$, one can interpret the T variable as a time parameter. Hence, the equation

$$H\chi = 0 \tag{85}$$

assumes the Schrödinger form

$$\iota\frac{\partial}{\partial T}\chi = \frac{1}{4}\left\{a^{(3\lambda-1)/2}\frac{\partial}{\partial a}\left[a^{(3\lambda-1)/2}\frac{\partial}{\partial a}\right]\right\}\chi - \left[\frac{a^{3\lambda-1}}{2}\int d^3x\frac{\delta^2}{\delta v^2} - \frac{a^{3\lambda+1}\lambda}{2}\int d^3x v^{,i}v_{,i}\right]\chi, \tag{86}$$

where we have chosen the factor ordering in a in order to yield a covariant Schrödinger equation under field redefinitions, and V and l have been absorbed in redefinitions of the fields.

B. Further developments using the Bohm-de Broglie interpretation

If one makes the ansatz

$$\chi[a, v, T] = \chi_{(0)}(a, T)\chi_{(2)}[v, T] \tag{87}$$

where $\chi_{(0)}(a, T)$ satisfies the equation

$$\iota\frac{\partial}{\partial T}\chi_{(0)}(a, T) = \frac{1}{4}\left\{a^{(3\lambda-1)/2}\frac{\partial}{\partial a}\left[a^{(3\lambda-1)/2}\frac{\partial}{\partial a}\right]\right\}\chi_{(0)}(a, T), \tag{88}$$

then we obtain for $\chi_{(2)}(a, v, T)$ the equation

$$\iota\frac{\partial}{\partial T}\chi_{(2)}(a, v, T) = -\frac{a^{(3\lambda-1)}}{2}\int d^3x\frac{\delta^2}{\delta v^2}\chi_{(2)}(a, v, T) + \frac{\lambda a^{(3\lambda+1)}}{2}\int d^3x v^{,i}v_{,i}\chi_{(2)}(a, v, T). \tag{89}$$

Solutions of the zeroth order equation (88) are known [14,16]. If one uses the ontological Bohm-de Broglie interpretation of quantum mechanics in order to obtain the Bohmian trajectories $a(T)$ from Eq. (88), this $a(T)$ can be viewed as a given function of time in the second equation (88). Going to conformal time $d\eta = a^{3\lambda-1}dT$, and performing the unitary transformation

$$U = e^{\{\iota[\int d^3x\gamma^{1/2}\dot{a}v/(2a)]\}}e^{-\{\iota[\int d^3x(v\pi+\pi v)/2\ln(a)]\}}, \tag{90}$$

the Schrödinger functional equation for the perturbations is transformed to

$$i\frac{\partial\chi_{(2)}[v, \eta]}{\partial\eta} = \int d^3x\left(-\frac{1}{2}\frac{\delta^2}{\delta v^2} + \frac{\lambda}{2}v_{,i}v^i - \frac{a''}{2a}v^2\right)\chi_{(2)}[v, \eta], \tag{91}$$

where we have gone to the new quantum variable $\bar{v} = av$, the Mukhanov-Sasaki variable defined in Ref. [1], after performing transformation (90), and we have omitted the bars.

The corresponding time evolution equation for the operator v in the Heisenberg picture is given by

$$v'' - \lambda v^{,i}{}_{,i} - \frac{a''}{a}v = 0, \tag{92}$$

where a prime means derivative with respect to conformal time. In terms of the normal modes v_k, the above equation reads

$$v_k'' + \left(\lambda k^2 - \frac{a''}{a}\right)v_k = 0. \tag{93}$$

These equations have the same form as the equations for scalar perturbations obtained in Ref. [1] (for one single fluid, the pump function z''/z obtained in [1] is exactly equal to a''/a obtained here, if we make use of the background equations). The difference is that the function $a(\eta)$ is no longer a classical solution of the background equations but a quantum Bohmian trajectory of the quantized background, which may lead to different power spectra.

VI. CONCLUSION

In this paper we have managed to obtain simple Hamiltonians for scalar perturbations when the matter content is described by a perfect fluid, without recurring to the background classical equations. Performing canonical transformations and redefining the homogeneous lapse functions with terms which do not alter the linear perturbation equations, the constraint connected to the inhomogeneous part of the lapse function is greatly simplified implying that the momentum canonically conjugate to the scalar perturbation ψ is weakly zero. The Hamiltonian constraint is also greatly simplified when written in terms of a new variable which is exactly equal to the usual Mukhanov-Sasaki variable [1].

This simplified Hamiltonian can now be used in the Dirac quantization procedure not only to quantize the perturbations but also the background, yielding a Wheeler-DeWitt equation much simpler to handle than the one of Ref. [5]. In the case of perfect fluids, where a preferred time variable appears and the Wheeler-DeWitt equation can be put in a Schrödinger form, and using the Bohm-de Broglie interpretation of quantum mechanics to perform a last unitary transformation, one obtains an equation for the modes which has the same form as in Ref. [1], where the pump field is obtained from a scale factor which now takes into account the quantum effects, the quantum Bohmian trajectory of the background.

In future publications, we will apply these results to specific models, and evaluate the power spectrum of scalar perturbations which arise on them in order to compare, when taken together with the results of Ref. [20], with observations. One application of our results is already published: see Ref [25].

The case of a scalar field was also calculated and the result shown in Appendix A. It is in accordance with the result previously obtained in Ref. [21], obtained through a different method. The further developments we have implemented in Sec. V for perfect fluids are much more involved in the case of scalar fields. First, the simplified Hamiltonian in the case of a scalar field [see Eq. (A40) below and Ref. [21]], although being a substantial improvement over the one obtained in [5], still contains coupling terms involving nontrivial powers of the canonical background momenta and the perturbation degrees of freedom, which renders the total Wheeler-DeWitt equation of the system (perturbations plus background) much more difficult to handle. Second, there is no preferred gauge time choice as in the perfect fluid case. Finally, the Bohmian trajectories obtained from the minisuperspace background quantum solutions, if they are still applicable when the perturbations are also considered, do not have an explicit form (see Ref. [15]), which imposes further technical difficulties. Hence, the calculation of scalar perturbations for the scalar field case when the background is also quantized demands much more work. This problem will be addressed in our future publications.

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APPENDIX A: THE SCALAR FIELD

In this appendix we implement the same simplifications we have done for hydrodynamical matter for the case of a scalar field.

The scalar field Lagrangian reads

$$\mathcal{L}_m = \tfrac{1}{2}\varphi_{;\mu}\varphi^{;\mu} - \tfrac{1}{2}U(\varphi). \tag{A1}$$

We write its perturbation as

$$\varphi = \varphi_0 + \delta\varphi, \tag{A2}$$

where φ_0 is the homogeneous scalar field depending only on time. Substituting (A2) in (A1) we obtain

$$\begin{aligned}\mathcal{L}_m = {} & \frac{\dot{\varphi}_0^2}{2N^2} - \frac{1}{2}U - \frac{\dot{\varphi}_0^2}{N^2}\phi + \frac{\dot{\varphi}_0}{N^2}\delta\dot{\varphi} - \frac{1}{2}U_\varphi\delta\varphi \\ & + \frac{2\dot{\varphi}_0^2}{N^2}\phi^2 - \frac{\dot{\varphi}_0^2}{2N^2}A^iA_i - \frac{2\dot{\varphi}_0}{N^2}\phi\delta\dot{\varphi} \\ & - \frac{\dot{\varphi}_0}{Na}A^i\delta\varphi_{|i} + \frac{1}{2N^2}\delta\dot{\varphi}^2 - \frac{1}{2a^2}\delta\varphi^i\delta\varphi_i \\ & - \frac{1}{4}U_{\varphi\varphi}\delta\varphi^2.\end{aligned} \tag{A3}$$

The total Lagrangian including the gravitational sector reads

$$
\begin{aligned}
L = &-\frac{\dot{a}^2 aV}{l^2 N} + \frac{NKaV}{l^2} + \frac{\dot{\varphi}_0{}^2 a^3 V}{2N} - \frac{Na^3 VU}{2} + \frac{Na}{6l^2}\int d^3x\gamma^{1/2}\Big[A^{i|j}A_{[i|j]} - \frac{1}{4}\epsilon^{ij|k}\epsilon_{ij|k} + \frac{a}{N}\dot{A}_i\epsilon^{ij}{}_{|j} + \frac{1}{2}\epsilon^{ij}{}_{|j}\epsilon_i{}^k{}_{|k} + \phi_{|i}\epsilon^{ij}{}_{|j} \\
&- \frac{1}{2}\epsilon_{|i}\epsilon^{ij}{}_{|j} - \phi_{|i}\epsilon^{|i} + \frac{1}{4}\epsilon_{|i}\epsilon^{|i} + K\Big(\frac{1}{4}\epsilon^2 - \epsilon^{ij}\epsilon_{ij} - \epsilon\phi + A^iA_i - 3\phi^2\Big)\Big] + \frac{a^3}{24l^2N}\int d^3x\gamma^{1/2}\dot{\epsilon}^{ij}\dot{\epsilon}_{ij} \\
&- \frac{a^3}{24l^2N}\int d^3x\gamma^{1/2}\dot{\epsilon}^2 + \frac{a\dot{a}^2}{6l^2N}\int d^3x\gamma^{1/2}\Big(-9\phi^2 - 3\epsilon\phi - \frac{3}{4}\epsilon^2 + 3A^iA_i + \frac{3}{2}\epsilon^{ij}\epsilon_{ij}\Big) \\
&- \frac{2a\dot{a}}{3l^2}\int d^3x\gamma^{1/2}\Big(\phi A^i{}_{|i} - \frac{1}{2}A_i\epsilon^{ij}{}_{|j}\Big) + \frac{a^2\dot{a}}{3l^2N}\int d^3x\gamma^{1/2}\Big(\epsilon^{ij}\dot{\epsilon}_{ij} - \frac{1}{2}\epsilon\dot{\epsilon} - \phi\dot{\epsilon}\Big) - \frac{a^2}{6l^2}\int d^3x\gamma^{1/2}\dot{\epsilon}A^i{}_{|i} \\
&- \frac{a^3\dot{\varphi}_0}{N}\int d^3x\gamma^{1/2}\Big(\phi + \frac{1}{2}\epsilon\Big)\delta\dot{\varphi} + a^2\dot{\varphi}_0\int d^3x\gamma^{1/2}\delta\varphi A^i{}_{|i} - \frac{Na^3U_\varphi}{2}\int d^3x\gamma^{1/2}\Big(\phi - \frac{1}{2}\epsilon\Big)\delta\varphi \\
&+ \frac{\dot{\varphi}_0{}^2a^3}{4N}\int d^3x\gamma^{1/2}\Big(3\phi^2 + \epsilon\phi - A^iA_i - \frac{1}{2}\epsilon^{ij}\epsilon_{ij} + \frac{1}{4}\epsilon^2\Big) + \frac{Na^3U}{4}\int d^3x\gamma^{1/2}\Big(\phi^2 + \epsilon\phi - A^iA_i + \frac{1}{2}\epsilon^{ij}\epsilon_{ij} - \frac{1}{4}\epsilon^2\Big) \\
&+ \frac{Na^3}{2}\int d^3x\gamma^{1/2}\Big(\frac{\delta\dot{\varphi}^2}{N^2} - \frac{\delta\varphi^{|i}\delta\varphi_{|i}}{a^2} - \frac{1}{2}U_{\varphi\varphi}\delta\varphi^2\Big).
\end{aligned}
\tag{A4}
$$

Its Hamiltonian is given by

$$
\begin{aligned}
H = N\Big\{&-\frac{l^2P_a^2}{4aV} - \frac{KaV}{l^2} + \frac{P_\varphi^2}{2a^3V} + \frac{a^3VU}{2} + \frac{l^2P_a^2}{8aV^2}\int d^3x\gamma^{1/2}\phi^2 + \frac{l^2P_a^2}{24aV^2}\int d^3x\gamma^{1/2}\epsilon\phi - \frac{l^2P_a^2}{8aV^2}\int d^3x\gamma^{1/2}A^iA_i \\
&+ \frac{5l^2P_a^2}{48aV^2}\int d^3x\gamma^{1/2}\epsilon^{ij}\epsilon_{ij} - \frac{l^2P_a^2}{32aV^2}\int d^3x\gamma^{1/2}\epsilon^2 + \frac{P_a}{6V}\int d^3x\gamma^{1/2}A^i{}_{|i}\phi + \frac{P_a}{6V}\int d^3x\gamma^{1/2}A_i\epsilon^{ij}{}_{|j} \\
&- \frac{P_\varphi^2}{4a^3V^2}\int d^3x\gamma^{1/2}\Big(\phi^2 - \epsilon\phi - A^iA_i - \frac{1}{2}\epsilon^{ij}\epsilon_{ij} - \frac{1}{4}\epsilon^2\Big) + \frac{P_\varphi}{a^3V}\int d^3x\Big(\phi + \frac{1}{2}\epsilon\Big)\pi_\varphi - \frac{P_\varphi}{aV}\int d^3x\gamma^{1/2}\delta\varphi A^i{}_{|i} \\
&+ \frac{6l^2}{a^3}\int d^3x\frac{\pi^{ij}\pi_{ij}}{\gamma^{1/2}} - \frac{3l^2}{a^3}\int d^3x\frac{\pi^2}{\gamma^{1/2}} - \frac{l^2P_a}{2a^2V}\int d^3x\pi\epsilon - \frac{a}{4l^2}\int d^3x\gamma^{1/2}A^i{}_{|i}A^j{}_{|j} - \frac{1}{a}\int d^3x\pi A^i{}_{|i} + \frac{l^2P_a}{a^2V}\int d^3x\pi\phi \\
&+ \frac{2l^2P_a}{a^2V}\int d^3x\pi^{ij}\epsilon_{ij} + \frac{1}{2a^3}\int d^3x\frac{\pi_\varphi^2}{\gamma^{1/2}} - \frac{a}{6l^2}\int d^3x\gamma^{1/2}\Big[A^{i|j}A_{[i|j]} - \frac{1}{4}\epsilon^{ij|k}\epsilon_{ij|k} + \frac{1}{2}\epsilon^{ij}{}_{|j}\epsilon_i{}^k{}_{|k} + \phi_{|i}\epsilon^{ij}{}_{|j} - \frac{1}{2}\epsilon_{|i}\epsilon^{ij}{}_{|j} \\
&- \phi_{|i}\epsilon^{|i} + \frac{1}{4}\epsilon^{|i}\epsilon_{|i} + K\Big(\frac{1}{4}\epsilon^2 - \epsilon^{ij}\epsilon_{ij} - \epsilon\phi + A^iA_i - 3\phi^2\Big)\Big] + \frac{a^3U_\varphi}{2}\int d^3x\gamma^{1/2}\Big(\phi - \frac{1}{2}\epsilon\Big)\delta\varphi \\
&- \frac{a^3U}{4}\int d^3x\gamma^{1/2}\Big(\phi^2 + \epsilon\phi - A^iA_i + \frac{1}{2}\epsilon^{ij}\epsilon_{ij} - \frac{1}{4}\epsilon^2\Big) + \frac{a^3}{2}\int d^3x\gamma^{1/2}\Big(\frac{1}{a^2}\delta\varphi^{|i}\delta\varphi_{|i} + \frac{1}{2}U_{\varphi\varphi}\delta\varphi^2\Big) \\
&+ \frac{P_a}{12V}\int d^3x\gamma^{1/2}\epsilon A^i{}_i\Big\}.
\end{aligned}
\tag{A5}
$$

Performing the canonical transformation generated by

$$
\begin{aligned}
F = &\,a\tilde{P}_a + \varphi_0\tilde{P}_\varphi - \int d^3x(\tilde{\phi}\pi_\phi + \tilde{A}_i\pi_A^i + \tilde{\epsilon}_{ij}\pi^{ij} + \delta\tilde{\varphi}\pi_\varphi) - \frac{a\tilde{P}_a}{12V}\int d^3x\gamma^{1/2}\Big(\tilde{\epsilon}^{ij}\tilde{\epsilon}_{ij} - \frac{1}{2}\tilde{\epsilon}^2\Big) \\
&- \frac{\tilde{P}_\varphi}{V}\int d^3x\gamma^{1/2}\Big(\tilde{\phi} + \frac{1}{2}\tilde{\epsilon}\Big)\delta\tilde{\varphi}
\end{aligned}
\tag{A6}
$$

which are

$$
\begin{aligned}
a &= \tilde{a} + \frac{\tilde{a}}{12V}\int d^3x\gamma^{1/2}\Big(\tilde{\epsilon}^{ij}\tilde{\epsilon}_{ij} - \frac{1}{2}\tilde{\epsilon}^2\Big), \qquad P_a = \tilde{P}_a - \frac{\tilde{P}_a}{12V}\int d^3x\gamma^{1/2}\Big(\tilde{\epsilon}^{ij}\tilde{\epsilon}_{ij} - \frac{1}{2}\tilde{\epsilon}^2\Big), \\
\varphi_0 &= \tilde{\varphi}_0 + \frac{1}{V}\int d^3x\gamma^{1/2}\Big(\tilde{\phi} + \frac{1}{2}\tilde{\epsilon}\Big)\delta\tilde{\varphi}, \qquad \pi_\phi = \tilde{\pi}_\phi - \frac{\tilde{P}_\varphi}{V}\gamma^{1/2}\delta\tilde{\varphi}, \\
\pi^{ij} &= \tilde{\pi}^{ij} - \frac{\tilde{a}\tilde{P}_a}{6V}\gamma^{1/2}\Big(\tilde{\epsilon}^{ij} - \frac{1}{2}\tilde{\epsilon}\gamma^{ij}\Big) - \frac{\tilde{P}_\varphi}{2V}\gamma^{1/2}\delta\tilde{\varphi}\gamma^{ij}, \qquad \pi_\varphi = \tilde{\pi}_\varphi - \frac{\tilde{P}_\varphi}{V}\gamma^{1/2}\Big(\tilde{\phi} + \frac{1}{2}\tilde{\epsilon}\Big),
\end{aligned}
\tag{A7}
$$

yields the new Hamiltonian

$$H = N\Big\{-\frac{l^2P_a^2}{4aV} - \frac{KaV}{l^2} + \frac{P_\varphi^2}{2a^3V} + \frac{a^3VU}{2} + \frac{l^2P_a^2}{8aV^2}\int d^3x\gamma^{1/2}\phi^2 + \frac{l^2P_a^2}{8aV^2}\int d^3x\gamma^{1/2}\epsilon\phi - \frac{l^2P_a^2}{8aV^2}\int d^3x\gamma^{1/2}A^iA_i$$
$$+ \frac{P_a}{6V}\int d^3x\gamma^{1/2}A^i{}_{|i}\phi + \frac{P_a}{6V}\int d^3x\gamma^{1/2}A_i\epsilon^{ij}{}_{|j} - \frac{P_\varphi^2}{4a^3V^2}\int d^3x\gamma^{1/2}(3\phi^2 + \epsilon\phi - A^iA_i) + \frac{l^2P_a}{a^2V}\int d^3x\pi\phi$$
$$+ \frac{3l^2P_a}{a^3V}\int d^3x\pi\delta\varphi + \frac{P_\varphi}{2aV}\int d^3x\gamma^{1/2}\delta\varphi A^i{}_{|i} + \frac{6l^2}{a^3}\int d^3x\frac{\pi^{ij}\pi_{ij}}{\gamma^{1/2}} - \frac{3l^2}{a^3}\int d^3x\frac{\pi^2}{\gamma^{1/2}} - \frac{a}{4l^2}\int d^3x\gamma^{1/2}A^i{}_{|i}A^j{}_{|j}$$
$$- \frac{1}{a}\int d^3x\pi A^i{}_{|i} - \frac{9l^2P_\varphi^2}{4a^3V^2}\int d^3\gamma^{1/2}\delta\varphi^2 - \frac{3l^2P_aP_\varphi}{2a^2V^2}\int d^3x\gamma^{1/2}\phi\delta\varphi + \frac{1}{2a^3}\int d^3x\frac{\pi_\varphi^2}{\gamma^{1/2}}$$
$$- \frac{a}{6l^2}\int d^3x\gamma^{1/2}\Big[A^{i|j}A_{[i|j]} - \frac{1}{4}\epsilon^{ij|k}\epsilon_{ij|k} + \frac{1}{2}\epsilon^{ij}{}_{|j}\epsilon_i{}^k{}_{|k} + \phi_{|i}\epsilon^{ij}{}_{|j} - \frac{1}{2}\epsilon_{|i}\epsilon^{ij}{}_{|j} - \phi_{|i}\epsilon^{|i} + \frac{1}{4}\epsilon^{|i}\epsilon_{|i}$$
$$+ K\Big(-\frac{1}{2}\epsilon^{ij}\epsilon_{ij} - \epsilon\phi + A^iA_i - 3\phi^2\Big)\Big] + a^3U_\varphi\int d^3x\gamma^{1/2}\phi\delta\varphi - \frac{a^3U}{4}\int d^3x\gamma^{1/2}(\phi^2 + \epsilon\phi - A^iA_i)$$
$$+ \frac{a^3}{2}\int d^3x\gamma^{1/2}\Big(\frac{1}{a^2}\delta\varphi^{|i}\delta\varphi_{|i} + \frac{1}{2}U_{\varphi\varphi}\delta\varphi^2\Big)\Big\}. \tag{A8}$$

Going back to its corresponding Lagrangian, and redefining N as

$$N = \tilde{N}\Big[1 + \frac{1}{2V}\int d^3x\gamma^{1/2}(\epsilon\phi + \phi^2 - A^iA_i)\Big], \tag{A9}$$

we obtain

$$L = -\frac{\dot{a}^2aV}{l^2N} + \frac{NKaV}{l^2} + \frac{\dot{\varphi_0}^2a^3V}{2N} - \frac{Na^3VU}{2} + \frac{Na}{6l^2}\int d^3x\gamma^{1/2}\Big[A^{i|j}A_{[i|j]} - \frac{1}{4}\epsilon^{ij|k}\epsilon_{ij|k} + \frac{a}{N}\dot{A}_i\epsilon^{ij}{}_{|j} + \frac{1}{2}\epsilon^{ij}{}_{|j}\epsilon_i{}^k{}_{|k} + \phi_{|i}\epsilon^{ij}{}_{|j}$$
$$- \frac{1}{2}\epsilon_{|i}\epsilon^{ij}{}_{|j} - \phi_{|i}\epsilon^{|i} + \frac{1}{4}\epsilon_{|i}\epsilon^{|i} + K\Big(-\frac{1}{2}\epsilon^{ij}\epsilon_{ij} + 2\epsilon\phi - 2A^iA_i\Big)\Big] + \frac{a^3}{24l^2N}\int d^3x\gamma^{1/2}\dot{\epsilon}^{ij}\dot{\epsilon}_{ij} - \frac{a^3}{24l^2N}\int d^3x\gamma^{1/2}\dot{\epsilon}^2$$
$$- \frac{a\dot{a}^2}{l^2N}\int d^3x\gamma^{1/2}\phi^2 - \frac{2a\dot{a}}{3l^2}\int d^3x\gamma^{1/2}\Big(\phi A^i{}_{|i} - \frac{1}{2}A_i\epsilon^{ij}{}_{|j}\Big) - \frac{a^2\dot{a}}{3l^2N}\int d^3x\gamma^{1/2}\phi\dot{\epsilon} - \frac{a^2}{6l^2}\int d^3x\gamma^{1/2}\dot{\epsilon}A^i{}_{|i}$$
$$+ \frac{a^3\dot{\varphi}_0}{N}\int d^3x\gamma^{1/2}\Big(\dot{\phi} + \frac{1}{2}\dot{\epsilon}\Big)\delta\varphi + a^2\dot{\varphi}_0\int d^3x\gamma^{1/2}\delta\varphi A^i{}_{|i} - Na^3U_\varphi\int d^3x\gamma^{1/2}\phi\delta\varphi + \frac{\dot{\varphi_0}^2a^3}{2N}\int d^3x\gamma^{1/2}\phi^2$$
$$+ \frac{Na^3}{2}\int d^3x\gamma^{1/2}\Big(\frac{\delta\dot{\varphi}^2}{N^2} - \frac{\delta\varphi^{|i}\delta\varphi_{|i}}{a^2} - \frac{1}{2}U_{\varphi\varphi}\delta\varphi^2\Big). \tag{A10}$$

Splitting as before the perturbations into their tensor, vector, and scalar parts,

$$A_i = B_{|i} + S_i,$$
$$\epsilon_{ij} = 2\psi\gamma_{ij} - 2E_{|i|j} - F_{i|j} - F_{j|i} + w_{ij}, \tag{A11}$$

with

$$S^i{}_{|i} = F^i{}_{|i} = 0, \qquad w^{ij}{}_{|j} = 0, \qquad w^i{}_i = 0, \tag{A12}$$

the Lagrangian also splits into tensor, vector, and scalar parts. The tensor part was already treated in Ref. [19].

The vector part reads

$$L^{(V)} = \frac{Na}{6l^2}\int d^3x\gamma^{1/2}S^{i|j}S_{[i|j]} - \frac{a^2}{6l^2}\int d^3x\gamma^{1/2}S_{i|j}\dot{F}^{i|j}$$
$$+ \frac{a^3}{12l^2N}\int d^3x\gamma^{1/2}\dot{F}^{i|j}\dot{F}_{i|j} + \frac{a^2K}{3l^2}\int d^3x\gamma^{1/2}S_i\dot{F}^i$$
$$- \frac{a^3K}{6l^2N}\int d^3x\gamma^{1/2}\dot{F}^i\dot{F}_i - \frac{NaK}{3l^2}\int d^3x\gamma^{1/2}S^iS_i. \tag{A13}$$

Using the gauge invariant quantity

$$V_i = S_i - \frac{a}{N}\dot{F}_i, \tag{A14}$$

this Lagrangian simplifies to

$$\frac{Na}{6l^2}\int d^3x\gamma^{1/2}(V^{[i|j]}V_{[i|j]}-2KV^iV_i). \tag{A15}$$

Its associated Hamiltonian reads

$$H^{(V)}=\frac{Na}{6l^2}\int d^3x\gamma^{1/2}V_i\left(\frac{1}{2}V^{i|j}{}_{|j}+KV_i\right)+\int d^3x\Lambda_i\pi^i, \tag{A16}$$

where we have the constraint

$$\pi_V^i\approx 0. \tag{A17}$$

Conservation of the constraint $\pi_V^i\approx 0$ leads to the secondary constraint

$$\tfrac{1}{2}V^{i|j}{}_{|j}+KV_i\approx 0 \tag{A18}$$

whose conservation fixes the Lagrange multiplier Λ_V^i, which means that both constraints are second class. Defining the associated Dirac brackets, they become strong equalities, yielding the well-known result for a universe filled only with a scalar field:

$$V^i=0. \tag{A19}$$

In the scalar sector we have

$$\begin{aligned}L^{(E)}=&\frac{Na}{6l^2}\int d^3x\gamma^{1/2}[2\psi^i\psi_{,i}-4\phi_{,i}\psi^i+K(-6\psi^2+12\phi\psi)]-\frac{a^3}{l^2N}\int d^3x\gamma^{1/2}\dot\psi^2-\frac{\dot a^2a}{l^2N}\int d^3x\gamma^{1/2}\phi^2\\&-\frac{2a^2\dot a}{l^2N}\int d^3x\gamma^{1/2}\phi\dot\psi+\frac{a^3\dot\varphi_0}{N}\int d^3x\gamma^{1/2}(\dot\phi+3\dot\psi)\delta\varphi-Na^3U_\varphi\int d^3x\gamma^{1/2}\phi\delta\varphi+\frac{\dot\varphi_0{}^2a^3}{2N}\int d^3x\gamma^{1/2}\phi^2\\&+\frac{Na^3}{2}\int d^3x\gamma^{1/2}\left(\frac{1}{N^2}\delta\dot\varphi^2-\frac{1}{a^2}\delta\varphi^{,i}\delta\varphi_{,i}-\frac{1}{2}U_{\varphi\varphi}\delta\varphi^2\right)-\frac{2a^2}{3l^2}\left[\dot\psi+\frac{\dot a}{a}\phi-\frac{3l^2\dot\varphi_0}{2}\delta\varphi-\frac{1}{2}\frac{N}{a}K\left(B-\frac{a}{N}\dot E\right)\right]\\&\times\left(B-\frac{a}{N}\dot E\right)^{,i}{}_{,i}.\end{aligned} \tag{A20}$$

When constructing the Hamiltonian, we obtain the primary constraints (here again we define $F=B-a\dot E/N$)

$$\phi_1=P_N\approx 0,\qquad \phi_2=\pi_\phi-\frac{a^3\dot\varphi_0}{N}\gamma^{1/2}\delta\varphi\approx 0,\qquad \phi_3=\pi_F\approx 0. \tag{A21}$$

The Hamiltonian reads

$$H=NH_0+\Lambda_NP_N+\int d^3x\Lambda_F\pi_F+\int d^3x\Lambda_\phi\left(\pi_\phi-\frac{P_\varphi}{V}\gamma^{1/2}\delta\varphi\right) \tag{A22}$$

where H_0 is given by

$$\begin{aligned}H_0=&-\frac{l^2P_a^2}{4aV}+\frac{P_\varphi^2}{2a^3V}-\frac{KaV}{l^2}+\frac{a^3VU}{2}+\frac{l^2P_a}{2a^2V}\int d^3\phi\pi_\psi-\frac{P_\varphi^2}{2a^3V^2}\int d^3x\gamma^{1/2}\phi^2+\frac{3l^2P_\varphi}{2a^3V}\int d^3\delta\varphi\pi_\psi\\&-\frac{a^3}{l^2}\int d^3x\gamma^{1/2}\left(\frac{l^2}{2a^3\gamma^{1/2}}\pi_\psi+\frac{1}{3a}F^{,i}{}_{,i}\right)^2+\left(-\frac{3l^2P_aP_\varphi}{2a^2V^2}+a^3U_\varphi\right)\int d^3x\gamma^{1/2}\phi\delta\varphi\\&-\frac{a}{3l^2}\int d^3x\gamma^{1/2}[\psi^i\psi_{,i}-2\phi^i\psi_i+K(-3\psi^2+6\phi\psi)]-\frac{aK}{3l^2}\int d^3x\gamma^{1/2}FF^{,i}{}_{,i}+\frac{1}{2a^3}\int d^3x\frac{\pi_\varphi^2}{\gamma^{1/2}}\\&+\frac{a}{2}\int d^3x\gamma^{1/2}\delta\varphi^{,i}\delta\varphi_{,i}+\left(-\frac{9l^2P_\varphi^2}{4a^3V^2}+\frac{a^3U_{\varphi\varphi}}{4}\right)\int d^3x\gamma^{1/2}\delta\varphi^2.\end{aligned} \tag{A23}$$

Performing the canonical transformation

$$\varphi_0=\tilde\varphi_0-\frac{1}{V}\int d^3x\gamma^{1/2}\tilde\phi\delta\tilde\varphi,\qquad \pi_\phi=\tilde\pi_\phi+\frac{\tilde P_\varphi}{V}\gamma^{1/2}\delta\tilde\varphi,\qquad \pi_\varphi=\tilde\pi_\varphi+\frac{\tilde P_\varphi}{V}\gamma^{1/2}\tilde\phi \tag{A24}$$

generated by

$$\mathcal{F}=\mathcal{I}-\frac{P_\varphi}{V}\int d^3x\gamma^{1/2}\tilde\phi\delta\varphi, \tag{A25}$$

where $\mathcal{I}$ represents the identity transformation, the new H reads

$$H = NH_0 + \Lambda_N P_N + \int d^3x \Lambda_F \pi_F + \int d^3x \Lambda_\phi \pi_\phi, \tag{A26}$$

where

$$\begin{aligned} H_0 = & -\frac{l^2 P_a^2}{4aV} + \frac{P_\varphi^2}{2a^3 V} - \frac{KaV}{l^2} + \frac{a^3 VU}{2} + \frac{l^2 P_a}{2a^2 V}\int d^3\phi \pi_\psi + \frac{3l^2 P_\varphi}{2a^3 V}\int d^3\delta\varphi \pi_\psi - \frac{a^3}{l^2}\int d^3x \gamma^{1/2}\left(\frac{l^2}{2a^3\gamma^{1/2}}\pi_\psi + \frac{1}{3a}F^{,i}{}_{,i}\right)^2 \\ & + \left(-\frac{3l^2 P_a P_\varphi}{2a^2 V^2} + \frac{a^3 U_\varphi}{2}\right)\int d^3x \gamma^{1/2}\phi\delta\varphi - \frac{a}{3l^2}\int d^3x\gamma^{1/2}[\psi^i\psi_{,i} - 2\phi^i\psi_i + K(-3\psi^2 + 6\phi\psi)] \\ & - \frac{aK}{3l^2}\int d^3x\gamma^{1/2}FF^{,i}{}_{,i} + \frac{P_\varphi}{a^3 V}\int d^3x\phi\pi_\varphi + \frac{1}{2a^3}\int d^3x\frac{\pi_\varphi^2}{\gamma^{1/2}} + \frac{a}{2}\int d^3x\gamma^{1/2}\delta\varphi^{,i}\delta\varphi_{,i} \\ & + \left(-\frac{9l^2 P_\varphi^2}{4a^3 V^2} + \frac{a^3 U_{\varphi\varphi}}{4}\right)\int d^3x\gamma^{1/2}\delta\varphi^2. \end{aligned} \tag{A27}$$

Conservation of the primary constraints (A21) leads to the secondary constraints

$$H_0 \approx 0, \tag{A28}$$

$$\phi_5 \equiv \frac{1}{3a}\pi_\psi + \frac{2a}{9l^2}\gamma^{1/2}F^{,i}{}_{,i} + \frac{2aK}{3l^2}\gamma^{1/2}F \approx 0, \tag{A29}$$

$$\begin{aligned} \phi_6 \equiv & -\frac{l^2 P_a}{2a^2 V}\pi_\psi + \left(\frac{3l^2 P_a P_\varphi}{2a^2 V^2} - \frac{a^3 U_\varphi}{2}\right)\gamma^{1/2}\delta\varphi \\ & + \frac{2a}{3l^2}\gamma^{1/2}\psi^i{}_{,i} + \frac{6aK}{3l^2}\gamma^{1/2}\psi - \frac{P_\varphi}{a^3 V}\pi_\varphi \approx 0. \end{aligned} \tag{A30}$$

Conservation of H_0 is identically satisfied. Conservation of ϕ_6 leads to a term proportional to H_0 up to second order terms. Finally, ϕ_5 fixes the Lagrange multiplier Λ_F:

$$a\Lambda_F = \psi - \phi + \frac{l^2 P_a}{aV}F. \tag{A31}$$

Substituting $\dot{F} = \{F, H\} = \Lambda_F$ into the above equation, we get for the gauge invariant Bardeen potentials Φ and Ψ

$$\Phi = \Psi. \tag{A32}$$

Calculating the non-null Poisson brackets among the constraints yields

$$\begin{aligned} \{\phi_3, \phi_5\} = & -\frac{2a}{9l^2}\gamma^{1/2}\delta^3(x - x')^{,i}{}_{,i} - \frac{2aK}{3l^2}\gamma^{1/2}\delta^3(x - x'), \\ \{\phi_5, \phi_6\} = & -\frac{2}{9l^2}\gamma^{1/2}\delta^3(x - x')^{,i}{}_{,i} - \frac{2K}{3l^2}\gamma^{1/2}\delta^3(x - x'). \end{aligned} \tag{A33}$$

The ϕ_3 and ϕ_5 constraints are second class, while

$$\bar{\phi}_6 =: \phi_6 + \frac{1}{a}\phi_3 \tag{A34}$$

is a first class constraint. Defining the Dirac brackets, the second class constraints can be substituted in the Hamiltonian.

Making $K = 0$, and performing the canonical transformations generated by

$$\begin{aligned} \mathcal{F}_1 = & a\tilde{P}_a + \varphi_0\tilde{P}_\varphi + \int d^3x\Big[a\pi\delta\varphi + \psi\tilde{\pi}_\psi - \frac{2\tilde{P}_\varphi}{l^2 P_a}\psi\pi \\ & + \frac{\alpha}{2}\gamma^{1/2}\delta\varphi^2\Big], \end{aligned} \tag{A35}$$

where

$$\alpha = \frac{3P_\varphi^2}{aP_a V} + \frac{al^2 P_a}{2V}, \tag{A36}$$

and

$$\begin{aligned} \mathcal{F}_2 = & a\tilde{P}_a + \varphi_0\tilde{P}_\varphi + \int d^3x\Big\{\psi\tilde{\pi}_\psi + \upsilon\tilde{\pi} \\ & + \left(\frac{2\tilde{P}_\varphi}{aV} - \frac{a^4 VU_\varphi}{l^2\tilde{P}_a} - \frac{6\tilde{P}_\varphi{}^3}{l^2 a^3\tilde{P}_a{}^2 V}\right)\gamma^{1/2}\upsilon\psi \\ & + \left(\frac{2\tilde{P}_a{}^2}{l^2 a\tilde{P}_a V} - \frac{a^4 VU_\varphi\tilde{P}_\varphi}{l^4\tilde{P}_a{}^2} - \frac{6\tilde{P}_\varphi{}^4}{l^4 a^3\tilde{P}_a{}^3 V}\right)\gamma^{1/2}\psi^2 \\ & + \frac{2a^3 V}{3l^4\tilde{P}_a}\gamma^{1/2}\psi\psi^i{}_{,i}\Big\}, \end{aligned} \tag{A37}$$

and making a redefinition of N, we finally obtain

$$\tilde{\phi}_6 = \pi_\psi, \tag{A38}$$

and

$$\begin{aligned} H = & NH_0 + \int d^3x\Big(-\frac{l^2 P_a}{2a^2 V}\phi + \frac{3P_\varphi^2}{a^4 P_a V}\psi \\ & + \frac{3l^2 P_\varphi}{2a^4 V}\upsilon\Big)\tilde{\phi}_6 + \Lambda_N P_N + \int d^3x\Lambda_\phi\pi_\phi, \end{aligned} \tag{A39}$$

where

$$H_0 = -\frac{l^2P_a^2}{4aV} + \frac{P_\varphi^2}{2a^3V} + \frac{a^3VU}{2} + \frac{1}{2a}\int d^3x\frac{\pi^2}{\gamma^{1/2}} + \frac{1}{2a}\int d^3x\gamma^{1/2}\upsilon^{,i}\upsilon_{,i} + \left(\frac{15l^2P_\varphi^2}{4a^5V^2} + \frac{aU_{\varphi\varphi}}{4} - \frac{3l^2Ua}{8} + \frac{9UP_\varphi^2}{4aP_a^2} - \frac{l^4P_a^2}{16a^3V^2} - \frac{27P_\varphi^4}{4a^7V^2P_a^2} - \frac{3P_\varphi U_\varphi}{P_a}\right)\times\int d^3x\gamma^{1/2}\upsilon^2. \tag{A40}$$

Using the background classical equations one can show that the coefficient of υ^2 can be written as z''/z as in [1]. Without their use, this is the simplest form the Hamiltonian of scalar perturbations can have in scalar field models.

APPENDIX B: THE EXPLICIT CANONICAL TRANSFORMATIONS

The explicit canonical transformations obtained from the generators $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ of Sec. III are, respectively,

$$\begin{aligned}
a &= \tilde{a}\Big[1 + \frac{2\sqrt{(\lambda+1)P_T}\sqrt{V}}{l^2\tilde{P}_a^2\sqrt{\lambda}}\tilde{a}^{(1/2)(1-3\lambda)}\int d^3x\tilde{\psi}\pi \\
&\quad + \frac{1}{2\tilde{a}}\frac{\partial\alpha}{\partial\tilde{P}_a}\int d^3x\gamma^{1/2}\Big(\sqrt{6}l\tilde{a}^{(1/2)(1-3\lambda)}\upsilon - \frac{2\sqrt{6}\sqrt{(\lambda+1)P_T}\sqrt{V}}{l\tilde{P}_a\sqrt{\lambda}}\tilde{a}^{2-3\lambda}\tilde{\psi}\Big)^2\Big], \\
P_a &= \tilde{P}_a - \frac{(1-3\lambda)}{2\sqrt{6}l}\tilde{a}^{-(3/2)(1-\lambda)}\int d^3x\Big(\sqrt{6}l\tilde{a}^{(1/2)(1-3\lambda)}\upsilon - \frac{2\sqrt{6}\sqrt{(\lambda+1)P_T}\sqrt{V}}{l\tilde{P}_a\sqrt{\lambda}}\tilde{a}^{2-3\lambda}\tilde{\psi}\Big)\pi \\
&\quad + \frac{3(1-\lambda)\sqrt{(\lambda+1)P_T}\sqrt{V}}{l^2\tilde{P}_a\sqrt{\lambda}}\tilde{a}^{(1/2)(1-3\lambda)}\int d^3x\tilde{\psi}\pi \\
&\quad - \frac{1}{2}\frac{\partial\alpha}{\partial a}\int d^3x\gamma^{1/2}\Big(\sqrt{6}l\tilde{a}^{(1/2)(1-3\lambda)}\upsilon - \frac{2\sqrt{6}\sqrt{(\lambda+1)P_T}\sqrt{V}}{l\tilde{P}_a\sqrt{\lambda}}\tilde{a}^{2-3\lambda}\tilde{\psi}\Big)^2, \\
\pi_{\varphi(c)} &= \frac{\tilde{a}^{-(1/2)}(1-3\lambda)}{\sqrt{6}l}\pi - \alpha\sqrt{6}l\tilde{a}^{(1/2)(1-3\lambda)}\gamma^{1/2}\upsilon + \frac{2\alpha\sqrt{6}\sqrt{(\lambda+1)P_T}\sqrt{V}}{l\tilde{P}_a\sqrt{\lambda}}\tilde{a}^{2-3\lambda}\gamma^{1/2}\tilde{\psi}, \\
\varphi_{(c)} &= \sqrt{6}l\tilde{a}^{(1/2)(1-3\lambda)}\upsilon - \frac{2\sqrt{6}\sqrt{(\lambda+1)P_T}\sqrt{V}}{l\tilde{P}_a\sqrt{\lambda}}\tilde{a}^{2-3\lambda}\tilde{\psi}, \\
\pi_{\psi(c)} &= \tilde{\pi}_\psi + \frac{2\sqrt{(\lambda+1)P_T}\sqrt{V}}{l^2\tilde{P}_a\sqrt{\lambda}}\tilde{a}^{(3/2)(1-\lambda)}\pi,
\end{aligned} \tag{B1}$$

$$\begin{aligned}
a &= \tilde{a}\Big\{1 + \frac{12\sqrt{V}[(\lambda+1)P_T]^{3/2}}{l^2\tilde{P}_a^3\sqrt{\lambda}}\tilde{a}^{(1/2)(91-9\lambda)}\int d^3x\gamma^{1/2}\upsilon\tilde{\psi} + \Big[-\frac{18V[(\lambda+1)P_T]^2}{l^4\tilde{P}_a^4\lambda}\tilde{a}^{2-6\lambda} + \frac{(1-3\lambda)(\lambda1)P_T}{2l^2\tilde{P}_a^2\lambda}\tilde{a}^{1-3\lambda}\Big] \\
&\quad\times\int d^3x\gamma^{1/2}\psi^2 + \frac{2\tilde{a}^2V}{3l^4\tilde{P}_a^2}\int d^3x\gamma^{1/2}\tilde{\psi}\tilde{\psi}^{,i}{}_{,i}\Big\} \\
P_a &= \tilde{P}_a + \Big[\frac{9(1-3\lambda)\sqrt{V}[(\lambda+1)P_T]^{3/2}}{l^2\tilde{P}_a^2\sqrt{\lambda}}\tilde{a}^{(1/2)(1-9\lambda)} - \frac{(1-9\lambda^2)\sqrt{(\lambda+1)P_T}}{4\sqrt{\lambda}\sqrt{V}}\tilde{a}^{-(1/2)(1+3\lambda)}\Big]\int d^3x\gamma^{1/2}\tilde{\upsilon}\,\tilde{\psi} \\
&\quad + \Big[-\frac{18V(1-2\lambda)[(\lambda+1)P_T]^2}{l^4\tilde{P}_a^3\lambda}\tilde{a}^{2-6\lambda} + \frac{(2+3\lambda-9\lambda^2)(\lambda+1)P_T}{2l^2\tilde{P}_a\lambda}\tilde{a}^{1-3\lambda}\Big]\int d^3x\gamma^{1/2}\tilde{\psi}^2 \\
&\quad + \frac{2\tilde{a}^2V}{l^4\tilde{P}_a}\int d^3x\gamma^{1/2}\tilde{\psi}\tilde{\psi}^i{}_{,i}, \\
\pi &= \tilde{\pi} + \Big[\frac{6\sqrt{V}[(\lambda+1)P_T]^{3/2}}{l^2\tilde{P}_a^2\sqrt{\lambda}}\tilde{a}^{(3/2)(1-3\lambda)} - \frac{(1+3\lambda)\sqrt{(\lambda+1)P_T}}{2\sqrt{\lambda}\sqrt{V}}\tilde{a}^{(1/2)(1-3\lambda)}\Big]\gamma^{1/2}\tilde{\psi}, \\
\pi_\psi &= \tilde{\pi}_\psi + \Big[\frac{6\sqrt{V}[(\lambda+1)P_T]^{3/2}}{l^2\tilde{P}_a^2\sqrt{\lambda}}\tilde{a}^{(3/2)(1-3\lambda)} - \frac{(1+3\lambda)\sqrt{(\lambda+1)P_T}}{2\sqrt{\lambda}\sqrt{V}}\tilde{a}^{(1/2)(1-3\lambda)}\Big]\gamma^{1/2}\tilde{\upsilon} \\
&\quad + \Big[-\frac{12V[(\lambda+1)P_T]^2}{l^4\tilde{P}_a^3\lambda}\tilde{a}^3 1 - 2\lambda + \frac{(1+3\lambda)(\lambda+1)P_T}{l^2\tilde{P}_a\lambda}\tilde{a}^{2-3\lambda}\Big]\gamma^{1/2}\tilde{\psi} + \frac{4\tilde{a}^3V}{3l^4\tilde{P}_a}\gamma^{1/2}\tilde{\psi}^i{}_i,
\end{aligned} \tag{B2}$$

$$a = \tilde{a} + \frac{l^2}{4\tilde{a}V}\int d^3x\gamma^{1/2}\tilde{v}^2, \qquad P_a = \tilde{P}_a - \frac{1}{\tilde{a}^2}\int d^3x v\pi + \frac{l^2\tilde{P}_a}{4V}\int d^3\gamma^{1/2}\tilde{v}^2,$$
$$\pi = \frac{1}{a}\tilde{\pi} - \frac{l^2\tilde{P}_a}{2V}\gamma^{1/2}\tilde{v}, \qquad v = a\tilde{v}. \tag{B3}$$

The intermediary Hamiltonian between $\mathcal{F}_2$ and $\mathcal{F}_3$ reads

$$\begin{aligned} H_0 = &-\frac{l^2P_a^2}{4aV} + \frac{P_T}{a^{3\lambda}} + \frac{1}{2a}\int d^3x\frac{\pi^2}{\gamma^{1/2}} + \frac{\lambda}{2a}\int d^3x\gamma^{1/2}v^{,i}v_{,i} - \frac{(1-3\lambda)l^2P_T}{4V}a^{-(2+3\lambda)}v^2 \\ &+ \left\{-\frac{\sqrt{\lambda}}{2}\frac{3l^2\sqrt{(\lambda+1)P_T}}{\sqrt{V}}a^{-(1/2)(5+3\lambda)}v + \frac{3(\lambda+1)P_T}{P_a}a^{-(1+3\lambda)}\psi\right\}\pi_\psi \\ &+ H_0^{(0)}\int d^3x\left\{-\frac{9(\lambda+1)P_T}{l^2P_a^2}a^{1-3\lambda}\gamma^{1/2}\psi^2 + \frac{6\sqrt{V}\sqrt{(\lambda+1)P_T}}{l^2P_a^2\sqrt{\lambda}}a^{(1/2)(1-3\lambda)}\psi\pi\right. \\ &+ \left[\frac{18[(\lambda+1)P_T]^{3/2}\sqrt{V}}{l^2P_a^3\sqrt{\lambda}}a^{(1/2)(1-9\lambda)} - \frac{3(1-3\lambda)\sqrt{(\lambda+1)P_T}}{2\sqrt{V}\sqrt{\lambda}P_a}a^{-(1/2)(1+3\lambda)}\right]\gamma^{1/2}v\psi + \frac{2a^2V}{l^4P_a^2}\psi\psi^{,i}{}_{,i} \\ &\left.+ \left[-\frac{9(1+2\lambda)[(\lambda+1)P_T]}{2P_a^2}a^{-(1+3\lambda)} + \frac{(2-9\lambda+9\lambda^2)l^2}{8Va^2}\right]\gamma^{1/2}v^2\right\}. \end{aligned} \tag{B4}$$

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