PHYSICAL REVIEW D **76,** 021501(R) (2007)

Equilibrium configurations of two charged masses in general relativity

G. A. Alekseev[1,*] and V. A. Belinski[2,†]

[1]*Steklov Mathematical Institute, Gubkina 8, Moscow 119991, Moscow, Russia*

[2]*INFN, Rome University "La Sapienza", 00185 Rome, Italy, ICRANet, Piazzale della Repubblica, 10, 65122 Pescara, Italy, and IHES, F-91440 Bures-sur-Yvette, France*

(Received 15 November 2006; published 24 July 2007)

An asymptotically flat static solution of Einstein-Maxwell equations which describes the field of two nonextreme Reissner-Nordström sources in equilibrium is presented. It is expressed in terms of physical parameters of the sources (their masses, charges, and separating distance). Very simple analytical forms were found for the solution as well as for the equilibrium condition which guarantees the absence of any struts on the symmetry axis. This condition shows that the equilibrium is not possible for two black holes or for two naked singularities. However, in the case when one of the sources is a black hole and another one is a naked singularity, the equilibrium is possible at some distance separating the sources. It is interesting that for appropriately chosen parameters even a Schwarzschild black hole together with a naked singularity can be "suspended" freely in the superposition of their fields.

DOI: 10.1103/PhysRevD.76.021501 PACS numbers: 04.20.Jb, 04.40.Nr, 04.40.−b, 04.70.Bw

I. INTRODUCTION

In the Newtonian physics two pointlike particles can be in equilibrium if the product of their masses is equal to the product of their charges (we use the units for which $G = c = 1$). Until now, in general relativity the equilibrium condition for two particlelike sources imposed on their physical masses, charges and separating distance was not known in explicit and reasonably simple analytical form which would admit a rigorous analysis without a need for numerical experiments. The only exceptional case was the Majumdar-Papapetrou solution [1,2], for which the charge of each source is equal to its mass. In this case, the equilibrium is independent of the distance between the sources. For each of the static sources of this sort its outer and inner Reissner-Nordström horizons coincide and such sources are called extreme ones. Accordingly, the sources with two separated horizons are called underextreme and the sources without horizons, superextreme.

The problem, which had been under investigation by many researchers and which we solve in the present paper, consists in the search of equilibrium configurations of nonextreme sources. Since the advent of solution generating techniques for stationary axisymmetric Einstein-Maxwell fields, a construction of an exact solution for two charged masses at rest does not represent any principal difficulty. However, in general, the asymptotically flat solutions of this kind contain conical singularities on the symmetry axis between the sources which can be interpreted as the presence of some extraneous struts preventing the sources from falling onto or running away from each other. The equilibrium condition just implies the absence of such struts. Naturally, if the metric is known, so is the equilibrium condition. In the static case, the latter means that the product of the metric coefficients g_{tt} and $g_{\rho\rho}$ (in cylindrical Weyl coordinates) should be equal to unity at the axis where $\rho = 0$. However, this equilibrium equation in such general form usually is expressed by a set of formal parameters, and it is so complicated that its analytical investigation appears to be very difficult. Therefore, it is desirable to have this equation expressed in terms of physical parameters and in a simple enough form, making it accessible for an analytical examination of a possibility of realization of equilibrium. Moreover, this realization should be compatible with a condition of a positive value of the distance between the sources. This task has not been accomplished yet, and up to now only some results achieved by numerical calculations were known.

The first researches of the equilibrium of nonextreme sources [3–8] led to contradictory conclusions. The authors of the indicated papers used both the exact techniques and post-Newtonian and post-post-Newtonian approximations. The common opinion expressed in [3,4,6–8] is that the equilibrium for nonextreme sources is impossible. Nevertheless, in [7] one can find a remark that the analysis performed was insufficient and the existence of equilibrium configurations for the nonextreme objects cannot be excluded. The arguments in favor of such a possibility can also be found in [5].

The next step which attracted attention to the problem again has been done by Bonnor in [9], where the equilibrium condition for a charged test particle in the Reissner-Nordström field was analyzed. Examination made there also suggested some plausible assumptions for the exact solutions. As has been indicated in [9] a charged test body can be at rest in the field of the Reissner-Nordström source only if theses two sources are both either extreme (for the test particle the degree of its extremality is defined just by the ratio between its charge and mass), balanced irrespective of distance, or one of them is superextreme and the other is underextreme, and in this case the equilibrium

*G.A.Alekseev@mi.ras.ru
†belinski@icra.it

depends on the distance. There is no way for equilibrium in cases when both sources are either superextreme or underextreme. It is worth mentioning that in the very recent papers [10] a new perturbative solution describing an equilibrium state of a two-body system consisting of a Reissner-Nordström black hole and a superextreme test particle has been presented. The whole set of combined Einstein-Maxwell equations has been solved there by using the first order perturbation approach developed in [11] and based on the tensor harmonic expansion of both the gravitational and electromagnetic fields adopting the Regge-Wheeler [12] gauge. (The basic equations for combined gravitational and electromagnetic perturbations of the Reissner-Nordström background in the decoupled form were found in other gauges in [13] and also in the decoupled Hamiltonian form in [14].) Both the electromagnetically induced gravitational perturbations and gravitationally induced electromagnetic perturbations [15] due to the mass as well as the charge of the particle have thus been taken into account. The expressions in a closed form for both the perturbed metric and electromagnetic field have been explicitly given [10]. It is interesting that the equilibrium equation (which arises in this case as a self-consistency condition for the set of differential equations for perturbations) remains the same as that of Bonnor [9].

Bonnor's analysis allows one to expect that, qualitatively, the same can also happen for two Reissner-Nordström sources. For two extreme sources this is indeed the case because it is known that such generalization exists and leads to the Majumdar-Papapetrou solution. Up to 1997 it remained unknown whether the analogous generalization for the nonextreme bodies could be found. The first solid arguments in favor of the existence of a static equilibrium configuration for the ‘‘black hole–naked singularity’’ system was presented in [16]. These results have been obtained there by numerical calculations and three examples of numerical solutions of the equilibrium equation have been demonstrated. These solutions can correspond to the equilibrium configurations free of struts. For the complete proof it would be necessary to show that such configurations indeed consist of two sources, separated by physically sensible distance between them. However, in [16] it was pointed out that the distance dependence for the equilibrium state is unknown. The authors of [16] also reported that a number of numerical experiments for two black holes and for two naked singularities showed the negative outcomes, i.e. all tested sets of parameters were not in power to satisfy the equilibrium equation. These findings are in agreement with Bonnor's test particle analysis. One year later a similar numerical analysis was made in [17].

In this paper, we present an exact solution of the Einstein-Maxwell equations which describes the field of two Reissner-Nordström sources in static equilibrium as well as the equilibrium condition itself which turns out to have an unexpectedly simple form expressed in terms of physical parameters of the sources. This simplicity permits us to prove a validity of conjectures of the papers [9,16] on an exact analytical level. It allows a direct analytical investigation of the physical properties of the equilibrium state of two nonextreme sources.

We precede a description of our results with a few remarks about the method we used for derivation of our solution. For derivation of this solution, an application of the electrovacuum soliton generating technique (developed in [18,19] and described in detail in the book [20]) does not lead to the most convenient parametrization of the solution. This gives rise to subsequent technical difficulties, although there are no principal obstacles to use this technique. Instead, we used the integral equation method [19,21] which opens a shorter way to the desirable results. The first step was to construct the solution for the two-pole structure of the monodromy data on the spectral plane with a special choice of parameters providing asymptotical flatness and the static character of the solution. This also corresponds to the two-pole structure of the Ernst potentials (as functions of the Weyl cylindrical coordinate z) on the symmetry axis. Then the expressions for physical masses and physical charges for both sources were found with the help of the Gauss theorem, and the notion of distance between these sources was also defined. We stress here that the physical character of masses and charges of the sources follows not only from their definition using the Gauss theorem, but also from the analysis of that limiting case in which one of the sources is a test particle [see the formulas (12) and (13) below and the text after them]. After that, we derived the equilibrium equation in terms of these five physical parameters. The miracle arises if one substitutes this equilibrium equation back into the solution. This results in the impressive simplification of all formulas. Below we expose the final outcome which is ready for use in practical purposes without the necessity of knowledge of any details of its derivation.

It is worthwhile to mention that a correctness of our solution has been confirmed also by its direct substitution into the Einstein-Maxwell field equations.

II. THE SOLUTION

For our static solution the metric and vector electromagnetic potential in cylindrical Weyl coordinates have the forms

$$ds^2 = H dt^2 - f(d\rho^2 + dz^2) - \frac{\rho^2}{H} d\varphi^2, \qquad (1)$$

$$A_t = \Phi, \qquad A_\rho = A_z = A_\varphi = 0, \qquad (2)$$

where H, f, and Φ are real functions of the coordinates ρ and z. These functions take the most simple form in bipolar coordinates which consist of two pairs of spheroidal variables (r_1, θ_1), (r_2, θ_2) defined by their relations to the Weyl coordinates:

$$\begin{cases} \rho = \sqrt{(r_1 - m_1)^2 - \sigma_1^2}\sin\theta_1, \\ z = z_1 + (r_1 - m_1)\cos\theta_1, \end{cases}$$
$$\begin{cases} \rho = \sqrt{(r_2 - m_2)^2 - \sigma_2^2}\sin\theta_2, \\ z = z_2 + (r_2 - m_2)\cos\theta_2. \end{cases} \quad (3)$$

Here and below, the indices 1 and 2 denote the coordinates and parameters related to the Reissner-Nordström sources located at the symmetry axis at the points $z = z_1$ and $z = z_2$, respectively. A positive constant ℓ defined as

$$\ell = z_2 - z_1 \quad (4)$$

characterizes the z distance separating these sources (for definiteness we take $z_2 > z_1$). The constants m_1 and m_2 are physical masses of the sources.

Each of the parameters σ_k ($k = 1, 2$) can be either real or pure imaginary and this property characterizes the corresponding Reissner-Nordström source to be either a black hole or a naked singularity: the real value of σ_k means that this is a black hole whose horizon in Weyl coordinates is $\{\rho = 0, z_k - \sigma_k \le z \le z_k + \sigma_k\}$ while the imaginary σ_k corresponds to a naked singularity whose critical spheroid $r_k = m_k$ is $\{0 \le \rho \le |\sigma_k|, z = z_k\}$. So we define the coordinate distance between two black holes (both σ_1 and σ_2 are real and positive) as the distance along the z axis between the nearest points of its intersections with two horizons, and this distance is $\ell - \sigma_1 - \sigma_2$. We define the distance between the black hole located at the point $z = z_2$ and the naked singularity at the point $z = z_1$ (σ_2 is real and positive but σ_1 is pure imaginary) as the distance between the nearest points of intersections of the symmetry axis with a black hole horizon and critical spheroid, and this distance is $\ell - \sigma_2$. The distance between two naked singularities (both σ_1 and σ_2 are pure imaginary) is simply ℓ, and it is the length of the segment between the nearest points of intersections of two critical spheroids with the z axis.

In terms of bipolar coordinates our solution reads

$$H = [(r_1 - m_1)^2 - \sigma_1^2 + \gamma^2\sin^2\theta_2] \times [(r_2 - m_2)^2 - \sigma_2^2 + \gamma^2\sin^2\theta_1]\mathcal{D}^{-2}, \quad (5)$$

$$\Phi = [(e_1 - \gamma)(r_2 - m_2) + (e_2 + \gamma)(r_1 - m_1) + \gamma(m_1\cos\theta_1 + m_2\cos\theta_2)]\mathcal{D}^{-1}, \quad (6)$$

$$f = [(r_1 - m_1)^2 - \sigma_1^2\cos^2\theta_1]^{-1} \times [(r_2 - m_2)^2 - \sigma_2^2\cos^2\theta_2]^{-1}\mathcal{D}^2, \quad (7)$$

where

$$\mathcal{D} = r_1 r_2 - (e_1 - \gamma - \gamma\cos\theta_2)(e_2 + \gamma - \gamma\cos\theta_1). \quad (8)$$

In these expressions the quantities e_1, e_2 represent physical charges of the sources. The parameter γ and the parameters σ_1, σ_2 are determined by the relations

$$\sigma_1^2 = m_1^2 - e_1^2 + 2e_1\gamma, \qquad \sigma_2^2 = m_2^2 - e_2^2 - 2e_2\gamma, \quad \gamma = (m_2 e_1 - m_1 e_2)(\ell + m_1 + m_2)^{-1}. \quad (9)$$

The formulas (1)–(9) give the exact solution of the Einstein-Maxwell equations if and only if the five parameters m_1, m_2, e_1, e_2, and ℓ satisfy the following condition:

$$m_1 m_2 = (e_1 - \gamma)(e_2 + \gamma). \quad (10)$$

The condition (10) guarantees the equilibrium without any struts on the symmetry axis between the sources.

III. PROPERTIES OF THE SOLUTION

First of all, one can see that the balance equation (10) does not admit two black holes ($\sigma_1^2 > 0$, $\sigma_2^2 > 0$) to be in equilibrium under the condition that there is some distance between them, that is, if $\ell - \sigma_1 - \sigma_2 > 0$. This is in agreement with a nonexistence of static equilibrium configurations of charged black holes proved under rather general assumptions in [22]. (To avoid confusion, we mention here that the results of [22] do not apply in the presence of naked singularities.) The equilibrium is also impossible if one of the sources is extreme and the other is a nonextreme one, and a positive distance exists between them, i.e. if $\ell - \sigma_2 > 0$ for the case $\sigma_1 = 0$ and $\sigma_2^2 > 0$ (a negative value for σ_2^2 is forbidden altogether if $\sigma_1 = 0$) [23]. The condition (10) also implies that σ_1^2 and σ_2^2 can never both be negative, that is, the equilibrium of two naked singularities is impossible. So, for separated sources, an equilibrium may exist either between a black hole and a naked singularity or between two extreme sources. The latter case can be realized only if $\sigma_1 = \sigma_2 = 0$, $\gamma = 0$, and it is easy to see that the formulas (1)–(9) reduce for this case to the Majumdar-Papapetrou solution.

At spatial infinity the variables r_1, r_2 coincide and one can choose either of them as the radial coordinate. In this region the fields, as can be seen from (5) and (6), acquire the standard Reissner-Nordström asymptotical form with the total mass $m_1 + m_2$ and the total charge $e_1 + e_2$.

At the symmetry axis $\cos^2\theta_1 = \cos^2\theta_2 = 1$ and the formulas (5) and (7) show that the condition $fH = 1$ is satisfied there automatically, i.e. there are no conical singularities. Besides the singularities inherent to the sources themselves, any other kinds of singularities (such as, for example, the off-axis singularities found in the double-Kerr solution in [24]) are also absent in our solution.

The constant γ vanishes in the limit $\ell \to \infty$, whence it follows from (10) that the equilibrium condition asymptotically reduces to the Newtonian form $m_1 m_2 = e_1 e_2$ for a large distance between the sources.

If one of the sources disappears, e.g. $m_1 = e_1 = 0$, our solution reduces to the exact Reissner-Nordström solution with the mass m_2 and the charge e_2 in the standard spherical coordinates r_2, θ_2.

Let us turn now to the limiting case in which one of the sources can be considered as a test particle. For this we assume that m_1 and e_1 are infinitesimally small but the ratio e_1/m_1 is finite. In this case, in the first nonvanishing order with respect to the constants m_1 and e_1, the equilibrium condition (10) gives

$$(\ell + m_2)(m_1 m_2 - e_1 e_2) = (m_1 e_2 - m_2 e_1)e_2. \quad (11)$$

We introduce instead of m_1 a new parameter μ_1 defined by the relation

$$m_1 = \mu_1[1 - 2m_2(\ell + m_2)^{-1} + e_2^2(\ell + m_2)^{-2}]^{1/2} + e_1 e_2(\ell + m_2)^{-1}. \quad (12)$$

Now the relation (11) takes the form

$$m_2 - e_2^2(\ell + m_2)^{-1} = e_1 e_2 \mu_1^{-1}[1 - 2m_2(\ell + m_2)^{-1} + e_2^2(\ell + m_2)^{-2}]^{1/2}. \quad (13)$$

This last equation is nothing more than Bonnor's balance condition [9] for the test particle of the rest mass μ_1 and the charge e_1 in the Reissner-Nordström field of the mass m_2 and the charge e_2. The particle is at rest on the symmetry axis at the point $R = \ell + m_2$ where R is the radius of the standard spherical coordinates of the Reissner-Nordström solution. If we calculate from (6) the potential Φ in the linear approximation with respect to the small parameters m_1 and e_1 for the particular case $e_2 = 0$ (i.e. for the Schwarzschild background), the result will coincide exactly with the potential which was found first in [25–27] in the form of multipole expansion and then in [28] in closed analytical form.

The relation (12) is important since it exhibits clearly the physical nature of the mass m_1 and gives its correct interpretation. This relation shows that the parameters m_1, m_2 are not the rest masses but they represent the total relativistic energy of each source in the external field produced by its partner.

Finally it is worth mentioning that our exact solution remains physically sensible also in the case $e_2 = 0$. This corresponds to a Schwarzschild black hole of the mass m_2 hovering freely in the field of a naked singularity of the mass m_1 and the charge e_1. Such configuration exists due to the repulsive nature of gravity in the vicinity of the naked Reissner-Nordström singularity.

ACKNOWLEDGMENTS

G. A. A. is thankful to ICRAnet for the financial support and hospitality during his visit to ICRAnet (Pescara, Italy) during May 2006, when this paper was started. The work of G. A. A. was also supported in part by the Russian Foundation for Basic Research (Grant No. 05-01-00219, No. 05-01-00498, No. 06-01-92057-CE) and the programs ''Mathematical Methods of Nonlinear Dynamics'' of the Russian Academy of Sciences, and ''Leading Scientific Schools'' of the Russian Federation (Grant No. NSh-4710.2006.1). We are especially grateful to R. Price for useful comments which urged us to essentially improve this paper.

[1] S. D. Majumdar, Phys. Rev. **72**, 390 (1947).
[2] A. Papapetrou, Proc. R. Irish Acad., Sect. A **LI**, 191 (1947).
[3] B. M. Barker and R. F. O'Connell, Phys. Lett. **61A**, 297 (1977).
[4] T. Kimura and T. Ohta, Phys. Lett. **63A**, 193 (1977).
[5] W. B. Bonnor, Phys. Lett. **83A**, 414 (1981).
[6] T. Ohta and T. Kimura, Prog. Theor. Phys. **68**, 1175 (1982).
[7] A. Tomimatzu, Prog. Theor. Phys. **71**, 409 (1984).
[8] T. Azuma and T. Koikawa, Prog. Theor. Phys. **92**, 1095 (1994).
[9] W. B. Bonnor, Classical Quantum Gravity **10**, 2077 (1993).
[10] D. Bini, A. Geralico, and R. J. Ruffini, Phys. Rev. D **75**, 044012 (2007); Phys. Lett. A **360**, 515 (2007).
[11] F. J. Zerilli, Phys. Rev. D **9**, 860 (1974).
[12] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[13] N. R. Sibgatullin and G. A. Alekseev, Zh. Eksp. Teor. Fiz. **67**, 1233 (1974) [Sov. Phys. JETP **40**, 613 (1975)].
[14] V. Moncrief, Phys. Rev. D **9**, 2707 (1974); **10**, 1057 (1974).
[15] M. Johnston, R. Ruffini, and F. J. Zerilli, Phys. Rev. Lett. **31**, 1317 (1973); Phys. Lett. **49B**, 185 (1974).
[16] G. P. Perry and F. I. Cooperstock, Classical Quantum Gravity **14**, 1329 (1997).
[17] N. Breton, V. S. Manko, and J. A. Sanches, Classical Quantum Gravity **15**, 3071 (1998).
[18] G. A. Alekseev, Pis'ma Zh. Eksp. Teor. Fiz. **32**, 301 (1980) [JETP Lett. **32**, 277 (1980)].
[19] G. A. Alekseev, Proc. Steklov Inst. Math. **3**, 215 (1988).
[20] V. Belinski and E. Verdaguer, *Gravitational Solitons* (CUP, Cambridge, 2001).
[21] G. A. Alekseev, Sov. Phys. Dokl. **30**, 565 (1985).
[22] P. Chrusciel and P. Tod, Commun. Math. Phys. **271**, 577 (2007).
[23] The cases of nonseparated objects for which two horizons overlap or the horizon intersects with the critical spheroid may be possible also, but a consideration of such cases is not in the scope of this paper.
[24] J. Bičák and C. Hoenselaers, Phys. Rev. D **31**, 2476 (1985).
[25] R. S. Hanni and R. Ruffini, Bull. Am. Phys. Soc. **16**, 34 (1971).
[26] J. Cohen and R. Wald, J. Math. Phys. (N.Y.) **12**, 1845 (1971).
[27] R. Hanni and R. Ruffini, Phys. Rev. D **8**, 3259 (1973).
[28] B. J. Linet, J. Phys. A **9**, 1081 (1976).