White dwarf stars in *D* dimensions

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We derive the mass-radius relation of relativistic white dwarf stars (modeled as a self-gravitating degenerate Fermi gas at $T = 0$) in a D-dimensional universe and study the influence of the dimension of space on the laws of physics when we combine quantum mechanics, special relativity, and gravity. We exhibit characteristic dimensions $D = 1$, $D = 2$, $D = 3$, $D = (3 + \sqrt{17})/2$, $D = 4$, $D = 2(1 + \sqrt{2})$ and show that quantum mechanics cannot balance gravitational collapse for $D \geq 4$. This is similar to a result found by Ehrenfest (1917) at the atomic level for Coulomb forces (in Bohr's model) and for the Kepler problem. This makes the dimension of our universe $D = 3$ very particular with possible implications regarding the anthropic principle. We discuss some historic aspects concerning the discovery of the Chandrasekhar (1931) limiting mass in relation to previous investigations by Anderson (1929) and Stoner (1930). We also propose different derivations of the stability limits of polytropic distributions and consider their application to classical and relativistic white dwarf stars.

I. INTRODUCTION: SOME HISTORIC ELEMENTS

The study of stars is one of the most fascinating topics in astrophysics because it involves many areas of physics: gravitation, thermodynamics, hydrodynamics, statistical mechanics, quantum mechanics, relativity, ... and it has furthermore a very interesting history. In his classical monograph *The Internal Constitution of the Stars*, Eddington (1926) [1] lays down the foundations of the subject and describes in detail the basic processes that govern the structure of ordinary stars. At that stage, only elements of "classical" physics are used and difficulties with such theories to account for the structure of high density stars such as the companion of Sirius are pointed out. Soon after the discovery of the quantum statistics by Fermi (1926) [2] and Dirac (1926) [3], Fowler (1926) [4] uses this "new thermodynamics" to explain the puzzling nature of white dwarf stars. He understands that low mass white dwarf stars owe their stability to the quantum pressure of the degenerate electron gas [5]. The resulting structure is equivalent to a polytrope of index $n = 3/2$ so that the mass-radius relation of classical white dwarf stars behaves like $MR^3 \sim 1$ (Chandrasekhar 1931a [6]). The next step was made by Chandrasekhar, aged only 19, who was accepted by the University of Cambridge to work with Fowler. In the boat that took him from Madras to Southampton [7], Chandrasekhar understands that relativistic effects are important in massive white dwarf stars and that Einstein kinematic must be introduced in the problem. In his first treatment (Chandrasekhar 1931b [8]), he considers the ultrarelativistic limit and shows that the resulting structure is equivalent to a polytrope of index $n = 3$. Applying the theory of polytropic gas spheres (Emden 1907 [9]), this leads to a unique value of the mass M_c that he interprets as a limiting mass, nowadays

called the Chandrasekhar limit [10]. The complete mass-radius relation of relativistic white dwarf stars was given later (Chandrasekhar 1935 [11]) and departs from Fowler's sequence as we approach the limiting mass. At this critical mass, the radius of the configuration vanishes. Above the critical mass, the equation of state of the relativistic degenerate Fermi gas of electrons is not able to balance gravitational forces and, when considering the final evolution of such a star, Chandrasekhar (1934) [12] "left speculating on other possibility." Chandrasekhar's result was severely criticized by Eddington (1935) [13] who viewed this result as *a reductio ad absurdum* of the relativistic formula and considered the combination of special relativity and nonrelativistic quantum theory as an "unholy alliance." Because of the quarrel with Eddington, it took some time to realize the physical implication of Chandrasekhar's results. However, progressively, his early investigations on white dwarf stars were extended in general relativity to the case of neutron stars (Oppenheimer and Volkoff 1939 [14]) and finally led to the concept of "black holes," a term coined by Wheeler in 1967. Without the dispute with Eddington, this ultimate stage of matter resulting from gravitational collapse could have been predicted much earlier from the discovery of Chandrasekhar [15].

Although these historic elements are well known, it is less well known that the concept of a maximum mass for relativistic white dwarf stars had been introduced earlier by Anderson (1929) [16] and Stoner (1930) [17,18]. These studies are mentioned in the early works of Chandrasekhar but they have been progressively forgotten and are rarely quoted in classical textbooks of astrophysics. These authors investigated the equation of state of a relativistic degenerate Fermi gas and predicted an upper limit for the mass of white dwarf stars. Stoner (1930) [17] uses a uniform mass density to model the star while Chandrasekhar (1931) [8] considers a more realistic $n = 3$

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polytrope. However, as noted in Chandrasekhar (1931) [8], the value of the limiting mass found by Stoner with his simplified model is relatively close to that obtained with the improved treatment. Interestingly, Nauenberg (1972) [19] introduced long after a simplified treatment of relativistic white dwarf stars in order to obtain an analytical approximation of the mass-radius relation. It turns out that this model, which gives a very good agreement with Chandrasekhar's numerical results (up to some normalization factors), is similar to that introduced by Stoner.

Leaving aside these interesting historical remarks, the result of Chandrasekhar [8] concerning the existence of a limiting mass is very profound because this mass can be expressed in terms of fundamental constants, similar to the Bohr radius of the hydrogen atom [20]. Hence, the mass of stars is determined typically by the following combination

$$\left[\frac{hc}{G}\right]^{3/2}\frac{1}{H^2} \simeq 29.2 M_\odot, \qquad (1)$$

where G is the constant of gravity, h the Planck constant, c the velocity of light, and H the mass of the hydrogen atom ($M_\odot$ is the solar mass). This formula results from the combination of quantum mechanics (h), special relativity (c), and gravity (G). Since dimensional analysis plays a fundamental role in physics, it is of interest to investigate how the preceding results depend on the dimension of space D of the universe. In our previous investigations [21–23], we considered the case of classical white dwarf stars in D dimensions and found that they become unstable in a space of dimension $D \geq 4$. In that case, the star either evaporates or collapses. Therefore, quantum mechanics cannot stabilize matter against gravitational collapse in $D \geq 4$ contrary to what happens in $D = 3$ [4,24,25]. Interestingly, this is similar to a result found by Ehrenfest [26] at the atomic level for Coulomb forces in Bohr's model and for the planetary motion (Kepler problem). The object of this paper is to extend these results to the case of relativistic white dwarf stars and exhibit particular dimensions of space which play a special role in the problem when we combine Newtonian gravity, quantum mechanics, and special relativity. We shall see that the problem is very rich and interesting in its own right. It shows to which extent the dimension $D = 3$ of our universe is particular, with possible implications regarding the anthropic principle [27]. We note that a similar problem has been considered in [28] on the basis of dimensional analysis. Our approach is more precise since we generalize the *exact* mathematical treatment of Chandrasekhar [11] to a space of dimension D. A connection with other works investigating the role played by the dimension of space on the laws of physics is made in the conclusion. In Appendix B, we propose different derivations of the stability limits of polytropic spheres and consider applications of these results to classical and relativistic white dwarf stars.

II. THE EQUATION OF STATE

Following Chandrasekhar [11], we model a white dwarf star as a degenerate gas sphere in hydrostatic equilibrium. The pressure is due to the quantum properties of the electrons and the density to the protons. In the completely degenerate limit, the electrons have momenta less than a threshold value p_0 (Fermi momentum) and their distribution function is $f = 2/h^D$ where h is the Planck constant. There can only be two electrons in a phase space element of size h^D on account of the Pauli exclusion principle. Therefore, the number of electrons per unit volume is

$$n = \int f d\mathbf{p} = \frac{2S_D}{h^D}\int_0^{p_0} p^{D-1} dp = \frac{2S_D}{Dh^D} p_0^D, \quad (2)$$

where $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the surface of a unit sphere in D dimensions. The mean kinetic energy per electron is given by

$$\kappa = \frac{1}{n}\int f\epsilon(p) d\mathbf{p} = \frac{2S_D}{nh^D}\int_0^{p_0} \epsilon(p) p^{D-1} dp, \quad (3)$$

where $\epsilon(p)$ is the energy of an electron with impulse p. In relativistic mechanics,

$$\epsilon = mc^2\left\{\left(1 + \frac{p^2}{m^2c^2}\right)^{1/2} - 1\right\}. \qquad (4)$$

The pressure of the electrons is

$$P = \frac{1}{D}\int fp\frac{d\epsilon}{dp} d\mathbf{p} = \frac{2S_D}{Dh^D}\int_0^{p_0} p^D\frac{d\epsilon}{dp} dp. \quad (5)$$

Using Eq. (4), the pressure can be rewritten

$$P = \frac{2S_D}{Dmh^D}\int_0^{p_0} \frac{p^{D+1}}{(1 + \frac{p^2}{m^2c^2})^{1/2}} dp. \qquad (6)$$

Finally, the mass density of the star is

$$\rho = n\mu H, \qquad (7)$$

where H is the mass of the proton and μ the molecular weight. If we consider a pure gas of fermions (like, e.g., massive neutrinos in dark matter models), we just have to replace μH by their mass m.

Introducing the notation $x = p_0/mc$, we can write the density of state parametrically as follows

$$P = A_2 f(x), \qquad \rho = Bx^D, \qquad (8)$$

where

$$A_2 = \frac{S_D m^{D+1} c^{D+2}}{4Dh^D}, \qquad B = \frac{2S_D m^D c^D \mu H}{Dh^D}, \quad (9)$$

$$f(x) = 8\int_0^x \frac{t^{D+1}}{(1 + t^2)^{1/2}} dt. \qquad (10)$$

The function $f(x)$ has the asymptotic behaviors

$$f(x) \simeq \frac{8}{D+2} x^{D+2} \qquad (x \ll 1), \qquad (11)$$

$$f(x) \simeq \frac{8}{D+1} x^{D+1} \qquad (x \gg 1). \qquad (12)$$

The classical limit corresponds to $x \ll 1$ and the ultrarelativistic limit to $x \gg 1$. Explicit expressions of the function $f(x)$ are given in Appendix A for different dimensions of space.

III. THE CHANDRASEKHAR EQUATION

For a spherically symmetric distribution of matter, the equations of hydrostatic equilibrium are

$$\frac{dP}{dr} = -\frac{GM(r)}{r^{D-1}} \rho, \qquad (13)$$

$$M(r) = \int_0^r \rho S_D r^{D-1} dr. \qquad (14)$$

They can be combined to give

$$\frac{1}{r^{D-1}} \frac{d}{dr} \left(\frac{r^{D-1}}{\rho} \frac{dP}{dr} \right) = -S_D G \rho. \qquad (15)$$

Expressing ρ and P in terms of x and setting $y^2 = 1 + x^2$, we obtain

$$\frac{1}{r^{D-1}} \frac{d}{dr} \left(r^{D-1} \frac{dy}{dr} \right) = -\frac{S_D G B^2}{8A_2} (y^2 - 1)^{D/2}. \qquad (16)$$

We denote by x_0 and y_0 the values of x and y at the center. Furthermore, we define

$$r = a\eta, \qquad y = y_0 \phi, \qquad (17)$$

$$a = \left(\frac{8A_2}{S_D G} \right)^{1/2} \frac{1}{B y_0^{(D-1)/2}}, \qquad y_0^2 = 1 + x_0^2. \qquad (18)$$

Note that the scale of length a is independent on y_0 for $D = 1$. Substituting these transformations in Eq. (16), we obtain the D-dimensional generalization of Chandrasekhar's differential equation

$$\frac{1}{\eta^{D-1}} \frac{d}{d\eta} \left(\eta^{D-1} \frac{d\phi}{d\eta} \right) = -\left(\phi^2 - \frac{1}{y_0^2} \right)^{D/2}, \qquad (19)$$

with the boundary conditions

$$\phi(0) = 1, \qquad \phi'(0) = 0. \qquad (20)$$

The radius R of the star is such that $\rho(R) = 0$. This yields

$$\phi(\eta_1) = \frac{1}{y_0}. \qquad (21)$$

The density can be expressed as

$$\rho = \rho_0 \frac{y_0^D}{(y_0^2 - 1)^{D/2}} \left(\phi^2 - \frac{1}{y_0^2} \right)^{D/2}, \qquad (22)$$

where the central density

$$\rho_0 = B x_0^D = B(y_0^2 - 1)^{D/2}. \qquad (23)$$

Finally, we find that the mass is related to y_0 by

$$M = -S_D \left(\frac{8A_2}{S_D G} \right)^{D/2} \frac{1}{B^{D-1}} y_0^{D(3-D)/2} \left(\eta^{D-1} \frac{d\phi}{d\eta} \right)_{\eta=\eta_1}. \qquad (24)$$

Note that y_0 does not *explicitly* enter in this expression for $D = 3$ but it is of course present implicitly.

IV. THE CLASSICAL LIMIT

In the classical case $x \ll 1$, we find that the equation of state takes the form

$$P = K_1 \rho^{1+2/D}, \qquad (25)$$

with

$$K_1 = \frac{1}{D+2} \left(\frac{D}{2S_D} \right)^{2/D} \frac{h^2}{m(\mu H)^{(D+2)/D}}. \qquad (26)$$

Therefore a classical white dwarf star is equivalent to a polytrope of index [21,29]:

$$n_{3/2} = \frac{D}{2}. \qquad (27)$$

Polytropic stars are described by the Lane-Emden equation [9]. This can be recovered as a limit of the Chandrasekhar equation. For $x \ll 1$, we have $y_0 \simeq 1 + \frac{1}{2} x_0^2$. We define $\theta = \phi^2 - 1/y_0^2$. To leading order, $\phi = 1 - (x_0^2 - \theta)/2$. Setting $\xi = \sqrt{2}\eta$ and combining the foregoing results, we find that Eq. (19) reduces to the Lane-Emden equation with index $D/2$:

$$\frac{1}{\xi^{D-1}} \frac{d}{d\xi} \left(\xi^{D-1} \frac{d\theta}{d\xi} \right) = -\theta^{D/2}, \qquad (28)$$

$$\theta(0) = x_0^2, \qquad \theta'(0) = 0. \qquad (29)$$

Note that the condition at the origin is $\theta(0) = x_0^2$ instead of $\theta(0) = 1$ as in the ordinary Lane-Emden equation. However, using the homology theorem for polytropic spheres [30], we can easily relate θ to $\theta_{D/2}$, the solution of the Lane-Emden equation with index $n_{3/2} = D/2$ and condition at the origin $\theta(0) = 1$.

The structure and the stability of polytropic spheres in various dimensions of space has been studied by Chavanis and Sire [21]. For $D > 2$, this study exhibits two important indices:

$$n_3 = \frac{D}{D-2}, \qquad n_5 = \frac{D+2}{D-2}. \qquad (30)$$

According to this study, a classical white dwarf star is self-confined (complete) if $n_{3/2} < n_5$, i.e. $D < 2(1 + \sqrt{2}) = 4.828\,427\,1\ldots$. In that case, the density vanishes at a finite

radius R identified as the radius of the star. On the other hand, it is nonlinearly dynamically stable with respect to the Euler-Poisson system if $n_{3/2} < n_3$, and linearly unstable otherwise. Therefore, classical white dwarf stars are stable for $D < 4$ and unstable for $D \geq 4$ (see Fig. 1 and Appendix B). For $D \leq 2$, classical white dwarf stars are always self-confined and stable. Using the results of [21], the mass-radius relation for complete polytropes with index $n_{3/2} = D/2$ in D dimensions is

$$M^{(D-2)/D}R^{4-D} = \frac{K_1(D+2)}{2GS_D^{2/D}}\,\omega_{D/2}^{(D-2)/D}, \qquad (31)$$

where we have defined

$$\omega_{D/2} = -\xi_1^{(D+2)/(D-2)}\theta'_{D/2}(\xi_1), \qquad (32)$$

where ξ_1 is such that $\theta_{D/2}(\xi_1) = 0$. Using Eq. (26), we find that the mass-radius relation for classical white dwarf stars in D dimensions is

$$M^{(D-2)/D}R^{4-D} = \frac{1}{2}\left(\frac{D}{2S_D^2}\right)^{2/D}\frac{h^2}{mG(\mu H)^{(D+2)/D}}\,\omega_{D/2}^{(D-2)/D}. \qquad (33)$$

For $2 < D < 4$, the mass M decreases as the radius R increases while for $D < 2$ and for $4 < D < 2(1 + \sqrt{2})$ it increases with the radius (see Fig. 2).

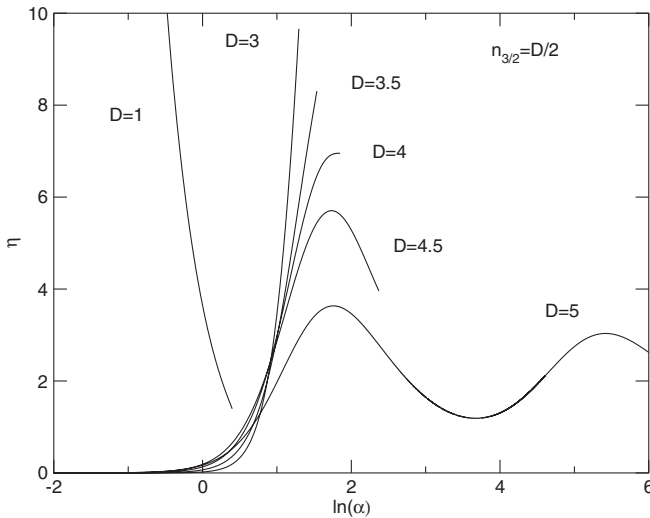For $D = 4$ the mass is independent on the radius and given in terms of fundamental constants by

FIG. 1. Relation between the mass (ordinate) and the central density (abscissa) of box-confined polytropes with index $n_{3/2} = D/2$ in an appropriate system of coordinates (see [21,23] for details). Complete polytropes correspond to the terminal point in the series of equilibria. The series becomes dynamically unstable with respect to the Euler-Poisson system (saddle point of the energy functional) after the turning point of mass η which appears for $D = 4$. Thus, classical white dwarf stars are stable for $D < 4$ and unstable for $D \geq 4$.
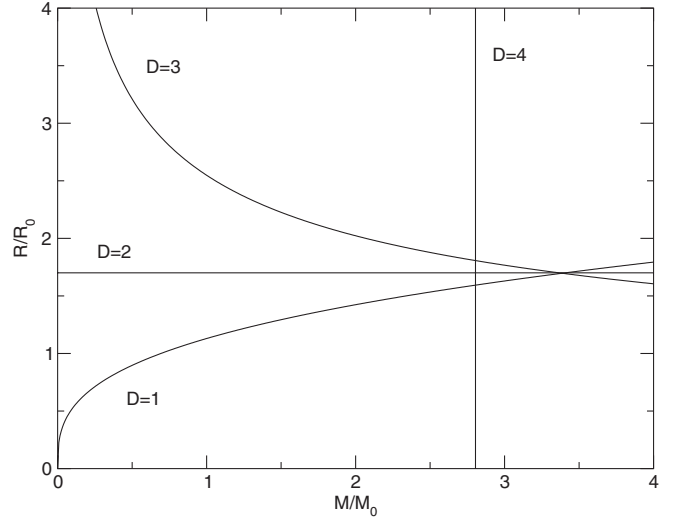

FIG. 2. Mass-radius relation for classical white dwarf stars in different dimensions of space. The radius is independent on mass in $D = 2$ and the mass is independent on radius in $D = 4$. The dimensional factors M_0 and R_0 are defined in Sec. VI.

$$M = \frac{\omega_2}{2S_4^2}\frac{h^4}{m^2G^2(\mu H)^3} \simeq 0.0143\,958\ldots\frac{h^4}{m^2G^2(\mu H)^3}. \qquad (34)$$

We recall that the value of the gravitational constant G depends on the dimension of space so that we cannot give an explicit value to this limiting mass. The central density is related to the radius by

$$\rho_0 R^4 = \frac{\xi_1^4}{16\pi^6}\frac{h^4}{m^2G^2(\mu H)^3} \simeq 0.105\,468\ldots\frac{h^4}{m^2G^2(\mu H)^3}. \qquad (35)$$

For $D = 2$, the radius is independent on mass and given in terms of fundamental constants by

$$R = \frac{\xi_1}{2\sqrt{2}\pi}\frac{h}{(Gm)^{1/2}\mu H} \simeq 0.270\,638\ldots\frac{h}{(Gm)^{1/2}\mu H}. \qquad (36)$$

Furthermore, for $D = 2$, the Lane-Emden equation (28) with index $n_{3/2} = 1$ can be solved analytically, yielding $\theta_1 = J_0(\xi)$ and $\xi_1 = \alpha_{0,1}$ where $\alpha_{0,1} = 2.404\,826\ldots$ is the first zero of Bessel function J_0. We obtain a density profile

$$\rho(r) = \rho_0 J_0(\xi_1 r/R), \qquad (37)$$

where the central density is related to the total mass by

$$M = -\frac{\rho_0}{4\pi}\frac{h^2}{Gm(\mu H)^2}\xi_1\theta'_1 = 0.099\,349\,2\ldots\frac{\rho_0 h^2}{Gm(\mu H)^2}. \qquad (38)$$

Finally, for $D = 3$, using the mass-radius relation (33), we find that the average density is related to the total mass by

$\bar{\rho} = 2.162 \ 10^6 (M/M_\odot)^2$ g/cm^2 (for $\mu = 2.5$). Historically, this result was obtained by Chandrasekhar (1931) [6] who first applied the theory of polytropic gas spheres with index $n = 3/2$ to classical white dwarf stars. It improves an earlier result $\bar{\rho} = 3.977 \ 10^6 (M/M_\odot)^2$ g/cm^2 obtained by Stoner (1929) [31] on the basis of his model of stars with uniform density (see Sec. VII).

V. THE ULTRARELATIVISTIC LIMIT

In the ultrarelativistic limit $x \gg 1$, we find that the equation of state takes the form

$$P = K_2 \rho^{1+1/D}, \qquad (39)$$

with

$$K_2 = \frac{1}{D+1} \left(\frac{D}{2S_D} \right)^{1/D} \frac{hc}{(\mu H)^{(D+1)/D}}. \qquad (40)$$

Therefore, an ultrarelativistic white dwarf star is equivalent to a polytrope of index

$$n_3' = D. \qquad (41)$$

This also directly results from the Chandrasekhar equation. For $x \gg 1$, it reduces to

$$\frac{1}{\xi^{D-1}} \frac{d}{d\xi} \left(\xi^{D-1} \frac{d\theta}{d\xi} \right) = -\theta^D, \qquad (42)$$

$$\theta(0) = 1, \qquad \theta'(0) = 0, \qquad (43)$$

where we have set $\theta = \phi$ and $\xi = \eta$. This is the Lane-Emden equation with index $n_3' = D$.

Using the results of [21] for $D > 2$, we deduce that an ultrarelativistic white dwarf star is self-confined if $n_3' < n_5$, i.e. $D < (3 + \sqrt{17})/2 = 3.561\,552\,8\dots$. In addition, it is nonlinearly dynamically stable with respect to the Euler-Poisson system if $n_3' < n_3$ and linearly unstable otherwise. Therefore, ultrarelativistic white dwarf stars are stable for $D \leq 3$ and unstable for $D > 3$ (see Fig. 3 and Appendix B). For $D \leq 2$, ultrarelativistic white dwarf stars are self-confined and stable. On the other hand, using the results of [21], the mass-radius relation for complete polytropes with index $n_3' = D$ in D dimensions is

$$M^{(D-1)/D} R^{3-D} = \frac{K_2(D+1)}{G S_D^{1/D}} \omega_D^{(D-1)/D}, \qquad (44)$$

where we have defined

$$\omega_D = -\xi_1^{(D+1)/(D-1)} \theta_D'(\xi_1). \qquad (45)$$

Using Eq. (40), we find that the mass-radius relation for ultrarelativistic white dwarf stars in D dimensions is

$$M^{(D-1)/D} R^{3-D} = \left(\frac{D}{2S_D^2} \right)^{1/D} \frac{hc}{G(\mu H)^{(D+1)/D}} \omega_D^{(D-1)/D}. \qquad (46)$$
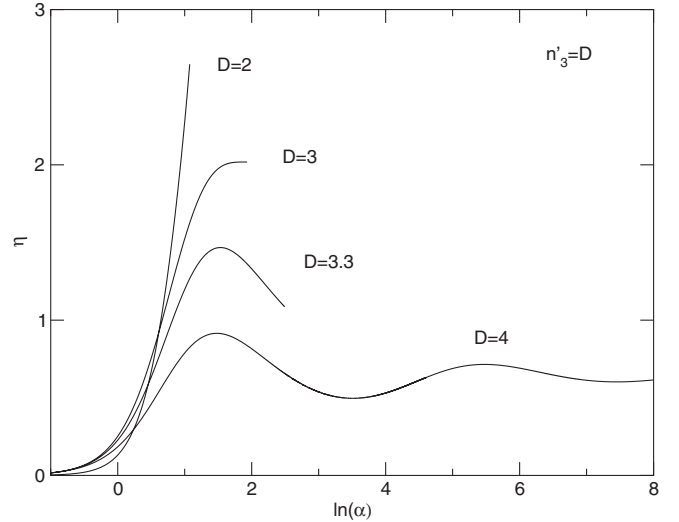

FIG. 3. Relation between the mass (ordinate) and the central density (abscissa) of box-confined polytropes with index $n_3' = D$ in an appropriate system of coordinates (see [21,23] for details). Complete polytropes correspond to the terminal point in the series of equilibria. The series becomes dynamically unstable with respect to the Euler-Poisson system (saddle point of the energy functional) after the turning point of mass η which appears for $D = 3$. Thus, ultrarelativistic white dwarf stars are stable for $D \leq 3$ and unstable for $D > 3$.

For $3 < D < (3 + \sqrt{17})/2$, the mass M increases as the radius R increases while for $1 < D < 3$ it decreases with the radius (see Fig. 4).

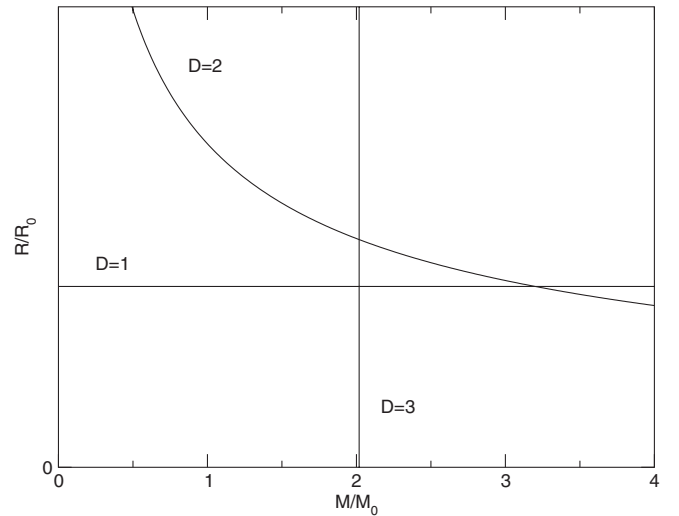For $D = 1$, the radius is independent on mass and given in terms of fundamental constants by

FIG. 4. Mass-radius relation for ultrarelativistic white dwarf stars in different dimensions of space. The radius is independent on mass in $D = 1$ and the mass is independent on radius in $D = 3$ (Chandrasekhar's mass).

$$R = \frac{\xi_1}{\sqrt{2}S_1}\left(\frac{hc}{G}\right)^{1/2}\frac{1}{\mu H} = 0.555\,360\ldots\left(\frac{hc}{G}\right)^{1/2}\frac{1}{\mu H}. \tag{47}$$

Furthermore, for $D = 1$, the Lane-Emden equation (42) with index $n'_3 = 1$ can be solved analytically, yielding $\theta_1 = \cos(\xi)$ and $\xi_1 = \pi/2$. We obtain a density profile

$$\rho(r) = \rho_0 \cos(\xi_1 r/R), \tag{48}$$

where the central density is related to the total mass by

$$M = \frac{\rho_0}{\sqrt{2}}\left(\frac{hc}{G}\right)^{1/2}\frac{1}{\mu H}. \tag{49}$$

For $D = 3$, the mass is independent on radius and given in terms of fundamental constants by

$$M = \left(\frac{3}{32\pi^2}\right)^{1/2}\omega_3\left(\frac{hc}{G}\right)^{3/2}\frac{1}{(\mu H)^2}. \tag{50}$$

This is the Chandrasekhar mass

$$M = 0.196\,701\ldots\left(\frac{hc}{G}\right)^{3/2}\frac{1}{(\mu H)^2} \simeq 5.76 M_\odot/\mu^2. \tag{51}$$

Coming back to Eq. (19), we can show that for this limiting value, the radius R of the configuration tends to zero (see Sec. VI). Historically, the existence of a maximum mass for relativistic white dwarf stars was first published by Anderson (1929) [16] who considered a relativistic extension of the model of Stoner (1929) [31] for classical white dwarf stars. He obtained a limiting mass $M = 1.37\,10^{33}$ g (for $\mu = 2.5$). The relativistic treatment of Anderson was criticized and corrected by Stoner (1930) [17] who obtained a value of the limiting mass $M = 2.19\,10^{33}$ g. The uniform density model of Stoner was in turn criticized and corrected by Chandrasekhar (1931) [8] who applied the theory of polytropic gas spheres with index $n = 3$ to relativistic white dwarf stars and obtained the value (51) of the limiting mass $M = 1.822\,10^{33}$ g. It seems that these historical details are not well known because the references to the works of Anderson and Stoner progressively disappeared from the literature.

VI. THE GENERAL CASE

Collecting together the results of Sec. III, the mass-radius relation for relativistic white dwarf stars in D dimensions can be written in the general case under the parametric form

$$\frac{M}{M_0} = y_0^{D(3-D)/2}\Omega(y_0), \qquad \frac{R}{R_0} = \frac{1}{y_0^{(D-1)/2}}\eta_1, \tag{52}$$

where we have defined

$$M_0 = S_D\left(\frac{8A_2}{S_D G}\right)^{D/2}\frac{1}{B^{D-1}}, \qquad R_0 = \left(\frac{8A_2}{S_D G}\right)^{1/2}\frac{1}{B}, \tag{53}$$

and

$$\Omega(y_0) = -\left(\eta^{D-1}\frac{d\phi}{d\eta}\right)_{\eta=\eta_1}. \tag{54}$$

The mass M_0 and the radius R_0 can be expressed in terms of fundamental constants as

$$R_0 = \left(\frac{D}{2S_D^2}\right)^{1/2}\frac{h^{D/2}}{G^{1/2}m^{(D-1)/2}c^{(D-2)/2}\mu H}, \tag{55}$$

$$M_0 = \left(\frac{D}{2S_D^2}\right)^{(D-2)/2}\frac{h^{D(D-2)/2}c^{(4-D)D/2}}{m^{(D-3)D/2}G^{D/2}}\frac{1}{(\mu H)^{D-1}}. \tag{56}$$

We can now obtain the mass-radius curve M-R by the following procedure. We fix a value of the parameter y_0 and solve the differential equation (19) with initial condition (20) until the point $\eta = \eta_1$, determined by Eq. (21), at which the density vanishes. The radius and the mass of the corresponding configuration are then given by Eq. (52). By varying y_0, we can obtain the full curve $R(y_0)$-$M(y_0)$ parametrized by the value of the central density ρ_0 given by Eq. (23). To solve the differential equation (19), we need the behavior of ϕ at the origin. Expanding $\phi(\eta)$ in Taylor series and substituting this expansion in Eq. (19) we obtain for $\eta \to 0$:

$$\phi = 1 - \frac{q^D}{2D}\eta^2 + \frac{1}{8(D+2)}q^{2(D-1)}\eta^4 + \ldots \tag{57}$$

where

$$q^2 = 1 - \frac{1}{y_0^2}. \tag{58}$$

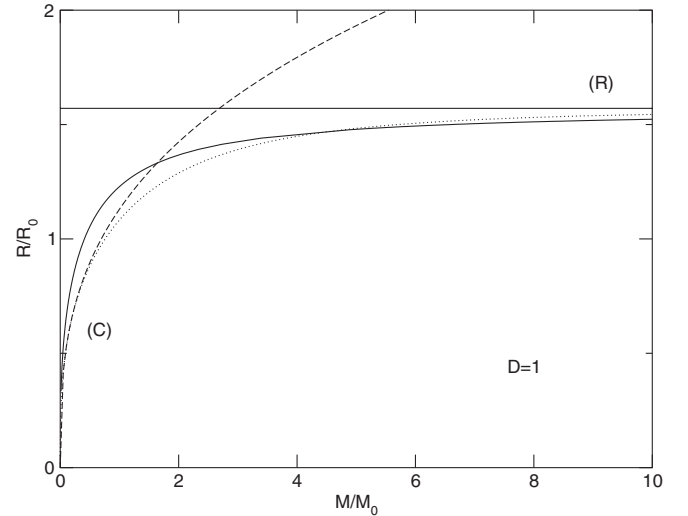We note, in particular, that $\phi''(0) = -q^D/D$.

FIG. 5. Mass-radius relation in $D = 1$. The radius increases with the mass. The configurations are always stable and there exists a maximum radius achieved in the ultrarelativistic limit for $M \to +\infty$.
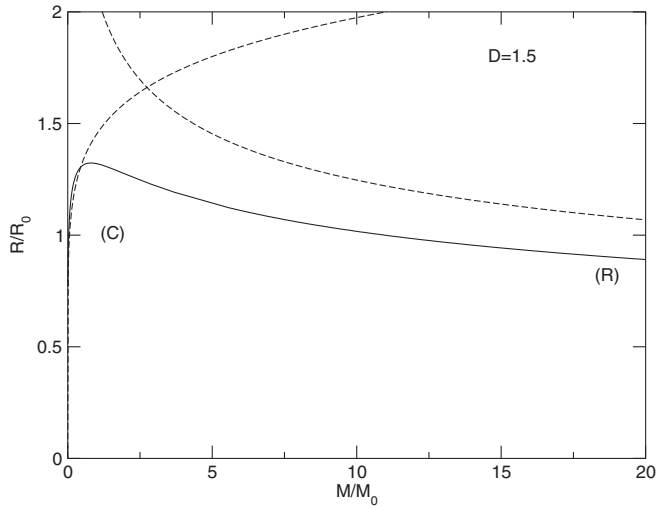
FIG. 6. Mass-radius relation in $D = 1.5$. The configurations are always stable and there exists a maximum radius for partially relativistic distributions.

Finally, we give the asymptotic expressions of the mass-radius relation. In the classical limit, using Eq. (33), we obtain

$$\left(\frac{M}{M_0}\right)^{(D-2)/D}\left(\frac{R}{R_0}\right)^{4-D} = \frac{1}{2}\omega_{D/2}^{(D-2)/D}. \qquad (59)$$

In the ultrarelativistic limit, using Eq. (46), we get

$$\left(\frac{M}{M_0}\right)^{(D-1)/D}\left(\frac{R}{R_0}\right)^{3-D} = \omega_D^{(D-1)/D}. \qquad (60)$$

In Figs. 5–12, we plot the mass-radius relation of relativistic white dwarf stars (full line) for different dimensions of space. The asymptotic relations (59) and (60) valid in the
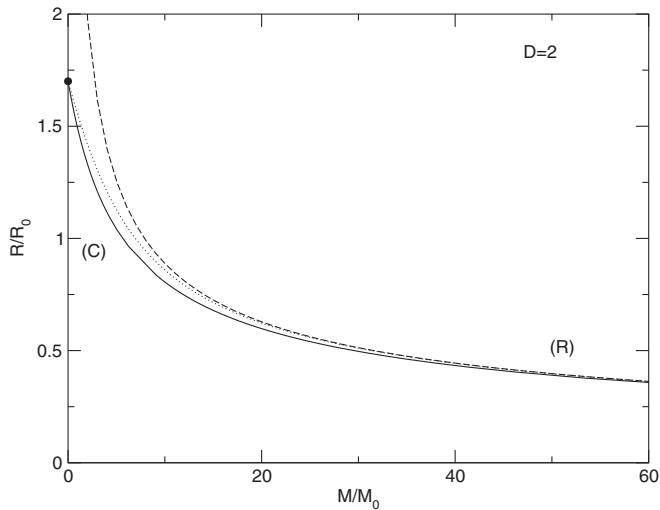

FIG. 7. Mass-radius relation in $D = 2$. The radius decreases with the mass. The configurations are always stable and there exists a maximum radius achieved in the classical limit for $M \rightarrow 0$.
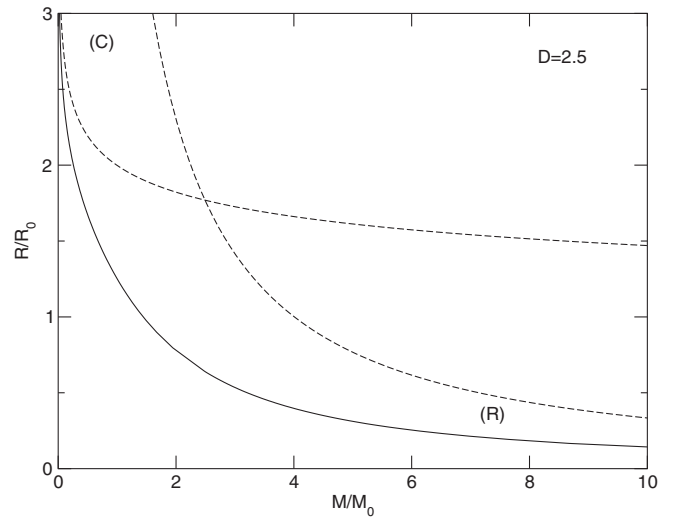

FIG. 8. Mass-radius relation in $D = 2.5$. The radius decreases with the mass. The configurations are always stable.

classical (C) and ultrarelativistic (R) limits are also shown for comparison (dashed line) together with the analytical approximation (dotted line) derived in Sec. VII.

VII. ANALYTICAL APPROXIMATION OF THE MASS-RADIUS RELATION

We present here an analytical approximation of the mass-radius relation in various dimensions of space based on the treatment by Nauenberg [19] in $D = 3$. This treatment amounts to considering that the density is uniform in the star and the mass-radius relation is obtained by mini-
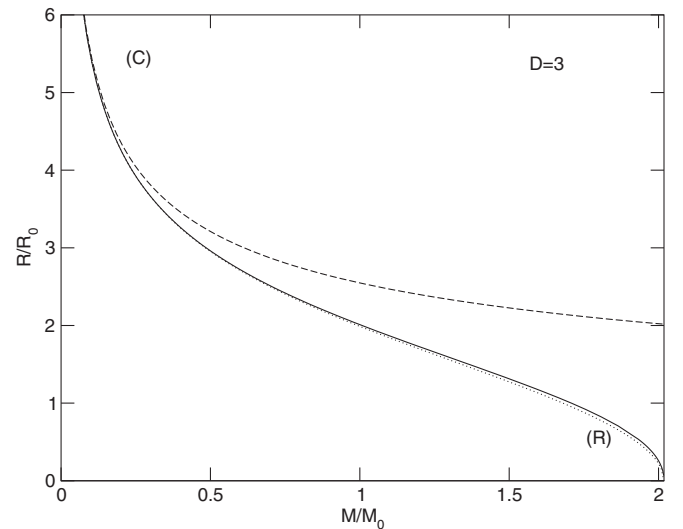

FIG. 9. Mass-radius relation in $D = 3$. The radius decreases with the mass. The configurations are always stable and there exists a limiting mass (Chandrasekhar's mass) achieved in the ultrarelativistic limit for $R = 0$. For $M > M_{\text{Chandra}}$, quantum mechanics cannot arrest gravitational collapse.
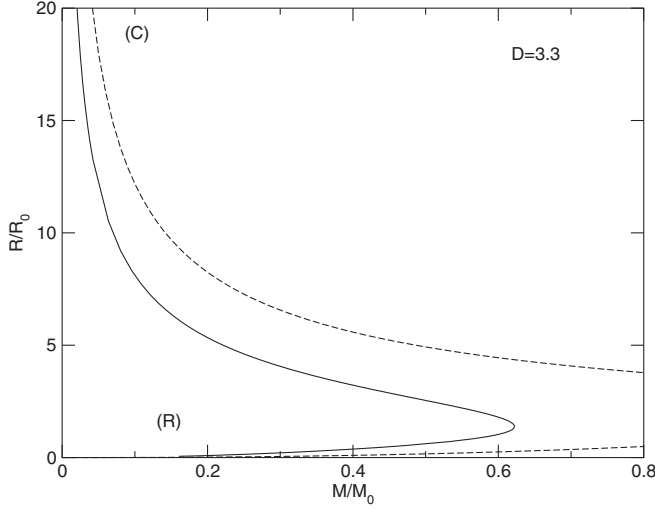
FIG. 10. Mass-radius relation in $D = 3.3$. There exists a limiting mass for partially relativistic distributions. Classical configurations are stable and the series of relativistic equilibria becomes unstable after the turning point of mass. Thus, highly relativistic configurations in $D > 3$ cannot be in hydrostatic equilibrium.

FIG. 12. Mass-radius relation in $D = 4$. There exists a limiting mass achieved in the classical limit for $R \rightarrow +\infty$. In fact, for $D \geq 4$, all the configurations (classical and relativistic) are unstable. Quantum mechanics cannot balance gravitational attraction even in the classical limit.

mizing the energy functional (see Appendix C) with respect to the radius (or to the density) at fixed mass. As mentioned in the introduction, this is similar to the simplified model of relativistic white dwarf stars made by
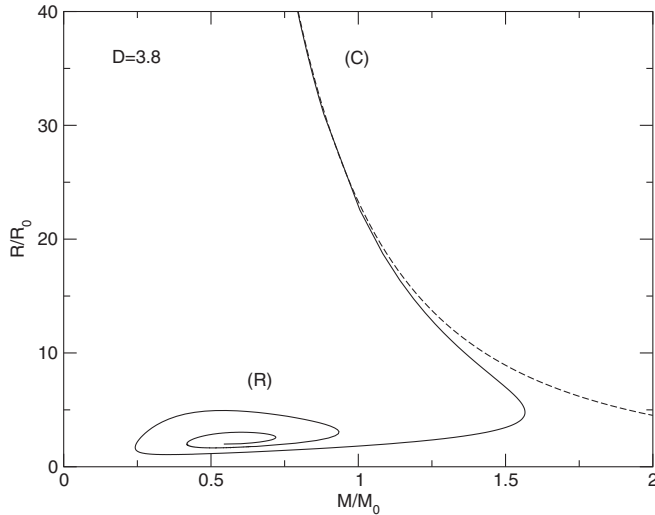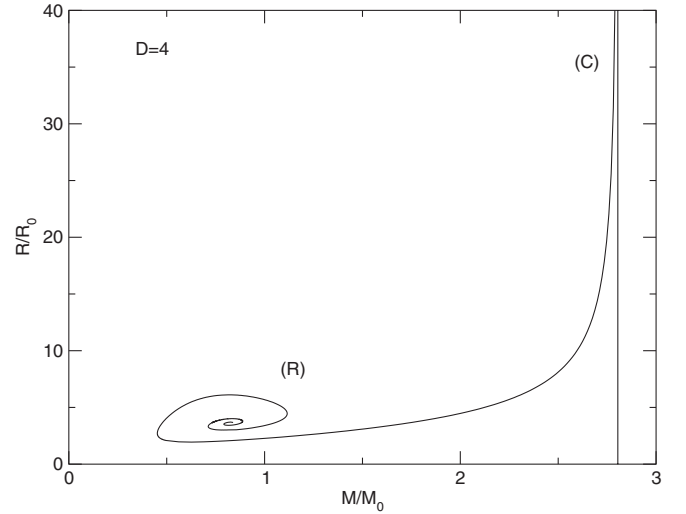

FIG. 11. Mass-radius relation in $D = 3.8$. For $D > (3 + \sqrt{17})/2$ the ultrarelativistic configurations (equivalent to polytropes of index $n'_3 = D$) are not self-confined anymore and a spiral develops in the mass-radius relation. This is somehow similar to the classical spiral occurring in the (E, β) plane in the thermodynamics of isothermal self-gravitating systems [38–41] and to the spiral occurring in the (M, R) plane in the general relativistic treatment of neutron stars [46,55]. The series of equilibria becomes unstable at the first turning point of mass and new modes of instability occur at the secondary turning points [33].

Stoner [17] before Chandrasekhar's treatment [8]. Following Nauenberg [19], we approximate the kinetic energy K by the form

$$K = Nmc^2 \left\{ \left(1 + \frac{p^2}{m^2 c^2} \right)^{1/2} - 1 \right\}, \qquad (61)$$

where N is the number of electrons and p is an average over the star of the momentum of the electrons [32]. We assume that it is determined by an appropriate average value of the density by the relation

$$\rho = \frac{2S_D}{Dh^D} \mu H p^D, \qquad (62)$$

based on the Pauli exclusion principle. Now, for the density, we write

$$\rho = \zeta \frac{M}{S_D R^D}, \qquad (63)$$

where ζ is a dimensionless parameter. We also write the potential energy in the form

$$W = -\frac{\nu}{D - 2} \frac{GM^2}{R^{D-2}}, \qquad (64)$$

where ν is another dimensionless parameter. By writing Eq. (64), we have assumed that $D \neq 2$ but we shall see that the following results pass to the limit for $D \rightarrow 2$. We introduce two dimensionless variables n and r and two fixed constants $M_* = N_* \mu H$ and R_* such that

$$M = nM_*, \qquad R = rR_*. \qquad (65)$$

We determine M_* and R_* by the requirement that the relativity parameter $x = p/mc$ have the form

$$x = \frac{n^{1/D}}{r}, \qquad (66)$$

and that the potential energy can be written

$$W = -\frac{1}{D-2} mc^2 N_* \frac{n^2}{r^{D-2}}. \qquad (67)$$

This yields

$$R_* = \left(\frac{D\zeta}{2\nu S_D^2}\right)^{1/2} \frac{h^{D/2}}{G^{1/2} m^{(D-1)/2} c^{(D-2)/2} \mu H}, \qquad (68)$$

$$M_* = \frac{1}{\nu^{D/2}} \left(\frac{D\zeta}{2S_D^2}\right)^{(D-2)/2} \frac{h^{D(D-2)/2} c^{D(4-D)/2}}{m^{D(D-3)/2} G^{D/2}} \frac{1}{(\mu H)^{D-1}}. \qquad (69)$$

Comparing with Eqs. (55) and (56), we find that

$$M_* = \frac{1}{\nu^{D/2}} \zeta^{(D-2)/2} M_0, \qquad R_* = \left(\frac{\zeta}{\nu}\right)^{1/2} R_0. \qquad (70)$$

Now, the energy $E = K + W$ of the star can be written

$$E = N_* mc^2 n \left\{ (1+x^2)^{1/2} - 1 - n^{2/D} \frac{x^{D-2}}{D-2} \right\}. \qquad (71)$$

In the classical limit $x \ll 1$, we get

$$E_{\text{class}} = N_* mc^2 \left\{ \frac{n^{1+2/D}}{2r^2} - \frac{n^2}{(D-2)r^{D-2}} \right\}, \qquad (72)$$

and in the ultrarelativistic limit $x \gg 1$, we obtain

$$E_{\text{relat}} = N_* mc^2 \left\{ \frac{n^{1+1/D}}{r} - \frac{n^2}{(D-2)r^{D-2}} \right\}. \qquad (73)$$

We shall consider the energy as a function of the radius R, with the mass M fixed. Thus, the mass-radius relation will be obtained by minimizing the energy E versus x with fixed n. Writing $\partial E / \partial x = 0$, we obtain the equations

$$n = \frac{x^{D(4-D)/2}}{(1+x^2)^{D/4}}, \qquad (74)$$

$$x = \frac{n^{1/D}}{r}, \qquad (75)$$

defining the mass-radius relation in parametric form in the framework of the simplified model. Note that x^D represents the value of the density in units of M_*/R_*^D, i.e. $\rho = x^D M_*/R_*^D$. Therefore, Eq. (74) can be viewed as the relation between the mass and the density. Eliminating the relativity parameter x (or density) between Eqs. (74) and (75), we explicitly obtain

$$n = \frac{r^3}{\sqrt{1-r^4}} \qquad (D=1), \qquad (76)$$

$$n = \frac{1-r^4}{r^2} \qquad (D=2), \qquad (77)$$

$$r = \frac{(1-n^{4/3})^{1/2}}{n^{1/3}} \qquad (D=3), \qquad (78)$$

$$r = \frac{n^{3/4}}{\sqrt{1-n}} \qquad (D=4). \qquad (79)$$

In $D=1$, there exists a maximum radius $r=1$ achieved for $n \to +\infty$, and we have the scaling

$$n \sim \tfrac{1}{2}(1-r)^{-1/2}. \qquad (80)$$

In $D=2$, there exists a maximum radius $r=1$ achieved for $n \to 0$, and we have the scaling

$$n \sim 4(1-r). \qquad (81)$$

In $D=3$, there exists a maximum mass $n=1$ (Chandrasekhar's mass) achieved for $r \to 0$, and we have the scaling

$$r \sim \tfrac{2}{\sqrt{3}}(1-n)^{1/2}. \qquad (82)$$

In $D=4$, there exists a maximum mass $n=1$ achieved for $r \to +\infty$, and we have the scaling

$$r \sim (1-n)^{-1/2}. \qquad (83)$$

In order to compare Eqs. (76)–(79) with the exact results, we need to estimate the values of the constants ζ and ν. This can be done by considering the limiting forms of Eqs. (74) and (75). In the classical limit $x \ll 1$, we obtain

$$n^{(D-2)/D} r^{4-D} = 1, \qquad (84)$$

and in the ultrarelativistic limit $x \gg 1$, we get

$$n^{(D-1)/D} r^{3-D} = 1. \qquad (85)$$

Comparing with Eqs. (59) and (60), we find that

$$\zeta = \left(\frac{1}{2}\right)^D \frac{\omega_{D/2}^{D-2}}{\omega_D^{D-1}}, \qquad \nu = \frac{1}{2} \frac{\omega_{D/2}^{(D-2)/D}}{\omega_D^{2(D-1)/D}}, \qquad (86)$$

where the quantities ω_D and $\omega_{D/2}$ can be deduced from the numerical study of the Lane-Emden equation. The analytical approximations of the mass-radius relation (76)–(79) are plotted in dotted lines in Figs. 5, 7, and 9, and they give a fair agreement with the exact results (full line). Of course, they cannot reproduce the spiral in $D=4$ which requires the resolution of the full differential equation (19).

Let us address the stability of the configurations within this simplified analytical model. If we view a white dwarf star as a gas of electrons at statistical equilibrium at temperature T, stable configurations are those that minimize the free energy $F = E - TS$ at fixed mass, where E is the total energy (kinetic + potential) and S is the Fermi-Dirac entropy (see Appendix C). For a completely degenerate gas

at $T = 0$, stable configurations are those that minimize the energy E at fixed mass. As we have seen previously, the first order variations $E'(x) = 0$ determine the mass-radius relation. Then, the configurations are stable if the energy is minimum, i.e. $E''(x) > 0$. Now, we have

$$E'' = \frac{N_* mc^2 n}{(1 + x^2)^{3/2}} [4 - D - (D - 3)x^2]. \quad (87)$$

For $D \leq 3$, all the solutions are stable while for $D \geq 4$, all the solutions are unstable. For $3 < D < 4$, the solutions with $x < x_c$ are stable and the solutions with $x > x_c$ are unstable where

$$x_c = \left(\frac{4 - D}{D - 3}\right)^{1/2}. \quad (88)$$

We now show that the onset of instability precisely corresponds to the turning point of mass, i.e. to the point where the mass is maximum in the series of equilibria $M(R)$. In terms of reduced variables, this corresponds to $dn/dr = 0$, or equivalently $n'(x) = 0$. Taking the logarithmic derivative of Eq. (74), we have

$$\frac{dn}{n} = \frac{D}{2}(4 - D)\frac{dx}{x} - \frac{D}{2}\frac{x}{1 + x^2}dx, \quad (89)$$

so that the condition $n'(x) = 0$ yields $x = x_c$. Thus, instability sets in precisely at the maximum mass as could have been directly inferred from the turning point criterion [33].

It is also useful to discuss the classical and the ultrarelativistic limits specifically. In the classical limit, using Eq. (72), we get

$$E''_{\text{class}} = N_* mc^2 n(4 - D), \quad (90)$$

so that a classical white dwarf star is stable for $D < 4$ and unstable for $D \geq 4$. In the ultrarelativistic limit, using Eq. (73), we get

$$E''_{\text{relat}} = \frac{N_* mc^2 n}{x}(3 - D), \quad (91)$$

so that an ultrarelativistic white dwarf star is stable for $D \leq 3$ and unstable for $D > 3$. This returns the results obtained in Secs. IV and V.

Finally, considering the function $E(R)$ given by Eq. (71) and assuming that the system evolves so as to minimize its energy (this requires some source of dissipation), we have the following results [34]: for $D < 3$, there exists a stable equilibrium state (global minimum of E) with radius R for all mass M. For $D = 3$, there exists a maximum mass M_c so that: (i) for $M < M_c$, there exists a stable equilibrium state (global minimum of E) with radius $R > 0$; (ii) for $M = M_c$, the system collapses to a point ($R = 0$) but its energy remains finite (lower bound); (iii) for $M > M_c$, the system collapses to a point $R \rightarrow 0$ and $E \rightarrow -\infty$. For $3 < D < 4$, there exists a maximum mass M_c so that: (i) for $M < M_c$, there exists a metastable equilibrium state (local minimum of E) with $x < x_c$ and an unstable equilibrium

state (global maximum of E) with $x > x_c$. The system can either reach the metastable state or collapse to a point ($R \rightarrow 0$, $E \rightarrow -\infty$); the choice probably depends on a notion of basin of attraction; (ii) for $M \geq M_c$, the system collapses to a point ($R \rightarrow 0$, $E \rightarrow -\infty$). For $D = 4$, there exists a critical mass M_c so that: (i) for $M < M_c$, there exists an unstable equilibrium state (global maximum of E) so the system either collapses ($R \rightarrow 0$ and $E \rightarrow -\infty$) or evaporates ($R \rightarrow +\infty$ and $E \rightarrow 0$); (ii) for $M > M_c$, the system collapses to a point ($R \rightarrow 0$ and $E \rightarrow -\infty$). For $D > 4$, there exists an unstable equilibrium state (global maximum of E) for all mass M so the system either collapses ($R \rightarrow 0$ and $E \rightarrow -\infty$) or evaporates ($R \rightarrow +\infty$ and $E \rightarrow 0$).

VIII. CONCLUSION

"Why is our universe three dimensional? Does the dimension $D = 3$ play a special role among other space dimensions?"

Several scientists have examined the role played by the dimension of space in determining the form of the laws of physics. This question goes back to Ptolemy who argues in his treatise *On dimensionality* that no more than three spatial dimensions are possible in nature. In the 18th century, Kant realizes the deep connection between the inverse square law of gravitation and the existence of three spatial dimensions. Interestingly, he regards the three spatial dimensions as a consequence of Newton's inverse square law rather than the converse. In the twentieth century, Ehrenfest [26], in a paper called ''In what way does it become manifest in the fundamental laws of physics that space has three dimensions?'' argues that planetary orbits, atoms, and molecules would be unstable in a space of dimension $D \geq 4$. This idea has been followed more recently by Gurevich and Mostepanenko [35] who argue that if the universe is made of metagalaxies with a various number of dimensions, atomic matter and life are possible only in three-dimensional space. Other investigations on dimensionality are reviewed in the paper of Barrow [36]. We have found that the relativistic self-gravitating Fermi gas at $T = 0$ (a white dwarf star) possesses a rich structure as a function of the dimension of space. We have exhibited several characteristic dimensions $D = 1$, $D = 2$, $D = 3$, $D = (3 + \sqrt{17})/2$, $D = 4$, and $D = 2(1 + \sqrt{2})$. For $D < 3$, there exists stable configurations for any value of the mass. For $D = 3$, the sequence of equilibrium configurations is stable but there exists a maximum mass (Chandrasekhar's limit) above which there is no equilibrium state. For $3 < D < 4$, the sequence of equilibrium configurations is stable for classical white dwarf stars but it becomes unstable for relativistic white dwarf stars with high density after the turning point of mass. Therefore, the dimension $D = 3$ is special because it is the largest dimension at which the sequence of equilibrium configurations is stable all the way long; for $D > 3$, a turning point of mass appears so that ultrarelativistic white dwarf stars become

unstable. Therefore, the dimension $D = 3$ is *marginal* in that respect. Finally, for $D \geq 4$, the whole sequence of equilibrium configurations is unstable. Therefore, as already noted in our previous papers [21–23], the dimension $D = 4$ is critical because at that dimension quantum mechanics cannot stabilize matter against gravitational collapse, even in the classical regime, contrary to the situation in $D = 3$ [4,24,25]. Interestingly, this result is similar to that of Ehrenfest [26] although it applies to white dwarf stars instead of atoms.

Our exact description of D-dimensional white dwarf stars based on Chandrasekhar's seminal paper [11] shows that relativistic white dwarf stars become unstable in $D > 3$ and that classical white dwarf stars become unstable in $D \geq 4$. Therefore, for $D \geq 4$, a self-gravitating Fermi gas forms a black hole or evaporates (see Appendix B). These conclusions have also been reached by Bechhoefer and Chabrier [28] on the basis of simple dimensional analysis. This suggests that a $D \geq 4$ universe is not viable (see also Appendix D) and gives insight why our universe in apparently three dimensional. We note that extra dimensions can appear at the microscale, an idea originating from Kaluza-Klein theory. This idea took a renaissance in modern theories of grand unification which are formulated in higher-dimensional spaces [37]. Our approach shows that already at a simple level, the coupling between Newton's equations (gravitation), Fermi-Dirac statistics (quantum mechanics), and special relativity reveals a rich structure as a function of D. In this respect, it is interesting to note that the critical masses (34) and (50) and radii (36) and (47) that we have found occur for simple *integer* dimensions $D = 1, 2, 3,$ and 4, which was not granted *a priori*.

It is interesting to develop a parallel between the mass-radius relation $M(R)$ of white dwarf stars and the caloric curve $T(E)$ giving the temperature as a function of the energy in the thermodynamics of self-gravitating systems [38–41]. In this analogy, the Chandrasekhar mass in $D = 3$ is the counterpart of the critical temperature for isothermal systems in $D = 2$ (in both cases, the equilibrium density profile is a Dirac peak containing all the mass) [42]. On the other hand, for $D > 3$, the mass-radius relation for white dwarf stars exhibits turning points, and even a spiraling behavior for $D > \frac{1}{2}(3 + \sqrt{17})$, which is similar to the spiraling behavior of the caloric curve for isothermal systems in $D = 3$. In this analogy, the maximum mass, corresponding to a critical value of the central density which parametrizes the series of equilibria, is the counterpart of the Antonov energy (in the microcanonical ensemble) [38] or of the Emden temperature (in the canonical ensemble) [41]. The series of equilibria becomes unstable after this turning point. In addition, there is no equilibrium state above this maximum mass, or below the minimum energy or minimum temperature in the thermodynamic problem. In that case, the system is expected to undergo gravitational collapse.

As a last comment (notified by the referee), it should be emphasized that the conclusions reached in this paper concerning dimensionality implicitly assume that the laws of physics that we know remain the same in a universe of arbitrary dimension D. This is of course not granted at all. There may be a new cosmological theory, a new theory of star formation and stellar evolution in higher dimensions. We also emphasize that our approach does not take into account general relativistic effects that can sensibly modify the results [43].

APPENDIX A: THE EQUATION OF STATE

We provide here the explicit expression of the function $f(x)$ defined by Eq. (10) for different dimensions of space $D = 1, 2, 3,$ and 4, respectively:

$$f(x) = 4x(1 + x^2)^{1/2} - 4\sinh^{-1}x, \qquad (A1)$$

$$f(x) = \tfrac{16}{3} + \tfrac{8}{3}(x^2 - 2)(1 + x^2)^{1/2}, \qquad (A2)$$

$$f(x) = x(2x^2 - 3)(1 + x^2)^{1/2} + 3\sinh^{-1}x, \qquad (A3)$$

$$f(x) = -\tfrac{64}{15} + \tfrac{8}{15}(3x^4 - 4x^2 + 8)(1 + x^2)^{1/2}. \qquad (A4)$$

APPENDIX B: STABILITY CRITERIA FOR POLYTROPIC SPHERES IN D DIMENSIONS

We generalize in D dimensions the usual stability criteria for polytropic gaseous spheres, and apply them to classical and ultrarelativistic white dwarf stars.

1. The Euler-Poisson system

Let us consider the Euler-Poisson system [44]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad (B1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla \Phi, \qquad (B2)$$

$$\Delta \Phi = S_D G \rho, \qquad (B3)$$

describing the dynamical evolution of a barotropic gas with an equation of state $P = P(\rho)$. The Euler-Poisson system conserves the mass M and the energy [23]:

$$\mathcal{W} = \int \rho \int^{\rho} \frac{P(\rho')}{\rho'^2} d\rho' d\mathbf{r} + \frac{1}{2}\int \rho \Phi d\mathbf{r} + \frac{1}{2}\int \rho \mathbf{u}^2 d\mathbf{r}.$$

$$(B4)$$

In the following, we shall essentially consider the case of a polytropic equation of state $P(\rho) = K\rho^\gamma$ where $\gamma = 1 + 1/n$. For $\gamma \neq 1$, the energy functional $\mathcal{W}$ is given by

$$\mathcal{W} = \frac{K}{\gamma - 1} \int \rho^\gamma d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r} + \frac{1}{2} \int \rho \mathbf{u}^2 d\mathbf{r}.$$
(B5)

It can be rewritten

$$\mathcal{W} = \frac{1}{\gamma - 1} \int P d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r} + \frac{1}{2} \int \rho \mathbf{u}^2 d\mathbf{r}. \quad (B6)$$

For an isothermal equation of state $P = \rho k_B T / m$, corresponding to $\gamma \rightarrow 1$ or $n \rightarrow +\infty$, we have

$$\mathcal{W} = k_B T \int \frac{\rho}{m} \ln \frac{\rho}{m} d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r} + \frac{1}{2} \int \rho \mathbf{u}^2 d\mathbf{r}.$$
(B7)

For a general polytropic equation of state $P = K\rho^\gamma$ with arbitrary γ, it is convenient to write the energy of the gas in the form

$$\mathcal{W} = \frac{K}{\gamma - 1} \int (\rho^\gamma - \rho) d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r} + \frac{1}{2} \int \rho \mathbf{u}^2 d\mathbf{r}.$$
(B8)

We have added a constant term $-\frac{K}{\gamma-1} \int \rho d\mathbf{r}$ proportional to the total mass (which is a conserved quantity) so as to recover the energy (B7) of an isothermal gas in the limit $\gamma \rightarrow 1$ [45].

2. The eigenvalue equation

We first consider the linear dynamical stability of a polytropic star described by the Euler-Poisson system and generalize the approach developed in Chavanis [21,23,41,46–49]. We consider a steady solution of the Euler-Poisson system satisfying $\mathbf{u} = \mathbf{0}$ and the condition of hydrostatic balance $\nabla P + \rho \nabla \Phi = \mathbf{0}$. For a polytropic equation of state $P = K\rho^{1+1/n}$, the equilibrium density profile is solution of the D-dimensional Lane-Emden equation [21]. We consider *complete polytropes* such that the density vanishes at a finite radius R. For $n \neq n_3$, there exists a unique steady state for any mass M. The mass-radius relation is [21]:

$$M^{(n-1)/n} R^{(D-2)(n_3-n)/n} = \frac{K(1+n)}{G S_D^{1/n}} \omega_n^{(n-1)/n}, \quad (B9)$$

where

$$\omega_n = -\xi_1^{(n+1)/(n-1)} \theta_n'(\xi_1), \quad (B10)$$

is a constant given in terms of the solution $\theta_n(\xi)$ of the Lane-Emden equation of index n in D dimensions. For the critical index $n = n_3$, steady state solutions exist only for a unique value of the mass

$$M_c = \left[\frac{K(1+n)}{G S_D^{1/n}}\right]^{n/(n-1)} \omega_n. \quad (B11)$$

These solutions have the same mass M_c but an arbitrary radius R. We already anticipate that the index $n = n_3$ will correspond to a case of marginal stability separating stable and unstable solutions.

Linearizing the equations of motion (B1)–(B3) around a stationary solution in hydrostatic balance and writing the perturbation in the form $\delta\rho \sim e^{\lambda t}$, we obtain after some calculations [41,47,49] the eigenvalue equation

$$\frac{d}{dr}\left(\frac{P'(\rho)}{S_D \rho r^{D-1}} \frac{dq}{dr}\right) + \frac{Gq}{r^{D-1}} = \frac{\lambda^2}{S_D \rho r^{D-1}} q, \quad (B12)$$

where we have restricted ourselves to spherically symmetric perturbations and defined $q(r) = \int_0^r \delta\rho S_D r^{D-1} dr$. For a polytropic gas with an equation of state $P = K\rho^\gamma$, the foregoing equation becomes [47]:

$$K\gamma \frac{d}{dr}\left(\frac{\rho^{\gamma-2}}{S_D r^{D-1}} \frac{dq}{dr}\right) + \frac{Gq}{r^{D-1}} = \frac{\lambda^2}{S_D \rho r^{D-1}} q. \quad (B13)$$

The polytrope is stable if $\lambda^2 < 0$ (yielding oscillatory modes with pulsation $\omega = \sqrt{-\lambda^2}$) and unstable if $\lambda^2 > 0$ (yielding exponentially growing modes with growth rate $\gamma = \sqrt{\lambda^2}$). Considering the point of marginal stability ($\lambda = 0$) and introducing the Emden variables [21,30,47], Eq. (B13) reduces to

$$\frac{d}{dr}\left(\frac{\theta^{1-n}}{\xi^{D-1}} \frac{dF}{d\xi}\right) + \frac{nF}{\xi^{D-1}} = 0. \quad (B14)$$

This equation has the exact solution [21,47]:

$$F(\xi) = c_1\left[\xi^D \theta^n + \frac{(D-2)n - D}{n-1} \xi^{D-1} \theta'\right]. \quad (B15)$$

The point of marginal stability is then determined by the boundary conditions. One can show [47,49] that the velocity perturbation is given by $\delta u = -\lambda q/(S_D \rho r^{D-1})$. Therefore, if the density of the configuration vanishes at $r = R$, one must have $q(R) = 0$ to avoid unphysical divergences. Thus, if ξ_1 denotes the value of the normalized radius R of the star [21,30,47] such that $\theta(\xi_1) = 0$, the natural boundary condition for the eigenvalue equation (B13) is $F(\xi_1) = 0$. Substituting this condition in Eq. (B15), we obtain the critical index

$$n = \frac{D}{D-2} \equiv n_3, \quad (B16)$$

corresponding to a marginally stable gaseous polytrope ($\lambda = 0$). Our method gives the form (B15) of the neutral perturbation $\delta\rho$ at the critical index $n = n_3$. We conclude that the infinite family of steady state solutions with equal mass M_c and arbitrary radius R that exists for $n = n_3$ is marginally stable.

We now consider the nonlinear dynamical stability problem. Since the Euler-Poisson system (B1)–(B3) conserves the mass M and the energy $\mathcal{W}$, a maximum or a minimum of the energy functional $\mathcal{W}[\rho, \mathbf{u}]$ at fixed mass $M[\rho] = M$ determines a steady state of the Euler-Poisson system that is formally nonlinearly dynamically stable [23]. Because of the presence of the kinetic term $\Theta[\mathbf{u}] = (1/2) \int \rho \mathbf{u}^2 d\mathbf{r}$, the functional $\mathcal{W}[\rho, \mathbf{u}]$ has no absolute maximum. Thus, we need to investigate the possible existence of a minimum of $\mathcal{W}[\rho, \mathbf{u}]$ at fixed mass $M[\rho] = M$. A barotropic star that minimizes the energy functional $\mathcal{W}$ at fixed mass M is nonlinearly dynamically stable with respect to the Euler-Poisson system. The cancellation of the first order variations $\delta \mathcal{W} - \alpha \delta M = 0$, where α is a Lagrange multiplier, yields $\mathbf{u} = \mathbf{0}$ and the condition of hydrostatic balance $\nabla P + \rho \nabla \Phi = \mathbf{0}$. Then, the condition of nonlinear dynamical stability is

$$\delta^2 \mathcal{W} = \int \frac{P'(\rho)}{2\rho} (\delta\rho)^2 d\mathbf{r} + \frac{1}{2} \int \delta\rho \, \delta\Phi \, d\mathbf{r} \geq 0, \quad \text{(B17)}$$

for all perturbations that conserve mass, i.e. $\int \delta\rho \, d\mathbf{r} = 0$. After some calculations [23], this can be put in the quadratic form

$$\delta^2 \mathcal{W} = -\frac{1}{2} \int_0^R \left[\frac{d}{dr}\left(\frac{P'(\rho)}{S_D \rho r^{D-1}} \frac{dq}{dr} \right) + \frac{Gq}{r^{D-1}} \right] q \, dr. \quad \text{(B18)}$$

We are led therefore to consider the eigenvalue problem

$$\left[\frac{d}{dr}\left(\frac{P'(\rho)}{S_D \rho r^{D-1}} \frac{d}{dr} \right) + \frac{G}{r^{D-1}} \right] q_\lambda(r) = \lambda q_\lambda(r). \quad \text{(B19)}$$

If all the eigenvalues λ are negative, then $\delta^2 \mathcal{W} > 0$ and the configuration is a minimum of $\mathcal{W}$ at fixed mass. This implies that it is nonlinearly dynamically stable. If at least one eigenvalue λ is positive, the configuration is a saddle point of $\mathcal{W}$ and the star is dynamically unstable. The marginal case is when the largest eigenvalue λ is equal to zero. Now, for $\lambda = 0$, Eqs. (B12) and (B19) coincide. This implies that the conditions of linear and nonlinear dynamical stability are the same. In the case of polytropic stars, the case of marginal stability corresponds to the critical index $n = n_3$. At that index, the equilibrium configurations with mass M_c and radius R *all* have the same value of energy $\mathcal{W} = 0$ (see Eq. (B43) later). This is therefore a very degenerate situation.

The nonlinear dynamical stability of gaseous polytropes can also be investigated by plotting the series of equilibria $M(\rho_0)$ (mass vs central density) of box-confined configurations and using the turning point argument of Poincaré [21,23,47]. We can thus determine whether the last point on the series of equilibria, which corresponds to a *complete polytrope* whose density vanishes precisely at the box radius, is stable or not. For $D \leq 2$, there is no turning point of mass (we restrict ourselves to $n \geq 0$), implying that the gaseous polytropes are always stable. For $D > 2$, a turning

point of mass $M(\rho_0)$ appears precisely for $n = n_3$ (see Fig. 5 of [21]). This method shows that, for $D > 2$, complete polytropes with $n < n_3$ are nonlinearly dynamically stable (they are minima of $\mathcal{W}$ at fixed mass M) while complete polytropes with $n > n_3$ are dynamically unstable (they are saddle points of $\mathcal{W}$ at fixed mass M). Complete polytropes with $n = n_3$ and $M = M_c$ are marginally stable.

3. The Ledoux criterion

We can also investigate the linear dynamical stability of gaseous polytropic spheres by using the method introduced by Eddington [50] and Ledoux [51]. If we introduce the radial displacement

$$\xi(r) = -\frac{\delta u}{\lambda r} = \frac{q}{S_D \rho r^D} \propto \frac{\delta r}{r}, \quad \text{(B20)}$$

we can rewrite the eigenvalue equation (B13) in the form [49,52]:

$$\frac{d}{dr}\left(P\gamma r^{D+1} \frac{d\xi}{dr} \right) + r^D (D\gamma + 2 - 2D) \frac{dP}{dr} \xi$$
$$= \lambda^2 \rho r^{D+1} \xi. \quad \text{(B21)}$$

This is the Eddington equation of pulsations, which has been written here in the form of a Sturm-Liouville problem. It must be supplemented by the boundary conditions

$$\delta r = \xi r = 0, \quad \text{in } r = 0, \quad \text{(B22)}$$

$$dP = \lambda \gamma P \left(D\xi + r \frac{d\xi}{dr} \right) = 0, \quad \text{in } r = R, \quad \text{(B23)}$$

where $dP/dt = \partial \delta P / \partial t + \delta u \, dP / dr$ is the Lagrangian derivative of the pressure. Since $P = 0$ at the surface of the star, it is sufficient to demand that ξ and $d\xi/dr$ be finite in $r = R$. Multiplying Eq. (B21) by ξ and integrating between 0 and R, we obtain

$$\lambda^2 \int_0^R \rho r^{D+1} \xi^2 dr = -\int_0^R dr \left\{ P\gamma r^{D+1} \left(\frac{d\xi}{dr} \right)^2 \right.$$
$$\left. - \xi^2 r^D (D\gamma + 2 - 2D) \frac{dP}{dr} \right\}. \quad \text{(B24)}$$

The system is linearly dynamically stable if $\lambda^2 < 0$ and unstable otherwise. Since $dP/dr < 0$, a sufficient condition of stability is $D\gamma + 2 - 2D > 0$, i.e.

$$\gamma > \gamma_{4/3} \equiv \frac{2(D-1)}{D}, \quad \frac{1}{n} > \frac{1}{n_3} \equiv \frac{D-2}{D}. \quad \text{(B25)}$$

It can be shown furthermore (see below) that the system is unstable if $\gamma < \gamma_{4/3}$ so that the criterion (B25) is a necessary and sufficient condition of dynamical stability (the case $\gamma = \gamma_{4/3}$ is marginal). In terms of the index n, a complete polytrope is stable with respect to the Euler-Poisson system in $D = 1$ for $n \geq 0$ and for $n < -1$, in

$D = 2$ for $n > 0$, in $D = 3$ for $0 \leq n < 3$, and in $D > 2$ for $0 \leq n < n_3$.

From the theory of Sturm-Liouville problems, it is known that expression (B24), which can be written $\lambda^2 = I[\xi]$, forms the basis of a variational principle. The function $\xi(r)$ which maximizes the functional $I[\xi]$ is the fundamental eigenfunction and the maximum value of this functional gives the fundamental eigenvalue λ^2. Furthermore, any trial function underestimates the value of λ^2 so this variational principle may prove the existence of instability but can only give approximate information concerning stability. As shown by Ledoux and Pekeris [51], we can get a good approximation of the fundamental eigenvalue by taking $\xi(r)$ to be a constant (note that $\xi = $ Cst, i.e. $\delta r \propto r$, is the *exact* solution of the Sturm-Liouville equation (B21) at the point of marginal stability $\lambda = 0$ for a polytropic equation of state). For the trial function $\xi = $ Cst, expression (B24) gives

$$\lambda^2 \int \rho r^2 d\mathbf{r} = (D\gamma + 2 - 2D) \int r \frac{dP}{dr} d\mathbf{r}. \quad \text{(B26)}$$

According to the condition of hydrostatic balance,

$$\frac{dP}{dr} = -\rho \frac{d\Phi}{dr}, \quad \text{(B27)}$$

we have

$$\int r \frac{dP}{dr} d\mathbf{r} = -\int \rho \mathbf{r} \cdot \nabla \Phi d\mathbf{r}, \quad \text{(B28)}$$

where we recognize the Virial

$$W_{ii} \equiv -\int \rho \mathbf{r} \cdot \nabla \Phi d\mathbf{r}. \quad \text{(B29)}$$

Inserting

$$\nabla \Phi = G \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^D} d\mathbf{r}', \quad \text{(B30)}$$

in Eq. (B29), interchanging the dummy variables $\mathbf{r}$ and $\mathbf{r}'$ and taking the half-sum of the resulting expressions, we get

$$W_{ii} = -\frac{G}{2} \int \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^{D-2}} d\mathbf{r} d\mathbf{r}'. \quad \text{(B31)}$$

Therefore, the Virial can be written

$$W_{ii} = (D - 2)W, \qquad (D \neq 2), \quad \text{(B32)}$$

$$W_{ii} = -\frac{GM^2}{2}, \qquad (D = 2), \quad \text{(B33)}$$

where

$$W = \frac{1}{2} \int \rho \Phi d\mathbf{r}, \quad \text{(B34)}$$

is the potential energy and

$$\Phi = -\frac{G}{D - 2} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^{D-2}} d\mathbf{r}', \quad \text{(B35)}$$

is the gravitational potential. Therefore, we can rewrite Eq. (B26) in the form

$$\lambda^2 = (D\gamma + 2 - 2D) \frac{W_{ii}}{I}, \quad \text{(B36)}$$

where

$$I = \int \rho r^2 d\mathbf{r}, \quad \text{(B37)}$$

is the moment of inertia. Since $W_{ii} < 0$, we conclude that the system is unstable if $D\gamma + 2 - 2D < 0$, which completes the proof above. On the other hand, Eq. (B36) provides an estimate of the pulsation period $\omega = \sqrt{-\lambda^2}$ when the system is stable. This is the D-dimensional generalization of the Ledoux stability criterion.

Note finally that, for a spherically symmetric system, using the Gauss theorem

$$\nabla \Phi = \frac{GM(r)}{r^{D-1}} \mathbf{e}_r, \quad \text{(B38)}$$

we get

$$W_{ii} = -S_D G \int \rho(r) M(r) r dr = -\int \frac{GM(r)}{r^{D-2}} dM(r). \quad \text{(B39)}$$

This expression may be useful to calculate the potential energy.

4. Virial theorem and Poincaré argument

The Virial theorem associated with the barotropic Euler-Poisson system (B1)–(B3) is [44,49]:

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2\Theta + \Pi + W_{ii}, \quad \text{(B40)}$$

where $\Theta = (1/2) \int \rho u^2 d\mathbf{r}$ is the kinetic energy of the macroscopic motion and $\Pi = D \int P d\mathbf{r}$. For a polytropic equation of state, the energy functional (B4), which is a conserved quantity, can be written

$$\mathcal{W} = U + W + \Theta, \quad \text{(B41)}$$

where

$$U = \frac{1}{\gamma - 1} \int P d\mathbf{r}, \quad \text{(B42)}$$

is the internal energy. We note that $\Pi = D(\gamma - 1)U = (D/n)U$ for a polytrope. At equilibrium, $\ddot{I} = \Theta = 0$, we get $(D/n)U + W_{ii} = 0$ and $\mathcal{W} = U + W$. For $D = 2$, this implies that $U = nGM^2/4$. For $D \neq 2$, this implies that

$$\mathcal{W} = \left(1 - \frac{n}{n_3}\right)W, \quad \text{(B43)}$$

where we recall that

$$W = -\frac{G}{2(D-2)} \int \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^{D-2}} d\mathbf{r} d\mathbf{r}'. \qquad (B44)$$

For $D > 2$ and $n \geq 0$, the system is dynamically stable if $\mathcal{W} < 0$ and unstable otherwise (Poincaré argument [30]). Since $W < 0$, we find from Eq. (B43) that the polytropic star is stable if and only if $n < n_3$. At the point of marginal stability $n = n_3$, where several steady configurations exist with the same mass M_c given by Eq. (B11) and an arbitrary radius R, the energy of these configurations is $\mathcal{W} = 0$ for all R.

Using $\mathcal{W} = (n/D)\Pi + W + \Theta$ and eliminating the pressure term in Eq. (B40), the Virial theorem for $D \neq 2$ can be put in the form

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \left(2 - \frac{D}{n}\right)\Theta + \frac{D}{n}\mathcal{W} + \left(D - 2 - \frac{D}{n}\right)W. \quad (B45)$$

Alternatively, eliminating the kinetic energy in Eq. (B40), the Virial theorem for $D \neq 2$ can be written

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2\mathcal{W} + \left(\frac{D}{n} - 2\right)U + (D-4)W. \qquad (B46)$$

For $n = n_{3/2}$, corresponding to classical white dwarf stars, the Virial theorem becomes

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2\mathcal{W} + (D-4)W. \qquad (B47)$$

For $n = n_3'$, corresponding to relativistic white dwarf stars, we have

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \Theta + \mathcal{W} + (D-3)W. \qquad (B48)$$

Finally, considering the polytropic index $n = n_3$ of marginal stability, the Virial theorem (B45) takes the form

$$\frac{1}{2}\frac{d^2 I}{dt^2} = (4-D)\Theta + (D-2)\mathcal{W}. \qquad (B49)$$

If we consider the dimension $D = 4$, it reduces to

$$\frac{d^2 I}{dt^2} = 4\mathcal{W}. \qquad (B50)$$

This equation describes the case of classical white dwarf stars at the critical dimension $D = 4$ where $n_{3/2} = n_3 = 2$. It can be integrated into

$$I(t) = 2\mathcal{W}t^2 + C_1 t + C_2. \qquad (B51)$$

For $\mathcal{W} > 0$, we find that $I(t) \rightarrow +\infty$ for $t \rightarrow +\infty$ so that the system evaporates. Alternatively, for $\mathcal{W} < 0$, we find that $I(t) \rightarrow 0$ in a finite time, so that the system collapses and forms a Dirac peak in a finite time. If we consider the dimension $D = 3$, Eq. (B49) reduces to

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \Theta + \mathcal{W}. \qquad (B52)$$

This equation describes the case of relativistic white dwarf stars at the critical dimension $D = 3$ where $n_3' = n_3 = 3$. For $\mathcal{W} > 0$, we find that $I(t) \rightarrow +\infty$ for $t \rightarrow +\infty$ indicating that the system evaporates.

5. Dimensional analysis

Finally, we show that the instability criterion for polytropic stars can be obtained from simple dimensional analysis. We shall approximate the internal energy (B42) and the potential energy (B34) by

$$U = \frac{K\zeta}{\gamma - 1}\left(\frac{M}{R^D}\right)^\gamma R^D, \qquad (B53)$$

$$W = -\frac{\nu}{D-2}\frac{GM^2}{R^{D-2}}, \qquad (B54)$$

where M is the total mass of the configuration and R its radius (ν and ζ are dimensionless parameters). For homogeneous spheres, the values of ζ and ν are given by Eq. (B70). With these expressions, the energy functional (B5) becomes

$$\mathcal{W} = \frac{K\zeta}{\gamma - 1}\left(\frac{M}{R^D}\right)^\gamma R^D - \frac{\nu}{D-2}\frac{GM^2}{R^{D-2}} + \Theta. \quad (B55)$$

We now need to minimize this functional with respect to R for a given mass M (a minimum necessarily requires $\Theta = 0$). We first look for the existence of critical points (extrema). The cancellation of the first order variations

$$\frac{d\mathcal{W}}{dR} = 0 = -K\zeta DM^\gamma R^{D(1-\gamma)-1} + G\nu M^2 R^{1-D},$$
$$(B56)$$

yields the mass-radius relation

$$M^{(n-1)/n} R^{(D-2)(n_3-n)/n} = \frac{K\zeta D}{G\nu}. \qquad (B57)$$

This relation determines the radius R of the star as a function of its mass M. This expression is consistent with the exact mass-radius relation (B9) deduced from the Lane-Emden equation (of course, our simple approach can only model compact density profiles corresponding to $n < n_5$ for $D > 2$) [21]. For $n \neq n_3$, there is one, and only one, extremum of $\mathcal{W}(R)$ for each mass M. In order to have a true minimum, we need to impose

$$\frac{d^2\mathcal{W}}{dR^2} = -G\nu DM^2 R^{-D}\left(\frac{1}{n_3} - \frac{1}{n}\right) > 0. \qquad (B58)$$

Therefore, the system is stable if

$$\frac{1}{n} > \frac{1}{n_3}, \qquad (B59)$$

and unstable otherwise. This simple dimensional analysis returns the exact stability criterion (B25).

We can be a little more precise. For $1/n > 1/n_3$, the functional $\mathcal{W}(R)$ has a global minimum reached for a finite, and nonzero, value of R. This solution is stable. For $1/n < 1/n_3$, the functional $\mathcal{W}(R)$ has an unstable global maximum: for $D < 2$, $\mathcal{W} \rightarrow -\infty$ when $R \rightarrow +\infty$ (evaporation) and $\mathcal{W} \rightarrow 0$ when $R \rightarrow 0$ (collapse); for $D = 2$, $\mathcal{W} \rightarrow -\infty$ when $R \rightarrow 0$ and $R \rightarrow +\infty$; for $D > 2$ and $n < 0$, $\mathcal{W} \rightarrow -\infty$ when $R \rightarrow 0$ and $R \rightarrow +\infty$; for $D > 2$ and $n > n_3$, $\mathcal{W} \rightarrow -\infty$ when $R \rightarrow 0$ and $\mathcal{W} \rightarrow 0$ when $R \rightarrow +\infty$. Since we have constructed a particular configuration (homogeneous sphere) which makes the energy $\mathcal{W}$ diverge to $-\infty$, the above arguments prove that the exact functional $\mathcal{W}[\rho, \mathbf{u}]$ given by Eq. (B4) has no absolute minimum at fixed mass for $1/n < 1/n_3$. Since the functional $\mathcal{W}[\rho, \mathbf{u}]$ has only one critical point (cancelling the first variations) at fixed mass, we conclude that, for $1/n < 1/n_3$, this critical point is a saddle point. Therefore, there is no stable steady state of polytropic spheres for $1/n < 1/n_3$. For the critical index $n = n_3$, the relation (B57) shows the existence of a critical mass

$$M_c = \left(\frac{K\zeta D}{G\nu}\right)^{D/2}. \tag{B60}$$

The functional (B55) can be rewritten

$$\mathcal{W} = K\zeta \frac{D}{D-2} \frac{M^{2(D-1)/D}}{R^{D-2}} \left[1 - \left(\frac{M}{M_c}\right)^{2/D}\right] + \Theta. \tag{B61}$$

For $M = M_c$, $\mathcal{W}(R) = 0$ for all R (at equilibrium $\Theta = 0$). This determines an infinite family of solutions with the same mass M_c and different radii. These solutions have the same energy and are marginally stable. This returns the result of the exact model where the configurations are the solution of the Lane-Emden equation (see Sec. B 2). For $M < M_c$, the function $\mathcal{W}(R)$ is monotonically decreasing with R (so the system tends to evaporate): for $D < 2$, $\mathcal{W}(R)$ goes from 0 to $-\infty$ and for $D > 2$, $\mathcal{W}(R)$ goes from $+\infty$ to 0. For $M > M_c$, the function $\mathcal{W}(R)$ is monotonically increasing with R (so the system tends to collapse): for $D < 2$, $\mathcal{W}(R)$ goes from 0 to $+\infty$ and for $D > 2$, $\mathcal{W}(R)$ goes from $-\infty$ to 0.

We can obtain a simple dynamical model by using the Virial theorem (B40). At equilibrium ($\ddot{I} = \Theta = 0$), we have

$$\frac{D}{n} U + (D-2)W = 0. \tag{B62}$$

Inserting the expressions (B53) and (B54) in Eq. (B62), we recover the mass-radius relation (B57). For $n \neq n_3$ and for any given mass M, there is only one steady state. Its radius R_0 is given by Eq. (B57) and its energy $\mathcal{W}_0$ is given by Eq. (B43). It corresponds to the extremum value of $\mathcal{W}(R)$. Estimating the moment of inertia by

$$I = \alpha M R^2, \tag{B63}$$

and inserting the expressions (B53) and (B54) in Eq. (B46), we obtain

$$\frac{1}{2}\alpha M \frac{d^2 R^2}{dt^2} = 2\mathcal{W} + (D - 2n)K\zeta M^{1+1/n} R^{-D/n}$$
$$- \frac{\nu(D-4)}{D-2} \frac{GM^2}{R^{D-2}}. \tag{B64}$$

This equation determines the evolution of the radius of the star for a fixed mass M and a fixed energy $\mathcal{W}$. The evolution of the kinetic energy is then given by $\Theta = \mathcal{W} - U - W$. The solution of Eq. (B64) depends on two control parameters M and $\mathcal{W}$ and on the initial condition $R(0)$ and $\dot{R}(0)$. We shall consider the case where $\mathcal{W}$ is equal to the value $\mathcal{W}_0$ corresponding to the steady state, such that the right-hand side (r.h.s.) of the above equation is equal to zero at equilibrium. Then, we can rewrite Eq. (B64) as

$$\frac{1}{2}\alpha M \frac{d^2 R^2}{dt^2} = (D - 2n)K\zeta M^{1+1/n}(R^{-D/n} - R_0^{-D/n})$$
$$- \frac{\nu(D-4)}{D-2}\left(\frac{GM^2}{R^{D-2}} - \frac{GM^2}{R_0^{D-2}}\right), \tag{B65}$$

where R_0 is the radius of the star at equilibrium. Since the dynamics is nondissipative [53], the system does not evolve towards the minimum of $\mathcal{W}(R)$ (unless $R = R_0$ initially). The system can either oscillate around the minimum (stable) or evolve away from it (unstable). Considering small perturbations around equilibrium, writing $R = R_0(1 + \epsilon)$ with $\epsilon \ll 1$, and linearizing the foregoing equation, we obtain

$$\frac{d^2\epsilon}{dt^2} + (D\gamma + 2 - 2D)\frac{\nu GM}{\alpha R_0^D}\epsilon = 0, \tag{B66}$$

where we have used the mass-radius relation (B57) to simplify the expression. This is the equation for a harmonic oscillator with pulsation

$$\omega^2 = (D\gamma + 2 - 2D)\frac{\nu GM}{\alpha R^D}. \tag{B67}$$

The star is stable if $\omega^2 > 0$ and unstable otherwise. This returns the exact stability criterion (B25). Furthermore, using Eqs. (B54) and (B63), the pulsation can be rewritten in the form

$$\omega^2 = -(D\gamma + 2 - 2D)\frac{W_{ii}}{I}, \tag{B68}$$

which exactly coincides with the Ledoux formula (B36). Therefore, our simple dimensional model allows to obtain a lot of interesting results. The case of arbitrary perturbations around equilibrium will be considered in a future work. Finally, we note that for $n = n_3$, Eq. (B64) becomes

$$\frac{1}{2}\alpha M \frac{d^2 R^2}{dt^2} = 2\mathcal{W} + K\zeta \frac{D(D-4)}{D-2} \frac{M^{2(D-1)/D}}{R^{D-2}}$$
$$\times \left[1 - \left(\frac{M}{M_c}\right)^{2/D} \right], \qquad (B69)$$

and Eq. (B65) corresponds to $\mathcal{W} = 0$ and $M = M_c$ yielding $d^2 R^2/dt^2 = 0$. Hence, $R^2 = C_1 t + C_2$ corresponding to a marginal evolution.

In our dimensional analysis, the constants ζ, ν, and α are dimensionless parameters that could be chosen to fit the exact results at best. Alternatively, we can try to obtain quantitative predictions by calculating their values for homogeneous spheres. This yields

$$\zeta = \left(\frac{D}{S_D}\right)^{1/n}, \qquad \nu = \alpha = \frac{D}{D+2}. \qquad (B70)$$

Note that the potential energy of a homogeneous sphere in D dimensions can be easily calculated from Eq. (B39). On the other hand, in $D = 2$, a direct calculation gives $W = (1/2)GM^2 \ln(R/L) - GM^2/8$ where L is a reference radius where $\Phi(L) = 0$. All the results given above pass to the limit $D \to 2$ provided that we take $\nu = 1/2$. Using Eq. (B70), we find that the approximate value of the pulsation (B67) becomes

$$\omega^2 = (D\gamma + 2 - 2D)\frac{GM}{R^D}. \qquad (B71)$$

On the other hand, if we compare the approximate mass-radius relation (B57) with the exact mass-radius relation (B9), we obtain an estimate of the constant ω_n in the form

$$\omega_n^{\text{approx}} = \left(\frac{D+2}{n+1}\right)^{n/(n-1)} D^{1/(n-1)}. \qquad (B72)$$

This has to be compared with the exact value (B10) given in terms of the solution $\theta_n(\xi)$ of the Emden equation of index n in D dimensions [21]. Let us consider the case $D = 3$. For $n = 3/2$, corresponding to classical white dwarf stars, we find $\omega_{3/2}^{\text{approx}} = 72$ instead of the exact value $\omega_{3/2} = 132.3843\ldots$. For $n = 3$, corresponding to relativistic white dwarf stars, we find $\omega_3^{\text{approx}} = 2.42\ldots$ instead of the exact value $\omega_{3/2} = 2.018\,24\ldots$ This suggests that the homogeneous star model will provide a fair description of relativistic white dwarf stars and a poorer description of classical white dwarf stars. We shall come back to these different issues (static and dynamics) in a future work.

APPENDIX C: ENERGY FUNCTIONALS

In this appendix, we show that the condition of thermodynamical stability in the canonical ensemble is equivalent to the condition of nonlinear dynamical stability with respect to the barotropic Euler-Poisson system. We apply this result to white dwarf stars.

1. Energy of a barotropic gas

We consider a barotropic gas with an equation of state $P = P(\rho)$ described by the Euler-Poisson system [44]. We introduce the energy functional

$$\mathcal{W} = \int \rho \int^\rho \frac{P(\rho')}{\rho'^2} d\rho' d\mathbf{r} + \frac{1}{2}\int \rho\Phi d\mathbf{r} + \frac{1}{2}\int \rho\mathbf{u}^2 d\mathbf{r}. \qquad (C1)$$

The first term $\mathcal{W}_1$ is the work $-P(\rho)d(1/\rho)$ done in compressing the system from infinite dilution, the second term W is the gravitational energy, and the third term Θ is the kinetic energy associated with the mean motion. For a gas in local thermodynamic equilibrium, the equation of state is $P = P(\rho, T)$ or $P = P(\rho, s)$ and the first law of thermodynamics can be written $d(u/\rho) = -Pd(1/\rho) + Td(s/\rho)$ where s is the entropy density and u is the density of internal energy. For a gas without interaction (apart from the long-range gravitational attraction), the internal energy is equal to the kinetic energy. There are two important cases where the gas is barotropic. The first case is when $Td(s/\rho) = 0$. This concerns either adiabatic (or isentropic) fluids so that $d(s/\rho) = 0$ or fluids at zero temperature so that $T = 0$. When $Td(s/\rho) = 0$, the first law of thermodynamics reduces to $d(u/\rho) = -Pd(1/\rho)$. It can be integrated into $u = \rho \int^\rho [P(\rho')/\rho'^2]d\rho'$. Then, the work $\mathcal{W}_1$ done by the pressure force (first term in Eq. (C1)) coincides with the internal energy U of the gas. In that case, we get $\mathcal{W}_1 = U$ and the energy functional (C1) can be written

$$\mathcal{W} = U + W + \Theta = E. \qquad (C2)$$

Thus, at $T = 0$ or for an adiabatic evolution, the total energy of the gas E is conserved by the Euler-Poisson system (since $\mathcal{W}$ is conserved). Alternatively, for an isothermal gas $dT = 0$, the first law of thermodynamics $d(u/\rho) = -Pd(1/\rho) + Td(s/\rho)$ can be written $d((u - Ts)/\rho) = -Pd(1/\rho)$. It can be integrated into $u - Ts = \rho \int^\rho [P(\rho')/\rho'^2]d\rho'$. Therefore, the work $\mathcal{W}_1$ done by the pressure force (first term in Eq. (C1)) coincides with the free energy $U - TS$ of the gas. In that case, we get $\mathcal{W}_1 = U - TS$ and the energy functional (C1) can be written

$$\mathcal{W} = U - TS + W + \Theta = E - TS = F. \qquad (C3)$$

Thus, for an isothermal evolution, the free energy of the system F is conserved by the Euler-Poisson system (since $\mathcal{W}$ is conserved). At $T = 0$, we recover the conservation of the energy E.

Let us apply these results to white dwarf stars. We can view a white dwarf star as a barotropic gas described by an equation of state $P = P(\rho)$. According to the discussion of Appendix B 2 (see also [23]), it is nonlinearly dynamically stable with respect to the Euler-Poisson system if it is a minimum of the energy functional $\mathcal{W}$ at fixed mass. If a minimum exists, it is necessary that $\mathbf{u} = 0$. As a result, a

steady state of the Euler-Poisson system is nonlinearly dynamically stable if, and only if, it is a minimum of $\tilde{\mathcal{W}}[\rho] = \mathcal{W}[\rho, \mathbf{u}] - \Theta[\mathbf{u}]$ at fixed mass M. In conclusion, the condition of nonlinear dynamical stability can be written

$$\min\{\tilde{\mathcal{W}}[\rho]|M[\rho] = M\}. \tag{C4}$$

For a white dwarf star at zero temperature ($T = 0$), according to Eqs. (C2) and (C3), the functional $\tilde{\mathcal{W}}$ reduces to the energy $E = U + W$, where U is the kinetic energy.

2. Free energy of self-gravitating fermions

We can also view a white dwarf star as a gas of self-gravitating relativistic fermions at statistical equilibrium in the canonical ensemble. It is thermodynamically stable if, and only if, it is a minimum of free energy F at fixed mass M. The free energy is given by

$$F = E - TS$$
$$= \int f\epsilon(p)d\mathbf{p}d\mathbf{r} + \frac{1}{2}\int \rho\Phi d\mathbf{r}$$
$$+ \frac{T\eta_0}{m}\int \left[\frac{f}{\eta_0}\ln\frac{f}{\eta_0} + \left(1 - \frac{f}{\eta_0}\right)\ln\left(1 - \frac{f}{\eta_0}\right)\right]d\mathbf{r}d\mathbf{p}, \tag{C5}$$

where $\eta_0 = 2/h^D$ is the maximum value of the distribution function fixed by the Pauli exclusion principle. For a white dwarf star at zero temperature ($T = 0$), the free energy F reduces to the energy $E = U + W$. The condition of thermodynamical stability in the canonical ensemble can be written

$$\min\{F[f]|M[f] = M\}. \tag{C6}$$

The critical points of free energy at fixed mass, determined by the variational principle $\delta F - \alpha \delta M = 0$, correspond to the relativistic mean-field Fermi-Dirac distribution

$$f = \frac{\eta_0}{1 + e^{\beta[\epsilon(p) + \mu H\Phi(\mathbf{r}) + \lambda_0]}}, \tag{C7}$$

where λ_0 is a Lagrange multiplier (related to α) determined by the mass M and $\epsilon(p)$ is given by Eq. (4). Using Eqs. (2), (5), and (C7), the density and the pressure are of the form $\rho = \rho(\mu H\Phi(\mathbf{r}) + \lambda_0)$ and $P = P(\mu H\Phi(\mathbf{r}) + \lambda_0)$. Eliminating $\mu H\Phi(\mathbf{r}) + \lambda_0$ between these two expressions, we find that $P = P(\rho)$ so that the gas is barotropic. The equation of state is parametrized by T and is fully determined by the entropic functional in Eq. (C5), which is here the Fermi-Dirac entropy. At $T = 0$ we obtain the explicit relations of Sec. II. Furthermore, the condition that $f(\mathbf{r}, \mathbf{p})$ is a function of the energy $e = \epsilon(p) + \mu H\Phi(\mathbf{r})$ implies that the corresponding barotropic gas is at hydrostatic equilibrium. Indeed, using Eq. (6),

$$\nabla P = \frac{1}{D}\int \frac{\partial f}{\partial \mathbf{r}} p\epsilon'(p)d\mathbf{p} = \mu H\nabla\Phi \frac{1}{D}\int f'(e)p\epsilon'(p)d\mathbf{p}$$
$$= \mu H\nabla\Phi \frac{1}{D}\int \left(\mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{p}}\right)d\mathbf{p} = -\mu H\nabla\Phi \int f d\mathbf{p}, \tag{C8}$$

so that

$$\nabla P = -\rho\nabla\Phi. \tag{C9}$$

The relativistic mean-field Fermi-Dirac distribution (C7) is just a critical point of free energy at fixed mass M. To determine whether it corresponds to a true minimum of free energy, we can proceed in two steps. We first minimize $F[f]$ for a fixed density profile $\rho(\mathbf{r})$. Since the potential energy $W[\rho]$ and the mass $M[\rho]$ are entirely determined by the density profile, this is equivalent to minimizing $U[f] - TS[f]$ at fixed $\rho(\mathbf{r})$, where U is the kinetic energy. This gives a distribution

$$\tilde{f} = \frac{\eta_0}{1 + e^{\beta[\epsilon(p) + \lambda(\mathbf{r})]}}, \tag{C10}$$

where $\lambda(\mathbf{r})$ is a local Lagrange multiplier determined by the density $\rho(\mathbf{r})$, using $\rho = \mu H\int f d\mathbf{p}$. Since $\delta^2(U - TS) \geq 0$, the distribution (C10) is a true minimum of $F[f]$ at fixed $\rho(\mathbf{r})$. Substituting the optimal distribution function (C10) in Eq. (C5), we obtain a functional $\tilde{F}[\rho] \equiv F[\tilde{f}]$ of the density $\rho(\mathbf{r})$. Using Eqs. (2), (5), and (C10), we note that the density and the pressure are of the form $\rho(\mathbf{r}) = \rho(\lambda(\mathbf{r}))$ and $P(\mathbf{r}) = P(\lambda(\mathbf{r}))$. Eliminating $\lambda(\mathbf{r})$ between these expressions, we find that $P = P(\rho)$ where the equation of state is the same as the one determined from the Fermi-Dirac distribution (C7) at equilibrium. Now, we can show that

$$\tilde{F}[\rho] = \tilde{\mathcal{W}}[\rho], \tag{C11}$$

where $\tilde{\mathcal{W}}[\rho]$ is the functional defined in Sec. C 1. The relation (C11) can be obtained by an explicit calculation which extends the proof given in Appendix B of [23] for classical particles (this will be shown in a future work; see also particular cases in Sec. C 3) but it also results from a straightforward argument. We note that the distribution function (C10) corresponds to a condition of local thermodynamical equilibrium with uniform temperature and zero average velocity. Thus, it locally satisfies the first law of thermodynamic $d(u/\rho) = -Pd(1/\rho) + Td(s/\rho)$ with $dT = 0$. Integrating this relation like in Sec. C 1, we find that $U[\rho] - TS[\rho] = \mathcal{W}_1[\rho]$ where $U[\rho] = U[\tilde{f}]$ and $S[\rho] = S[\tilde{f}]$. This directly yields the identity (C11).

We can now conclude that $F[f]$ has a minimum $f_*(\mathbf{r}, \mathbf{p})$ at fixed mass M if, and only if, $\tilde{F}[\rho] = \tilde{\mathcal{W}}[\rho]$ has a minimum $\rho_*(\mathbf{r})$ at fixed mass M. In that case, $f_*(\mathbf{r}, \mathbf{p})$ is given by Eq. (C10) where $\lambda_*(\mathbf{r})$ is determined by $\rho_*(\mathbf{r})$, writing $\rho_* = \mu H\int f_* d\mathbf{p}$. Therefore, a system is thermodynamically stable in the canonical ensemble if, and only

if, the corresponding barotropic gas with the same equilibrium distribution is nonlinearly dynamically stable with respect to the barotropic Euler-Poisson system. Said differently, a system that minimizes the functional (C11) is (i) thermodynamically stable in the canonical ensemble and (ii) nonlinearly dynamically stable with respect to the barotropic Euler-Poisson system. This result applies to white dwarf stars at arbitrary temperature.

3. Application to white dwarf stars at $T = 0$

Although the above results are general, it may be useful to explicitly compute the functionals (C1) and (C5) for white dwarf stars at $T = 0$ and check the relation (C11). If we view a white dwarf star as a barotropic gas described by an equation of state $P(\rho)$, its energy $\tilde{W}$ can be written

$$\tilde{W} = \int \rho \Gamma(\rho) d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r}, \qquad (C12)$$

with

$$\Gamma(\rho) = \int^{\rho} \frac{P(\rho')}{\rho'^2} d\rho'. \qquad (C13)$$

A white dwarf star is nonlinearly dynamically stable with respect to the Euler-Poisson system if it is a minimum of $\tilde{W}$ at fixed mass M. At $T = 0$, the equation of state is given by Eq. (8). In the classical limit (polytrope $n_{3/2} = D/2$) we have

$$\tilde{W} = \frac{DK_1}{2} \int \rho^{(D+2)/D} d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r}, \qquad (C14)$$

and in the ultrarelativisitic limit (polytrope $n_3' = D$) we have

$$\tilde{W} = DK_2 \int \rho^{(D+1)/D} d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r}. \qquad (C15)$$

In the general case, using Eq. (8), the function (C13) can be written

$$\Gamma(\rho) = \frac{A_2 D}{B} \int^x \frac{f(x')}{x'^{D+1}} dx'. \qquad (C16)$$

Integrating by parts and using Eqs. (9) and (10), we find after straightforward calculations that

$$\frac{\mu H}{mc^2} \Gamma(\rho) = \sqrt{1 + x^2} - \frac{1}{x^D} \int_0^x \frac{t^{D+1}}{(1 + t^2)^{1/2}} dt. \qquad (C17)$$

Alternatively, we can view a white dwarf star as a gas of self-gravitating relativistic fermions at statistical equilibrium in the canonical ensemble. At $T = 0$, its free energy $F = E - TS$ coincides with its energy E. Using Eq. (C5), it is given by

$$F = E = \int \frac{\rho}{\mu H} \kappa(\rho) d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r}, \qquad (C18)$$

with

$$\frac{\rho}{\mu H} \kappa(\rho) = \int f \epsilon(p) d\mathbf{p}. \qquad (C19)$$

A white dwarf star at $T = 0$ is thermodynamically stable if it is a minimum of the energy E at fixed mass M. In the classical limit, $\epsilon = p^2/2m$ and $P = \frac{1}{D} \int (f/m) p^2 d\mathbf{p}$ so $\rho \kappa/\mu H = (D/2)P$. Therefore, the free energy can be written

$$\tilde{F} = \frac{D}{2} \int P d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r}. \qquad (C20)$$

Using Eq. (25), this is equivalent to Eq. (C14). In the ultrarelativistic limit, $\epsilon = pc$ and $P = \frac{1}{D} \int f pc d\mathbf{p}$ so $\rho \kappa/\mu H = DP$. Therefore, the free energy can be written

$$\tilde{F} = D \int P d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r}. \qquad (C21)$$

Using Eq. (39), this is equivalent to Eq. (C15). In the general case, we have

$$\kappa(\rho) = \frac{2S_D}{nh^D} \int_0^{p_0} \epsilon(p) p^{D-1} dp. \qquad (C22)$$

Using Eq. (4), it can be put in the form

$$\frac{\kappa(\rho)}{mc^2} = \frac{D}{x^D} \int_0^x \sqrt{1 + t^2} t^{D-1} dt. \qquad (C23)$$

Now, it is straightforward to check that the two expressions in the r.h.s. of Eqs. (C17) and (C23) are equal so that $\Gamma(\rho) = \kappa(\rho)/\mu H$ implying the relation (C11).

Finally, if we consider a classical isothermal self-gravitating gas at temperature T with an equation of state $P = \rho k_B T/m$, its free energy (C5) can be written

$$\tilde{F} = \frac{D}{2} N k_B T + k_B T \int \frac{\rho}{m} \ln \frac{\rho}{m} d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r}. \quad (C24)$$

Comparing with Eq. (B7), we find that relation (C11) is indeed satisfied. Therefore, a classical isothermal self-gravitating gas that minimizes the functional (B7) or (C24) at fixed mass is (i) thermodynamically stable in the canonical ensemble and (ii) nonlinearly dynamically stable with respect to the barotropic Euler-Poisson system. We had already made this observation in [41]. In [23], this result was extended to an arbitrary form of entropic functional, including the Fermi-Dirac entropy, for nonrelativistic systems. The present paper shows that, due to relation (C11), the equivalence between nonlinear dynamical stability with respect to the barotropic Euler-Poisson system and thermodynamical stability in the canonical ensemble is general.

APPENDIX D: NONVIABILITY OF A $D \geq 4$ UNIVERSE

If we consider, in a D-dimensional universe, a Hamiltonian system of self-gravitating classical point masses whose dynamics is described by the Newton equa-

tions

$$\ddot{\mathbf{r}}_\alpha = \sum_{\beta \neq \alpha} \frac{Gm(\mathbf{r}_\beta - \mathbf{r}_\alpha)}{|\mathbf{r}_\beta - \mathbf{r}_\alpha|^D}, \tag{D1}$$

the scalar Virial theorem reads [54]:

$$\frac{1}{2}\ddot{I} = 2K + W_{ii}, \tag{D2}$$

where K is the kinetic energy and W_{ii} the trace of the potential energy tensor for the N-body system [54]. For $D \neq 2$, using Eq. (B32) and introducing the total energy $E = K + W$, which is a constant of the motion for an isolated system, we have [54]:

$$\frac{1}{2}\ddot{I} = 2K + (D-2)W = 2E + (D-4)W. \tag{D3}$$

We note that the dimension $D = 4$ is critical. In that case, $\ddot{I} = 4E$ which yields after integration $I = 2Et^2 + C_1 t + C_2$. For $E > 0$, $I \rightarrow +\infty$ indicating that the system evaporates. For $E < 0$, I goes to zero in a finite time, indicating that the system forms a Dirac peak ("black hole") in a finite time. More generally, for $D \geq 4$, since $(D-4)W \leq 0$, we have $I \leq 2Et^2 + C_1 t + C_2$ so that the system forms a Dirac peak in a finite time if $E < 0$. Therefore, self-gravitating systems with $E < 0$ are not stable in a space of dimension $D \geq 4$. The study of the present paper indicates that this observation remains true if quantum (Pauli exclusion principle for fermions) and relativistic effects are taken into account. In this sense, a universe with $D \geq 4$ is not viable.

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