

Matrix elements of the complete set of $\Delta B = 2$ and $\Delta C = 2$ operators in heavy meson chiral perturbation theory

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Using heavy meson chiral perturbation theory, we consider the light quark mass and spatial volume dependence of the matrix elements of $\Delta B = 2$ and $\Delta C = 2$ four-quark operators relevant for $B_{(s)}^0 - \bar{B}_{(s)}^0$ and $D^0 - \bar{D}^0$ mixing, and the B_s meson width difference. Our results for these matrix elements are obtained in the $N_f = 2 + 1$ partially quenched theory, which becomes full QCD in the limit where sea and valence quark masses become equal. They can be used in extrapolation of lattice calculations of these matrix elements to the physical light quark masses and to infinite volume. An important conclusion of this paper is that the chiral extrapolations for matrix elements of heavy-light meson mixing beyond the standard model, and those relevant for the B_s width difference are more complicated than that for the standard model mixing matrix elements.

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I. INTRODUCTION

Neutral heavy-light meson mixing systems play a crucial role in precision tests of the standard model and the search for new physics. With the recently measured Δm_s [1], we can hope to obtain stringent constraints on the unitarity triangle of the Cabibbo-Kobayashi-Maskawa (CKM) matrix, provided that the hadronic matrix elements of the $B^0 - \bar{B}^0$ and $B_s^0 - \bar{B}_s^0$ mixing processes are reliably calculated. On the other hand, the $D^0 - \bar{D}^0$ mixing system is a good channel to search for new physics [2], because the standard model contribution is strongly suppressed.

In the standard model, the short-distance contribution to the mass differences of the heavy neutral meson mixing systems ($B^0 - \bar{B}^0$, $B_s^0 - \bar{B}_s^0$ and $D^0 - \bar{D}^0$) is predominantly determined by the matrix elements of a single set of four-quark operators:

$$\mathcal{O}_{1,aa} = \bar{h}^\alpha \gamma_\mu (1 - \gamma_5) q_a^\alpha \bar{h}^\beta \gamma_\mu (1 - \gamma_5) q_a^\beta, \quad (1)$$

where h is a heavy quark field (either a b or a c quark), q_a is a light quark field with flavor a (a is not summed over), and α and β are color indices. Models containing flavor-changing currents other than the $V - A$ form (arising in supersymmetric extensions of the standard model and other scenarios) usually result in mass differences that additionally depend on matrix elements of the four-quark operators [3]

$$\begin{aligned} \mathcal{O}_{2,aa} &= \bar{h}^\alpha (1 - \gamma_5) q_a^\alpha \bar{h}^\beta (1 - \gamma_5) q_a^\beta, \\ \mathcal{O}_{3,aa} &= \bar{h}^\alpha (1 - \gamma_5) q_a^\beta \bar{h}^\beta (1 - \gamma_5) q_a^\alpha, \\ \mathcal{O}_{4,aa} &= \bar{h}^\alpha (1 - \gamma_5) q_a^\alpha \bar{h}^\beta (1 + \gamma_5) q_a^\beta, \\ \mathcal{O}_{5,aa} &= \bar{h}^\alpha (1 - \gamma_5) q_a^\beta \bar{h}^\beta (1 + \gamma_5) q_a^\alpha, \end{aligned} \quad (2)$$

(the right-handed analogues of $\mathcal{O}_{i,aa}$ for $i = 1, 2, 3$ can also contribute but their matrix elements are the same as those

above as the strong interaction conserves parity). Generically we can represent these operators as

$$\mathcal{O}_{i,aa} = \bar{h} \Gamma_1 q \bar{h} \Gamma_2 q, \quad (3)$$

for the appropriate choice of spin and color matrices, $\Gamma_{1,2}$. In lattice calculations, it is convenient to perform a Fierz transformation which renders linear combinations of the operators in Eq. (2) into products of color-singlet currents. We choose to work in the basis of Eq. (2).

A subset of the operators in Eqs. (1) and (2) are also relevant for calculation of the width difference in the B_s system, $\Delta\Gamma_{B_s}/\Gamma_{B_s}$. This difference is the largest amongst the beauty hadrons ($\Delta\Gamma_{B_s}/\Gamma_{B_s} = 0.31_{-0.13}^{+0.11}$ [4]) and, following an operator product expansion, is given by [5],

$$\begin{aligned} \frac{\Delta\Gamma_{B_s}}{\Gamma_{B_s}} &= \frac{G_F^2 m_b^2}{12\pi M_{B_s}} |V_{cb} V_{cs}|^2 \tau_{B_s} [G(z) \langle \bar{B}_s^0 | \mathcal{O}_{1,ss} | B_s^0 \rangle \\ &+ G_S(z) \langle \bar{B}_s^0 | \mathcal{O}_{2,ss} | B_s^0 \rangle] + \mathcal{O}(1/M_b), \end{aligned} \quad (4)$$

where the functions $G(z)$ and $G_S(z)$ are known at next-to-leading order (NLO) in perturbative QCD [6]. At $\mathcal{O}(1/M_b)$, matrix elements of $\mathcal{O}_{3,ss}$ also enter [7].

Lattice QCD is the only method for calculating the $B^0 - \bar{B}^0$ and $D^0 - \bar{D}^0$ matrix elements of the operators in Eqs. (1) and (2) from first principles, and much effort has gone into such calculations (see Ref. [8] for a recent review). However the existing lattice calculations have been performed at light quark masses significantly larger than the physical values and necessarily in finite volumes. The effects of these approximations need to be understood. In this paper we consider the light quark mass extrapolation to the physical values for the lattice calculations of these matrix elements. Our framework is heavy meson chiral perturbation theory (HM χ PT) [9–11] at finite volume [12]. The standard model $\Delta B = 2$ operator $\mathcal{O}_{1,aa}$ has been considered in this context in Ref. [12] and here we

extend that analysis to the full set of operators discussed above. As appropriate for current and foreseeable lattice calculations, we work in the isospin limit of SU(3) heavy meson chiral perturbation theory and give results in the SU(6|3) partially quenched extension. Primarily, we treat the heavy quark as static throughout but consider the leading effects of the splitting between the heavy-light vector and pseudoscalar mesons.

An important conclusion of this work is that the chiral extrapolation for matrix elements of $\mathcal{O}_{1,aa}$ is considerably less complicated than that for matrix elements of the operators in Eq. (2). Generically, the chiral expansion for $\langle \bar{B}_{(s)}^0 | \mathcal{O}_{1,aa} | B_{(s)}^0 \rangle$ takes the form

$$\langle \bar{B}_{(s)}^0 | \mathcal{O}_{1,aa} | B_{(s)}^0 \rangle \xrightarrow{\text{chiral}} \gamma_1(1 + L) + \text{analytic terms}, \quad (5)$$

where γ_1 is the leading-order low-energy constant (LEC), L denotes the nonanalytic one-loop contributions (chiral logarithms), and the analytic terms are from the next-to-leading-order counterterms in the chiral expansion. However, for the operators in Eq. (2), the chiral expansion has the generic feature:

$$\langle \bar{B}_{(s)}^0 | \mathcal{O}_{i,aa} | B_{(s)}^0 \rangle \xrightarrow{\text{chiral}} \gamma_i(1 + L) + \gamma'_i L' + \text{analytic terms}, \quad (6)$$

where $i = 2, 3, 4, 5$, γ_i and γ'_i are unknown leading-order LECs, and L and L' are different one-loop chiral logarithms. Again, the analytic terms are from the next-to-leading-order counterterms in the chiral expansion. The appearance of the second nonanalytic term complicates the chiral extrapolation in Eq. (6) because an additional unknown parameter must be determined. The origin of this complication is discussed in detail in Sec. III.

This paper is structured as follows: in Sec. II we briefly discuss heavy meson chiral perturbation theory before turning to the inclusion of the four-quark operators in HM χ PT in Sec. III. We present the results of the next-to-leading-order light quark mass and lattice volume dependence of the relevant matrix elements in Sec. IV before concluding (Sec. V). Various technical details are relegated to the appendices.

While this work was being completed, a preprint describing similar work appeared [13]. The conclusions of the revised version of that work agree with those presented herein, specifically the forms of the chiral extrapolations in Eqs. (5) and (6).

II. HEAVY MESON CHIRAL PERTURBATION THEORY

The inclusion of the heavy-light mesons in chiral perturbation theory (HM χ PT) was first proposed in Refs. [9–11], with the generalization to quenched¹ and partially

¹We do not consider the quenched theory here as quenched quantities are unrelated to those in QCD [14].

quenched theories given in Refs. [15,16]. The $1/M_P$ (M_P is the mass of the heavy-light pseudoscalar meson) and chiral corrections were studied by Boyd and Grinstein [17] in full QCD and by Booth [18] in quenched QCD. The field appearing in this effective theory is

$$H_a^{(Q)} = \frac{1 + \not{p}}{2} (P_{a,\mu}^{*(Q)} \gamma^\mu - P_a^{(Q)} \gamma_5), \quad (7)$$

where $P_a^{(Q)}$ and $P_{a,\mu}^{*(Q)}$ annihilate pseudoscalar and vector mesons containing a heavy quark Q and a light antiquark of flavor a . In the heavy particle formalism, such mesons have momentum $p^\mu = M_P v^\mu + k^\mu$ with $|k^\mu| \ll M_P$ and v^μ is the velocity of the particle. Under a heavy quark spin SU(2) transformation S and a generic light flavor transformation U [i.e., $U \in \text{SU}(3)$ for full QCD and $U \in \text{SU}(6|3)$ for partially quenched QCD (PQQCD)],

$$H_a^{(Q)} \rightarrow S H_b^{(Q)} U_{ba}^\dagger. \quad (8)$$

The conjugate field, which creates heavy-light mesons containing a heavy quark Q and a light antiquark of flavor a , is defined as

$$\bar{H}_a^{(Q)} = \gamma^0 H^{(Q)\dagger} \gamma_0 = (P_{a,\mu}^{*(Q)\dagger} \gamma^\mu + P_a^{(Q)\dagger} \gamma_5) \frac{1 + \not{p}}{2}, \quad (9)$$

which transforms under S and U as

$$\bar{H}_a^{(Q)} \rightarrow U_{ab} \bar{H}_b^{(Q)} S^\dagger. \quad (10)$$

The chiral Lagrangian for the Goldstone particles is

$$\mathcal{L}_{\text{GP}} = \frac{f^2}{8} (s) \text{tr}[(\partial_\mu \Sigma^\dagger)(\partial^\mu \Sigma) + \Sigma^\dagger \chi + \chi^\dagger \Sigma], \quad (11)$$

where $\Sigma = \exp(2i\Phi/f)$ is the nonlinear Goldstone field, with Φ being the matrix containing the standard Goldstone fields. We use $f = 132$ MeV. In this work, we follow the supersymmetric formulation of partially quenched chiral perturbation theory [PQ χ PT] [19,20]. Therefore Σ transforms linearly under $\text{SU}(3)_L \otimes \text{SU}(3)_R$ and $\text{SU}(6|3)_L \otimes \text{SU}(6|3)_R$ in full QCD and PQQCD, respectively. The symbol “(s) tr” in the above equation means “trace” in chiral perturbation theory (χ PT) and “supertrace” in PQ χ PT where the flavor group is graded. The variable χ is defined as

$$\chi = 2B_0 \mathcal{M}_q = \frac{-2\langle 0 | \bar{u}u + \bar{d}d | 0 \rangle}{f^2} \mathcal{M}_q, \quad (12)$$

where the quark mass matrix \mathcal{M}_q is

$$\mathcal{M}_q^{(\text{QCD})} = \text{diag}(m_u, m_u, m_s), \quad (13)$$

in full QCD, and

$$\mathcal{M}_q^{(\text{PQQCD})} = \text{diag}(\underbrace{m_u, m_u, m_s}_{\text{valence}}, \underbrace{m_j, m_j, m_r}_{\text{sea}}, \underbrace{m_w, m_w, m_s}_{\text{ghost}}), \quad (14)$$

in PQQCD. We keep the strange quark mass different from that of the (degenerate) up and down quarks in the valence, sea and ghost sectors. Notice that the flavor singlet state $\Phi_0 = (s) \text{tr}(\Phi)/\sqrt{6}$ is rendered heavy by the $U(1)_A$ anomaly in QCD and PQQCD [14,21] and has been integrated out.

Furthermore, the Goldstone mesons appear in the HM χ PT Lagrangian via the field

$$\xi \equiv e^{i\Phi/f}, \quad (15)$$

which transforms as

$$\xi \rightarrow U_L \xi U^\dagger = U \xi U_R^\dagger, \quad (16)$$

where $U_{L(R)}$ is an element of the left-handed (right-handed) $SU(3)$ and $SU(6|3)$ groups for QCD and PQQCD, respectively. The HM χ PT Lagrangian, to lowest order in the chiral and $1/M_P$ expansion, for mesons containing a heavy quark Q and a light antiquark of flavor a is then

$$\begin{aligned} \mathcal{L}_{\text{HM}\chi\text{PT}} = & -i \text{tr}_D(\bar{H}_a^{(Q)} v_\mu \partial^\mu H_a^{(Q)}) \\ & + \frac{i}{2} \text{tr}_D(\bar{H}_a^{(Q)} v_\mu [\xi^\dagger \partial^\mu \xi + \xi \partial^\mu \xi^\dagger]_{ab} H_b^{(Q)}) \\ & + \frac{i}{2} g \text{tr}_D(\bar{H}_a^{(Q)} \gamma_\mu \gamma_5 [\xi^\dagger \partial^\mu \xi - \xi \partial^\mu \xi^\dagger]_{ab} H_b^{(Q)}) \\ & + B_{\eta'} \frac{i}{2} \gamma \text{tr}_D(\bar{H}_a^{(Q)} H_a^{(Q)} \gamma_\mu \gamma_5)(s) \\ & \times \text{tr}[\xi^\dagger \partial^\mu \xi - \xi \partial^\mu \xi^\dagger], \end{aligned} \quad (17)$$

where $B_{\eta'} = 0$ for full QCD, and $B_{\eta'} = 1$ for PQQCD.² The flavor (super-)trace (s) tr is taken in the appropriate flavor space and tr_D is the trace over Dirac space. The low-energy constant g occurring in this Lagrangian is common to both HM χ PT and partially quenched HM χ PT. Note that factors of $\sqrt{M_Q}$ and $\sqrt{M_Q^*}$ have been absorbed into the heavy meson fields so the $H_b^{(Q)}$ are of mass dimension 3/2.

The HM χ PT Lagrangian for mesons containing a heavy antiquark \bar{Q} and a light quark of flavor a is obtained by applying the charge conjugation operation to the above Lagrangian [22]. The field that annihilates such mesons is

$$H_a^{(\bar{Q})} = (P_{a,\mu}^{*(\bar{Q})} \gamma^\mu - P_a^{(\bar{Q})} \gamma_5) \frac{1 - \not{v}}{2}, \quad (18)$$

which transforms under S and U as

$$H_a^{(\bar{Q})} \rightarrow U_{ab} H_b^{(\bar{Q})} S^\dagger. \quad (19)$$

The effects of chiral and heavy quark symmetry breakings have been systematically studied at next-to-leading order in full [17] and quenched HM χ PT [18]. Amongst them, the only relevant feature necessary for our calculations are the shifts to the masses of the heavy-light mesons. These shifts are from the heavy quark spin breaking term

²However, since we integrate out the η' in PQQCD [14], the coupling γ does not appear in the results presented in this paper.

$$\frac{\lambda_2}{M_P} \text{tr}_D(\bar{H}_a^{(Q)} \sigma_{\mu\nu} H_a^{(Q)} \sigma^{\mu\nu}), \quad (20)$$

and the chiral symmetry breaking terms

$$\begin{aligned} & \lambda_1 B_0 \text{tr}_D(\bar{H}_a^{(Q)} [\xi \mathcal{M}_q \xi + \xi^\dagger \mathcal{M}_q \xi^\dagger]_{ab} H_b^{(Q)}) \\ & + \lambda'_1 B_0 \text{tr}_D(\bar{H}_a^{(Q)} H_a^{(Q)}) [\xi \mathcal{M}_q \xi + \xi^\dagger \mathcal{M}_q \xi^\dagger]_{bb}. \end{aligned} \quad (21)$$

We choose to use a field redefinition that allows us to work with the effective theory in which the heavy-light pseudo-scalar mesons that contain a heavy quark and a u or d valence antiquark are massless. Notice that the term proportional to λ'_1 in Eq. (21) causes a universal shift to all the heavy-light meson masses. This means that the propagators of the heavy mesons are as follows

$$\begin{aligned} & \frac{i}{2(v \cdot k + i\epsilon)}, \quad \frac{-i(g_{\mu\nu} - v_\mu v_\nu)}{2(v \cdot k - \Delta_* + i\epsilon)}, \\ & \frac{i}{2(v \cdot k - \delta_{us} + i\epsilon)}, \quad \text{and} \quad \frac{-i(g_{\mu\nu} - v_\mu v_\nu)}{2(v \cdot k - \Delta_* - \delta_{us} + i\epsilon)}, \end{aligned} \quad (22)$$

for P , P^* , P_s , and P_s^* , respectively. The mass shifts can be written in terms of the couplings in Eqs. (20) and (21):

$$\Delta_* = -8 \frac{\lambda_2}{M_P}, \quad (23)$$

and

$$\delta_{us} = 2\lambda_1 B_0 (m_s - m_u). \quad (24)$$

In the partially quenched extension, there are two additional mass shifts because the sea quarks masses differ from those of the valence and ghost quarks:

$$\delta_{jr} = M_{\bar{P}_s} - M_{\bar{P}} = 2\lambda_1 B_0 (m_r - m_j), \quad (25)$$

and

$$\delta_{uj} = M_{\bar{P}} - M_P = 2\lambda_1 B_0 (m_j - m_u). \quad (26)$$

where $\bar{P}(\bar{P}_s)$ is the heavy-light pseudoscalar meson with a $j(r)$ sea antiquark. The propagators of the heavy mesons containing sea antiquarks are:

$$\begin{aligned} & \frac{i}{2(v \cdot k - \delta_{uj} + i\epsilon)}, \quad \frac{-i(g_{\mu\nu} - v_\mu v_\nu)}{2(v \cdot k - \Delta_* - \delta_{uj} + i\epsilon)}, \\ & \frac{i}{2(v \cdot k - \delta_{uj} - \delta_{jr} + i\epsilon)}, \\ & \text{and} \quad \frac{-i(g_{\mu\nu} - v_\mu v_\nu)}{2(v \cdot k - \Delta_* - \delta_{uj} - \delta_{jr} + i\epsilon)}, \end{aligned} \quad (27)$$

for \bar{P} , \bar{P}^* (vector meson with a j sea antiquark), \bar{P}_s , and \bar{P}_s^* (vector meson with an r sea antiquark), respectively.

III. FOUR-FERMION OPERATORS IN HEAVY MESON CHIRAL PERTURBATION THEORY

A. Construction of the $\Delta B = 2$ and $\Delta C = 2$ operators

Under a chiral transformation, the four-quark operators in Eqs. (1) and (2) fall into two categories:

$$\mathcal{O}_{LL} = \bar{h}\Gamma_{LL}q_L\bar{h}\Gamma_{LL}q_L, \quad \mathcal{O}_{LR} = \bar{h}\Gamma_{LR}^{(1)}q_L\bar{h}\Gamma_{LR}^{(2)}q_R, \quad (28)$$

where

$$q_{L,R} = \frac{1 \pm \gamma_5}{2} q. \quad (29)$$

Operators $\mathcal{O}_{1,aa}$, $\mathcal{O}_{2,aa}$ and $\mathcal{O}_{3,aa}$ are of the first type and transform in the symmetric $(\mathbf{6}_L, \mathbf{1}_R)$ representation built from the direct product $(\mathbf{3}_L, \mathbf{1}_R) \otimes (\mathbf{3}_L, \mathbf{1}_R) = (\mathbf{6}_L, \mathbf{1}_R) \oplus (\bar{\mathbf{3}}_L, \mathbf{1}_R)$ under chiral rotations while $\mathcal{O}_{4,aa}$ and $\mathcal{O}_{5,aa}$ are of the second type and transform in the $(\mathbf{3}_L, \mathbf{3}_R)$ representation. Here we refer to the SU(3) flavor transformation properties, leaving the partially quenched extension to

$$\begin{aligned} \mathcal{O}_{i,aa}^{\text{HM}\chi\text{PT}} = & \sum_x \{ \alpha_{i,x}^{(1)} \text{tr}_D[(\xi\bar{H}^{(h)})_a\Gamma_1\Xi_x] \text{tr}_D[(\xi^\dagger H^{(\bar{h})})_a\Gamma_2\Xi'_x] + \alpha_{i,x}^{(2)} \text{tr}_D[(\xi\bar{H}^{(h)})_a\Gamma_2\Xi_x] \text{tr}_D[(\xi^\dagger H^{(\bar{h})})_a\Gamma_1\Xi'_x] \\ & + \alpha_{i,x}^{(3)} \text{tr}_D[(\xi\bar{H}^{(h)})_a\Gamma_1\Xi_x(\xi^\dagger H^{(\bar{h})})_a\Gamma_2\Xi'_x] + \alpha_{i,x}^{(4)} \text{tr}_D[(\xi\bar{H}^{(h)})_a\Gamma_2\Xi_x(\xi^\dagger H^{(\bar{h})})_a\Gamma_1\Xi'_x] \\ & + \alpha_{i,x}^{(5)} \text{tr}_D[(\xi^\dagger\bar{H}^{(h)})_a\Gamma_1\Xi_x] \text{tr}_D[(\xi H^{(\bar{h})})_a\Gamma_2\Xi'_x] + \alpha_{i,x}^{(6)} \text{tr}_D[(\xi^\dagger\bar{H}^{(h)})_a\Gamma_2\Xi_x] \text{tr}_D[(\xi H^{(\bar{h})})_a\Gamma_1\Xi'_x] \\ & + \alpha_{i,x}^{(7)} \text{tr}_D[(\xi^\dagger\bar{H}^{(h)})_a\Gamma_1\Xi_x(\xi H^{(\bar{h})})_a\Gamma_2\Xi'_x] + \alpha_{i,x}^{(8)} \text{tr}_D[(\xi^\dagger\bar{H}^{(h)})_a\Gamma_2\Xi_x(\xi H^{(\bar{h})})_a\Gamma_1\Xi'_x] \}. \end{aligned} \quad (32)$$

The positions in the above operators in which the arbitrary Dirac structures, Ξ_x and Ξ'_x , are inserted is constrained by the heavy quark spin symmetry [24,25]. Notice that in general both single and double Dirac trace operators must be considered.

Performing the Dirac traces for the particular $\Gamma, \Gamma_{1,2}$ in Eqs. (1) and (2), and keeping only the terms that will contribute to the matrix elements we consider, leads to the following set of operators involving the individual heavy meson fields:

$$\begin{aligned} \mathcal{O}_{1,aa}^{\text{HM}\chi\text{PT}} &= \beta_1 [(\xi P^{(h)\dagger})_a(\xi P^{(\bar{h})})_a + (\xi P_\mu^{*(h)\dagger})_a(\xi P^{*(\bar{h}),\mu})_a], \\ \mathcal{O}_{2(3),aa}^{\text{HM}\chi\text{PT}} &= \beta_{2(3)}(\xi P^{(h)\dagger})_a(\xi P^{(\bar{h})})_a + \beta'_{2(3)}(\xi P_\mu^{*(h)\dagger})_a(\xi P^{*(\bar{h}),\mu})_a, \\ \mathcal{O}_{4(5),aa}^{\text{HM}\chi\text{PT}} &= \beta_{4(5)}(\xi P^{(h)\dagger})_a(\xi^\dagger P^{(\bar{h})})_a + \hat{\beta}_{4(5)}(\xi^\dagger P^{(h)\dagger})_a(\xi P^{(\bar{h})})_a + \beta'_{4(5)}(\xi P_\mu^{*(h)\dagger})_a(\xi^\dagger P^{*(\bar{h}),\mu})_a + \hat{\beta}'_{4(5)}(\xi^\dagger P_\mu^{*(h)\dagger})_a(\xi P^{*(\bar{h}),\mu})_a, \end{aligned} \quad (33)$$

where the $\beta_i, \beta'_i, \hat{\beta}_i$ and $\hat{\beta}'_i$ are linear combinations of the various $\alpha_{i,x}^{(j)}$ appearing in Eqs. (31) and (32).

It is important to note that in the above equation, the operator $\mathcal{O}_{1,aa}^{\text{HM}\chi\text{PT}}$ behaves somewhat differently from the other operators as only a single LEC, β_1 , occurs. This greatly simplifies any chiral extrapolation of corresponding lattice data for neutral heavy-light meson mixing in the standard model, as confirmed by the one-loop results presented in the next section. We stress that this simplification is not obvious from the operator structure in Eq. (31) and is particular to the $V - A$ structure of the standard model currents. In general, one would expect from Eqs. (31) and (32) that operators for pseudoscalar and vector meson

the following subsection. Note that the color indices in Eq. (2) are relevant to short-distance physics, and hence play no role in the chiral properties of these operators [23]. Treating $\Gamma_{LL}, \Gamma_{LR}^{(1)}$ and $\Gamma_{LR}^{(2)}$ as spurions transforming as

$$\begin{aligned} \Gamma_{LL} &\rightarrow S\Gamma_{LL}U_L^\dagger, & \Gamma_{LR}^{(1)} &\rightarrow S\Gamma_{LR}^{(1)}U_L^\dagger, \\ \Gamma_{LR}^{(2)} &\rightarrow S\Gamma_{LR}^{(2)}U_R^\dagger, \end{aligned} \quad (30)$$

the operators in Eq. (28) remain invariant under heavy quark spin and chiral rotations. We then find that the bosonization of the operators in Eqs. (1) and (2) is given by

$$\begin{aligned} \mathcal{O}_{i,aa}^{\text{HM}\chi\text{PT}} = & \sum_x \{ \alpha_{i,x}^{(1)} \text{tr}_D[(\xi\bar{H}^{(h)})_a\Gamma\Xi_x] \text{tr}_D[(\xi H^{(\bar{h})})_a\Gamma\Xi'_x] \\ & + \alpha_{i,x}^{(3)} \text{tr}_D[(\xi\bar{H}^{(h)})_a\Gamma\Xi_x(\xi H^{(\bar{h})})_a\Gamma\Xi'_x] \}, \end{aligned} \quad (31)$$

for $i = 1, 2, 3$ where $\Gamma = \Gamma_1 = \Gamma_2$ in Eq. (3) and Ξ_x and Ξ'_x are all possible pairs of Dirac structures.³ For $i = 4, 5$ the HM χ PT operators are

mixing processes are accompanied by different LECs. This is the case for all the nonstandard model operators, as shown in Eq. (33).

To understand the origin of the above simplification in the standard model operator $\mathcal{O}_{1,aa}^{\text{HM}\chi\text{PT}}$, we turn to heavy quark effective theory (HQET) [26–28]. In this effective theory, the operators that produce the same matrix ele-

³An overcomplete list of the possible pairs of structures is: $\{\Xi_x, \Xi'_x\} = \{\{1, 1\}, \{1, \not{p}\}, \{\not{p}, \not{p}\}, \{\gamma_\mu, \gamma^\mu\}, \{\gamma_\mu \not{p}, \gamma^\mu\}, \{\gamma_\mu \not{p}, \gamma^\mu \not{p}\}, \{\sigma_{\mu\nu}, \sigma^{\mu\nu}\}, \{\sigma_{\mu\nu} \not{p}, \sigma^{\mu\nu}\}, \{\sigma_{\mu\nu} \not{p}, \sigma^{\mu\nu} \not{p}\}\}$, their permutations, and possible combinations with γ_5 . There is some redundancy here as the equations of motion of the heavy meson fields, $\not{p}H_a^{(Q)} = H_a^{(Q)}$ etc., relate various terms in Eqs. (31) and (32).

ments as those in Eq. (3) are [29]

$$\mathcal{O}_{i,aa}^{\text{HQET}} = \tilde{Q}\Gamma_1 q_a Q^\dagger \Gamma_2 q_a + Q^\dagger \Gamma_1 q_a \tilde{Q}\Gamma_2 q_a, \quad (34)$$

where $\Gamma_{1,2}$ are the appropriate Dirac and color structures from Eq. (2). Here, Q and \tilde{Q} denote fields annihilating a heavy quark and heavy antiquark, respectively, (these fields do not create the corresponding antiparticles). Additional terms in HQET which create two heavy quarks or annihilate two heavy antiquarks will not contribute to neutral heavy meson mixing and are ignored.

The standard model operator in HQET, $\mathcal{O}_{1,aa}^{\text{HQET}}$, satisfies the relation

$$\{S_Q^3, \mathcal{O}_{1,aa}^{\text{HQET}}\}|P\rangle = \{S_Q^3, \mathcal{O}_{1,aa}^{\text{HQET}}\}|\bar{P}\rangle = 0, \quad (35)$$

where $|P\rangle$ is pseudoscalar heavy-light meson state, and $|\bar{P}\rangle$ is the state of its antiparticle. The operator

$$S_Q^3 = \epsilon^{ij3}[Q^\dagger \sigma_{ij} Q - \tilde{Q} \sigma_{ij} \tilde{Q}^\dagger] \quad (36)$$

is the heavy quark spin operator [30] that changes the spin of the heavy-light meson state by one. Therefore, Eq. (35) implies that the mixing matrix elements for vector and pseudoscalar heavy-light mesons are equal and opposite [22] in the heavy quark limit. This symmetry is reflected in HM χ PT, leading to the result for $\mathcal{O}_{1,aa}^{\text{HM}\chi\text{PT}}$ in Eq. (33).

For the nonstandard model operators, it is straightforward to show that

$$\{S_Q^3, \mathcal{O}_{i,aa}^{\text{HQET}}\}|P\rangle \neq 0, \quad \{S_Q^3, \mathcal{O}_{i,aa}^{\text{HQET}}\}|\bar{P}\rangle \neq 0, \quad (37)$$

and

$$[S_Q^3, \mathcal{O}_{i,aa}^{\text{HQET}}]|P\rangle \neq 0, \quad [S_Q^3, \mathcal{O}_{i,aa}^{\text{HQET}}]|\bar{P}\rangle \neq 0, \quad (38)$$

where $i = 2, 3, 4, 5$. This means that the pseudoscalar and vector meson mixing processes via these operators are not proportional to each other, hence the appearance of the terms accompanied by $\beta'_{2,3,4,5}$ and $\hat{\beta}'_{4,5}$ in Eq. (33).

We end this subsection by noting that equations of motion for the heavy quark [31] result in $\mathcal{O}_{3,aa} = -\mathcal{O}_{1,aa}/2 - \mathcal{O}_{2,aa}$, and can be used to relate some of the LECs in Eq. (33).

B. Partially quenched extensions

In the partially quenched theory, the operator matching is analogous because the QCD operators considered on the lattice involve only valence quarks. Since HM χ PT is contained within partially quenched HM χ PT, the LECs occurring in the four-quark operators of both theories are the same for the quantities we consider.

In the partially quenched case, the $\Delta B = 2$ and $\Delta C = 2$ operators transform in the symmetric tensor product of two fundamental representations of $\text{SU}(6|3)$, a **42**-dimensional representation. The operators arising from QCD can be simply embedded in this larger representation with no mixing into different representations. In most cases (and

herein) it is sensible to choose the quark ‘‘charges’’ such that the operators are purely valence, but any other element of this representation suffices.

IV. NEUTRAL MESON MIXING MATRIX ELEMENTS

Calculations at NLO in the chiral expansion require the evaluation of the one-loop diagrams shown in Fig. 1. We perform these calculations both at infinite volume and in a cubic spatial box of dimensions L^3 (the time extent is assumed to be infinite). In this section we summarize the results, relegating details of the calculations to Appendix A.

For the standard model operator we find the following matrix elements

$$\begin{aligned} \langle \bar{B}^0 | \mathcal{O}_{1,dd} | B^0 \rangle &= \beta_1 \left(1 + \mathcal{T}_d^{(1)} + \frac{\mathcal{W}_{\bar{B}^0} + \mathcal{W}_{B^0}}{2} + \mathcal{Q}_d^{(1)} \right) \\ &\quad + \text{analytic terms,} \\ \langle \bar{B}_s^0 | \mathcal{O}_{1,ss} | B_s^0 \rangle &= \beta_1 \left(1 + \mathcal{T}_s^{(1)} + \frac{\mathcal{W}_{\bar{B}_s^0} + \mathcal{W}_{B_s^0}}{2} + \mathcal{Q}_s^{(1)} \right) \\ &\quad + \text{analytic terms.} \end{aligned} \quad (39)$$

The wave function contributions, \mathcal{W}_M , and tadpole- and sunset-type operator renormalizations, $\mathcal{T}_a^{(i)}$ and $\mathcal{Q}_a^{(i)}$ (diagrams (b) and (c) in Fig. 1, respectively), are non-analytic functions of the light quark mass and lattice volume and are defined in Appendix B. The ‘‘analytic terms’’ here include Goldstone meson mass squared terms, a term $\sim \alpha_s(M_b)/4\pi$ (arising from mixing) and a term $\sim \Lambda_{\text{QCD}}/M_b$. The $\sim \Lambda_{\text{QCD}}/M_b$ term arises from $1/M_b$ terms in the Lagrangian (those from Eq. (20) are included) and from additional $1/M_b$ -suppressed HM χ PT operators that match onto the QCD operators. We note that at higher orders, the simple relation between the B and B^* matrix elements of the standard model operator will break down. Although we present results specifically in the B -meson systems, note that they are also applicable to D -meson systems, under the assumption that the charm-quark mass is large enough compared to Λ_{QCD} .

Parametrizing the $\Delta B = 2$ matrix elements in the standard form [3] (f_{B_q} is the weak decay constant defined through $\langle 0 | \bar{b} \gamma_\mu \gamma_5 q | B_q(\vec{p}) \rangle = i p_\mu f_{B_q}$),

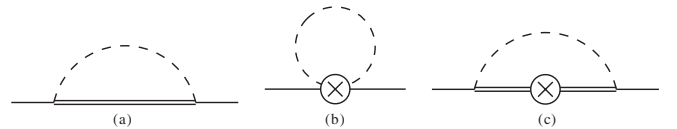


FIG. 1. Diagrams contributing to the matrix elements of four-quark operators at NLO in the chiral expansion. Solid, double and dashed lines correspond to propagators of pseudoscalar and vector heavy-light mesons, and Goldstone mesons, respectively. The crossed circle denotes the four-quark operator and diagram (a) is the wave function renormalization.

$$\langle \bar{B}_q^0 | \mathcal{O}_{1,qq} | B_q^0 \rangle = \frac{8}{3} M_{B_q}^2 f_{B_q}^2 B_{B_q}^{(1)}(\mu), \quad (40)$$

$$\langle \bar{B}_q^0 | \mathcal{O}_{i,qq} | B_q^0 \rangle = \eta_i R^2 M_{B_q}^2 f_{B_q}^2 B_{B_q}^{(i)}(\mu) \quad \text{for } i = 2, \dots, 5, \quad (41)$$

$$(R = \frac{M_{B_q}}{m_b(\mu) + m_q(\mu)} \text{ and } \eta_2 = -\frac{5}{3}, \eta_3 = \frac{1}{3}, \eta_4 = 2, \eta_5 = \frac{2}{3})$$

$$\begin{aligned} \langle \bar{B}^0 | \mathcal{O}_{2(3),dd} | B^0 \rangle &= \beta_{2(3)} \left(1 + \mathcal{T}_d^{(2(3))} + \frac{\mathcal{W}_{\bar{B}^0} + \mathcal{W}_{B^0}}{2} \right) + \beta'_{2(3)} \mathcal{Q}_d^{(2(3))} + \text{analytic terms}, \\ \langle \bar{B}_s^0 | \mathcal{O}_{2(3),ss} | B_s^0 \rangle &= \beta_{2(3)} \left(1 + \mathcal{T}_s^{(2(3))} + \frac{\mathcal{W}_{\bar{B}_s^0} + \mathcal{W}_{B_s^0}}{2} \right) + \beta'_{2(3)} \mathcal{Q}_s^{(2(3))} + \text{analytic terms}, \\ \langle \bar{B}^0 | \mathcal{O}_{4(5),dd} | B^0 \rangle &= [\beta_{4(5)} + \hat{\beta}_{4(5)}] \left(1 + \mathcal{T}_d^{(4(5))} + \frac{\mathcal{W}_{\bar{B}^0} + \mathcal{W}_{B^0}}{2} \right) + [\beta'_{4(5)} + \hat{\beta}'_{4(5)}] \mathcal{Q}_d^{(4(5))} + \text{analytic terms}, \\ \langle \bar{B}_s^0 | \mathcal{O}_{4(5),ss} | B_s^0 \rangle &= [\beta_{4(5)} + \hat{\beta}_{4(5)}] \left(1 + \mathcal{T}_s^{(4(5))} + \frac{\mathcal{W}_{\bar{B}_s^0} + \mathcal{W}_{B_s^0}}{2} \right) + [\beta'_{4(5)} + \hat{\beta}'_{4(5)}] \mathcal{Q}_s^{(4(5))} + \text{analytic terms}. \end{aligned} \quad (42)$$

The terms $\sim \mathcal{Q}_q^{(i)}$ arising from the sunset diagrams [Fig. 1(c)] involve the neutral heavy-light vector meson mixing amplitudes. As discussed in the preceding section, it is only in the case of $\mathcal{O}_{1,qq}$ that these amplitudes are related to those of the pseudoscalar heavy-light mesons. For $i = 2, 3, 4, 5$ these terms are consequently accompanied by different LECs. The analytic terms in the above expressions depend on the renormalization scale in such a way as to cancel the scale dependence of the nonanalytic loop contributions.

The matrix elements in Eqs. (39) and (42) also determine the B_s decay width differences [Eq. (4)]. The extrapolation in light quark mass and lattice volume is more involved here than for the matrix elements determining standard model oscillations and have not been accounted for in the existing unquenched lattice calculations [32,33]. Direct calculations of the ratio of the matrix element of $\mathcal{O}_{1,ss}$ to that of $\mathcal{O}_{2,ss}$ do not help in this regard as the nonanalytic behavior does not simplify.

At present, we can only study the finite volume and light quark mass effects described in these formulas cursorily as there is very little lattice data to use for such a task in any reliable manner. This will hopefully change in the near

future; the recent calculations of [34] are encouraging (we note, however, that our results imply that more than three light quark masses are needed for the NLO chiral extrapolation of matrix elements of $\mathcal{O}_{i,aa}$ for $i = 2, 3, 4, 5$). As a guide to the importance of such effects, we present representative results for the $\mathcal{O}_{2,4}$ matrix elements. Results for the standard model operators have been discussed in Ref. [12]. Taking the QCD limit for definiteness, Figs. 2 and 3 explore the dependence of the finite volume shifts in the matrix elements of $\mathcal{O}_{2,dd}$, $\mathcal{O}_{2,ss}$, $\mathcal{O}_{4,dd}$ and $\mathcal{O}_{4,ss}$, normalized by their tree-level values, on the pion mass for two different volumes, $L = 2.5, 3.5$ fm. In each figure we plot the ratio

$$\begin{aligned} \langle \bar{B}_{(s)}^0 | \mathcal{O}_{i,ff} | B_{(s)}^0 \rangle_{\text{FV}} \\ = \frac{\langle \bar{B}_{(s)}^0 | \mathcal{O}_{i,ff} | B_{(s)}^0 \rangle(L) - \langle \bar{B}_{(s)}^0 | \mathcal{O}_{i,ff} | B_{(s)}^0 \rangle(\infty)}{\langle \bar{B}_{(s)}^0 | \mathcal{O}_{i,ff} | B_{(s)}^0 \rangle_{\text{tree}}}. \end{aligned} \quad (43)$$

We fix $f = 0.132$ GeV and $\Delta_* = 50$ MeV (variation with Δ_* is small and similar to that found for the standard model operator [12], with the finite volume (FV) effect decreasing with increasing Δ_*). Using recent CLEO measurements [35,36], the coupling g is taken to be $0.3 < g^2 < 0.5$ with a

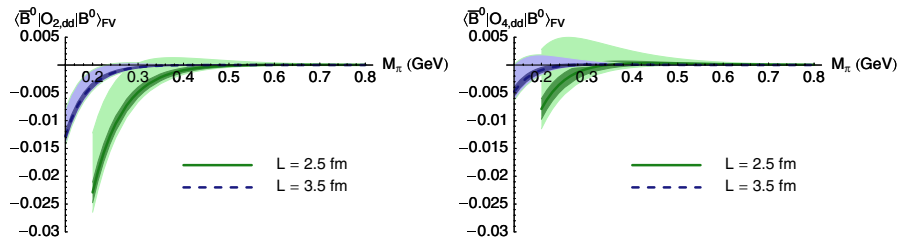


FIG. 2 (color online). Finite volume effects in mixing matrix elements of the operators $\mathcal{O}_{i,dd}$ for $i = 2, 4$ for two lattice volumes, $L = 2.5, 3.5$ fm. The central curve corresponds to $g = 0.4$ and $\beta'_i/\beta_i = 1$ while the inner (darker) and outer (lighter) shaded regions correspond to variation of $0.3 < g < 0.5$ and $|\beta'_i/\beta_i| < 2$. The curves terminate at $m_\pi L = 2.5$ where p -regime chiral perturbation theory becomes unreliable.

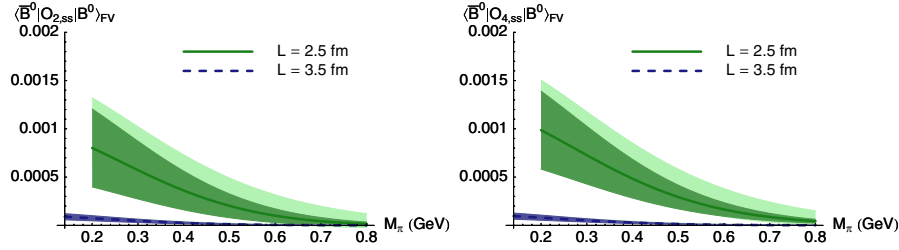


FIG. 3 (color online). Finite volume effects in mixing matrix elements of the operators $\mathcal{O}_{i,ss}$ for $i = 2, 4$ for two lattice volumes, $L = 2.5, 3.5$ fm. Details are as in Fig. 2.

central value of $g^2 = 0.4$; variation in g is indicated in the figures by the inner (darker) shaded regions. For each quantity, we vary the ratio of B^* to B LECs over a reasonable range, taking $|\beta'_2/\beta_2| < 2$ and $|(\beta'_4 + \hat{\beta}'_4)/(\beta_4 + \hat{\beta}_4)| < 2$ (naturalness would suggest these ratios should be of order unity, but for simplicity we also allow smaller values). In each figure, the central curve corresponds to a ratio of unity and the outer (lighter) shaded region to this variation. As can be seen, effects of the finite volume on $\mathcal{O}_{i,dd}$ are similar in size to those found for the standard model operator in Ref. [12]. Effects for the strange operators are considerably suppressed as pion loops do not contribute to these matrix elements in the QCD limit.

Before concluding, it is useful to consider how lattice spacing artifacts will enter the above expressions. Since calculations here involve both light and heavy quarks, there are two types of discretization effects. The effects of light quark discretization are very simple to incorporate; at a particular lattice spacing, a , the masses of the various Goldstone mesons in chiral loops are shifted from their continuum values (and these shifted masses should be used in fits to lattice data using the above partially quenched expressions) and the various counterterms become polynomials in the lattice spacing. In general this polynomial will contain all powers of a , but if both the light quark action and the four-quark operator are improved (or a discretization satisfying the Ginsparg-Wilson relation [37] is used for the valence quarks [38]), the leading corrections in a can be eliminated. For most foreseeable calculations, this is the extent of discretization effects at the order to which we have worked (we assume that $a\Lambda_{\text{QCD}} \lesssim m_q/\Lambda_{\text{QCD}}$). However, if a heavy quark action is used that breaks heavy quark spin symmetry at $\mathcal{O}(a)$, additional complications will arise as this symmetry can no longer be used to constrain the form of the effective field theory operator, $\mathcal{O}_{1,aa}^{\text{HM}\chi\text{PT}}$; $B_{(s)} - \bar{B}_{(s)}$ and $B_{(s)}^* - \bar{B}_{(s)}^*$ matrix elements are no longer related. The resulting mass and volume dependence of matrix elements of this operator will become more complicated, resembling instead that of the matrix elements of the other operators in Eqs. (31) and (32).⁴

⁴Additional complications beyond the scope of this work may arise if the Kogut-Susskind fermion action is used for the light quarks.

V. CONCLUSION

We have considered the matrix elements of four-quark operators relevant for heavy-light neutral meson mixing and decay width differences in partially quenched, finite volume heavy meson chiral perturbation theory relevant for lattice computations. For heavy-light neutral meson mixing, inclusion of operators beyond those in the standard model complicates the chiral extrapolation as two LECs appear at leading order rather than one for the standard model operator. The matrix elements relevant for lifetime ratios and decay width differences have similarly complicated light quark mass and spatial volume dependencies. This arises due to the fact that the operators for these processes do not guarantee that $B - \bar{B}$ and $B^* - \bar{B}^*$ mixing amplitudes are proportional to each other in the heavy quark limit. Our results are useful for current and future lattice calculations of these matrix elements, which are needed in high-precision tests of the standard model and the search for new physics.

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APPENDIX A: INTEGRALS AND SUMS

We have regularized ultraviolet divergences that appear in the various loop integrals using dimensional regularization, and subtracted the term

$$\bar{\lambda} = \frac{2}{4-d} - \gamma_E + \log(4\pi) + 1. \quad (\text{A1})$$

The integrals appearing in the full QCD calculation are defined by

$$\begin{aligned} I_{\bar{\lambda}}(m) &\equiv \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2 + i\epsilon} \\ &= \frac{im^2}{16\pi^2} \left[\bar{\lambda} - \log\left(\frac{m^2}{\mu^2}\right) \right], \end{aligned} \quad (\text{A2})$$

$$\begin{aligned}
H_{\bar{\lambda}}(m, \Delta) &\equiv (g^{\rho\nu} - v^\rho v^\nu) \mu^{4-d} \frac{\partial}{\partial \Delta} \int \frac{d^d k}{(2\pi)^d} \\
&\quad \times \frac{k_\rho k_\nu}{(k^2 - m^2 + i\epsilon)(v \cdot k - \Delta + i\epsilon)} \\
&= 3 \frac{\partial}{\partial \Delta} F_{\bar{\lambda}}(m, \Delta), \tag{A3}
\end{aligned}$$

where

$$\begin{aligned}
F_{\bar{\lambda}}(m, \Delta) &= \frac{i}{16\pi^2} \left\{ \left[\bar{\lambda} - \log\left(\frac{m^2}{\mu^2}\right) \right] \left(\frac{2\Delta^2}{3} - m^2 \right) \Delta \right. \\
&\quad \left. + \left(\frac{10\Delta^2}{9} - \frac{4m^2}{3} \right) \Delta + \frac{2(\Delta^2 - m^2)}{3} m R\left(\frac{\Delta}{m}\right) \right\}, \tag{A4}
\end{aligned}$$

with

$$R(x) \equiv \sqrt{x^2 - 1} \log\left(\frac{x - \sqrt{x^2 - 1 + i\epsilon}}{x + \sqrt{x^2 - 1 + i\epsilon}}\right), \tag{A5}$$

and μ is the renormalization scale. For the partially quenched calculations, we also need the integrals

$$I_{\bar{\lambda}}^{(\eta')} \equiv \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i\epsilon)^2} = \frac{\partial I_{\bar{\lambda}}(m)}{\partial m^2}, \tag{A6}$$

and

$$\begin{aligned}
H_{\bar{\lambda}}^{\eta'}(m, \Delta) &\equiv (g^{\rho\nu} - v^\rho v^\nu) \mu^{4-d} \frac{\partial}{\partial \Delta} \int \frac{d^d k}{(2\pi)^d} \\
&\quad \times \frac{k_\rho k_\nu}{(k^2 - m^2 + i\epsilon)^2 (v \cdot k - \Delta + i\epsilon)} \\
&= \frac{\partial}{\partial m^2} H_{\bar{\lambda}}(m, \Delta). \tag{A7}
\end{aligned}$$

In a cubic spatial box of side length L with periodic boundary condition, the three-momenta are quantized as

$$\vec{k} = \left(\frac{2\pi}{L}\right) \vec{i}, \tag{A8}$$

and one instead obtains the sums (after subtracting the ultraviolet divergences)

$$I(m) \equiv \frac{1}{L^3} \sum_{\vec{k}} \int \frac{dk_0}{2\pi} \frac{1}{k^2 - m^2 + i\epsilon} = I(m) + I_{\text{FV}}(m) \tag{A9}$$

and

$$\begin{aligned}
\mathcal{H}(m, \Delta) &\equiv (g^{\rho\nu} - v^\rho v^\nu) \left(\frac{1}{L^3}\right) \sum_{\vec{k}} \frac{\partial}{\partial \Delta} \int \frac{dk_0}{2\pi} \\
&\quad \times \frac{k_\rho k_\nu}{(k^2 - m^2 + i\epsilon)(v \cdot k - \Delta + i\epsilon)} \\
&= H(m, \Delta) + H_{\text{FV}}(m, \Delta) \tag{A10}
\end{aligned}$$

for the full QCD calculation, where

$$I(m) = I_{\bar{\lambda}}(m)|_{\bar{\lambda}=0} \tag{A11}$$

and

$$H(m) = H_{\bar{\lambda}}(m, \Delta)|_{\bar{\lambda}=0} \tag{A12}$$

are the infinite volume limits of I and \mathcal{H} , and ($n = |\vec{n}|$)

$$\begin{aligned}
I_{\text{FV}}(m) &= \frac{-i}{4\pi^2} m \sum_{\vec{n} \neq \vec{0}} \frac{1}{nL} K_1(nmL) \xrightarrow{mL \gg 1} \frac{-i}{4\pi^2} \sum_{\vec{n} \neq \vec{0}} \sqrt{\frac{m\pi}{2nL}} \left(\frac{1}{nL}\right) \\
&\quad \times e^{-nmL} \left\{ 1 + \frac{3}{8nmL} - \frac{15}{128(nmL)^2} \right. \\
&\quad \left. + \mathcal{O}\left(\left[\frac{1}{nmL}\right]^3\right) \right\} \tag{A13}
\end{aligned}$$

is the finite volume correction to $I(m)$. The function H_{FV} is the finite volume correction to $H(m, \Delta)$ and can be obtained via

$$\begin{aligned}
H_{\text{FV}}(m, \Delta) &= i[(m^2 - \Delta^2)K_{\text{FV}}(m, \Delta) - 2\Delta J_{\text{FV}}(m, \Delta) \\
&\quad + iI_{\text{FV}}(m)], \tag{A14}
\end{aligned}$$

where $J_{\text{FV}}(m, \Delta)$ and $K_{\text{FV}}(m, \Delta)$ are given by ($w_{\vec{k}} = \sqrt{|\vec{k}|^2 + m^2}$)

$$\begin{aligned}
J_{\text{FV}}(m, \Delta) &= \left(\frac{1}{2\pi}\right)^2 \sum_{\vec{n} \neq \vec{0}} \int_0^\infty d|\vec{k}| \left(\frac{|\vec{k}|}{w_{\vec{k}}(w_{\vec{k}} + \Delta)}\right) \\
&\quad \times \left(\frac{\sin(|\vec{k}||\vec{n}|L)}{|\vec{n}|L}\right) \tag{A15}
\end{aligned}$$

and

$$K_{\text{FV}}(m, \Delta) = \frac{\partial J_{\text{FV}}(m, \Delta)}{\partial \Delta}. \tag{A16}$$

In the asymptotic limit where $mL \gg 1$ it can be shown that (with $n \equiv |\vec{n}|$)

$$J_{\text{FV}}(m, \Delta) = \sum_{\vec{n} \neq \vec{0}} \left(\frac{1}{8\pi nL}\right) e^{-nmL} \mathcal{A}, \tag{A17}$$

where

$$\begin{aligned}
\mathcal{A} &= e^{(z^2)} [1 - \text{Erf}(z)] \\
&\quad + \left(\frac{1}{nmL}\right) \left[\frac{1}{\sqrt{\pi}} \left(\frac{z}{4} - \frac{z^3}{2}\right) + \frac{z^4}{2} e^{(z^2)} [1 - \text{Erf}(z)] \right] \\
&\quad - \left(\frac{1}{nmL}\right)^2 \left[\frac{1}{\sqrt{\pi}} \left(\frac{9z}{64} - \frac{5z^3}{32} + \frac{7z^5}{16} + \frac{z^7}{8}\right) \right. \\
&\quad \left. - \left(\frac{z^6}{2} + \frac{z^8}{8}\right) e^{(z^2)} [1 - \text{Erf}(z)] \right] + \mathcal{O}\left(\left[\frac{1}{nmL}\right]^3\right). \tag{A18}
\end{aligned}$$

with

$$z \equiv \left(\frac{\Delta}{m}\right) \sqrt{\frac{nmL}{2}}. \quad (\text{A19})$$

The quantity \mathcal{A} is the alteration of finite volume effects due to the presence of a nonzero Δ . See Ref. [12] for further discussion.

For the PQ χ PT calculations, one also needs

$$\begin{aligned} I^{\eta'}(m) &\equiv \frac{1}{L^3} \sum_{\vec{k}} \int \frac{dk_0}{2\pi} \frac{1}{(k^2 - m^2 + i\epsilon)^2} \\ &= \frac{\partial I(m)}{\partial m^2} + \frac{\partial I_{\text{FV}}(m)}{\partial m^2} \end{aligned} \quad (\text{A20})$$

and

$$\begin{aligned} \mathcal{H}^{\eta'}(m, \Delta) &\equiv \frac{\partial}{\partial \Delta} \left[(g^{\rho\nu} - v^\rho v^\nu) \left(\frac{1}{L^3}\right) \sum_{\vec{k}} \int \frac{dk_0}{2\pi} \right. \\ &\quad \left. \times \frac{k_\rho k_\nu}{(k^2 - m^2 + i\epsilon)^2 (v \cdot k - \Delta + i\epsilon)} \right] \\ &= \frac{\partial H(m, \Delta)}{\partial m^2} + \frac{\partial H_{\text{FV}}(m, \Delta)}{\partial m^2}. \end{aligned} \quad (\text{A21})$$

APPENDIX B: LOOP CONTRIBUTIONS AND BAG PARAMETERS

In this appendix, we present results for the various contributions in Eqs. (39) and (42), $\mathcal{W}_{B(s)}$, $\mathcal{W}_{\bar{B}(s)}$, $\mathcal{T}_{d(s)}^{(i)}$ and $\mathcal{Q}_{d(s)}^{(i)}$ ($i = 1, 2, 3, 4, 5$). These results are given in the sea and valence isospin limit of SU(6|3) partially quenched HM χ PT, with the quark masses given in Eq. (14). The QCD limit, where sea and valence quark masses are equal, is easily taken by setting $m_j = m_u$ and $m_r = m_s$. We also present the bag parameters defined in Eqs. (40) and (41).

1. Loop contributions in SU(6|3) partially quenched heavy meson chiral perturbation theory

To compactly express the partially quenched expressions, it is useful to define the following quantities:

$$A_{u,u} = \frac{2(\delta_{\text{VS}}^2 - M_\pi^2 + M_X^2)\delta_{\text{VS}}^2}{(M_\pi^2 - M_X^2)^2} + \frac{3}{2}, \quad (\text{B1})$$

$$B_{u,u} = 1 - A_{u,u}, \quad (\text{B2})$$

$$C_{u,u} = 3\delta_{\text{VS}}^2 - \frac{2\delta_{\text{VS}}^4}{M_\pi^2 - M_X^2}, \quad (\text{B3})$$

$$A_{s,s} = \frac{3(8\delta_{\text{VS}}^4 + (2\delta_{\text{VS}}^2 - M_\pi^2 + M_{s,s}^2)^2)}{(2\delta_{\text{VS}}^2 + 4\delta_{\text{VS}}^2 - M_\pi^2 + M_{s,s}^2)^2}, \quad (\text{B4})$$

$$B_{s,s} = 1 - A_{s,s}, \quad (\text{B5})$$

$$C_{s,s} = \frac{6\delta_{\text{VS}}^2(2\delta_{\text{VS}}^2 - M_\pi^2 + M_{s,s}^2)}{(2\delta_{\text{VS}}^2 + 4\delta_{\text{VS}}^2 - M_\pi^2 + M_{s,s}^2)^2}, \quad (\text{B6})$$

with $M_{a,b}^2 = B_0(m_a + m_b)$, $\delta_{\text{VS}}^2 = M_\pi^2 - M_{u,j}^2$, $\delta_{\text{VS}s}^2 = M_{s,s}^2 - M_{s,r}^2$, $M_\pi = M_{u,u}$, and $M_X^2 = \frac{1}{3}(M_\pi^2 + 2M_{s,s}^2 - 2\delta_{\text{VS}}^2 - 4\delta_{\text{VS}s}^2)$. For ease of use, we note that in the QCD limit (setting valence and sea masses to be identical),

$$\begin{aligned} A_{u,u}^{\text{QCD}} &= \frac{3}{2}, & B_{u,u}^{\text{QCD}} &= -\frac{1}{2}, & C_{u,u}^{\text{QCD}} &= 0, \\ A_{s,s}^{\text{QCD}} &= 3, & B_{s,s}^{\text{QCD}} &= -2, & C_{s,s}^{\text{QCD}} &= 0. \end{aligned}$$

The various loop contributions can then be written as

$$\begin{aligned} \mathcal{W}_{B^0} &= \mathcal{W}_{\bar{B}^0} \\ &= -\frac{ig^2}{3f^2} [B_{u,u} \mathcal{H}(M_X, \Delta_*) - 6\mathcal{H}(M_{u,j}, \Delta_* + \delta_{uj}) \\ &\quad - 3\mathcal{H}(M_{u,r}, \Delta_* + \delta_{sr} + \delta_{us}) \\ &\quad + A_{u,u} \mathcal{H}(M_{u,u}, \Delta_*) + C_{u,u} \mathcal{H}^{\eta'}(M_{u,u}, \Delta_*)], \end{aligned} \quad (\text{B7})$$

$$\begin{aligned} \mathcal{W}_{B_s^0} &= \mathcal{W}_{\bar{B}_s^0} \\ &= -\frac{ig^2}{3f^2} [B_{s,s} \mathcal{H}(M_X, \Delta_*) \\ &\quad - 6\mathcal{H}(M_{s,j}, \Delta_* + \delta_{uj} - \delta_{us}) \\ &\quad - 3\mathcal{H}(M_{s,r}, \Delta_* + \delta_{sr}) + A_{s,s} \mathcal{H}(M_{s,s}, \Delta_*) \\ &\quad + C_{s,s} \mathcal{H}^{\eta'}(M_{s,s}, \Delta_*)] \end{aligned} \quad (\text{B8})$$

for the wave-function renormalizations;

$$\begin{aligned} \mathcal{T}_d^{(1,2,3)} &= \frac{i}{3f^2} [2B_{u,u} I(M_X) - 6I(M_{u,j}) - 3I(M_{u,r}) \\ &\quad + (2A_{u,u} - 3)I(M_{u,u}) + 2C_{u,u} I^{\eta'}(M_{u,u})], \end{aligned} \quad (\text{B9})$$

$$\begin{aligned} \mathcal{T}_s^{(1,2,3)} &= \frac{i}{3f^2} [2B_{s,s} I(M_X) - 6I(M_{s,j}) - 3I(M_{s,r}) \\ &\quad + (2A_{s,s} - 3)I(M_{s,s}) + 2C_{s,s} I^{\eta'}(M_{s,s})], \end{aligned} \quad (\text{B10})$$

$$\mathcal{T}_d^{(4,5)} = -\frac{i}{f^2} [2I(M_{u,j}) + I(M_{u,r}) - I(M_{u,u})], \quad (\text{B11})$$

$$\mathcal{T}_s^{(4,5)} = -\frac{i}{f^2} [2I(M_{s,j}) + I(M_{s,r}) - I(M_{s,s})] \quad (\text{B12})$$

for tadpole integrals [Fig. 1(b)] and;

$$\begin{aligned} \mathcal{Q}_d^{(i)} &= \frac{ig^2}{3f^2} (B_{u,u} \mathcal{H}(M_X, \Delta) + (A_{u,u} - 3)\mathcal{H}(M_{u,u}, \Delta) \\ &\quad + C_{u,u} \mathcal{H}^{\eta'}[M_{u,u}, \Delta]), \end{aligned} \quad (\text{B13})$$

$$\begin{aligned} \mathcal{Q}_s^{(i)} = & \frac{ig^2}{3f^2} (B_{s,s} \mathcal{H}(M_X, \Delta) + (A_{s,s} - 3) \mathcal{H}(M_{s,s}, \Delta) \\ & + C_{s,s} \mathcal{H}^{\eta'}[M_{s,s}, \Delta]) \end{aligned} \quad (\text{B14})$$

for “sunset” integrals [Fig. 1(c)].

2. Bag parameters

For completeness, the bag parameters defined in Eqs. (40) and (41) are given by:

$$B_{B_d}^{(1)}(\mu) = \frac{3\beta_1}{8\kappa^2} \left(1 + \frac{X_{L,u}}{f^2} + \frac{g^2 X_{H,u}}{f^2} \right), \quad (\text{B15})$$

$$B_{B_s}^{(1)}(\mu) = \frac{3\beta_1}{8\kappa^2} \left(1 + \frac{X_{L,s}}{f^2} + \frac{g^2 X_{H,s}}{f^2} \right), \quad (\text{B16})$$

$$B_{B_d}^{(2/3)}(\mu) = \frac{\beta_{2/3}}{\kappa^2 \eta_{2/3} R^2} \left(1 + \frac{X_{L,u}}{f^2} + \frac{g^2 \beta_{2/3}^l X_{H,u}}{f^2 \beta_{2/3}} \right), \quad (\text{B17})$$

$$B_{B_s}^{(2/3)}(\mu) = \frac{\beta_{2/3}}{\kappa^2 \eta_{2/3} R^2} \left(1 + \frac{X_{L,s}}{f^2} + \frac{g^2 \beta_{2/3}^l X_{H,s}}{f^2 \beta_{2/3}} \right), \quad (\text{B18})$$

$$B_{B_d}^{(4/5)}(\mu) = \frac{\beta_{4/5} + \hat{\beta}_{4/5}}{\kappa^2 \eta_{4/5} R^2} \left(1 + \frac{X_{L,u}}{f^2} + \frac{g^2 (\beta_{4/5}^l + \hat{\beta}_{4/5}^l) X_{H,u}}{f^2 (\beta_{4/5} + \hat{\beta}_{4/5})} \right), \quad (\text{B19})$$

$$B_{B_s}^{(4/5)}(\mu) = \frac{\beta_{4/5} + \hat{\beta}_{4/5}}{\kappa^2 \eta_{4/5} R^2} \left(1 + \frac{X_{L,s}}{f^2} + \frac{g^2 (\beta_{4/5}^l + \hat{\beta}_{4/5}^l) X_{H,s}}{f^2 (\beta_{4/5} + \hat{\beta}_{4/5})} \right), \quad (\text{B20})$$

where κ is the LEC governing the heavy-light axial current [12], and for convenience we have defined

$$\begin{aligned} X_{H,u} = & \frac{i}{3} (B_{u,u} \mathcal{H}(M_X, \Delta) + (A_{u,u} - 3) \mathcal{H}(M_{u,u}, \Delta) \\ & + C_{u,u} \mathcal{H}^{\eta'}[M_{u,u}, \Delta]), \end{aligned} \quad (\text{B21})$$

$$\begin{aligned} X_{H,s} = & \frac{i}{3} (B_{s,s} \mathcal{H}(M_X, \Delta) + (A_{s,s} - 3) \mathcal{H}(M_{s,s}, \Delta) \\ & + C_{s,s} \mathcal{H}^{\eta'}[M_{s,s}, \Delta]), \end{aligned} \quad (\text{B22})$$

$$\begin{aligned} X_{L,u} = & \frac{i}{3} (B_{u,u} I(M_X) + (A_{u,u} - 3) I(M_{u,u}) \\ & + C_{u,u} I^{\eta'}[M_{u,u}]), \end{aligned} \quad (\text{B23})$$

$$X_{L,s} = \frac{i}{3} (B_{s,s} I(M_X) + (A_{s,s} - 3) I(M_{s,s}) + C_{s,s} I^{\eta'}[M_{s,s}]). \quad (\text{B24})$$

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- [1] A. Abulencia *et al.* (CDF), Phys. Rev. Lett. **97**, 242003 (2006).
[2] S. Bergmann, Y. Grossman, Z. Ligeti, Y. Nir, and A. A. Petrov, Phys. Lett. B **486**, 418 (2000).
[3] F. Gabbiani, E. Gabrielli, A. Masiero, and L. Silvestrini, Nucl. Phys. **B477**, 321 (1996).
[4] W.M. Yao *et al.* (Particle Data Group), J. Phys. G **33**, 1 (2006).
[5] M. Beneke, G. Buchalla, and I. Dunietz, Phys. Rev. D **54**, 4419 (1996).
[6] M. Beneke, G. Buchalla, C. Greub, A. Lenz, and U. Nierste, Phys. Lett. B **459**, 631 (1999).
[7] A. Lenz and U. Nierste, arXiv:hep-ph/0612167.
[8] T. Onogi, Proc. Sci., LAT2006 (2006) 017 [arXiv:hep-lat/0610115].
[9] G. Burdman and J.F. Donoghue, Phys. Lett. B **280**, 287 (1992).
[10] M. B. Wise, Phys. Rev. D **45**, R2188 (1992).
[11] T.-M. Yan *et al.*, Phys. Rev. D **46**, 1148 (1992).
[12] D. Arndt and C. J. D. Lin, Phys. Rev. D **70**, 014503 (2004).
[13] D. Becirevic, S. Fajfer, and J. Kamenik, arXiv:hep-ph/0612224.
[14] S. R. Sharpe and N. Shores, Phys. Rev. D **64**, 114510 (2001).
[15] M. J. Booth, Phys. Rev. D **51**, 2338 (1995).
[16] S. R. Sharpe and Y. Zhang, Phys. Rev. D **53**, 5125 (1996).
[17] C. G. Boyd and B. Grinstein, Nucl. Phys. **B442**, 205 (1995).
[18] M. J. Booth, arXiv:hep-ph/9412228.
[19] C. W. Bernard and M. F. L. Golterman, Phys. Rev. D **46**, 853 (1992).
[20] C. W. Bernard and M. F. L. Golterman, Phys. Rev. D **49**, 486 (1994).
[21] S. R. Sharpe and N. Shores, Phys. Rev. D **62**, 094503 (2000).
[22] B. Grinstein, E. Jenkins, A. V. Manohar, M. J. Savage, and M. B. Wise, Nucl. Phys. **B380**, 369 (1992).
[23] D. Becirevic and G. Villadoro, Phys. Rev. D **70**, 094036 (2004).
[24] N. Isgur and M. B. Wise, Phys. Lett. B **237**, 527 (1990).
[25] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989).
[26] B. Grinstein, Nucl. Phys. **B339**, 253 (1990).
[27] E. Eichten and B. Hill, Phys. Lett. B **234**, 511 (1990).
[28] H. Georgi, Phys. Lett. B **240**, 447 (1990).
[29] J. M. Flynn, O. F. Hernandez, and B. R. Hill, Phys. Rev. D **43**, 3709 (1991).
[30] M. J. Savage and M. B. Wise, Phys. Lett. B **248**, 177 (1990).

- [31] D. Becirevic, V. Gimenez, G. Martinelli, M. Papinutto, and J. Reyes, *J. High Energy Phys.* **04** (2002) 025.
- [32] V. Gimenez and J. Reyes, *Nucl. Phys. B, Proc. Suppl.* **94**, 350 (2001).
- [33] S. Hashimoto and N. Yamada (JLQCD), arXiv:hep-ph/0104080.
- [34] E. Dalgic *et al.*, arXiv:hep-lat/0610104.
- [35] S. Ahmed *et al.* (CLEO), *Phys. Rev. Lett.* **87**, 251801 (2001).
- [36] A. Anastassov *et al.* (CLEO), *Phys. Rev. D* **65**, 032003 (2002).
- [37] P.H. Ginsparg and K.G. Wilson, *Phys. Rev. D* **25**, 2649 (1982).
- [38] J.-W. Chen, D. O'Connell, and A. Walker-Loud, *Phys. Rev. D* **75**, 054501 (2007).